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Ergodic Control
Applied to Car Suspension Design

F. Campillo, F. Cérou, E. Pardoux

INRIA Sophia Antipolis
Routes des Lucioles
F–06565 Valbonne Cedex

Abstract *Stochastic ergodic control is used to compute feedback laws for semi-active vehicle suspensions. We present an overview of a study supported by* RENAULT[1]. *For the practical implementation we can numerically solve the Hamilton-Jacobi-Bellman equation, which gives an approximation of the optimal law. An other possibility is to seek for the best feedback in a given class of a parametrized feedbacks via a stochastic gradient algorithm.*

1 Introduction

In many applications we need to control the *long-time* behavior of a dynamical system in the sense that we want to minimize a cost function of the form

$$J(u) = \lim_{T\to\infty} \frac{1}{T}\, E \int_0^T f(u(X_t), X_t)\, dt \qquad (1)$$

where X_t is solution of the following stochastic system

$$dX_t = b(u(X_t), X_t)\, dt + \sigma\, dW_t \qquad (2)$$

and $u(\cdot)$ is a feedback control.

The problem addressed here is to compute a numerical approximation of the optimal feedback $\hat{u}(\cdot)$, i.e. a feedback which minimizes the cost function $J(u)$.

First, we present the semi-active vehicle suspension problem. Then we propose two approaches : the first one, called here "optimal", consists in solving numerically the Hamilton-Jacobi-Bellman equation (cf. Campillo [4]), the second, called here "sub-optimal", consists in looking for the best feedback control in a given class of parametrized controls (cf. Campillo [7]).

The first approach, tractable only in small state space dimension, results in a parallel algorithm (and so, could be treated on a parallel computer), the second one is essentially sequential.

[1] under contract H6.10.601/INRIA/17

2 Semi–active vehicle suspension

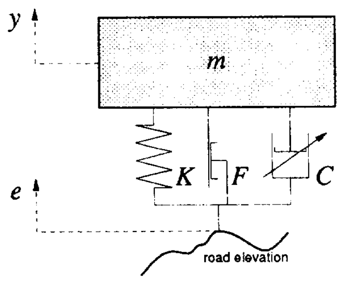
In this section we present a damping control method for a nonlinear suspension of road vehicle. The aim is to improve the ride comfort (or a trade-off between ride comfort and road holding).

Active suspension system is an alternative to classical suspension systems (passive systems). It consists in force elements in addition to a spring and a damper assembly. Force elements continuously vary the force according to some control law. In general, an active system is expensive, complicated, and requires an external power source (cf. Gooddall [9]). To overcome theses difficulties, the semi-active suspension system was developed. It requires no hydraulic power supply, and its hardware implementation is simpler and cheaper than a fully active system. A semi-active suspension system acts only on damping or spring laws, so it can only dissipate or store energy.

Here we consider a system with control on the damping law. The forces in the damper are generated by modulating its orifice for fluid flow (cf. Alanoly [1]).

2.1 One degree of freedom model (1–DOF)

We use the simplest model which consists in a



y	*absolute vertical displacement of mass m*
x	*relative vertical displacement* ($x = y - e$)
e	*stochastic input (road elevation)*
m	*mass*
C	*viscous damping constant (variable)*
K	*suspension stiffness*
F	*dry friction constant*

Figure 1: 1–DOF model

one degree–of–freedom model comprising a spring, a shock absorber, a mass, and taking into account the dry friction (cf. Figure 1).

The equation of motion for a one degree–of–freedom model is

$$m\,\ddot{y} = -C\,\dot{x} - K\,x - F\,\mathrm{sign}(\dot{x})\,. \qquad (3)$$

The restoring force $K\,x + F\,\mathrm{sign}(\dot{x})$, has a linear part $K\,x$, and a nonlinear part $F\,\mathrm{sign}(\dot{x})$ which describes the dry friction force. The damping force is $C\,\dot{x}$ where $C > 0$ is the instantaneous damping coefficient (the control is acting on this term).

The problem is to compute a feedback law $C = C(x, \dot{x})$ such that the solution of the system (3) minimizes a criterion — related to the vibration comfort

$$J = \lim_{T\to\infty} \frac{1}{T}\, E \int_0^T |\ddot{y}|^2\, dt$$

$\dot{e}$ is supposed to be a white Gaussian noise process, $\dot{e} = -\sigma\, dW/dt$ where W is a standard Wiener process.

