V - STOCHASTIC SYSTEMS

Approximation of a Stochastic Ergodic Control Problem

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Abstract

We study a degenerate non linear optimal stochastic control problem of ergodic type. We first prove that for each feedback control law, there exists a unique invariant measure which is equivalent to Lebesgue measure. This is proved using an accessibility property of the stochastic differential equation, after the discontinuous part of the drift has been removed via a change of probability measure. We then approximate the problem by ergodic control problems for finite state, continuous time Markov chains. We finally prove that the cost functionals of the approximate problems converge pointwise towards that of the continuous problem.

All the study is done for a particular problem introduced in [1], which is motivated by the optimal control of the shock-absorber of a road vehicle. The numerical results can be found in [1].

1 Introduction

The aim of this paper is the study and the approximation of a class of ergodic control problems. For clarity we will work on a particular problem already introduced in [1], which comes from a problem of optimization of controlled shock–absober. This involves three difficulties — which are met in most applied problems — : the diffusion we want to control is degenerate, some coefficients are discontinuous and the problem is strongly nonlinear.

Let us consider the following stochastic system

$$dX(t) = b(u(X(t)), X(t))\, dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW(t) \ , \tag{1}$$

where X is a process which takes values in $\mathbb{R}^2$, W is a real standard Wiener process and $\sigma > 0$. b maps $\mathbb{R} \times \mathbb{R}^2$ in $\mathbb{R}^2$ and is defined by

$$b(u,x) \triangleq \begin{pmatrix} b_1(u,x) \\ b_2(u,x) \end{pmatrix} \triangleq \begin{pmatrix} x_2 \\ -u\,x_2 - \beta\,x_1 - \gamma\,\mathrm{sign}(x_2) \end{pmatrix} , \quad x \triangleq \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ,$$

where β, γ are strictly positive constants. In (1), u is a feedback control which belongs to the class $\mathcal{U}$ of admissible controls defined by (fix $\underline{u}$, $\overline{u}$ such that $0 < \underline{u} < \overline{u} < \infty$)

$u \in \mathcal{U} \iff$ $u : \mathbb{R}^2 \to [\underline{u}, \overline{u}]$ and there exists a finite number of submanifolds of $\mathbb{R}^2$ with dimension less than or equal to 1 outside of which u is continuous.

We are concerned with an ergodic type control problem, whose cost functional is

$$J(u) \triangleq \lim_{T\to\infty} \frac{1}{T} E \int_0^T f(u(X(t)), X(t))\, dt\ , \quad \forall u \in \mathcal{U}\ , \tag{2}$$

where the instantaneous cost function f is defined by

$$f(u,x) \triangleq (u\,x_2 + \beta\,x_1 + \gamma\,\mathrm{sign}(x_2))^2\ . \tag{3}$$

From now on, we denote

$$b^u(x) \triangleq b(u(x),x)\ , \quad f^u(x) \triangleq f(u(x),x)\ , \quad \forall u \in \mathcal{U}\ .$$

The physical interpretation of this problem is the following: $y = X_1(t)$ is a solution of the equation

$$m\,\ddot{y} + v\,\dot{y} + K\,y + F\,\mathrm{sign}(\dot{y}) = m\,\ddot{e}\ . \tag{4}$$

(m, K, $F > 0$) which describes a one–degree–of–freedom shock–absober system with dry friction. y is the relative displacement, v is the shock–absorber damping constant (the controlled parameter). $K\,y + F\,\mathrm{sign}(\dot{y})$ represents the restoring force (including the dry friction term). $\ddot{e}$ is the random input of the system (i.e. the road surface displacement) which is supposed to be a white noise. Taking $u = v/m$, $\beta = K/m$, $\gamma = F/m$, (4) can be rewritten as (1). The problem is to improve vehicle riding comfort by the choice of an adequate feedback u, i.e. to minimize

$$J(u) \triangleq \lim_{T\to\infty} \frac{1}{T} E \int_0^T f(u,y,\dot{y})\, dt\ , \tag{5}$$

where the instantaneous cost function $f(u,y,\dot{y})$ is the absolute acceleration squared, that is

$$f(u,y,\dot{y}) \triangleq |\ddot{a}|^2 = |\ddot{y} - \ddot{e}|^2 = \left| \frac{1}{m}\,(v\,\dot{y} + K\,y + F\,\mathrm{sign}(\dot{y})) \right|^2 .$$

In [1], we present a numerical approach based on finite difference techniques [7,10]. For the discretized problem, we use the policy iteration algorithm for which we state a convergence property. In the present paper we give some results on the following properties

- existence and uniqueness of the invariant measure μ_u associated with (1),
- convergence result of the approximation when the discretizing parameter goes to 0.

The Hamilton–Jacobi–Bellman equation related to the ergodic control problem (1,2) can be formally stated as

$$\min_{u \in [\underline{u}, \overline{u}]} (\mathcal{L}_u v(\cdot) + f(u, \cdot)) = \rho \quad \text{on } \mathbb{R}^2, \tag{6}$$

where $v : \mathbb{R}^2 \to \mathbb{R}$ is defined up to an additive constant. ρ is a constant and $\mathcal{L}_u$ is the infinitesimal generator of (1)

$$\mathcal{L}_u \phi(x) \triangleq b_1^u(x) \frac{\partial \phi(x)}{\partial x_1} + b_2^u(x) \frac{\partial \phi(x)}{\partial x_2} + \frac{\sigma^2}{2} \frac{\partial^2 \phi(x)}{\partial^2 x_2} \,. \tag{7}$$

Numerical approximation of the Hamilton–Jacobi–Bellman equation, in the nonergodic case, may be found in [4,13], as well as the study of the convergence of the approximation. Here, we are not studying directly the Hamilton–Jacobi–Bellman equation for which there seems to be no result proved in the present context. We want to study — using probabilistic techniques — the convergence of the approximation.

The ergodic control problem has been studied in [5] for discrete state space Markov processes, and in [2,3,9,11,15,16] for diffusion processes. Most of these last works are based on a strong ellipticity assumption, or establish a recurrence property with a different set of hypotheses than ours.

The bound $\underline{u} > 0$ is important both for mathematical and physical reasons in order to ensure the stability of the system (cf. the proof of lemma 2.1).

