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# Optimal Ergodic Control for a Class of Nonlinear Stochastic Systems Application to Semi-Active Vehicle Suspensions*

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**Abstract** *We study a class of ergodic stochastic control problems for diffusion processes. We present a numerical approximation to the optimal feedback control based on the discretization of the infinitesimal generator using finite difference schemes. Finally, we apply these techniques to the control of semi-active suspensions for road vehicle.*

This paper deals with a numerical procedure for optimal stochastic control problems and its application to a non trivial example. This procedure consists in approximating the non linear Hamilton–Jacobi–Bellman partial differential equation which is formally satisfied by the minimal cost function. We use finite difference techniques and with a suitable choice of the schemes, the resulting discrete equation can be viewed as the dynamic programming equation for the minimal cost function for the optimal control of a certain Markov process with finite state space [14].

In section 1, we introduce a particular class — denoted by $\mathcal{C}$ — of ergodic control problems. Some characteristics of this problem are non classical (the diffusion is degenerate, the coefficients are non linear and discontinuous) and there is no available result concerning the HJB equation. This class of problems derives from a particular application in control of suspension systems [3].

In section 2, the approximation procedure is detailed in a more general context than the class $\mathcal{C}$. For the special case of the class $\mathcal{C}$ we have already stated two types of results [3]: existence and uniqueness property for the discrete HJB equation (with convergence of the algorithm used for solving it) and a convergence property of the approximation as the discretization step tends to 0. Finally, we apply these techniques to the suspension problem [3,2] and perform some numerical tests; related suboptimal and adaptive techniques may be found in [2].

## 1 A Class of Ergodic Stochastic Control Problems

We present a class of models which derive from a control problem for semi-active suspension systems. In these models — like in most realistic models — difficulties of the following type are met: the coefficients of the diffusion which we want to control are discontinuous and strongly nonlinear. In section 1.1 we introduce the class $\mathcal{C}$ of problems. In section 1.2, we present the original semi-active suspensions problem.

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### 1.1 The problem

Let us consider the following stochastic system

$$dX_t = b(u(X_t), X_t)\,dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW_t\,, \tag{1}$$

where $X$ is a process which takes values in $\mathbb{R}^2$, $W$ is a scalar standard Wiener process and $\sigma > 0$. $b$ maps $\mathbb{R} \times \mathbb{R}^2$ in $\mathbb{R}^2$ and is defined by

$$b(u,x) \triangleq \begin{pmatrix} x_2 \\ -u\,x_2 - \gamma_1\,x_1 - \gamma_2\,\mathrm{sign}(x_2) \end{pmatrix}, \quad x \triangleq \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

where $\gamma_1$, $\gamma_2$ are strictly positive constants. In (1), $u$ is a feedback control which belongs to the class $\mathcal{U}$ of admissible controls defined by (fix $\underline{u}$, $\overline{u}$ such that $0 < \underline{u} < \overline{u} < \infty$)

$u \in \mathcal{U} \iff$ $u : \mathbb{R}^2 \to [\underline{u}, \overline{u}]$ and there exists a finite number of submanifolds of $\mathbb{R}^2$ with dimension less than or equal to 1 outside of which $u$ is continuous.

We are concerned with an ergodic type control problem, whose cost functional is

$$J(u) \triangleq \lim_{T\to\infty} \frac{1}{T} E \int_0^T f(u(X_t), X_t)\,dt\,,\ \forall u \in \mathcal{U}\,, \tag{2}$$

where the instantaneous cost function $f$ is defined by

$$f(u,x) \triangleq (u\,x_2 + \gamma_1\,x_1 + \gamma_2\,\mathrm{sign}(x_2))^2\,. \tag{3}$$

We denote $b^u(x) \triangleq b(u(x),x)$, $f^u(x) \triangleq f(u(x),x)$.

The Hamilton–Jacobi–Bellman equation for the ergodic control problem (1,2) can be formally written as (cf. [4,15,16])

$$\min_{u\in[\underline{u},\overline{u}]} (\mathcal{L}^u\, v(x) + f(u,x)) = \rho\ \forall x \in \mathbb{R}^2, \tag{4}$$

where $v : \mathbb{R}^2 \to \mathbb{R}$ is defined up to an additive constant, $\rho$ is a constant and $\mathcal{L}^u$ is the infinitesimal generator associated with (1)

$$\mathcal{L}^u\,\phi(x) \triangleq b_1^u(x)\,\frac{\partial\phi(x)}{\partial x_1} + b_2^u(x)\,\frac{\partial\phi(x)}{\partial x_2} + \frac{\sigma^2}{2}\,\frac{\partial^2\phi(x)}{\partial^2 x_2}\,.$$

**Remark 1.1** The techniques presented in this paper may be applied to a wider class of problems. Indeed, we can consider a system of the form

$$d\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} b_1(X_t) \\ b_2(u, X_t) \end{pmatrix} dt + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} dW_t\,,$$

where $X_t^i$ takes values in $\mathbb{R}^{n_i}$ ($i = 1,2$) and $W$ is a standard $R^{n_2}$-valued Wiener process. The main hypotheses are

(i) the discontinuous terms appear only in the "noisy part" of the system, that is $b_1(x)$ is smooth and $\sigma\sigma^* > 0$,

(ii) the system satisfies a stability property (e.g. $E|X_t|^2 \le C$, $\forall t \ge 0$).

