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Numerical methods in Ergodic Optimal Stochastic Control Application to Semi–Active Vehicle Suspensions*

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Abstract *In [4] we presented a numerical algorithm for the computation of the optimal feedback law in an ergodic stochastic optimal control problem. This method, based on the discretization of the associated Hamilton–Jacobi–Bellman equation, can be used only in low dimension (2,4, or 6 in a parallel computer). For higher dimensional problems, we propose here to use a stochastic gradient algorithm in order to find the optimal feedback in a given subclass of parametrized controls. As is [4], we apply these techniques to the control of semi-active suspensions for road vehicle.*

In this paper we consider numerical procedures for stochastic control problems. Given a real case study (here we consider semi-active control of vehicle suspensions) we can use different modelization. For low dimensional modelization, we can pretend to use optimal methods : we discretize the Hamilton–Jacobi–Bellman equation (this approach is proposed in [4]). In higher dimension this approach is cumbersome or even impossible to implement, in this case we can look for the optimal feedback in a given subclass of parametrized controls using a stochastic gradient algorithm. The aim of this paper is to compute the stochastic gradient in the simplest two dimensional model relevant in our application.

In section 1 we introduce the stochastic control problem of ergodic type in $\mathbb{R}^2$ motivated by the application to the control of suspension systems [3].

In section 2, we derive the stochastic gradient for the above problem.

1 A stochastic control problem

1.1 The problem

Let $\{X_t(\theta) : t \geq 0\}$ be the solution of the following stochastic system in $\mathbb{R}^2$

$$\begin{aligned} dX_t^1(\theta) &= X_t^2(\theta)\,dt\,, & (1)\\ dX_t^2(\theta) &= -\Big[u(\theta, X_t(\theta))\,X_t^2(\theta) + \beta X_t^1(\theta) & (2)\\ &\qquad +\gamma\,\mathrm{sign}(X_t^2(\theta))\Big]\,dt + \sigma\,dW_t \end{aligned}$$

where

- β, γ are strictly positives constants.
- W is a standard brownian motion.
- $u(\theta, x)$ is a feedback control parametrized by $\theta \in \Theta$ (Θ is an open set of $\mathbb{R}^d$). $u(\theta, x)$ takes values in $U = [u_1, u_2]$, $0 < u_1 < u_2 < \infty$.

For each $\theta \in \Theta$, we consider the following long time average cost functional

$$J(\theta) \triangleq \lim_{T\to\infty} \frac{1}{T} E \int_0^T f(\theta, X_t(\theta))\,dt \qquad (3)$$

with

$$f(\theta, x) \triangleq \left[u(\theta, x)\,x^2 + \beta x^1 + \gamma\,\mathrm{sign}(x^2)\right]^2 .$$

A class of feedback controls Let $\mathcal{U}$ denote the set of feedback functions $u(\theta, x)$ such that

- $u : \Theta \times \mathbb{R}^2 \to U = [u_1, u_2]$.
- $\forall x \in \mathbb{R}^2$, $\theta \to u(\theta, x)$ is C^1.
- $\forall \theta \in \Theta$, $x \to u(\theta, x)\,x^2$ is C^1 on $\mathbb{R} \times \mathbb{R} \setminus \{0\}$.

This last condition allows u to be discontinuous at $x^2 = 0$, it is the case for the optimal control computed which we have computed in a previous work [4].

In (2) we group the nonlinear terms : we define

$$\alpha(\theta, x) \triangleq -u(\theta, x)\,x^2 - \gamma\,\mathrm{sign}(x^2)\,,$$

then the system (1), (2) reads

$$\begin{aligned} dX_t^1 &= X_t^2\,dt\,, & (4)\\ dX_t^2 &= -\beta X_t^1\,dt + \alpha(\theta, X_t)\,dt + \sigma\,dW_t\,. & (5) \end{aligned}$$

Note that $\alpha(\theta, x)$ is C^1 on $\mathbb{R} \times \mathbb{R} \setminus \{0\}$, and it is discontinuous in $x^2 = 0$ with a jump of amplitude -2γ.

1.2 An example: a semi–active suspension system

In this section we present a damping control method for a nonlinear suspension of road vehicle (comprising a spring, a shock absorber, a mass, and taking into account the dry friction, cf. figure 1). The aim is to improve the ride comfort.

