Algorithms for Constructing Stable Manifolds of Stationary Solutions

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Abstract. Algorithms for computing stable manifolds of hyperbolic stationary solutions of autonomous systems are of two types: either the aim is to compute a single point on the manifold or the entire (local) manifold. Traditionally only indirect methods have been considered, i.e. first the continuous problem is discretised by a one-step scheme and then the Liapunov-Perron or Hadamard graph transform are applied to the resulting discrete dynamical system. We will consider different variants of these indirect methods but also algorithms of the above two types which are applied directly to the continuous problem.

1. Introduction

We consider an autonomous system

$$\dot{\mathbf{u}} = \mathbf{F}(\mathbf{u}) \qquad \mathbf{F} : \Re^m \mapsto \Re^m \tag{1.1}$$

possessing a hyperbolic stationary solution, which without loss of generality we assume to be the origin; i.e. $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, $\mathbf{F}$ is differentiable at the origin with $A \equiv F'(\mathbf{0})$, and the $m \times m$ matrix A has no zero or purely imaginary eigenvalues. Our basic condition on the growth of F is that $\exists\, \gamma > 0$ such that

$$\mathbf{G}(\mathbf{x}) \equiv \mathbf{F}(\mathbf{x}) - A\mathbf{x} = O(\|\mathbf{x}\|^{1+\gamma}), \tag{1.2}$$

where $\|.\|$ denotes the Euclidean norm. [The precise conditions required for different results are stated in section 6.] Hence $\gamma = 1$ for a generic smooth $\mathbf{F}$, but we may have $\gamma > 1$ if certain higher-order terms are missing or $\gamma < 1$ if $\mathbf{F}$ lacks smoothness.

Our aim in this paper is to introduce new algorithms for approximating the local stable and unstable manifolds of zero for (1.1). We shall also describe the behaviour of certain algorithms in terms of γ. For the rest of this introduction, we briefly describe the elementary properties of these manifolds, for more details see [12]. They are denoted by $\mathcal{W}^s_{loc}$ & $\mathcal{W}^u_{loc}$ respectively and defined, for a neighbourhood $\mathcal{U}$ of the origin, by

$$\mathcal{W}^s_{loc} \equiv \{\mathbf{x} \in \mathcal{U} : S(t)\mathbf{x} \in \mathcal{U} \quad \forall t \geq 0 : \lim_{t\to\infty} S(t)\mathbf{x} = \mathbf{0}\}$$

and

$$\mathcal{W}^u_{loc} \equiv \{\mathbf{x} \in \mathcal{U} : S(t)\mathbf{x} \in \mathcal{U} \quad \forall t \leq 0 : \lim_{t\to-\infty} S(t)\mathbf{x} = \mathbf{0}\},$$

where $S(t)\mathbf{x}$ is the solution of (1.1) at time t for initial value $\mathbf{x}$. We make frequent use of the decomposition

$$\Re^m = \mathcal{E}^s \oplus \mathcal{E}^u;$$

where the stable subspace $\mathcal{E}^s$ is the invariant subspace of A corresponding to those eigenvalues with negative real part, while the unstable subspace $\mathcal{E}^u$ is the invariant subspace of A corresponding to those eigenvalues with positive real part. These subspaces have dimensions p^s & $p^u \equiv m - p^s$ respectively, and we use P^s & P^u to denote the projections induced by the above direct sum. $\mathcal{W}^s_{loc}$ & $\mathcal{W}^u_{loc}$ are p^s & p^u dimensional manifolds respectively and can be parametrised by $\mathcal{E}^s$ & $\mathcal{E}^u$ near the origin: hence we may write

$$\mathcal{W}^s_{loc} \equiv \{\boldsymbol{\xi} + \mathbf{z}^s(\boldsymbol{\xi}) \quad \text{where } \boldsymbol{\xi} \in \mathcal{E}^s_\varepsilon \text{ and } \mathbf{z}^s : \mathcal{E}^s_\varepsilon \mapsto \mathcal{E}^u\}$$

and

$$\mathcal{W}^u_{loc} \equiv \{\boldsymbol{\eta} + \mathbf{z}^u(\boldsymbol{\eta}) \quad \text{where } \boldsymbol{\eta} \in \mathcal{E}^u_\varepsilon \text{ and } \mathbf{z}^u : \mathcal{E}^u_\varepsilon \mapsto \mathcal{E}^s\},$$