Because the system (1) is degenerate (the noise appears only in the second component), the uniqueness of the invariant measure is related to a controllability type property, but, due to the nonregularity of the coefficients, the standard techniques fail in proving this last property. However, this can be done via a change of probability law.

In section 2 we establish an existence and uniqueness result for the invariant measure corresponding to system (1), for any u in $\mathcal{U}$. In 3 we present the approximation of the problem using finite difference techniques. The convergence of the approximate cost functionals to the original cost functional is studied in section 4.

2 The Invariant Probability Measure

The cost function (2) can be rewritten as

$$J(u) = \langle f^u \,, \mu_u \rangle \,, \quad \forall u \in \mathcal{U} \,, \tag{8}$$

where μ_u is the invariant probability measure associated with system (1). In this section we establish an existence and uniqueness property for μ_u.

2.1 Existence

Lemma 2.1 *There exists a constant C such that*

$$E\,|X(t)|^2 \leq C\ , \quad \forall t \geq 0\,,\ \forall u \in \mathcal{U}\ . \tag{9}$$

Proof We define

$$\mathcal{V}(x) \triangleq \beta\, x_1^2 + \varepsilon\, x_1\, x_2 + x_2^2\ , \qquad \text{and} \quad V(t) \triangleq E\,\mathcal{V}(X(t))\ .$$

There exists $\varepsilon_0 > 0$ such that for any $\varepsilon_0 > \varepsilon > 0$

$$\mathcal{V}(x) \geq \frac{1}{2}\,\left(\beta\, x_1^2 + x_2^2\right)\ .$$

Hence, it is sufficient to show that $V(t) \leq$ Cte for any $t \geq 0$. From (1),

$$\begin{aligned}
\frac{d}{dt}V(t) \;=\; & E\left[2\,\beta\, X_1(t)\, X_2(t) + \varepsilon\, X_2^2(t) - \varepsilon\,\beta\, X_1^2(t) - \varepsilon\, u(X(t))\, X_1(t)\, X_2(t)\right.\\
& -\varepsilon\,\gamma\, X_1(t)\,\mathrm{sign}(X_2(t)) - 2\,\beta\, X_1(t)\, X_2(t) - 2\, u(X(t))\, X_2^2(t)\\
& \left.-2\,\gamma\, X_2^2(t)\right] + \sigma^2\ .
\end{aligned}$$

Using $\underline{u} \leq u(x) \leq \overline{u}$ and the following inequalities

$$\begin{aligned}
-\varepsilon\, u(x)\, x_1\, x_2 \;&\leq\; \frac{\varepsilon\,\beta}{2}\, x_1^2 + \frac{\varepsilon\,\overline{u}^2}{2\,\beta}\, x_2^2\ ,\\
-\varepsilon\,\gamma\, x_1\,\mathrm{sign}(x_2) \;&\leq\; \frac{\varepsilon\,\delta\,\gamma}{2}\, x_1^2 + \frac{\varepsilon\,\gamma}{2\,\delta}\ , \quad (\forall \delta > 0)\ ,
\end{aligned}$$

we get

$$\frac{d}{dt}V(t) \leq E\left[-\left(\frac{1}{2}\,\varepsilon\,\beta - \frac{1}{2}\,\varepsilon\,\delta\,\gamma\right) X_1^2(t) - \left(2\,\underline{u} + 2\,\gamma - \frac{1}{2\,\beta}\,\varepsilon\,\overline{u}^2\right) X_2^2(t)\right] + \frac{\varepsilon\,\gamma}{2\,\delta} + \sigma^2\ ,$$

so there exists strictly positive constants ε and δ such that

$$\frac{d}{dt}V(t) \leq -C(\varepsilon, \delta)\, V(t) + \frac{\varepsilon}{2\,\delta} + \sigma^2\ ,$$

where $C(\varepsilon, \delta) > 0$. Applying Gronwall's lemma to this last inequality yields the conclusion. □

Lemma 2.2 *The process $X(t)$ solution of (1) has the Feller property, i.e. for any $u \in \mathcal{U}$, $t \geq 0$ and $\phi \in C_b(\mathbb{R}^2)$, the function*

$$\mathbb{R}^2 \ni x \longrightarrow E\phi\,(X^x(t)) \tag{10}$$

is continuous. $X^x(t)$ denotes the solution of (1) starting from x at time $t = 0$.

Proof In (1), the drift coefficient can be written as

$$b(u,x) = \overline{B}\,x + \begin{pmatrix} 0 \\ -u\,x_2 - \gamma\,\mathrm{sign}(x_2) \end{pmatrix} \triangleq \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -u\,x_2 - \gamma\,\mathrm{sign}(x_2) \end{pmatrix}.$$

Let

$$\begin{aligned} \overline{W}(t) &\triangleq W(t) + \int_0^t \psi(X^x(s))\,ds\ , \\ \psi(x) &\triangleq -\frac{1}{\sigma}\,(u(x)\,x_2 + \gamma\,\mathrm{sign}(x_2))\ , \\ Z^x(t) &\triangleq \exp\left(\int_0^t \psi(X^x(s))\,d\overline{W}(s) - \frac{1}{2}\int_0^t \psi(X^x(s))^2\,ds\right)\ . \end{aligned} \tag{11}$$

We define a new probability law

$$\left.\frac{d\overline{P}}{dP}\right|_{\mathcal{F}_t} \triangleq (Z^x(t))^{-1}\ .$$

X satisfies

$$dX(t) = \overline{B}\,X(t)\,dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} d\overline{W}(t)\ , \tag{12}$$

where — from Girsanov's theorem — $\overline{W}(t)$ is a real standard Wiener process under the probability law $\overline{P}$.