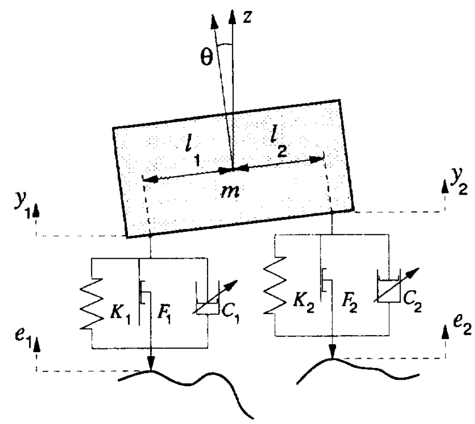
Using the state vector is $X_t = [x, \dot{x}]^*$, we can see that this problem is of the form (1) and (2).

2.2 Two degree of freedom models (2–DOF)

2–DOF in parallel

The equations of motion for the two degree–of–freedom model described in figure 2 are

$$\begin{aligned} m\ddot{z} \;&=\; -C_1\,(\dot{z} - l_1\dot{\theta} - \dot{e}_1) - K_1\,(z - l_1\theta - e_1) \\ &\quad -F_1\,\mathrm{sign}(\dot{z} - l_1\dot{\theta} - \dot{e}_1) \\ &\quad -C_2\,(\dot{z} + l_2\dot{\theta} - \dot{e}_2) - K_2\,(z + l_2\theta - e2) \\ &\quad -F_2\,\mathrm{sign}(\dot{z} + l_2\dot{\theta} - \dot{e}_2) \end{aligned}$$



z	*absolute vertical displacement of mass m*
θ	*angular displacement of the mass*
m	*mass*
I	*inertial tensor w.r.t. the gravity center*
suspension i :	
K_i	*stiffness*
F_i	*dry friction constant*
C_i	*viscous damping constant (variable)*
l_i	*distance to the center of gravity*
e_i	*stochastic input*

Figure 2: 2–DOF model (in parallel)

and

$$\begin{aligned} I\ddot{\theta} \;&=\; -l_1[-C_1\,(\dot{z} - l_1\dot{\theta} - \dot{e}_1) - K_1\,(z - l_1\theta - e_1) \\ &\qquad d - F_1\,\mathrm{sign}(\dot{z} - l_1\dot{\theta} - \dot{e}_1)] \\ &\quad +l_2[-C_2\,(\dot{z} + l_2\dot{\theta} - \dot{e}_2) - K_2\,(z + l_2\theta - e2) \\ &\qquad -F_2\,\mathrm{sign}(\dot{z} + l_2\dot{\theta} - \dot{e}_2)] \end{aligned}$$

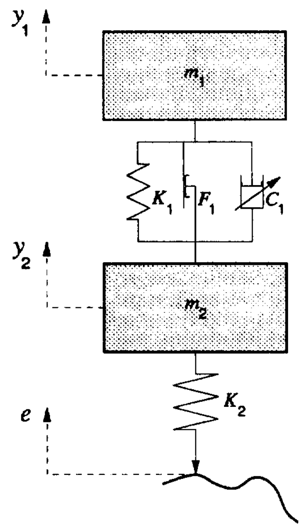
For the ride comfort we choose the following cost function which takes into account. not only the absolute vertical acceleration, but also the angular acceleration :

$$J = \lim_{T\to\infty} \frac{1}{T}\, E \int_0^T [\rho_1\,|\ddot{z}|^2 + \rho_2\,|\ddot{\theta}|^2]\, dt\,.$$

where ρ_1 and ρ_2 are weighting constants.

Let x_i be the relative displacement of the suspension i. It is given by $x_1 = z - l_1\theta - e_1$ and $x_2 = z + l_2\theta - e_2$. We suppose that $\dot{e}_1$ and $\dot{e}_1$ are independent standard Gaussian white noises $\dot{e}_i = -\sigma\, dW_i/dt$.