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where $\mathcal{E}^{s/u}_\varepsilon \equiv \{\mathbf{x} \in \mathcal{E}^{s/u} : ||\mathbf{x}|| \leq \varepsilon\}$. The manifolds, and thus also the functions $\mathbf{z}^s$ & $\mathbf{z}^u$, are as smooth as $\mathbf{F}$. In addition, the invariant subspaces are tangent to the corresponding manifolds at the origin and so the linearisations of both $\mathbf{z}^s$ & $\mathbf{z}^u$ are zero at the origin. Since the invariant subspaces and manifolds interchange if we replace $\mathbf{F}$ with $-\mathbf{F}$ in (1.1), it is sufficient to consider only the computation of $\mathcal{W}^s_{loc}$.

The contents of the paper are as follows. In section 2 we consider the traditional constructive methods for obtaining $\mathcal{W}^s_{loc}$, upon which the theorems for its existence are based. Where possible we also introduce new algorithms which make sense in the continuous framework of (1.1). In the past, however, $\mathcal{W}^s_{loc}$ has usually been approximated *indirectly*, by first using a difference scheme to replace (1.1) by a discrete dynamical system and then applying the traditional algorithms. In section 3, therefore, we compare both classical and new indirect methods. An alternative to this approach is to consider the *direct* [6,32] approximation of $\mathcal{W}^s_{loc}$, i.e. without first introducing a discrete analogue of (1.1). We split algorithms of this type between sections 4 and 5; the former consisting of methods which compute a single point on the manifold, while the latter is devoted to constructing the whole manifold. Finally, the proofs of various convergence results have been grouped together in section 6.

2. Classical Algorithms and Variants

In this section we consider the traditional methods for computing a single point on the manifold or constructing the whole manifold. All these procedures are iterative in character, and we are particularly concerned with how the convergence rates depend on the *nonlinearity* of $\mathbf{F}$ and the *dynamic* behaviour of (1.1). The former is measured by the constant γ in (1.2), while the latter depends on the constants α, β defined by

(2.1)
- $-\Re e(\lambda) > \alpha > 0$ for all eigenvalues λ of A with negative real part,
- $\Re e(\lambda) > \beta > 0$ for all eigenvalues λ of A with positive real part.

[As with (1.2), we shall define α, β more precisely in section 6.]

The classical method for determining a particular point on the stable manifold is the variation-of-constants approach of Liapunov & Perron [26]. The point $\mathbf{z}^s(\boldsymbol{\xi})$ on the manifold is obtained by solving (1.1) with boundary conditions

$$P^s\mathbf{u}(0) = \boldsymbol{\xi} \qquad \text{and} \qquad \lim_{t\to\infty} P^u\mathbf{u}(t) = \mathbf{0}, \tag{2.2}$$

which forces $\lim_{t\to\infty}\mathbf{u}(t) = \mathbf{0}$ and so $\mathbf{z}^s(\boldsymbol{\xi})$ is given by $P^u\mathbf{u}(0)$. By using the equivalent equation

$$\dot{\mathbf{u}} - A\mathbf{u} = \mathbf{G}(\mathbf{u}),$$

we may re-write (1.1/2.2) in the form

$$\begin{aligned} \dot{\mathbf{w}} - A\mathbf{w} &= P^s\mathbf{G}(\mathbf{w}+\mathbf{y}) & \mathbf{w}(0) &= \boldsymbol{\xi} \\ \dot{\mathbf{y}} - A\mathbf{y} &= P^u\mathbf{G}(\mathbf{w}+\mathbf{y}) & \lim_{t\to\infty}\mathbf{y}(t) &= \mathbf{0}, \end{aligned} \tag{2.3}$$

where $\mathbf{w} \equiv P^s\mathbf{u}$ and $\mathbf{y} \equiv P^u\mathbf{u}$ is a notation used throughout this paper. Hence an approximate Newton iteration method for solving (1.1/2.2) is

$$\begin{aligned} \dot{\mathbf{w}}^{(k)} - A\mathbf{w}^{(k)} &= P^s\mathbf{G}(\mathbf{w}^{(k-1)}+\mathbf{y}^{(k-1)}) & \mathbf{w}^{(k)}(0) &= \boldsymbol{\xi} \\ \dot{\mathbf{y}}^{(k)} - A\mathbf{y}^{(k)} &= P^u\mathbf{G}(\mathbf{w}^{(k-1)}+\mathbf{y}^{(k-1)}) & \lim_{t\to\infty}\mathbf{y}^{(k)}(t) &= \mathbf{0}, \end{aligned} \tag{2.4}$$