Among alternatives to classical suspension systems (passive systems) we distinguish between active and semi active techniques. An active suspension system consists in force elements in addition to a spring and a damper assembly. Force elements continuously vary the force according to some control law. In general, an active system is expensive, complicated, and requires an external power source [5]. In contrast, a semi-active system requires no hydraulic power supply, and its hardware implementation is simpler and cheaper than a fully active system. A semi active suspension system acts only on damping or spring laws, so it can only dissipate or store energy.

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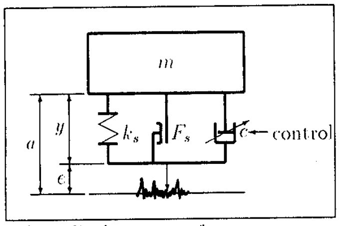

a absolute displacement of mass m
y absolute displacement ($y = a - \epsilon$)
ϵ stochastic input (surface road acceleration)
m spring mass
c shock-absorber damping constant (controlled)
k_s spring constant
F_s dry friction constant

Figure 1: One degree-of-freedom model.

Here we consider a system with control on the damping law, the forces in the damper are generated by modulating its orifice for fluid flow [1,8]. We use the simplest model which consists in a one degree-of-freedom model.

The equation of motion for a one degree-of-freedom model is (cf. figure 1 for the exact definition of the terms)

$$m\,\ddot{y} + c\,\dot{y} + k_s\,y + F_s\,\mathrm{sign}(\dot{y}) = -m\,\ddot{\epsilon}\ . \tag{6}$$

$\ddot{\epsilon}$ denotes the input acceleration. The restoring force $k_s\,y + F_s\,\mathrm{sign}(\dot{y})$, has a linear part $k_s\,y$, and a nonlinear part $F_s\,\mathrm{sign}(\dot{y})$ which describes the dry friction force. The damping force is $c\,\dot{y}$ where $c > 0$ is the instantaneous damping coefficient (the control is acting on this term).

The problem is to compute a feedback law $c = c(y,\dot{y})$ such that the solution of the system (6) minimizes a criterion — related to the vibration comfort

$$J(u) \triangleq \lim_{T\to\infty} \frac{1}{T} E \int_0^T |\ddot{a}|^2\,dt = \lim_{T\to\infty} \frac{1}{T} E \int_0^T |\ddot{y} + \ddot{\epsilon}|^2\,dt\ .$$

$\ddot{\epsilon}$ is supposed to be a white Gaussian noise process, $\ddot{\epsilon} = -\sigma\,dW/dt$ where W is a standard Wiener process.

Using

$$u = \frac{c}{m},\quad \beta = \frac{k_s}{m},\quad \gamma = \frac{F_s}{m},\quad \text{and } X \triangleq \begin{pmatrix} y \\ \dot{y} \end{pmatrix}\ .$$

equation (6) can be rewritten as (4), (5).

1.3 The "optimal" approach

This approach — presented in [4] — consists in discretizing the Hamilton-Jacobi-Bellman (HJB) equation associated with the ergodic stochastic control problem for the diffusion (1), (2) with the costs function (3) (but now the feedback function u is not parametrized)

The HJB equation is of the form

$$\min_{u\in[u_1,u_2]} (\mathcal{L}^u\,v(x) + f(u,x)) = \rho\ . \quad \forall x \in \mathbb{R}^2$$

where $v : \mathbb{R}^2 \to \mathbb{R}$ is defined up to an additive constant, ρ is a constant and $\mathcal{L}^u$ is the infinitesimal generator associated with the diffusion process (1), (2) (with $u(\theta,x)$ replaced by $u \in \mathcal{U}$.

2 Stochastic approximation algorithms

The problem is to find θ^* which minimizes the cost function (3). In [4] we have already proved that the cost function is of the form

$$J(\theta) = \int_{\mathbb{R}^2} f(\theta,x)\,\mu_\theta^X(dx) \tag{7}$$

where μ_θ^X is the invariant measure of the process $X_t(\theta)$. As noted by Ladelli [6], the gradient of $J(\theta)$ is *not* equal to

$$\int_{\mathbb{R}^2} \nabla_\theta f(\theta,x)\,\mu_\theta^X(dx)\ .$$

It is possible to use a Kiefer–Wolfowitz-type algorithm in order to minimize $J(\theta)$, thus avoiding the computation of the gradient of J. However, it seems that stochastic gradient algorithms often converge faster than Kiefer–Wolfowitz algorithms. Motivated by this remark, we shall compute here the gradient of $J(\theta)$, which involves differentiating $X_t(\theta)$.