For any sequence $x_n \to x$, we want to prove that

$$E\phi(X^{x_n}(t)) = \overline{E}\,[\phi(X^{x_n}(t))\,Z^{x_n}(t)] \underset{n\to\infty}{\longrightarrow} E\phi(X^x(t)) = \overline{E}\,[\phi(X^x(t))\,Z^x(t)]\ , \tag{13}$$

where $\overline{E}$ denotes the expectation with respect to $\overline{P}$. So, it is sufficient to check that

$$X^{x_n}(t) \underset{n\to\infty}{\longrightarrow} X^x(t) \quad \overline{P}\text{-a.s.}\ , \tag{14}$$

$$Z^{x_n}(t) \underset{n\to\infty}{\longrightarrow} Z^x(t) \quad \text{in } \overline{P}\text{-probability.} \tag{15}$$

Assume for a moment that (14) and (15) hold. Then, $\overline{E}Z^{x_n}(t) = \overline{E}Z^x(t) \equiv 1$ and (15) imply that $Z^{x_n}(t) \to Z^x(t)$ in $L^1(\overline{P})$, so

$$\begin{aligned} \left|\overline{E}\,(\phi(X^{x_n}(t))\,Z^{x_n}(t) - \phi(X^x(t))\,Z^x(t))\right| \ \le\ & \overline{E}\,(|\phi(X^{x_n}(t)) - \phi(X^x(t))|\,Z^x(t)) \\ & + C\,\overline{E}\,|Z^{x_n}(t) - Z^x(t)|\ , \end{aligned}$$

the second term tends to 0, the first one also by dominated convergence.

We now prove (14) and (15). Under the probability law $\overline{P}$, $X(t)$ is the solution of a *linear* stochastic differential system, so (14) is obvious. For (15), we show that

$$\overline{E}\int_0^t [u(X^{x_n}(s))\,X_2^{x_n}(s) - u(X^x(s))\,X_2^x(s)]^2\,ds \underset{n\to\infty}{\longrightarrow} 0\ , \tag{16}$$

$$\overline{E}\int_0^t [\mathrm{sign}(X_2^{x_n}(s)) - \mathrm{sign}(X_2^x(s))]^2\,ds \underset{n\to\infty}{\longrightarrow} 0\ . \tag{17}$$

For any $s \in [0,t]$ and $\varepsilon > 0$

$$\begin{aligned}\overline{E}[\mathrm{sign}(X_2^{x_n}(s)) - \mathrm{sign}(X_2^x(s))]^2 &= 4\overline{P}[X_2^{x_n}(s)\,X_2^x(s) < 0] \\ &\leq 4\overline{P}[|X_2^x(s)| < \varepsilon] + 4\overline{P}[|X_2^{x_n}(s) - X_2^x(s)| \geq \varepsilon] \\ &\longrightarrow 4\overline{P}[|X_2^x(s)| < \varepsilon]\ , \quad \text{as } n \to \infty \text{ (using (14))},\end{aligned}$$

so

$$\overline{E}[\mathrm{sign}(X_2^{x_n}(s)) - \mathrm{sign}(X_2^x(s))]^2 \underset{n\to\infty}{\longrightarrow} 0\ .$$

For (16), using (14) and the dominated convergence theorem, it is sufficient to state the following convergence in probability

$$\overline{P}\left(|u(X^{x_n}(s)) - u(X^x(s))| > \varepsilon\right) \underset{n\to 0}{\longrightarrow} 0\ , \quad \forall \varepsilon > 0\ , \quad \forall s \in [0,t]\ . \tag{18}$$

As $u \in \mathcal{U}$, for any $\delta > 0$ there exists a closed subset $D_\delta \subset \mathbb{R}^2$ and for any $\rho > 0$ there exists $C_\rho(\delta) \in [0,1]$ such that

$$\begin{aligned}&(i) \quad \overline{P}\left(X^x(s) \in D_\delta^c \cap B(0,\rho)\right) \leq C_\rho(\delta)\ , \quad \forall \rho, \delta > 0\ , \\ &(ii) \quad C_\rho(\delta) \longrightarrow 0\ , \quad \text{as } \delta \to 0\ , \quad \forall \rho > 0\ , \\ &(iii) \quad u \text{ is continuous on } D_\delta, \quad \forall \delta > 0\ ,\end{aligned}$$

where $B(0,\rho) \triangleq \{x; |x| < \rho\}$. We have the following inequality

$$\begin{aligned}&\overline{P}\left(|u(X^{x_n}(s)) - u(X^x(s))| > \varepsilon\right) \\ &\leq \overline{P}\left(X^x(s) \in B(0,\rho)^c\right) \\ &\quad + \overline{P}\left(|u(X^{x_n}(s)) - u(X^x(s))| > \varepsilon\ ;\ X^x(s) \in D_\delta \cap B(0,\rho)\right) \\ &\quad + \overline{P}\left(X^x(s) \in D_\delta^c \cap B(0,\rho)\right)\ .\end{aligned} \tag{19}$$

Hence from (14) and because $u(x)$ is uniformly continuous on $D_\delta \cap B(0,\rho)$, we get

$$\begin{aligned}&\lim_{n\to\infty} \overline{P}\left(|u(X^{x_n}(s)) - u(X^x(s))| > \varepsilon\right) \\ &\qquad \leq \overline{P}\left(X^x(s) \in B(0,\rho)^c\right) + \overline{P}\left(X^x(s) \in D_\delta^c \cap B(0,\rho)\right)\ .\end{aligned}$$

Let $\delta \to 0$ first and then $\rho \to \infty$, so we get (18), which proves the lemma. □

By means of usual techniques (e.g. [6] th. 9.3 ch. 4), lemmas 2.1, 2.2 yield

Proposition 2.3 *For any $u \in \mathcal{U}$, the diffusion process (1) admits an invariant probability measure μ_u.*

2.2 Uniqueness

In this section μ denotes a fixed invariant probability measure associated with system (1), and $X(t)$ is the solution of this system with μ as initial law (i.e. $X(0)$ has law μ). We also define $Z(t)$ by (11) where X^x is replaced by X.