starting from $\mathbf{u}^{(0)} = \mathbf{0}$. For ε sufficiently small, a solution of (2.3) is the trajectory in $\mathcal{W}^s_{loc}$ starting from $\mathbf{u}(0) \equiv \boldsymbol{\xi} + \mathbf{z}^s(\boldsymbol{\xi})$ and iteration (2.4) converges to it at an $O(||\boldsymbol{\xi}||^\gamma)$ rate, as is proved in section 6.1. The traditional way of presenting the initial value problems (2.3) and (2.4) is to apply the variation-of-constants formula and thus obtain

$$\begin{aligned} \mathbf{w}(t) &= e^{At}\boldsymbol{\xi} + \int_0^t e^{A(t-s)}P^s\mathbf{G}(\mathbf{w}(s)+\mathbf{y}(s))\,ds \\ \mathbf{y}(t) &= -\int_t^\infty e^{A(t-s)}P^u\mathbf{G}(\mathbf{w}(s)+\mathbf{y}(s))\,ds \end{aligned}$$

or

$$\mathbf{u}(t) = e^{At}\boldsymbol{\xi} + \int_0^t e^{A(t-s)}P^s\mathbf{G}(\mathbf{u}(s))\,ds - \int_t^\infty e^{A(t-s)}P^u\mathbf{G}(\mathbf{u}(s))\,ds$$

and

$$\mathbf{w}^{(k)}(t) = e^{At}\boldsymbol{\xi} + \int_0^t e^{A(t-s)}P^s\mathbf{G}(\mathbf{w}^{(k-1)}(s) + \mathbf{y}^{(k-1)}(s))\,ds$$
$$\mathbf{y}^{(k)}(t) = -\int_t^\infty e^{A(t-s)}P^u\mathbf{G}(\mathbf{w}^{(k-1)}(s) + \mathbf{y}^{(k-1)}(s))\,ds$$

or

$$\mathbf{u}^{(k)}(t) = e^{At}\boldsymbol{\xi} + \int_0^t e^{A(t-s)}P^s\mathbf{G}(\mathbf{u}^{(k-1)}(s))\,ds - \int_t^\infty e^{A(t-s)}P^u\mathbf{G}(\mathbf{u}^{(k-1)}(s))\,ds.$$

In section 6.2 we also measure the error induced by replacing (2.2) with the more practical truncated boundary conditions

$$P^s\mathbf{u}(0) = \boldsymbol{\xi} \qquad \text{and} \qquad P^u\mathbf{u}(T) = \mathbf{0}, \tag{2.5}$$

for some $T > 0$. This follows from establishing that the solution of (1.1) and (2.2) satisfies

$$||P^u\mathbf{u}(T)|| = O(||\boldsymbol{\xi}||^{1+\gamma}e^{-[1+\gamma]\alpha T}).$$

Hence, if $\check{\mathbf{u}}$ is the solution of (1.1) and (2.5), we have

$$\frac{d}{dt}[\mathbf{u} - \check{\mathbf{u}}] = \mathbf{F}(\mathbf{u}) - \mathbf{F}(\check{\mathbf{u}})$$

with boundary conditions

$$P^s[\mathbf{u} - \check{\mathbf{u}}](0) = \mathbf{0} \qquad \text{and} \qquad P^u[\mathbf{u} - \check{\mathbf{u}}](T) = O(||\boldsymbol{\xi}||^{1+\gamma}e^{-[1+\gamma]\alpha T}),$$

which leads to

$$[\mathbf{u} - \check{\mathbf{u}}](0) = O(||\boldsymbol{\xi}||^{1+\gamma}e^{-([1+\gamma]\alpha+\beta)T}).$$