Point (i) permits us to use a Girsanov transformation to remove the discontinuous terms.

**Remark 1.2** In this case the choice of the value of the function "sign" at point 0 is not important. Indeed, in (1) the noise is added to the second component, so we can prove that $P(X_t^2 = 0) = 0$, $\forall t$. This property implies that, if we change the value of sign(0), the (weak) solution of (1) will not be changed. If it was false, we should use differential inclusion techniques to give a meaning to the stochastic differential equation (1).

**Remark 1.3** The cost function (2) can be rewritten as $J(u) = \langle f^u, \mu_u \rangle$, where $\mu_u$ is the invariant probability measure associated with system (1). In [9] we established an existence and uniqueness property for $\mu_u$ which gives a meaning to expression (2).

## 1.2 An example: a semi–active suspension system

In this section we present a damping control method for a nonlinear suspension of road vehicle (comprising a spring, a shock absorber, a mass, and taking into account the dry friction, cf. figure 1). The aim is to improve the ride comfort.

Among alternatives to classical suspension systems (passive systems) we distinguish between active and semi–active techniques. An active suspension system consists in force elements in addition to a spring and a damper assembly. Force elements continuously vary the force according to some control law. In general, an active system is expensive, complicated, and requires an external power source [12]. In contrast, a semi–active system requires no hydraulic power supply, and its hardware implementation is simpler and cheaper than a fully active system. A semi–active suspension system acts only on damping or spring laws, so it can only dissipate or store energy.

Here we consider a system with control on the damping law, the forces in the damper are generated by modulating its orifice for fluid flow [1,17]. We use the simplest model which consists in a one degree–of–freedom model (this model can be represented as a problem of the class $\mathcal{C}$).

The equation of motion for a one degree–of–freedom model is (cf. figure 1 for the exact definition of the terms)

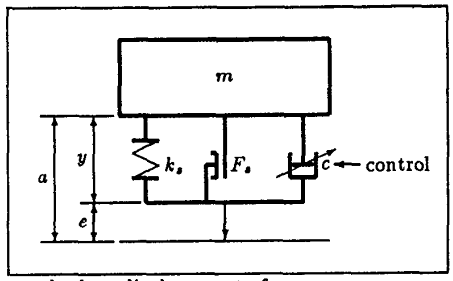
$$m\,\ddot{y} + c\,\dot{y} + k_s\,y + F_s\,\mathrm{sign}(\dot{y}) = -m\,\ddot{e}\,. \tag{5}$$

$\ddot{e}$ denotes the input acceleration. The restoring force $k_s\,y + F_s\,\mathrm{sign}(\dot{y})$, has a linear part $k_s\,y$, and a nonlinear part $F_s\,\mathrm{sign}(\dot{y})$ which describes the dry friction force (Coulomb friction force) [6,7]. The damping force is $c\,\dot{y}$ where $c > 0$ is the instantaneous damping coefficient (the control is acting on this term).

The problem is to compute a feedback law $c = c(y, \dot{y})$ such that the solution of the system (5) minimizes a criterion — related to the vibration comfort

$$J(u) \triangleq \lim_{T\to\infty} \frac{1}{T} E \int_0^T |\ddot{a}|^2\,dt = \lim_{T\to\infty} \frac{1}{T} E \int_0^T |\ddot{y} + \ddot{e}|^2\,dt\,.$$

$\ddot{e}$ is supposed to be a white Gaussian noise process, $\ddot{e} = -\sigma\,dW/dt$ where $W$ is a standard Wiener process.




| | |
|---|---|
| $a$ | absolute displacement of mass $m$ |
| $y$ | absolute displacement ($y = a - e$) |
| $e$ | stochastic input (surface road acceleration) |
| $m$ | sprung mass |
| $c$ | shock–absorber damping constant (controlled) |
| $k_s$ | spring constant |
| $F_s$ | dry friction constant |

Figure 1: One degree–of–freedom model.