Lemma 2.4 *Under $\overline{P}$, for any $t > 0$, the law of $X(t)$ has a density $\overline{p}(t,x)$ such that*

$$\overline{p}(t,x) > 0\ , \quad \forall x\ .$$

Proof From now on we are working under $\overline{P}$. Consider the system (12) where $d\overline{W}$ is replaced by $v\,dt$ ($v \in L^2(\mathbb{R}^+)$), we get

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -\beta\, x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} v \;, \quad x(0) = x \;. \tag{20}$$

Let $x^{x,v}(t)$ denote the solution of this last equation. We define the set of reachability

$$\mathcal{A}(t,x) \triangleq \left\{ x^{x,v}(t) \;;\; \forall v : \mathbb{R}^+ \to \mathbb{R} \,,\; \dot{v} \in L^2(\mathbb{R}^+) \right\} \;.$$

(20) can be rewritten as $\dot{x} = A\,x + B\,v$ and the matrix $[B|A\,B]$ has full rank. Hence this system is controllable. So

$$\forall t > 0 \;, \quad \forall x \in \mathbb{R}^2 \;, \quad \mathcal{A}(t,x) = \mathbb{R}^2 \;. \tag{21}$$

Using [12], we prove that — under $\overline{P}$ — the law of $X(t)$ is absolutely continuous with respect to Lebesgue measure and that its density $\overline{p}(t,x)$ is strictly positive for any $t > 0$ and x. □

Lemma 2.5 *Let μ be an invariant measure for $X(t)$ under P. Then μ has a density $p(x)$ with respect to Lebesgue measure, and $p(x) > 0$ for any x a.e. .*

Proof For any $\phi \in C_b(\mathbb{R}^2)$

$$\begin{aligned} \langle \mu \,,\, \phi \rangle &= \overline{E}[\phi(X(t))\, Z(t)] \;, \\ &= \overline{E}\left[\phi(X(t))\, \overline{E}[Z(t)|X(t)]\right] \;, \\ &= \int_{\mathbb{R}^2} \phi(x)\, \overline{E}[Z(t)|X(t) = x]\, \overline{p}(t,x)\, dx \;. \end{aligned}$$

Since $\overline{E}[Z(t)|X(t)] > 0$ $\overline{P}$-a.s. and under $\overline{P}$ the law of $X(t)$ is equivalent to Lebesgue measure, we get $\overline{E}[Z(t)|X(t) = x] > 0$ $\forall x$-a.e. . Using lemma 2.4 and the last inequality, we prove that μ has a density

$$q(x) \triangleq \overline{E}[Z(t)|X(t) = x]\, \overline{p}(t,x) \;,$$

and that this density is strictly positive for all $x \in \mathbb{R}^2$ a.e. . □

This lemma implies the following result: if there exists two invariant measures, they are equivalent. So there exits at most one extremal invariant measure. We can therefore state

Proposition 2.6 *For any $u \in \mathcal{U}$, the diffusion process (1) admits a unique invariant measure μ_u.*

3 Numerical Approximation

3.1 Approximation of the Control Problem

In a first step, the solution $X(t)$ of (1) is approximated by a controlled Markov process in continuous time and discrete (but infinite) state space. In a second step, it is approximated by a controlled Markov process in continuous time and finite state space.

a – *first step*

Let h_i be the finite difference interval to be used to approximate the derivative w.r.t. the spatial direction i $(i = 1, 2)$. We define the grid

$$\mathbb{R}_h^2 \triangleq \left\{x \in \mathbb{R}^2 \,;\, x = (n_1 h_1 + h_1/2\,,\, n_2 h_2 + h_2/2)\,,\ n_1,\, n_2 \in \mathbb{Z}\right\}\ ,\quad h \triangleq (h_1\,,\, h_2)\ .$$

We will use the finite difference approximation

$$b_i^u(x)\,\frac{\partial\phi(x)}{\partial x_i} \simeq \begin{cases} b_i^u(x)\,\dfrac{\phi(x+e_i\,h_i)-\phi(x)}{h_i}\,, & \text{if } b_i^u(x) > 0, \\ b_i^u(x)\,\dfrac{\phi(x)-\phi(x-e_i\,h_i)}{h_i}\,, & \text{if } b_i^u(x) < 0, \end{cases} \qquad (i = 1, 2) \quad (22)$$

$$\frac{\sigma^2}{2}\,\frac{\partial^2\phi(x)}{\partial x_2^2} \simeq \frac{\sigma^2}{2}\,\frac{\phi(x+e_2\,h_2) - 2\,\phi(x) + \phi(x-e_2\,h_2)}{h_2^2}\,, \qquad (23)$$

where e_i denotes the unit vector in the ith coordinate direction of $\mathbb{R}^2$ (the special choice for the finite difference approximation will be motivated in remark 3.2).

So $\mathcal{L}_u$ is approximated by a matrix $\mathcal{L}_u^h \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$

$$\mathcal{L}_u\phi(x) \simeq \mathcal{L}_u^h\phi(x) \triangleq \sum_{y\in\mathbb{R}_h^2} \mathcal{L}_u^h(x,y)\,\phi(y)\ ,\quad \forall x \in \mathbb{R}_h^2\ .$$

Because of the finite difference approximations (22,23) we use, $\mathcal{L}_u^h$ can be regarded as the infinitesimal generator of a Markov process in continuous time and discrete state space $\mathbb{R}_h^2$ [7,10]. Using classical definition, a Markov chain $\{\xi_k^h\,;\, k \in \mathbb{N}\}$ is associated with $X^h(t)$ [6].

We then have an ergodic stochastic control problem for a Markov process with infinitesimal generator $\mathcal{L}_u^h$. The cost function is

$$J_h(u) \triangleq \lim_{T\to\infty} E\,\frac{1}{T}\int_0^T f^u(X^h(t))\,dt\ . \qquad (24)$$

u is an element of the class $\tilde{\mathcal{U}}_h$ defined by

$$u \in \tilde{\mathcal{U}}_h \iff u \text{ is an application from } \mathbb{R}_h^2 \text{ to } [\underline{u}, \overline{u}].$$

b - *second step*

$X^h(t)$ has a discrete but infinite state space, and for numerical calculations we need to restrict ourselves to a *finite* state space. Let us then consider the following rectangular subset of $\mathbb{R}^2$

$$D \triangleq [-\overline{x}_1\,,\,\overline{x}_1] \times [-\overline{x}_2\,,\,\overline{x}_2]\;,\quad \overline{x}_i > 0 \quad (i = 1,2)\;, \tag{25}$$

from which we define the following new state space

$$\mathbb{R}^2_{h,D} \triangleq \mathbb{R}^2_h \cap D\;,\quad N \triangleq \mathrm{Card}\left(\mathbb{R}^2_{h,D}\right)\;. \tag{26}$$

Now we have to specify the boundary conditions. In practice, D is chosen to be large enough so that the process will rarely reach the border. Hence, the choice of the boundary conditions is not crucial. Nevertheless, they have to insure that all the states communicate. Example of such conditions (usually reflecting conditions) will be given later.