To recapitulate then, the Liapunov-Perron method is just an approximate Newton method, which may be written in a neat form because the linear differential operator

$$\dot{\mathbf{v}} - A\mathbf{v}$$

decouples stable and unstable components. Hence a new iteration method is obtained if one realises that (1.1) *approximately* decouples in the neighbourhood of the origin and consequently the nonlinear Gauss-Seidel iteration

$$\begin{aligned} \dot{\mathbf{w}}^{(k)} &= P^s\mathbf{F}(\mathbf{w}^{(k)} + \mathbf{y}^{(k-1)}) \qquad & \mathbf{w}^{(k)}(0) &= \boldsymbol{\xi} \\ \dot{\mathbf{y}}^{(k)} &= P^u\mathbf{F}(\mathbf{w}^{(k)} + \mathbf{y}^{(k)}) \qquad & \lim_{t\to\infty}\mathbf{y}^{(k)}(t) &= \mathbf{0}, \end{aligned} \tag{2.6}$$

starting from $\mathbf{y}^{(0)} \equiv \mathbf{0}$, is a good idea. (2.6) consists of solving an initial value problem for $\mathbf{w}$ and a final value problem for $\mathbf{y}$ at each iteration. The system of equations is of 2-cyclic form, with the coupling terms $O(||\boldsymbol{\xi}||^\gamma)$, and so the rate of convergence is $O(||\boldsymbol{\xi}||^{2\gamma})$, as is proved in section 6.3.

The classical algorithm for calculating the whole manifold is the Hadamard graph transform method [14], which computes a sequence of mappings (graphs) $\mathbf{z}^{(k)} : \mathcal{E}^s_\varepsilon \mapsto \mathcal{E}^u$ with $\lim_{k\to\infty}\mathbf{z}^{(k)} = \mathbf{z}^s$. It relies on the attractivity of $\mathcal{W}^s_{loc}$ as $t \to -\infty$ in (1.1). Usually this algorithm is described for one step of a discrete dynamical system [28], but we wish to consider more general forms. Thus given $\mathbf{z}^{(k)}$, and we start from $\mathbf{z}^{(0)} \equiv \mathbf{0}$, we compute $\mathbf{z}^{(k+1)}$ by, for each $\boldsymbol{\xi} \in \mathcal{E}^s_\varepsilon$, solving (1.1) with boundary conditions

$$P^s\mathbf{u}(0) = \boldsymbol{\xi} \qquad \text{and} \qquad P^u\mathbf{u}(\tilde{T}) = \mathbf{z}^{(k)}(P^s\mathbf{u}(\tilde{T})), \tag{2.7}$$

and setting $\mathbf{z}^{(k+1)}(\boldsymbol{\xi}) = P^u\mathbf{u}(0)$. Any $\tilde{T} > 0$ can be used but convergence is obviously more rapid for larger $\tilde{T}$, with $\tilde{T} = \infty$ giving $\mathbf{z}^{(1)} = \mathbf{z}^s$. Note that $\mathbf{z}^{(k)}(\boldsymbol{\xi})$ computed with $\tilde{T}$ is the same as $\mathbf{z}^{(1)}(\boldsymbol{\xi})$ computed with $k\tilde{T}$, which is also identical with the approximation obtained from the Liapunov-Perron approach with truncated boundary condition $P^u\mathbf{u}(T) = \mathbf{0}$ where $T = k\tilde{T}$. This then indicates that the rate of convergence of the Hadamard method is

$$|||\mathbf{z}^s - \mathbf{z}^{(k)}||| = O(e^{-([1+\gamma]\alpha+\beta)k\tilde{T}}), \tag{2.8}$$

where

$$|||\mathbf{z}||| \equiv \max_{\boldsymbol{\xi}\neq\mathbf{0}} \left\{ \frac{||\mathbf{z}(\boldsymbol{\xi})||}{||\boldsymbol{\xi}||^{1+\gamma}} \right\}.$$

Note that each iteration still involves the solution of nonlinear boundary value problems, although now over an interval of finite length $\tilde{T}$, and so an inner iteration is necessary.