Using $u = c/m$, $\gamma_1 = k_s/m$, $\gamma_2 = F_s/m$ and $X \triangleq \begin{pmatrix} y \\ \dot{y} \end{pmatrix}$, equation (5) can be rewritten as (1), with

$$b(u,x) \triangleq \begin{pmatrix} x_2 \\ -u\,x_2 - \gamma_1\,x_1 - \gamma_2\,\mathrm{sign}(x_2) \end{pmatrix},$$

$$f(u,x) \triangleq |\ddot{y} + \ddot{e}|^2 = |u\,x_2 + \gamma_1\,x_1 + \gamma_2\,\mathrm{sign}(x_2)|^2\,.$$

# 2 Numerical Approximation

We use the following procedure: we do not discretize directly the HJB equation but we transform the original ergodic control problem to a control problem for a Markov process in continuous time and finite state space (section 2.1). Then, for the discrete case, we can write a dynamic programming equation (section 2.2); this equation is solved numerically via an iterative algorithm (section 2.3).

We describe the approximation procedure in the case of a diffusion process defined by

$$dX_t = b(u(X_t), X_t)\,dt + \sigma(X_t)\,dW_t\,, \tag{6}$$

and with the following cost function

$$J(u) = \liminf_{T\to\infty} \frac{1}{T} E \int_0^T f(u(X_t), X_t)\,dt\,. \tag{7}$$

$X$ takes values in $\mathbb{R}^n$ and $W$ in $\mathbb{R}^d$. $u$ belongs to a given class $\mathcal{U}$ of applications from $\mathbb{R}^n$ to $U \subset \mathbb{R}^k$. We suppose that, for any $u \in \mathcal{U}$, the solution $X_t$ of (6) admits a unique invariant probability measure, so the cost function (7) is well defined.

The infinitesimal generator associated with (6) is

$$\mathcal{L}^u\phi(x) \triangleq \sum_{i=1}^n b_i^u(x)\,\frac{\partial\phi(x)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^n a_{ij}(x)\,\frac{\partial^2\phi(x)}{\partial x_i\,\partial x_j}\,, \tag{8}$$

where $a(x) \triangleq \sigma(x)\sigma^*(x)$, $b_i^u(x) = b_i(u,x)$ and $f^u(x) = f(u,x)$.

## 2.1 The finite state space problem

In a first step we approximate the solution $X_t$ of (6) by a controlled Markov process $X_t^h$ in continuous time and *discrete* (but infinite) state space. In a second step, $X_t$ is approximated by a controlled Markov process $X_t^{h,D}$ in continuous time and *finite* state space.

**First step: discrete state space** Let $h_i$ (resp. $e_i$) denote the finite difference interval (resp. the unit vector) in the $i$th coordinate direction and $h = (h_1, \dots, h_n)$. We define $\mathbb{R}_h^n$, the $h$-grid on $\mathbb{R}^n$, by

$$\mathbb{R}_h^n \triangleq \{x \in \mathbb{R}^n \, ; \, x_i = n_i \, h_i + h_i/2 \, , \; i = 1, \dots, n, \; n_i \in \mathbb{Z}\} \; .$$

The infinitesimal generator (8) is approximated using finite difference schemes given in table 1. The reason for the choices in the schemes will be explained below.

$\mathcal{L}^u$ is approximated by an infinite dimensional matrix $\mathcal{L}_h^u$ given as follows

$$\mathcal{L}^u \phi(x) \simeq \mathcal{L}_h^u \phi(x) \triangleq \sum_{y \in \mathbb{R}_h^n} \mathcal{L}_h^u(x,y) \, \phi(y) \, , \; \forall x \in \mathbb{R}_h^n$$

the terms $\mathcal{L}_h^u(x,y)$ of this matrix are given in table 2.

The matrix $\mathcal{L}_h^u$ satisfies : $\sum_{y \in \mathbb{R}_h^n} \mathcal{L}_h^u(x,y) = 0$, $\forall x \in \mathbb{R}_h^n$. Suppose that

$$a_{ii}(x) - \sum_{j;j \neq i} |a_{ij}(x)| \geq 0 \, , \; \forall x \in \mathbb{R}_h^n, \; i = 1, \dots, n \; , \qquad (9)$$

then $\mathcal{L}_h^u(x,y) \geq 0$, $\forall x, \, y \in \mathbb{R}_h^n, \; x \neq y$.

**Remark 2.1** The choice of the finite difference schemes we use (cf. table 1) depends on the sign of the drift coefficients of the diffusion process. The reason for this choice is the following: if (9) is true then $\{\mathcal{L}_h^u(x,y) \, ; \, x, \, y \in \mathbb{R}_h^n\}$ can be viewed as the infinitesimal generator of a controlled Markov process $X_t^h$ in continuous-time and discrete state space $\mathbb{R}_h^n$ [11,14]. We will see later why this is important.

We get a stochastic control problem for a Markov process $X_t^h$ with infinitesimal generator $\mathcal{L}_h^u$, and the following cost function

$$J_h(u) \triangleq \lim_{T \to \infty} E \frac{1}{T} \int_0^T f^u(X_t^h) \, dt \; ,$$

and $u \in \mathcal{U}_h \triangleq \{u : \text{application from } \mathbb{R}_h^n \text{ to } U\}$.