So we obtain $\mathcal{L}^{h,D}_u$, an approximation of $\mathcal{L}^h_u$. $\mathcal{L}^{h,D}_u$ is a $N \times N$ matrix, it can be interpreted as the generator of a controlled Markov process $X^{h,D}(t)$ in continuous time and finite state space; $\{\xi^{h,D}_k\,;\,k \in \mathbb{N}\}$ denotes the corresponding Markov chain.

The cost function is of the form

$$J_{h,D}(u) \triangleq \lim_{T\to\infty} E\,\frac{1}{T}\int_0^T f^u(X^{h,D}(t))\,dt = \sum_{x\in\mathbb{R}^2_{h,D}} f^u(x)\,\mu^{h,D}_u(x)\;, \tag{27}$$

where $\mu^{h,D}_u$ is the invariant measure[1] of the process $X^{h,D}(t)$ (more details can be found in [1]). This measure is a solution of the following linear system

$$\sum_{y\in\mathbb{R}^2_{h,D}} \mathcal{L}^{h,D}_u(y,x)\,\mu^{h,D}_u(y) = 0\;,\quad \forall x \in \mathbb{R}^2_{h,D}\;,\qquad \sum_{y\in\mathbb{R}^2_{h,D}} \mu^{h,D}_u(y) = 1\;. \tag{28}$$

where $u \in \tilde{\mathcal{U}}_{h,D}$, the class of policies which is defined by

$$u \in \tilde{\mathcal{U}}_{h,D} \quad\Longleftrightarrow\quad u \text{ is a mapping from } \mathbb{R}^2_{h,D} \text{ into } [\underline{u},\overline{u}].$$

A Hamilton–Jacobi–Bellman equation can be stated for this ergodic control problem

$$\min_{u\in[\underline{u},\overline{u}]}\left(\sum_{y\in\mathbb{R}^2_{h,D}} \mathcal{L}^{h,D}_u(x,y)\,v(y) + f^u(x)\right) = \rho\;,\quad \forall x \in \mathbb{R}^2_{h,D}\;, \tag{29}$$

where ρ is a positive constant and $v : \mathbb{R}^2_{h,D} \to \mathbb{R}$ (i.e. $v \in \mathbb{R}^N$) is defined up to an additive constant. In the first term of this equation, u has to be considered as an element of $[\underline{u},\overline{u}]$ ($f^u(x) = f(u,x)$). Equation (29) has been studied in [1], for

[1] in the discrete case we also use the notation $\sum_x f^u(x)\,\mu^{h,D}_u(x) = \langle f^u\,,\,\mu^{h,D}_u\rangle$.

fixed h and D. In particular, the existence and uniqueness property for a solution $(v, \rho) \in \mathbb{R}^N \times \mathbb{R}^+$ is established.

Equation (29) appears as an approximation for the Hamilton–Jacobi–Bellman equation (6), and leads to the solution of the ergodic control problem associated with the Markov process $X^{h,D}(t)$ in continuous time, finite state space, and infinitesimal generator $\mathcal{L}_u^{h,D}$.

3.2 The Policy Iteration Algorithm

In order to solve (29), we use the policy iteration algorithm [5,8]: suppose that $u^0 \in \mathcal{U}_{h,D}$ — the initial policy — is given. Starting with u^0 we generate a sequence $\{u^j; j \geq 1\}$. The iteration $u^j \rightarrow u^{j+1}$ proceeds in two steps

compute (v^j, ρ^j) | we compute $(v^j, \rho^j) \in \mathbb{R}^N \times \mathbb{R}^+$ the solution of the linear system

$$\sum_{y \in \mathbb{R}^2_{h,D}} \mathcal{L}_{u^j}^{h,D}(x, y)\, v(y) + f^{u^j}(x) = \rho \;, \quad \forall x \in \mathbb{R}^2_{h,D} \;.$$

compute u^{j+1} | we solve the N following optimization problems: for any $x \in \mathbb{R}^2_{h,D}$

$$u^{j+1}(x) \in \operatorname{Arg} \min_{u \in [\underline{u}, \overline{u}]} \left(\sum_{y \in \mathbb{R}^2_{h,D}} \mathcal{L}_u^{h,D}(x, y)\, v^j(y) + f^u(x) \right) \;.$$

The convergence of this algorithm is stated in [1] (where numerical results can be also found).

Remark 3.1 The first step of this algorithm leads to a linear system of dimension N. Let $\mathbb{R}^2_{h,D} = \{x^i\,;\, i = 1, \ldots, N\}$, then the unknown parameters are

$$v(x^2)\,,\; v(x^3)\,, \ldots,\; v(x^N)\,,\; \rho\,,$$

and we take $v(x^1) = 0$.

Remark 3.2 For the second step, the optimization problems are nonlinear and they are solved by means of iterative routines. The nonlinearity comes from the discretization technique we use. Indeed, the choice of finite difference approximation (22) depends on u. Instead of (22), we can use central difference approximation (so that it does not depend on u), in which case the second step becomes explicit because the functions to be optimized are now quadratic in u. On the other hand, with this kind of difference approximation, a certain condition on the parameter h has to be fulfilled (h must be small enough) for the matrix $\mathcal{L}_u^{h,D}$ to be the generator of a Markov process. See [10] p.175–179 for further considerations.

4 Convergence of the Cost Functions

We suppose that $h_1 = h_2$ and we denote it by h. In this section, we prove the convergence result

$$J_{h,D}(u) \to J(u) \ , \quad \forall u \in \mathcal{U}$$

as the discretization parameter h tends to 0 and the set D tends to $\mathbb{R}^2$. Let us fix $u \in \mathcal{U}$. All the results of this section — up to corollary 4.7 — are adapted from [7].

a – *a sequence of discretization sets*

We consider two strictly increasing sequences $\{\overline{x}_1^h; h > 0\}$ and $\{\overline{x}_2^h; h > 0\}$, such that $\overline{x}_i^h > 0$ and $\overline{x}_i^h \to \infty$ as $h \to 0$. We define

$$D_h = [-\overline{x}_1^h, \overline{x}_1^h] \times [-\overline{x}_2^h, \overline{x}_2^h] \ .$$

We suppose that

$$\lim_{h\to 0} h\,\delta_h = 0 \ , \quad \text{where} \quad \delta_h \triangleq \text{radius}(D_h) \ . \tag{30}$$