This last remark connects the Hadamard method with an alternative strategy, due to Perron [27], for computing a sequence of graphs whose limit is $\mathbf{z}^s$, and Perron's algorithm only requires the solution of initial value problems. Thus, to solve (1.1/2.7), it is natural to use the Gauss-Seidel approach of (2.6) and compute

$$\begin{aligned} &\text{a)} \quad \dot{\mathbf{w}}^{(\ell)} = P^s\mathbf{F}(\mathbf{w}^{(\ell)} + \mathbf{y}^{(\ell-1)}) \qquad \mathbf{w}^{(\ell)}(0) = \boldsymbol{\xi} \\ &\text{b)} \quad \dot{\mathbf{y}}^{(\ell)} = P^u\mathbf{F}(\mathbf{w}^{(\ell)} + \mathbf{y}^{(\ell)}) \qquad \mathbf{y}^{(\ell)}(\tilde{T}) = \mathbf{z}^{(k)}(\mathbf{w}^{(\ell)}(\tilde{T})), \end{aligned} \tag{2.9}$$

with $\lim_{\ell\to\infty}\mathbf{y}^{(\ell)}(0) = \mathbf{z}^{(k+1)}(\boldsymbol{\xi})$. A $\mathbf{y}^{(0)}$ is required in (2.9a) for $\ell = 1$, but this may be obtained implicitly since the best way of starting our inner iteration is by solving

$$\dot{\mathbf{w}}^{(1)} = P^s\mathbf{F}(\mathbf{w}^{(1)} + \mathbf{z}^{(k)}(\mathbf{w}^{(1)})) \qquad \mathbf{w}^{(1)}(0) = \boldsymbol{\xi}.$$

Perron's algorithm fits into this framework by applying only one step of (2.9) and replacing (2.9b) with its linear approximation

$$\dot{\mathbf{y}}^{(1)} - A\mathbf{y}^{(1)} = P^u\mathbf{G}(\mathbf{w}^{(1)}(t) + \mathbf{z}^{(k)}(\mathbf{w}^{(1)}(t))) \qquad \mathbf{y}^{(1)}(\tilde{T}) = \mathbf{z}^{(k)}(\mathbf{w}^{(1)}(\tilde{T})).$$

Thus, omitting superscripts, there is the explicit formula

$$\mathbf{z}^{(k+1)}(\boldsymbol{\xi}) = e^{-A\tilde{T}}\mathbf{z}^{(k)}(\mathbf{w}(\tilde{T})) - \int_0^{\tilde{T}} e^{-As}P^u\mathbf{G}(\mathbf{w}(s) + \mathbf{z}^{(k)}(\mathbf{w}(s)))ds,$$

where

$$\dot{\mathbf{w}} = P^s\mathbf{F}(\mathbf{w} + \mathbf{z}^{(k)}(\mathbf{w})) \qquad \mathbf{w}(0) = \boldsymbol{\xi},$$

and in section 6.4 it is proved that

$$|||\mathbf{z}^s - \mathbf{z}^{(k+1)}||| = O(e^{-([1+\gamma]\alpha+\beta)\tilde{T}} + \varepsilon^{\gamma})|||\mathbf{z}^s - \mathbf{z}^{(k)}|||.$$

Hence there is no point in using the full Hadamard iteration if ε^{γ} is comparable with $e^{-([1+\gamma]\alpha+\beta)\tilde{T}}$. In fact, previously, Perron's approach has only been described for $\tilde{T} = \infty$, i.e.

$$\mathbf{z}^{(k+1)}(\boldsymbol{\xi}) = -\int_0^{\infty} e^{-As}P^u\mathbf{G}(\mathbf{w}(s) + \mathbf{z}^{(k)}(\mathbf{w}(s)))ds$$

and convergence rate $O(\varepsilon^{\gamma})$, but we wish to contrast the performance of more general algorithms. A natural alternative to Perron's method, which we introduce and call the *nonlinear* Perron algorithm, is to apply one full iteration of (2.9), i.e. $\mathbf{z}^{(k+1)}(\boldsymbol{\xi}) = \mathbf{y}(0)$ where

$$\begin{aligned} &\text{a)} \quad \dot{\mathbf{w}} = P^s\mathbf{F}(\mathbf{w} + \mathbf{z}^{(k)}(\mathbf{w})) \qquad \mathbf{w}(0) = \boldsymbol{\xi} \\ &\text{b)} \quad \dot{\mathbf{y}} = P^u\mathbf{F}(\mathbf{w} + \mathbf{y}) \qquad \mathbf{y}(\tilde{T}) = \mathbf{z}^{(k)}(\mathbf{w}(\tilde{T})). \end{aligned}$$

Now, as expected, the convergence rate established in section 6.5 is

$$|||\mathbf{z}^s - \mathbf{z}^{(k+1)}||| = O(e^{-([1+\gamma]\alpha+\beta)\tilde{T}} + \varepsilon^{2\gamma})|||\mathbf{z}^s - \mathbf{z}^{(k)}|||.$$