**Second step: finite state space** $X_t^h$ has a discrete but infinite state space; if we want to perform computations it is necessary to work on a finite state space. We consider a bounded domain $D$ of $\mathbb{R}^n$. We define a new state space

$$\mathbb{R}_{h,D}^n \triangleq \mathbb{R}_h^n \cap D = \{x^1, \dots, x^N\} \, , \; N \triangleq \mathrm{Card}\left(\mathbb{R}_{h,D}^n\right) . \quad (10)$$

Because we are working on a bounded domain, we must specify boundary conditions. In practice, $D$ will be chosen large enough so that the process will rarely reach the border. Hence, the choice of the boundary conditions is of little importance, provided that all the states communicate. Example of such conditions (usually reflecting conditions) will be given later for the suspension problem.

So we get an approximation $\mathcal{L}_{h,D}^u$ to $\mathcal{L}_h^u$

$$\mathcal{L}_{h,D}^u \, \phi(x) = \sum_{y \in \mathbb{R}_{h,D}^n} \mathcal{L}_{h,D}^u(x,y) \, \phi(y) \; ,$$

$\mathcal{L}_{h,D}^u$ is a $N \times N$-matrix.

**Remark 2.2** The choice in the finite difference schemes (cf. table 1) imply that $\sum_{y \in \mathbb{R}_{h,D}^n} \mathcal{L}_{h,D}^u(x,y) = 0$ ($\forall x \in \mathbb{R}_{h,D}^n$), moreover, hypothesis (9) implies that $\mathcal{L}_{h,D}^u(x,y) \geq 0$, $\forall x, \, y \in \mathbb{R}_{h,D}^n, \; x \neq y$.

Hence $\mathcal{L}_{h,D}^u$ can be interpreted as the infinitesimal generator of a controlled Markov process $X_t^{h,D}$ in continuous time and finite state space (see [8,11] for more details).

With remark 2.2, the discretized problem can be viewed as a control problem for a Markov process $X_t^{h,D}$ in continuous time, finite state space, and infinitesimal generator $\mathcal{L}_{h,D}^u$. The cost function is

$$J_{h,D}(u) \triangleq \lim_{T \to \infty} E \frac{1}{T} \int_0^T f^u(X_t^{h,D}) \, dt \; , \qquad (11)$$

and $u \in \mathcal{U}_{h,D} \triangleq \left\{u : \text{application from } \mathbb{R}_{h,D}^n \text{ to } U\right\}$.

**Remark 2.3** Let $\mu_u^{h,D}$ be the invariant measure of the process $X_t^{h,D}$. Using $\mu_u^{h,D}$, the cost function (11) can be rewritten as

$$J_{h,D}(u) = \sum_{x \in \mathbb{R}_{h,D}^n} f^u(x) \, \mu_u^{h,D}(x) \; .$$

The measure $\mu_u^{h,D}$ is solution of the following linear system

$$\begin{cases} \displaystyle\sum_{y \in \mathbb{R}_{h,D}^n} \mathcal{L}_{h,D}^u(y,x) \, \mu_u^{h,D}(y) = 0 \, , & \forall x \in \mathbb{R}_{h,D}^n \, , \\ \displaystyle\sum_{y \in \mathbb{R}_{h,D}^n} \mu_u^{h,D}(y) = 1 \, . & \end{cases}$$

## 2.2 The "discrete" Hamilton–Jacobi–Bellman equation

Associated with the control problem defined in the last section we have the following dynamic programming equation

$$\min_{u \in U} \left[ \sum_{y \in \mathbb{R}_{h,D}^n} \mathcal{L}_{h,D}^u(x,y) \, v(y) + f^u(x) \right] = \rho \, , \; \forall x \in \mathbb{R}_{h,D}^n \, , \qquad (12)$$

where $\rho$ is a strictly positive constant and $v : \mathbb{R}_{h,D}^n \to \mathbb{R}$ (i.e. $v \in \mathbb{R}^N$) is defined up to an additive constant.

If $(v, \rho)$ is a solution to (12) then

$$\hat{u}(x) \in \mathrm{Arg} \min_{u \in U} \left[ \sum_{y \in \mathbb{R}_{h,D}^n} \mathcal{L}_{h,D}^u(x,y) \, v(y) + f^u(x) \right] \; , \qquad (13)$$

is an optimal feedback control law, and $\rho$ is the minimal cost : $\rho = J_{h,D}(\hat{u}) = \min_{u \in \mathcal{U}_{h,D}} J_{h,D}(u)$.

Equation (12) gives the solution to the ergodic control problem for the Markov process $X_t^{h,D}$. It can also be viewed as an approximation to the HJB equation (4).