The state variable is $X = [x_1, \dot{x}_1, x_2, \dot{x}_2]^*$. Here again the problem is of the form (1) and (2).



m_1	*quarter of the body mass*
m_2	*mass of the wheel with a semi-axle*
K_1	*suspension stiffness*
F_1	*dry friction coefficient*
C_1	*viscous damping term (variable)*
K_2	*tire stiffness*

Figure 3: 2–DOF model (in series)

2-DOF in series

In the two previous models, only a comfort criterion was taken into account. With this last example the cost function is a trade–off between ride comfort and road holding. The model presented in Figure 3 is a quarter car model.

The equation of motion gives

$$\begin{aligned} m_1\,\ddot{y}_1 &= -C_1\,(\dot{y}_1-\dot{y}_2)-K_1\,(y_1-y_2) \\ &\quad -F_1\,\mathrm{sign}(\dot{y}_1-\dot{y}_2)\;, \\ m_2\,\ddot{y}_2 &= +C_1\,(\dot{y}_1-\dot{y}_2)+K_1\,(y_1-y_2) \\ &\quad +F_1\,\mathrm{sign}(\dot{y}_1-\dot{y}_2)-K_2\,(y_2-e)\;. \end{aligned}$$

Let $x_1 = y_1 - y_2$ and $x_2 = y_2 - e$, we get

$$\begin{aligned} \ddot{x}_1 &= -(\tfrac{1}{m_1}+\tfrac{1}{m_2})\,[C_1\,\dot{x}_1+K_1\,x_1+F_1\,\mathrm{sign}(\dot{x}_1)] \\ &\quad +\frac{K_2}{m_2}\,x_2\;, \\ \ddot{x}_2 &= +\frac{1}{m_2}\,[C_1\,\dot{x}_1+K_1\,x_1+F_1\,\mathrm{sign}(\dot{x}_1)] \\ &\quad -\frac{K_2}{m_2}\,x_2-\ddot{e}\;. \end{aligned}$$

The cost function takes into account the following terms

- for the comfort : the passenger compartment acceleration $\ddot{y}_1$,
- for the road holding : the tire deflection $y_2 - e$,

and also, as a technical constraint,

- the suspension rattle space $y_1 - y_2$.

We can choose

$$\begin{aligned} J \;=\; \lim_{T\to\infty}\frac{1}{T}\,E\int_0^T [|\ddot{y}_1|^2 &+ \rho_1\,\psi_1(y_2-e) \\ &+\rho_2\,\psi_2(y_1-y_2)]\,dt \end{aligned}$$

for example : $\psi_1(y) = y^2$ and $\psi_2(y) = 0$ for $|y| \le R$, and $\psi_2(y) = |y - R|$ for $|y| > R$, where R is bounded by the maximum suspension rattle space. ρ_1 and ρ_2 are weighting constants which permit to give more or less importance to the comfort or to the road holding.

We suppose that $\ddot{e}$ is a standard Gaussian white noise $\ddot{e} = -\sigma\, dW/dt$. The state variable is $X = [x_1, \dot{x}_1, x_2, \dot{x}_2]^*$. Again, the problem is of the form (1) and (2).

3 Ergodic stochastic control problems

We consider the system (2) where X takes values in $\mathbb{R}^n$, W is a d–dimensional standard Wiener process, $b : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^n$, and $\sigma \in \mathbb{R}^{n\times d}$. $u(\cdot)$ is a feedback control $u : \mathbb{R}^n \to U \subset \mathbb{R}^p$ which belongs to a given class of admissible controls, say $\mathcal{U}$.

To each admissible control $u \in \mathcal{U}$, we associate the ergodic type cost (1) where $f : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}$ is a given instantaneous cost function.

The problem is to find an optimal feedback control $\hat{u} \in \mathcal{U}$, i.e.

$$\hat{u} \in \mathrm{Arg}\min_{u\in\mathcal{U}} J(u)\;.$$

The solution of this problem is given by the Hamilton–Jacobi–Bellman equation : we want to find $v : \mathbb{R}^n \to \mathbb{R}$ and $\rho \in \mathbb{R}$ such that

$$\min_{u\in U}\,[\mathcal{L}^u\,v(x)+f(u,x)] = \rho\;, \quad \forall x\in\mathbb{R}^n, \qquad (4)$$