2.1 The gradient process

Lemma 2.1 *$X_t(\theta)$ is differentiable w.r.t. parameter θ_i ($i = 1,\ldots,d$). let*

$$Y_t(i,\theta) \triangleq \frac{\partial}{\partial\theta_i} X_t(\theta)\ .$$

$Y(i,\theta)$ is solution of the following system

$$\begin{aligned} dY_t(i,\theta) \;=\; & \begin{pmatrix} Y_t^2(i,\theta) \\ -\beta Y_t^1(i,\theta) \end{pmatrix} dt \\ & + \begin{pmatrix} 0 \\ \alpha_{\theta_i}(\theta,X_t(\theta)) + \alpha_x(\theta,X_t(\theta))\,Y_t(i,\theta) \end{pmatrix} dt \\ & + \begin{pmatrix} 0 \\ -2\gamma \end{pmatrix} Y_t^2(i,\theta)\,dL_t\ , \end{aligned} \tag{8}$$

with $Y_0 \equiv 0$, and L is the local time of X^2 in 0. We note

$$Y_t(\theta) = [Y_t(1,\theta)|\cdots|Y_t(d,\theta)]\ .$$

Proof In the case of smooth coefficients, the result is essentially well known (see e.g. Stroock [7] where derivatives are computed with respect to the initial condition). The discontinuity of the sign function gives rise here to a local time term. ∎

Equation (8) reads

$$\begin{aligned} dY_t(i,\theta) \;=\; & A(\theta,X_t(\theta))\,Y_t(i,\theta)\,dt \\ & + B(\theta,X_t(\theta))\,Y_t(i,\theta)\,dL_t + C(i,\theta,X_t(\theta))\,dt\ . \end{aligned}$$

with

$$\begin{aligned} A(\theta,X_t(\theta)) &\triangleq \begin{pmatrix} 0 & 1 \\ -\beta + \alpha_{x^1}(\theta,X_t(\theta)) & \alpha_{x^2}(\theta,X_t(\theta)) \end{pmatrix} \\ B(\theta,X_t(\theta)) &\triangleq \begin{pmatrix} 0 & 0 \\ 0 & -2\gamma \end{pmatrix} \\ C(i,\theta,X_t(\theta)) &\triangleq \begin{pmatrix} 0 \\ \alpha_{\theta_i}(\theta,X_t(\theta)) \end{pmatrix} \end{aligned}$$

Let $\Phi_t(\theta)$ be the fundamental solution of this last equation

$$\begin{aligned} d\Phi_t(\theta) \;=\; & A(\theta,X_t(\theta))\,\Phi_t(\theta)\,dt \\ & + B(\theta,X_t(\theta))\,\Phi_t(\theta)\,dL_t\ , \quad \Phi_0(\theta) = I\ . \end{aligned} \tag{9}$$

We have

$$Y_t(i,\theta) = \Phi_t(\theta)\,Y_0(i,\theta) + \Phi_t(\theta)\int_0^t \Phi_s(\theta)^{-1}\,C(i,\theta,X_s(\theta))\,ds\ .$$

2.2 Asymptotic properties of $(X_t(\theta), Y_t(\theta))$

In [4] we showed that $X_t(\theta)$ admits an invariant measure μ_θ^X. We extend the process $X_t(\theta)$ for all $t \in \mathbb{R}$, such that it is stationary and μ_θ^X is the law of $X_t(\theta)$ for all $t \in \mathbb{R}$. We can then solve equation (9) and define $\Phi_t(\theta)$ for all $t \in \mathbb{R}$. It is easily seen that the Markov process $(X_t(\theta), Y_t(\theta))$ possesses a unique invariant measure μ_θ iff the following integral converges a.s.

$$\int_{-\infty}^{0} \Phi_t(\theta)^{-1} C(t, \theta, X_t(\theta))\, dt \ .$$

A sufficient condition for that fact is that there exists $\lambda < 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log \left\|\Phi_{-t}(\theta)^{-1}\right\| \leq \lambda \text{ a.s.}$$

Checking this amounts to checking that

$$\limsup_{t\to\infty} \frac{1}{t} \log \|\Phi_t(\theta)\| \leq \lambda \text{ a.s.}$$