Let Γ_h be the boundary of D_h, we define $\overline{\Gamma}_h$, the discretization of Γ_h as follows

$$x \in \overline{\Gamma}_h \Longleftrightarrow \begin{cases} x \in \Gamma_h \cap \mathbb{R}_h^2 \ , \\ \text{or} \\ x \in D_h \cap \mathbb{R}_h^2, \ \text{and } \exists y \in V_h(x) \text{ such that } y \notin D_h \cap \mathbb{R}_h^2 \ , \end{cases}$$

where $V_h(x) \triangleq \left\{y \in \mathbb{R}_h^2 ; \ |x-y| \leq \varepsilon\sqrt{2}\right\}$ is the set of points adjacent to x. We define

$$\overline{D}_h \triangleq D_h \cap \mathbb{R}_h^2 \ .$$

$\overline{D}_h$ is the state space for the discretized problem and $\overline{\Gamma}_h$ is the set of boundary points. We use the same set–up as in section 3 and we define

$$\begin{aligned} \overline{\mathcal{L}}_u^h &\triangleq \mathcal{L}^{h,D_h} \ , \\ \overline{X}^h(t) &\triangleq X^{h,D_h}(t) \ , \ \text{with initial law } \nu^h \ , \\ \overline{X}^{x,h}(t) &\triangleq X^{h,D_h}(t) \ , \ \text{with initial condition } \overline{X}^{x,h}(0) = x \ , \\ \overline{\xi}_k^h &\triangleq \xi_k^{h,D_h} \ . \end{aligned}$$

b – *the process* $\overline{X}^h(t)$

We can describe the process $\overline{X}^h(t)$ in the following way. We introduce

- a sequence $\{\Delta t_n^h; \ n \geq 0\}$, where Δt_n^h represents the elapsed time between the n–th and the $(n+1)$–th jump.

- a Markov chain $\{\overline{\xi}_n^h\,;\, n \geq 0\}$ with values in $\overline{D}^h$, where $\overline{\xi}_n^h$ represents the state of the process between the n–th and the $(n+1)$–th jump.

We consider

$$\begin{aligned} \lambda_h(x) &\triangleq -\overline{\mathcal{L}}_u^h(x,x) \geq 0\,, \qquad \forall x \in \overline{D}^h\,, \\ \pi^h(x,y) &\triangleq (\lambda_h(x))^{-1}\overline{\mathcal{L}}_u^h(x,y)\,,\ \forall x,y \in \overline{D}^h\,,\ x \neq y\,, \end{aligned}$$

(with $0/0 = 0$). We have the following properties

- the pair $(\Delta t_{n+1}^h, \overline{\xi}_{n+1}^h)$ depends only on $\overline{\xi}_n^h$,
- under the conditional law $P(\,\cdot\,|\overline{\xi}_n^h = x)$, the random variables Δt_{n+1}^h and $\overline{\xi}_{n+1}^h$ are independent.
- under the conditional law $P(\,\cdot\,|\overline{\xi}_n^h = x)$, Δt_{n+1}^h is exponentially distributed with parameter $(\lambda_h(x))^{-1}$,
- $\pi^h(x,y)$ is the transition probability of the chain $\{\overline{\xi}_n^h\,;\, n \geq 0\}$,

Now we give a representation for the process $\overline{X}^h(t)$. For this purpose we must specify the boundary conditions. These are of Neumann type (reflected) in order to simplify the proof of the convergence result (lemma 4.8).

We define

$$a \triangleq \begin{pmatrix} 0 \\ \sigma \end{pmatrix} (0\ \sigma) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix}\,, \quad a^h(x) \triangleq a + h \begin{pmatrix} |b_1^u(x)| & 0 \\ 0 & |b_2^u(x)| \end{pmatrix}\,,$$

and the stopping time

$$\tau^h \triangleq \inf\left\{t \geq 0\,;\, \overline{X}^h(t) \in \Gamma^h\right\}\,.$$

It is easely seen that for any $x \in \overline{D}^h \setminus \Gamma^h$

$$\begin{aligned} \sum_{y \in \overline{D}^h} \overline{\mathcal{L}}_u^h(x,y)\,(y-x) &= b^u(x)\,, \\ \sum_{y \in \overline{D}^h} \overline{\mathcal{L}}_u^h(x,y)\,(y-x) \otimes (y-x) &= a^h(x)\,, \end{aligned}$$

which yield the

Proposition 4.1 *The process $\{\overline{X}^h(t)\,;\, t \geq 0\}$ admits the following representation*[2]

$$\begin{aligned} \overline{X}^h(t) &= \overline{X}^h(0) + \int_0^t \begin{pmatrix} b_1^u(X(s))\,\mathbb{I}_{\{|X_1(s)|<\overline{x}_1^h\}} \\ b_2^u(X(s)) \end{pmatrix} ds + M^h(t) \qquad (31) \\ &\quad + \begin{pmatrix} h \\ 0 \end{pmatrix} N^{0-}(t) - \begin{pmatrix} h \\ 0 \end{pmatrix} N^{0+}(t) + \begin{pmatrix} 0 \\ h \end{pmatrix} N^{-}(t) - \begin{pmatrix} 0 \\ h \end{pmatrix} N^{+}(t)\,, \end{aligned}$$

[2] $\mathbb{I}_A$ denotes the indicator function of the event A.

with

$$N^{0-}(t) \triangleq \sum_{s\le t} \mathbb{I}_{\{X_1(s^-)=-\bar{x}_1^h\}} \mathbb{I}_{\{X_2(s^-)=h/2\}} ,$$
$$N^{0+}(t) \triangleq \sum_{s\le t} \mathbb{I}_{\{X_1(s^-)=\bar{x}_1^h\}} \mathbb{I}_{\{X_2(s^-)=-h/2\}} ,$$
$$N^{-}(t) \triangleq \sum_{s\le t} \mathbb{I}_{\{X_2(s^-)=-\bar{x}_2^h\}} ,$$
$$N^{+}(t) \triangleq \sum_{s\le t} \mathbb{I}_{\{X_2(s^-)=\bar{x}_2^h\}} ,$$

and $M^h(t)$ is a square-integrable martingale, with increasing process

$$\langle M^h , M^h\rangle_t = \int_0^t a^h(\overline{X}^h(s))\, ds .$$

c – *a priori estimations*

Proposition 4.2

$$E\left(\sup_{s\le t} |M^h(s)|\right)^2 \le 4\,K(h)\,t . \tag{32}$$

Proof Since $|b^u(x)| \le C\,(1+|x|)$, we deduce that for any x in $\overline{D}^h \setminus \Gamma^h$

$$|a^h(x)| = \mathrm{trace}(a^h(x)) \le K(h) , \quad \text{with } K(h) \triangleq \sigma^2 + h\sqrt{2}\,C\,(1+\delta_h) \underset{h\to 0}{\longrightarrow} \sigma^2 .$$