(v is defined up to an additive constant) where $\mathcal{L}^u$ is the infinitesimal generator associated with (2)

$$\mathcal{L}^u\,\phi = b_i(u,\cdot)\,\frac{\partial\phi}{\partial x_i}+[\sigma\sigma^*]_{ij}\,\frac{\partial^2\phi}{\partial x_i\,\partial x_j}\;. \qquad (5)$$

Given a solution (v, ρ) of (4)

$$\hat{u}(x) \in \operatorname{Arg}\min_{u \in U} [\mathcal{L}^u v(x) + f(u,x)] \ , \quad \forall x \in \mathbb{R}^n$$

is an optimal feedback control, and

$$\rho = \min_{u \in \mathcal{U}} J(u)$$

is the optimal cost. For a general presentation of this theory we refer to Bensoussan [2] and Borkar [3].

4 Numerical approximation

4.1 Optimal approach

Let $h_i > 0$ (resp. e_i) denote the finite difference interval (resp. the unit vector) in the ith coordinate direction and $h = (h_1, \ldots, h_n)$.

We define the h-grid on $\mathbb{R}^n$, by $\mathbb{R}^n_h = \{(k_i h_i + h_i/2 \ , \ i = 1, \ldots, n) ; (k_1, \ldots, k_n) \in \mathbf{Z}^n\}$.

The infinitesimal generator (5) is approximated using the following finite difference scheme

$$\frac{\partial \phi(x)}{\partial x_i} \simeq \begin{cases} \dfrac{\phi(x + e_i h_i) - \phi(x)}{h_i} & \text{if } b_i(u,x) > 0 \\ \dfrac{\phi(x) - \phi(x - e_i h_i)}{h_i} & \text{if } b_i(u,x) < 0 \end{cases}$$

$$\frac{\partial^2 \phi(x)}{\partial x_i^2} \simeq \frac{\phi(x + e_i h_i) - 2\phi(x) + \phi(x - e_i h_i)}{h_i^2}$$

(for notational convenience we suppose here that $[\sigma\sigma^*]_{ij} = 0$ for $i \neq j$).

Thus $\mathcal{L}^u$ is approximated by an infinite dimensional matrix $\mathcal{L}^u_h$ given as follows

$$\mathcal{L}^u \phi(x) \simeq \mathcal{L}^u_h \phi(x) = \sum_{y \in \mathbb{R}^n_h} \mathcal{L}^u_h(x,y)\, \phi(y)$$

for all $x \in \mathbb{R}^n_h$.

$\mathcal{L}^u_h$ can be interpreted as the infinitesimal generator of a controlled Markov process X^h_t in continuous time and discrete bu infinite state space.

X^h_t has a discrete but infinite state space. For actual computations it is necessary to work on a finite state space. So we consider a bounded domain D of $\mathbb{R}^n$. We define a new state space $\mathbb{R}^n_{h,D} = \mathbb{R}^n_h \cap D = \{x^1, x^2, \ldots, x^N\}$. $N = \operatorname{Card}[\mathbb{R}^n_{h,D}]$. At this level we must specify boundary conditions. We usually take reflecting conditions.

Thus we get an approximation $\mathcal{L}^u_{h,D}$ to $\mathcal{L}^u_h$

$$\mathcal{L}^u_{h,D}\, \phi(x) = \sum_{y \in \mathbb{R}^n_{h,D}} \mathcal{L}^u_{h,D}(x,y)\, \phi(y) \ ,$$

$\mathcal{L}^u_{h,D}$ is a $N \times N$-matrix. $\mathcal{L}^u_{h,D}$ can be interpreted as the infinitesimal generator of a controlled Markov process $X^{h,D}_t$ in continuous time and finite state space.

Therefore the discretized problem can be viewed as a control problem for a Markov process $X^{h,D}_t$ in continuous time, finite state space, and infinitesimal generator $\mathcal{L}^u_{h,D}$. The cost function is

$$J_{h,D}(u) = \lim_{T \to \infty} E \frac{1}{T} \int_0^T f(u(X^{h,D}_t), X^{h,D}_t)\, dt \ .$$

Associated with this control problem we have the following dynamic programming equation

$$\min_{u \in U} [\sum_{y \in \mathbb{R}^n_{h,D}} \mathcal{L}^u_{h,D}(x,y)\, v(y) + f(u,x)] = \rho \qquad (6)$$

for all $x \in \mathbb{R}^n_{h,D}$, where ρ is a strictly positive constant and $v : \mathbb{R}^n_{h,D} \to \mathbb{R}$ is defined up to an additive constant.