Lemma 2.2 *Suppose that for all (θ, x)*

$$u(\theta, x) + x^2 u_{x^2}(\theta, x) \geq 2 \ .$$
$$0 \leq u(\theta, x) + x^2 (u_{x^2}(\theta, x) - u_{x^1}(\theta, x)) \leq 2\sqrt{\beta} \ .$$

Then there exists $\lambda < 0$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log \|\Phi_t(\theta)\| \leq \lambda \text{ a.s.}$$

Proof Because of the form of the matrix B appearing in (9), it suffices to prove the result when (9) is replaced by the equation

$$\frac{d\Phi_t(\theta)}{dt} = \bar{A}(\theta, X_t(\theta))\,\Phi_t(\theta)$$

where $\bar{A} = \Lambda A \Lambda^{-1}$, Λ being an invertible matrix to be chosen below. It follows from the theory of Lyapunov exponents (see e.g. Arnold, Kleimann, Oeljeklaus [2]) that

$$\limsup_{t\to\infty} \frac{1}{t} \log \|\Phi_t(\theta)\| \leq \sup_{x\in\mathbb{R}^2} \sigma\left(\bar{A}(\theta, x)\right)$$

where $\sigma(\bar{A})$ denotes the largest eigenvalue of $(\bar{A} + \bar{A}^*)/2$. The result follows from routine calculations if we choose

$$\Lambda = \begin{pmatrix} \sqrt{\beta + 1} & 0 \\ 1 & 1 \end{pmatrix} .$$

■

Note that the assumptions of the lemma seem to be very reasonable in the case of our practical application. In any case, $0 < u \leq 2\sqrt{\beta}$ is a known stability condition for our problem, and we know from previous study that u and its derivatives take large values only when $|x^2|$ is small.

2.3 The gradient of the cost functional

Lemma 2.3

$$\begin{aligned} \nabla_\theta J(\theta) &= \lim_{T\to\infty} \frac{1}{T} E \int_0^T F(\theta, X_t(\theta), Y_t(\theta))\, dt \ . \qquad (10) \\ &= \int_{\mathbb{R}^2\times\mathbb{R}^{2\times d}} F(\theta, x, y)\, \mu_\theta(dx, dy) \end{aligned}$$

where

$$F(\theta, x, y) \triangleq f_\theta(\theta, x) + f_x(\theta, x)\, y \ .$$

Proof Suppose for the simplicity of the notations that θ is scalar. It suffices to show that, the pair $(X_t(\theta), Y_t(\theta))$ being understood as a stationary process, the following quantity tends to zero as $h \to 0$

$$\begin{aligned} &E\left(\frac{f(\theta + h, X_t(\theta + h)) - f(\theta, X_t(\theta))}{h} - F(\theta, X_t(\theta), Y_t(\theta))\right) \\ &= \int_0^1 [F(\theta + u\,h, X_t(\theta + u\,h), Y_t(\theta + u\,h)) \\ &\qquad - F(\theta, X_t(\theta), Y_t(\theta))]\, du \ . \end{aligned}$$

This follows from the continuity of μ_θ with respect to θ. ■

2.4 Stochastic gradient algorithm

In order to minimize (7), we want to find $\theta^* \in \Theta$ such that

$$h(\theta^*) = 0 \qquad (11)$$

where

$$h(\theta) \triangleq \int_{\mathbb{R}^2\times\mathbb{R}^{2\times d}} F(\theta, x, y)\, \mu_\theta(dx, dy) \qquad (12)$$

is the gradient of (7).

Given $\Delta t > 0$ and $t_n \triangleq n\,\Delta t$, we solve equations (1),(2),(8) with

$$\theta = \theta_n \text{ for } t_n \leq t < t_{n+1} \ .$$

and θ_n is given by

$$\theta_{n+1} = \theta_n - \rho_n\, F(\theta_n, X_{t_n}(\theta_n), Y_{t_n}(\theta_n)) \qquad (13)$$

where the sequence of positive gains $\{\rho_n\}$ satisfies appropriate conditions.

3 Conclusion

We have shown how to compute the gradient for the implementation of a stochastic gradient algorithm, aimed at solving an optimal stochastic ergodic control problem. We shall give conditions for the convergence of the algorithm and present numerical results in a forthcoming publication.

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