And proposition 4.1 yields to

$$\mathrm{trace}\langle M^h , M^h\rangle_t = \int_0^t \mathrm{trace}(a^h(\overline{X}^h(s))\, ds \le K(h)\,t .$$

From which (32) follows, using the Burkholder–Gungy inequality. □

Remark 4.3 The jumps of M^h and $\overline{X}^h$ coincide, and those of $\overline{X}^h$ are of amplitude less than h, so

$$\sup_{t\ge 0} |\Delta M_i^h(t)| \le h , \quad i = 1,2 .$$

Proposition 4.4

$$E\left(\sup_{s\le t} |\overline{X}^{x,h}(s)|\right)^2 \le 3\,\left(|x|^2 + (C\,t)^2 + 4\,K(h)\,t\right)\,\exp(2\,C\,t) . \tag{33}$$

Proof From (31)

$$|\overline{X}^h(t)| \le \psi_h(t) + C\int_0^t |\overline{X}^h(s)|\,ds\ , \quad \text{where } \psi_h(t) \triangleq |\overline{X}^h(0)| + C\,t + |M^h(t)|\ .$$

Using Gronwall's lemma, we get

$$|\overline{X}^h(t)| \le \psi_h(t) + C\int_0^t \psi_h(s)\,\exp(C\,(t-s))\,ds \le \left(\sup_{s\le t}\psi_h(s)\right)\,\exp(C\,t)\ .$$

Hence

$$\sup_{s\le t}|\overline{X}^h(s)| \le \left(|\overline{X}^h(0)| + C\,t + \sup_{s\le t}|M^h(s)|\right)\exp(C\,t)\ .$$

And (32) leads to (33). □

Remark 4.5 It follows from (33) that the sequence $\{\tau^h\,;\,h>0\}$ of random times tends to infinity in probability as $h \to 0$.

d – *convergence of invariant measures*

On the space $D^2 = \{\xi : \mathbb{R}^+ \to \mathbb{R}^2$; right continuous and with a left limit$\}$ with the Borel σ–field $\mathcal{B}(D^2)$, we define the following probability laws

$$\begin{aligned}
\mathbb{P}^h_{\nu^h} &= \text{law of process } \overline{X}^h \text{ with } \overline{X}^h(0) \sim \nu^h\ ,\\
\mathbb{P}_\nu &= \text{law of process } X \text{ with } X(0) \sim \nu\ ,\\
\mathbb{P}^h_x &= \text{law of process } \overline{X}^h \text{ starting from } x\ ,\\
\mathbb{P}_x &= \text{law of process } X \text{ starting from } x\ .
\end{aligned}$$

Proposition 4.6 *Suppose that* $\nu^h \underset{h\to 0}{\Longrightarrow} \nu$, *then*

$$\mathbb{P}^h_{\nu^h} \underset{h\to 0}{\Longrightarrow} \mathbb{P}_\nu \quad \textit{on } (D^2, \mathcal{B}(D^2))\ .$$

Proof Fix x in $\mathbb{R}^2$ and $\{x_h\,;\,h>0\}$ a sequence in $\mathbb{R}^2$ such that

$$\forall h>0\ , \quad x_h \in \overline{D}^h \quad \text{and } |x_h - x| \le \frac{h}{2}\sqrt{2}\ ,$$

(in particular $x_h \to x$ as $h \to 0$). Using the representation (31) in terms of semi-martingale for the process $\overline{X}^h$, the a priori estimation (33), the remark 4.3, and results from [14] (th. 5.8, ch. 2) we prove that

$$\mathbb{P}^h_{x_h} \underset{h\to 0}{\Longrightarrow} \mathbb{P}_x \quad \text{on } (D^2, \mathcal{B}(D^2))\ .$$

Furthermore, the convergence is uniform with respect to x on any compact subset of $\mathbb{R}^2$, the proof of the proposition follows. □

Corollary 4.7 *Let $\overline{\mu}_u^h \triangleq \mu_u^{h,D_h}$ be the invariant measure of the process $\overline{X}^h$ ($\forall h > 0$). If μ_u is a weak limit of some subsequence of $\{\overline{\mu}_u^h\}$, then μ_u is an invariant measure of the process X.*

We consider μ_u the invariant measure of X,

Lemma 4.8

$$\overline{\mu}_u^h \underset{h\to 0}{\Longrightarrow} \mu_u \ , \quad \forall u \in \mathcal{U} \ .$$

Proof In view of corollary 4.7, it is enough to prove that

$$\text{the sequence } \{\overline{\mu}_u^h;\, h > 0\} \text{ is tight,} \tag{34}$$

and a sufficient condition for (34) is that there exists a constant C independant of both t and h, such that

$$E\left|X^{h,D_h}(t)\right|^2 \leq C \ . \tag{35}$$

For notational convenience, we denote $X = X^{h,D_h}$, $X_i = X_i^{h,D_h}$, $b_i^u(\cdot) = b_i(\cdot)$. Starting from the representation (31), the proof is identical to that of lemma 2.1. We are concerned with the behavior of the function

$$V(t) \triangleq E\left(\beta\, X_1^2(t) + \varepsilon\, X_1(t)\, X_2(t) + X_2^2(t)\right) \ .$$

Since X is a pure jump process

$$\begin{aligned} V(t) - V(0) \ &= \ E \sum_{s\leq t} \big[\beta\,(X_1(s^-) + \Delta X_1(s))^2 - \beta\, X_1^2(s^-) \\ &\qquad +\varepsilon\,(X_1(s^-) + \Delta X_1(s))\,(X_2(s^-) + \Delta X_2(s)) \\ &\qquad -\varepsilon\, X_1(s^-)\, X_2(s^-) \\ &\qquad +(X_2(s^-) + \Delta X_2(s))^2 - X_2^2(s^-)\big] \\ &= \ E \sum_{s\leq t} \big[2\,\beta\, X_1(s^-)\,\Delta X_1(s) + \beta\,\Delta X_1(s)^2 + \varepsilon\, X_1(s^-)\,\Delta X_2(s) \\ &\qquad +\varepsilon\, X_2(s^-)\,\Delta X_1(s) + 2\, X_2(s^-)\,\Delta X_2(s) + \Delta X_2(s)^2\big] \ . \end{aligned}$$