Equation (6) can be viewed as an approximation to the HJB equation (4). For more details we can consult Campillo [5].

Numerical implementation

In order to solve numerically equation (6) we can use, as presented in [4], the *policy iteration* algorithm. But a better parallel implementation on a massively parallel architecture can be achieved using the *value iteration algorithm* :

$$v^{(k+1)}(x) = v^{(k)}(x) + \delta \times \min_{u \in U} H(u, v^{(k)}) \qquad (7)$$

where

$$H(u, v^{(k)}) = \sum_{y \in \mathbb{R}^n_{h,D}} \mathcal{L}^u_{h,D}(x,y)\, v^{(k)}(y) + f(u,x) \ .$$

Let

$$\lambda^{(k)}_{\min} = \min_{x \in \mathbb{R}^n_{h,D}} \frac{v^{(k+1)}(x) - v^{(k)}(x)}{\delta} \ ,$$

$$\lambda^{(k)}_{\max} = \max_{x \in \mathbb{R}^n_{h,D}} \frac{v^{(k+1)}(x) - v^{(k)}(x)}{\delta} \ .$$

We say that the algorithm has converged when $|\lambda^{(k)}_{\max} - \lambda^{(k)}_{\min}| \leq \varepsilon$. An approximation of the optimal cost is given by $(\lambda^{(k)}_{\max} + \lambda^{(k)}_{\min})/2$. Details are exposed in [6].

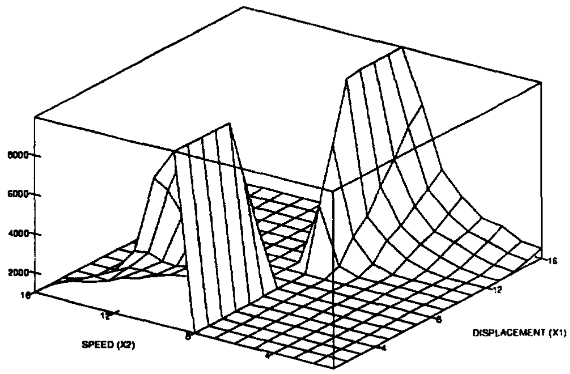


Figure 4: $(x_1, x_2) \rightarrow \hat{u}(x_1, x_2, x_3, x_4)$ for (x_3, x_4) fixed $(x_3 = -0.1 + 3\,h_3,\ x_4 = -1 + 4\,h_4)$

Application

We apply this discretization method to the 2–DOF model in parallel. In figure 4 we plot $(x_1, x_2) \rightarrow \hat{u}(x_1, x_2, x_3, x_4)$ for (x_3, x_4) fixed.

For the discretization we take

$$D = [-0.1, 0.1] \times [-1, 1] \times [-0.1, 0.1] \times [-1, 1] ,$$

$U = [1000, 10000]^2$, and a $16 \times 16 \times 8 \times 8$ points grid. The parameters are : $m = 550$, $I = 265$, $K_i = 15000$, $F_i = 40$, $l_i = 0.7$, $\sigma_i = 1.8$, $\pi_i = 1$ $(i = 1, 2)$, $\delta = 0.00001$.

We have performed numerical tests on a Connection Machine CM2 (16K processors) from TMC, using the C^* programming language. Each point of the grid is associated with a (virtual) processor.

The minimization step in (7) is 2–dimensional. On each processor it is performed using Powell's method of successive line minimizations (with the 1–dimensional golden section search algorithm).

4.2 Sub–optimal approach

We are given a parametrized feedback law $u : \Theta \times \mathbb{R}^n \rightarrow U$. The space of parameters Θ is finite dimensional : $\Theta \subset \mathbb{R}^d$. The problem is to find $\hat{\theta}$ such that

$$J(u(\hat{\theta}, \cdot)) = \min_{\theta \in \Theta} J(u(\theta, \cdot)) .$$

From now on $J(\theta)$ will denote $J(u(\hat{\theta}, \cdot))$.