## 2.3 The policy iteration algorithm

In order to solve (12), we use the policy iteration algorithm [10,13]: suppose that $u^0 \in \mathcal{U}_{h,D}$ — the initial policy — is given. Starting with $u^0$ we generate a sequence $\{u^j ; j \geq 1\}$. The iteration $u^j \to u^{j+1}$ proceeds in two steps (cf. table 3).

$$\frac{\partial \phi(x)}{\partial x_i} \simeq \begin{cases} \dfrac{\phi(x+e_i\,h_i)-\phi(x)}{h_i} & \text{if } b_i^u(x) > 0 \\ \dfrac{\phi(x)-\phi(x-e_i\,h_i)}{h_i} & \text{if } b_i^u(x) < 0 \end{cases}, \qquad \frac{\partial^2 \phi(x)}{\partial x_i^2} \simeq \frac{\phi(x+e_i\,h_i) - 2\,\phi(x) + \phi(x-e_i\,h_i)}{h_i^2}$$

$$\frac{\partial^2 \phi(x)}{\partial x_i\,\partial x_j} \simeq \begin{cases} \dfrac{2\,\phi(x)+\phi(x+e_i\,h_i+e_j\,h_j)+\phi(x-e_i\,h_i-e_j\,h_j)}{2\,h_i\,h_j} - \dfrac{\phi(x+e_i\,h_i)+\phi(x-e_i\,h_i)+\phi(x+e_j\,h_j)+\phi(x-e_j\,h_j)}{2\,h_i\,h_j} & \text{if } a_{ij}(x) > 0 \\ -\dfrac{2\,\phi(x)+\phi(x+e_i\,h_i-e_j\,h_j)+\phi(x-e_i\,h_i+e_j\,h_j)}{2\,h_i\,h_j} + \dfrac{\phi(x+e_i\,h_i)+\phi(x-e_i\,h_i)+\phi(x+e_j\,h_j)+\phi(x-e_j\,h_j)}{2\,h_i\,h_j} & \text{if } a_{ij}(x) < 0 \end{cases}$$

$i, j = 1, \ldots, n\,,\; i \neq j\,,\quad e_i$ unit vector in the $i$th coordinate direction

Table 1: Finite difference schemes.

$$\mathcal{L}_h^u(x,x) \triangleq -\sum_i \left(\frac{a_{ii}(x)}{h_i^2} - \frac{1}{2}\sum_{k;k\neq i}\frac{|a_{ik}(x)|}{h_i\,h_k}\right) - \sum_i \frac{|b_i^u(x)|}{h_i}$$

$$\mathcal{L}_h^u(x,x+e_i\,h_i) \triangleq \frac{1}{2}\left(\frac{a_{ii}(x)}{h_i^2} - \sum_{k;k\neq i}\frac{|a_{ik}(x)|}{h_i\,h_k}\right) + \frac{(b_i^u(x))^+}{h_i}\,, \quad \mathcal{L}_h^u(x,x-e_i\,h_i) \triangleq \frac{1}{2}\left(\frac{a_{ii}(x)}{h_i^2} - \sum_{k;k\neq i}\frac{|a_{ik}(x)|}{h_i\,h_k}\right) + \frac{(b_i^u(x))^-}{h_i}$$

$$\mathcal{L}_h^u(x,x+e_i\,h_i+e_j\,h_j) = \mathcal{L}_h^u(x,x-e_i\,h_i-e_j\,h_j) \triangleq \frac{a_{ij}^+(x)}{2\,h_i\,h_j}\,, \quad \mathcal{L}_h^u(x,x+e_i\,h_i-e_j\,h_j) = \mathcal{L}_h^u(x,x-e_i\,h_i+e_j\,h_j) \triangleq \frac{a_{ij}^-(x)}{2\,h_i\,h_j}$$

$i, j = 1, \ldots, n\,,\; i \neq j$

Table 2: The discrete infinitesimal generator.

**Remark 2.4** The first step of this algorithm leads to a linear system of dimension $N$. Let $\mathbb{R}^2_{h,D} = \{x^i\,;\, i = 1, \ldots, N\}$, then the unknown parameters are $v(x^2)$, $v(x^3), \ldots,\; v(x^N)$, $\rho$ and we take $v(x^1) = 0$.

**Remark 2.5** For the second step, the optimization problems are nonlinear and they are solved by means of iterative algorithms. The nonlinearity comes from the discretization technique we use. Indeed, the choice of finite difference approximation (cf. table 1) depends on $u$. Instead of the schemes of the table 1, we can use centered difference approximation (so that it does not depend on $u$), in which case the second step becomes explicit because the functions to be optimized are now quadratic in $u$. On the other hand, with this kind of difference approximation, a certain condition on the parameter $h$ has to be fulfilled ($h$ must be small enough) for the matrix $\mathcal{L}_u^{h,D}$ to be the generator of a Markov process. See [14] p.175–179 for further considerations.