This last equation and the representation (31), give

$$\begin{aligned} V(t) - V(0) &= 2\,\beta\, E \int_0^t X_1(s)\, X_2(s)\, \mathbb{I}_{\{|X_1(s)| < \overline{x}_1^h\}}\, ds \\ &\quad -2\,\beta\, h\, \overline{x}_1^h \sum_{s\leq t} \Big[P(\{X_2(s^-) = -\frac{h}{2}\} \cap \{X_1(s^-) = -\overline{x}_1^h\}) \\ &\qquad\qquad +P(\{X_2(s^-) = \frac{h}{2}\} \cap \{X_1(s^-) = \overline{x}_1^h\})\Big] \\ &\quad +\beta\, h^2 \sum_{s\leq t} \Big[P(\{X_2(s^-) = -\frac{h}{2}\} \cap \{X_1(s^-) = -\overline{x}_1^h\}) \\ &\qquad\qquad +P(\{X_2(s^-) = \frac{h}{2}\} \cap \{X_1(s^-) = \overline{x}_1^h\})\Big] \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon \frac{h^2}{2} \sum_{s\leq t} \Big[P(\{X_2(s^-) = -\frac{h}{2}\} \cap \{X_1(s^-) = -\overline{x}_1^h\}) \\
&\qquad\qquad\qquad + P(\{X_2(s^-) = \frac{h}{2}\} \cap \{X_1(s^-) = \overline{x}_1^h\})\Big] \\
&+\beta\, h\, E \int_0^t |b_1(X(s))|\, ds + \varepsilon\, E \int_0^t X_1(s)\, b_2(X(s))\, ds + \varepsilon\, E \int_0^t X_2(s)\, b_1(X(s))\, ds \\
&+\varepsilon\, h\, E \int_0^t X_1(s^-)\, dN^-(s) - \varepsilon\, h\, E \int_0^t X_1(s^-)\, dN^+(s) \\
&+2\, E \int_0^t X_2(s)\, b_2(X(s))\, ds + h\, E \int_0^t |b_2(X(s))|\, ds + \sigma^2 t \\
&+2\, h\, E \sum_{s\leq t} X_2(s^-)\, \left[\mathbb{I}_{\{X_2(s^-) = -\overline{x}_2^h\}} - \mathbb{I}_{\{X_2(s^-) = \overline{x}_2^h\}}\right] \\
&+h^2\, E \sum_{s\leq t} \left[\mathbb{I}_{\{X_2(s^-) = -\overline{x}_2^h\}} - \mathbb{I}_{\{X_2(s^-) = \overline{x}_2^h\}}\right] .
\end{aligned}
$$

the sum of terms 2, 3 and 4 is negative for h and ε small enough. The sum of the last two terms is negative as soon as $h < 2\,\overline{x}_2^h$.

We chose a law for $X(0)$ which is symmetrical with respect to 0, and the two axes. Then we get, using the symmetrical/dissymmetrical nature of the problem

$$
E \int_0^t X_1(s^-)\, dN^-(s) \leq 0\ , \qquad E \int_0^t X_1(s^-)\, dN^+(s) \geq 0\ .
$$

So

$$
\begin{aligned}
V(t) - V(0) \ \leq\ & 2\,\beta\, E \int_0^t X_1(s)\, X_2(s)\, \mathbb{I}_{\{|X_1(s)| < \overline{x}_1^h\}}\, ds + \varepsilon\, E \int_0^t X_1(s)\, b_2(X(s))\, ds \\
& + \varepsilon\, E \int_0^t X_2(s)\, b_1(X(s))\, ds + 2\, E \int_0^t X_2(s)\, b_2(X(s))\, ds + \sigma^2 t \\
& + h\, \beta\, E \int_0^t |b_1(X(s))|\, ds + h\, E \int_0^t |b_2(X(s))|\, ds\ ,
\end{aligned}
$$

which is the same expression as in lemma 2.1, except for the last two terms. But these terms are of linear growth with respect to X_1, X_2, with a multiplicative coefficient which tends to 0 as $h \to 0$. The first term is also different, but

$$
\begin{aligned}
2\,\beta\, E \int_0^t X_1(s)\, X_2(s)\, \mathbb{I}_{\{|X_1(s)| < \overline{x}_1^h\}}\, ds \ =\ & 2\,\beta\, E \int_0^t X_1(s)\, X_2(s)\, ds \\
& - 2\,\beta\, E \int_0^t X_1(s)\, X_2(s)\, \mathbb{I}_{\{|X_1(s)| = \overline{x}_1^h\}}\, ds\ ,
\end{aligned}
$$

and this last term is negative. This shows (35). □

Since μ_u has a density, the tools used for the proof of lemma 2.2 lead to

$$
\langle f^u\, ,\, \overline{\mu}_u^h \rangle \underset{h\to 0}{\longrightarrow} \langle f^u\, ,\, \mu_u \rangle\ ,
$$

which proves the

Theorem 4.1

$$
J_{h,D_h}(u) \underset{h\to 0}{\longrightarrow} J(u)\ , \quad \forall u \in \mathcal{U}\ .
$$

Remark 4.9 In [1], we proved the existence of an optimal feedback control law for the discretized problem. With such a control, we can associate a feedback control law $\hat{u}_h$ for the continuous state space problem, where $\hat{u}_h$ is piecewise constant. Using theorem 4.1 we can easely conclude that

$$\limsup_{h\to 0} J_{h,D_h}(\hat{u}_h) \leq \inf_{u\in\mathcal{U}} J(u) \ .$$

We would like to prove the stronger result that the sequence $\{\hat{u}_h; h > 0\}$ is a minimizing sequence for the functional J, i.e.

$$J(\hat{u}_h) \to \inf_{u\in\mathcal{U}} J(u) \ , \quad \text{quand } h \to 0 \ .$$

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