In order to get an approximation of $\hat{\theta}$ we can use a gradient algorithm

$$\theta^{(k+1)} = \theta^{(k+1)} - \delta_k\, G(\theta^{(k)}) .$$

where $G(\theta) = \nabla J(\theta)$ is the gradient of the cost function w.r.t. θ. The problem is to compute $G(\theta)$. Let

$$\begin{aligned} \tilde{f}(\theta, x) &= f(u(\theta, x), x) , \\ \tilde{b}(\theta, x) &= b(u(\theta, x), x) , \end{aligned}$$

and X_t^θ be the solution of (2).

Regular case

Suppose that in (1) and (2) the coefficients b, f, and u are regular. Then, from (1)

$$G(\theta) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T g(\theta, X_t^\theta, Y_t^\theta)\, dt \qquad (8)$$

where

$$g(\theta, x, y) = \tilde{f}'_\theta(\theta, x) + \tilde{f}'_x(\theta, x)\, y$$

and Y_t^θ is the gradient of X_t^θ w.r.t. θ :

$$\dot{Y}_t^\theta = g(\theta, X_t^\theta, Y_t^\theta) . \qquad (9)$$

The term $G(\theta)$ is not explicit so we must use an approximation of it. The final algorithm is a stochastic gradient algorithm :

$$\theta^{(k+1)} = \theta^{(k+1)} - \delta_k\, \bar{G}_k(\theta^{(k)}) \qquad (10)$$

where

$$\bar{G}_k(\theta) = \frac{1}{\Delta} \int_{T_k}^{T_{k+1}} g(\theta, \bar{X}_t^\theta, \bar{Y}_t^\theta)\, dt$$

and $\Delta > 0$ is given, $T_k = k\,\Delta$. $\bar{X}_t^\theta$, and $\bar{Y}_t^\theta$ are numerical approximations of X_t and Y_t solutions of (2) and (9) for a given θ.

The estimator of $\hat{\theta}$ at step k is given by

$$\bar{\theta}^{(k)} = \frac{1}{k} \sum_{l=1}^{k} \theta^{(l)} .$$

For the convergence of the algorithm we can consult Polyak [10].

Non regular case

In the examples presented in section 2 we can suppose that the feedback u is regular, but the functions f and b are not regular in x. Indeed, consider the simplest case of section 2.1. We have

$$\begin{aligned} b(u, x) &= \begin{pmatrix} x_2 \\ -\alpha_1\, x_1 - u\, x_2 - \alpha_2\, \mathrm{sign}(x_2) \end{pmatrix} , \\ f(u, x) &= (\alpha_1\, x_1 + u\, x_2 + \alpha_2\, \mathrm{sign}(x_2))^2 . \end{aligned}$$

Therefore equations (8) and (9) are not valid in this case. Let L_t^θ denote the local time of the process $X_t^{2,\theta}$ at point 0. The gradient of the cost function is

$$G(\theta) = \lim_{T\to\infty} \frac{1}{T}\left[\int_0^T g(\theta, X_t^\theta, Y_t^\theta)\,dt + \int_0^T 4\,\alpha_1\,\alpha_2\,X_t^{1,\theta}\,Y_t^{2,\theta}\,dL_t^\theta\right],$$

and Y_t^θ satisfies

$$dY_t^\theta = g(\theta, X_t^\theta, Y_t^\theta)\,dt + B\,Y_t^\theta\,dL_t^\theta\,. \qquad (11)$$

The stochastic gradient algorithm has the same form (10) but with

$$\bar{G}_k(\theta) = \frac{1}{\Delta}\left[\int_{T_k}^{T_{k+1}} g(\theta, \bar{X}_t^\theta, \bar{Y}_t^\theta)\,dt + \int_{T_k}^{T_{k+1}} 4\,\alpha_1\,\alpha_2\,\bar{X}_t^{1,\theta}\,\bar{Y}_t^{2,\theta}\,d\bar{L}_t^\theta\right]$$

instead, where $\bar{X}_t^\theta$ and $\bar{Y}_t^\theta$ are numerical approximations of (2) and (11) respectively, and $\bar{L}_t^\theta$ is an approximation of the process L_t^θ (cf. Campillo [8] for more details).

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