## 2.4 Application to the class of problem $\mathcal{C}$

### 2.4.1 The approximation

In this example, the discretized state space are $\mathbb{R}^2_h$ and $\mathbb{R}^2_{h,D}$ where $h = (h_1, h_2)$ and $D$ is of the form $D = [-\overline{x}_1, \overline{x}_1] \times [-\overline{x}_2, \overline{x}_2]$, so that

$$\mathbb{R}^2_{h,D} = \left\{x_1^{(1)},\, x_1^{(2)}, \ldots, x_1^{(N_1)}\right\} \times \left\{x_2^{(1)},\, x_2^{(2)}, \ldots, x_2^{(N_2)}\right\}\,,$$

with $x_i^{(k)} = -\overline{x}_i + 2\,\overline{x}_i\,(k-1)\,h_i$, $h_i = 1/(N_i - 1)$, ($k = 1, \ldots, N_i$, $i = 1, 2$).

The matrix $a(x) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix}$ is degenerate. Condition (9) is fulfilled.

For this example we give explicit boundary conditions. Let

$$\Gamma_{h,D} \triangleq \left\{x_1^{(1)}, x_1^{(N_1)}\right\} \times \left\{x_2^{(1)},\, x_2^{(2)}, \ldots, x_2^{(N_2)}\right\} \cup \left\{x_1^{(1)},\, x_1^{(2)}, \ldots, x_1^{(N_1)}\right\} \times \left\{x_2^{(1)}, x_2^{(N_2)}\right\}\,.$$

$\Gamma_{h,D}$ the set of points on the border. We choose very simple reflecting conditions and we obtain the matrix $\mathcal{L}^u_{h,D}$ described table 4.

### 2.4.2 The convergence results

We present two kinds of results. Firstly, considering the discrete HJB equation (12), we can prove that it admits a unique solution and that the policy iteration algorithm converges to this unique solution. Secondly, we can also prove a convergence result for the approximation as the discretization step $h$ tends to 0. These results are presented for the class $\mathcal{C}$.

#### Existence and uniqueness of a solution to the discrete HJB equation

**Theorem 2.6** *The HJB equation (12) (with $v(x^1) = 0$) admits a unique solution $(v, \rho) \in \mathbb{R}^N \times \mathbb{R}^+$.*

For the existence part of theorem (2.6), we use the following

**Lemma 2.7** *The policy iteration algorithm converge to an optimal feedback control.*

| | | |
|---|---|---|
| **1** | *compute* $(v^j, \rho^j)$ | we compute $(v^j, \rho^j) \in \mathbb{R}^N \times \mathbb{R}^+$ the solution of the linear system $$\sum_{y \in \mathbb{R}^n_{h,D}} \mathcal{L}^{u^j}_{h,D}(x,y)\, v^j(y) + f^{u^j}(x) = \rho^j \,, \ \forall x \in \mathbb{R}^n_{h,D}$$ |
| | *stopping test* | $\lvert \rho^{j+1} - \rho^j \rvert \le \varepsilon$. |
| **2** | *compute* $u^{j+1}$ | we solve the $N$ following optimization problems: for any $x \in \mathbb{R}^2_{h,D}$ $$u^{j+1}(x) \in \text{Arg} \min_{u \in [\underline{u}, \overline{u}]} \left( \sum_{y \in \mathbb{R}^2_{h,D}} \mathcal{L}^{h,D}_u(x,y)\, v^j(y) + f^u(x) \right) .$$ |

Table 3: The policy iteration algorithm, iteration $u^j \longrightarrow u^{j+1}$.

| | | |
|---|---|---|
| for $x \in \mathbb{R}^2_{h,D} \setminus \Gamma_{h,D}$ et $y \in \mathbb{R}^2_{h,D}$ | $\mathcal{L}^u_{h,D}(x,y) = \mathcal{L}^u_h(x,y)$ | |
| for $x \in \Gamma_{h,D}$ such that $x_1 = x_1^{(1)}$ | $\mathcal{L}^u_{h,D}(x,x) = \mathcal{L}^u_h(x,x)$ | $\mathcal{L}^u_{h,D}(x, x + h_1 e_1) = -\mathcal{L}^u_h(x,x)$ |
| for $x \in \Gamma_{h,D}$ such that $x_1 = x_1^{(N_1)}$ | $\mathcal{L}^u_{h,D}(x,x) = \mathcal{L}^u_h(x,x)$ | $\mathcal{L}^u_{h,D}(x, x - h_1 e_1) = -\mathcal{L}^u_h(x,x)$ |
| for $x \in \Gamma_{h,D}$ such that $x_2 = x_2^{(1)}$ | $\mathcal{L}^u_{h,D}(x,x) = \mathcal{L}^u_h(x,x)$ | $\mathcal{L}^u_{h,D}(x, x + h_2 e_2) = -\mathcal{L}^u_h(x,x)$ |
| for $x \in \Gamma_{h,D}$ such that $x_2 = x_2^{(N_2)}$ | $\mathcal{L}^u_{h,D}(x,x) = \mathcal{L}^u_h(x,x)$ | $\mathcal{L}^u_{h,D}(x, x - h_2 e_2) = -\mathcal{L}^u_h(x,x)$ |
| all other terms are null | | |

Table 4: Discrete infinitesimal generator $\mathcal{L}^u_{h,D}$ (class $\mathcal{C}$).

These results are proved in [3], but one can find the same kind of results in a more general setup in [5].

**Approximation: a convergence result** We present a convergence result concerning the approximation, when the discretization parameter $h$ tends to 0 and when the domain $D$ tends to $\mathbb{R}^2$ (for a complete proof of this result cf. [9]).

**Theorem 2.8** *Consider two strictly increasing sequences* $\{\overline{x}_1^h; h > 0\}$ *and* $\{\overline{x}_2^h; h > 0\}$ *such that* $\overline{x}_i^h > 0$ *and* $\overline{x}_i^h \to \infty$ *as* $h \to 0$. *Define* $D_h = [-\overline{x}_1^h, \overline{x}_1^h] \times [-\overline{x}_2^h, \overline{x}_2^h]$. *Suppose that* $\lim_{h \to 0} h\, \delta_h = 0$, *where* $\delta_h \triangleq \text{radius}(D_h)$. *Then, for any* $u \in \mathcal{U}$, $J_{h,D_h}(u) \to J(u)$ *as* $h \to 0$.

**Remark 2.9** Theorem 2.6 proves the existence of an optimal feedback control law for the discretized problem. With such a control, we can associate a feedback control law $\hat{u}_h$ for the continuous state space problem, where $\hat{u}_h$ is piecewise constant. Using theorem 2.8 we can easily conclude that

$$\limsup_{h \to 0} J_{h,D_h}(\hat{u}_h) \le \inf_{u \in \mathcal{U}} J(u) \,.$$

We would like to prove the stronger result that the sequence $\{\hat{u}_h; h > 0\}$ is a minimizing sequence for the functional $J$, i.e. $J(\hat{u}_h) \to \inf_{u \in \mathcal{U}} J(u)$, when $h \to 0$.

### 2.4.3 A numerical example

**Parameters** As an example, we use values which roughly correspond to a suspension system for the seat of a truck driver : $m = 60$(kg), $k_s = 3500$(N/m), $F_s = 40$(N). These values have already been used in [3]. We also set $\sigma = 0.5$.

We use the following discretization parameters $\overline{x}_1 = y_{max} = -y_{min} = 0.1$ (m), $n_1 = 30$, $\overline{x}_2 = \dot{y}_{max} = -\dot{y}_{min} = 1$ (m/s), $n_2 = 30$.

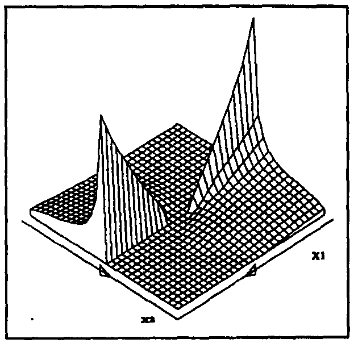



Figure 2: The optimal feedback control.

**Optimal feedback control[3]** The approximated optimal feedback control (13) (plotted on figure 2) is computed using the policy iteration algorithm. The value of the minimal cost is given below.

**Suboptimal feedback control #1** One possibility is to find a feedback control which minimizes the instantaneous cost function (3). We obtain $\tilde{u}(x) = (-k_s x_1 \text{sign}(x_2) - F_s)/\lvert x_2 \rvert$. To take into account the constraint $\underline{u} \le u \le \overline{u}$, we use the following control law : $u(x) = (\tilde{u}(x) \vee \underline{u}) \wedge \overline{u}$ (cf. figure 3) (we take $\underline{u} = 0$ and $\overline{u}$ large).

**Suboptimal feedback control #2** The previous results lead us to the class of suboptimal feedback controls —

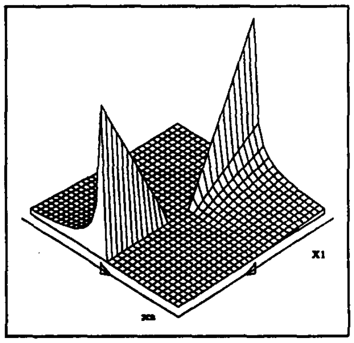


Figure 3: The suboptimal feedback control #1.

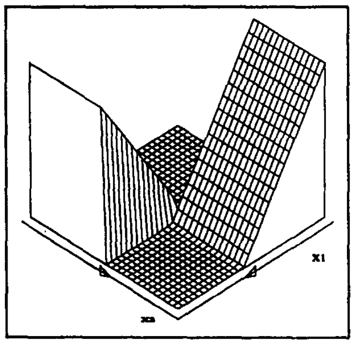


Figure 4: The suboptimal feedback control #2.

parametrized by $\theta \in \mathbb{R}^2$ — of the following form

$$u_\theta(x) \triangleq [(\theta_1 + \theta_2\, x_1\, \text{sign}(x_2)) \vee \underline{u}) \wedge \overline{u}\ ,$$

$\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$. The techniques presented above can also be applied to compute the suboptimal feedback control $u_{\hat\theta}$ such that $J_{h,D_h}(u_{\hat\theta}) = \min_{\theta\in\Theta} J_{h,D_h}(u_\theta)$, where $\Theta = \{\theta \in \mathbb{R}^2\,;\ u_\theta \in \mathcal{U}\}$. We get $\hat\theta_1 = 137.2$, $\hat\theta_2 = -12130$.

The control law $u_{\hat\theta}(x)$ is plotted on figure 4. A feedback control where the sign of the product $x_1\, x_2$ (i.e. $y\,\dot y$) appears has already been proposed in [17].

**Comparison of the feedback controls** Now we compare the three feedback controls presented above to the constant control $u(X) \equiv u_0$. The optimal constant $u_0$ (i.e. the constant which minimizes the cost) is 188. The different values of the cost are given in the following table

| *control type* | *cost* |
|---|---|
| constant control | 2.93 |
| suboptimal feedback control #1 | 2.68 |
| suboptimal feedback control #2 | 2.37 |
| optimal control | 2.22 |

# References


[1] J. ALANOLY and S. SANKAR. Semi-active force generators for shock isolation. *Journal of Sound and Vibration*, 126(1):145–156, 1988.

[2] S. BELLIZZI and R. BOUC. Adaptive suboptimal parametric control for nonlinear stochastic systems — Application to semi-active vehicle suspensions. In *Effective Stochastic, P. Krée & W. Wedig (eds.)*, 1989. To appear.

[3] S. BELLIZZI, R. BOUC, F. CAMPILLO, and E. PARDOUX. Contrôle optimal semi-actif de suspension de véhicule. In *Analysis and Optimization of Systems, A. Bensoussan and J.L. Lions (eds.)*, INRIA, Antibes, 1988. LNCIS 111, Srpinger Verlag, 1988.

[4] A. BENSOUSSAN. *Perturbation Methods in Optimal Control.* John Wiley & Sons, New–York, 1988.

[5] D.P. BERTSEKAS. *Dynamic Programming and Stochastic Control.* Academic Press, New–York, 1976.

[6] R. BOUC. Forced vibration of mechanical system with hysteresis. In *Proceedings of 4th conference ICNO*, Prague, 1967. Résumé.

[7] R. BOUC. Modèle mathématique d'hystérésis. *Acustica*, 24(3):16–25, 1971.

[8] F. CAMPILLO. Optimal ergodic control of nonlinear stochastic systems. In *Effective Stochastic, P. Krée & W. Wedig (eds.)*, 1989. To appear.

[9] F. CAMPILLO, F. LE GLAND, and E. PARDOUX. Approximation d'un problème de contrôle ergodique dégénéré. In *Colloque International Automatique Non Linéaire*, CNRS, Nantes, 13–17 Juin 1988. To appear.

[10] B.T. DOSHI. Continuous time control of Markov processes on an arbitrary state space: average return criterion. *Stochastic Processes and their Applications*, 4:55–77, 1976.

[11] F. LE GLAND. *Estimation de Paramètres dans les Processus Stochastiques, en Observation Incomplète.* Thèse, Université de Paris IX, 1981.

[12] R.M. GOODDALL and W. KORTUM. Active controls in ground transportation — A review of the state-of-the-art and future potential. *Vehicle systems dynamics*, 12:225–257, 1983.

[13] R.A. HOWARD. *Dynamic Programming and Markov Processes.* J. Wiley, New–York, 1960.

[14] H.J. KUSHNER. *Probability Methods for Approximations in Stochastic Control and for Elliptic Equations.* Academic Press, New–York, 1977.

[15] P.L. LIONS. On the Hamilton–Jacobi–Bellman equations. *Acta Applicandae Mathematicae*, 1:17–41, 1983.

[16] M. ROBIN. Long–term average cost control problems for continuous time Markov processes: a survey. *Acta Applicandae Mathematicae*, 1:281–299, 1983.

[17] S. TAKAHASHI, T. KANEKO, and K. TAKAHASHI. A damping force control method which reduce energy to the vehicle body. *JSAE Review*, 8(3):95–98, 1987.