3. Indirect Methods

The classical techniques described in the previous section, although constructive, remain infinite-dimensional and the invariant manifolds have usually been approximated by first discretising (1.1) [4]. Before considering this approximation, however, it is useful first to develop the corresponding terminology and briefly state the analogous results for the general discrete dynamical system

$$\mathbf{u}_n = \mathbf{f}(\mathbf{u}_{n-1}) \qquad \mathbf{f} : \Re^m \mapsto \Re^m, \tag{3.1}$$

where we assume that $\mathbf{0}$ is a hyperbolic fixed point of $\mathbf{f}$, i.e. $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, $\mathbf{f}$ is differentiable at the origin with $L \equiv f'(\mathbf{0})$ having no eigenvalues of unit modulus, and $\exists\, \gamma > 0$ such that

$$\mathbf{g}(\mathbf{x}) \equiv \mathbf{f}(\mathbf{x}) - L\mathbf{x} = O(||\mathbf{x}||^{1+\gamma}). \tag{3.2}$$

The properties of $\hat{\mathcal{W}}^s_{loc}$ & $\hat{\mathcal{W}}^u_{loc}$, the local stable and unstable manifolds of (3.1) defined by

$$\hat{\mathcal{W}}^s_{loc} \equiv \{\mathbf{x} \in \mathcal{U} : \mathbf{f}^n(\mathbf{x}) \in \mathcal{U} \quad \forall n \geq 0 : \lim_{n\to\infty} \mathbf{f}^n(\mathbf{x}) = \mathbf{0}\}$$

and

$$\hat{\mathcal{W}}^u_{loc} \equiv \{\mathbf{x} \in \mathcal{U} : \mathbf{f}^n(\mathbf{x}) \in \mathcal{U} \quad \forall n \leq 0 : \lim_{n\to-\infty} \mathbf{f}^n(\mathbf{x}) = \mathbf{0}\},$$

are analogous to those of $\mathcal{W}^s_{loc}$ & $\mathcal{W}^u_{loc}$ [12]. If L has $\hat{p}^s$ eigenvalues with modulus strictly less than 1 and $\hat{p}^u \equiv m - \hat{p}^s$ eigenvalues with modulus strictly greater than 1, then

$$\Re^m = \hat{\mathcal{E}}^s \oplus \hat{\mathcal{E}}^u,$$

where the stable subspace $\hat{\mathcal{E}}^s$ and the unstable subspace $\hat{\mathcal{E}}^u$ are the invariant subspaces corresponding to these sets of eigenvalues of L, and the respective projections are denoted by $\hat{P}^s$ & $\hat{P}^u$. $\hat{\mathcal{W}}^s_{loc}$ & $\hat{\mathcal{W}}^u_{loc}$ are $\hat{p}^s$ & $\hat{p}^u$ dimensional manifolds respectively and can be parametrised by $\hat{\mathcal{E}}^s$ & $\hat{\mathcal{E}}^u$ near the origin: hence we may write

$$\hat{\mathcal{W}}^s_{loc} \equiv \{\boldsymbol{\xi} + \hat{\mathbf{z}}^s(\boldsymbol{\xi}) \quad \text{where } \boldsymbol{\xi} \in \hat{\mathcal{E}}^s_\varepsilon \text{ and } \hat{\mathbf{z}}^s : \hat{\mathcal{E}}^s_\varepsilon \mapsto \hat{\mathcal{E}}^u\}$$

and

$$\hat{\mathcal{W}}^u_{loc} \equiv \{\boldsymbol{\eta} + \hat{\mathbf{z}}^u(\boldsymbol{\eta}) \quad \text{where } \boldsymbol{\eta} \in \hat{\mathcal{E}}^u_\varepsilon \text{ and } \hat{\mathbf{z}}^u : \hat{\mathcal{E}}^u_\varepsilon \mapsto \hat{\mathcal{E}}^s\},$$

where $\hat{\mathcal{E}}^{s/u}_\varepsilon \equiv \{\mathbf{x} \in \hat{\mathcal{E}}^{s/u} : ||\mathbf{x}|| \leq \varepsilon\}$. Again the manifolds, and thus also the functions $\hat{\mathbf{z}}^s$ & $\hat{\mathbf{z}}^u$, are as smooth as $\mathbf{f}$. In addition, the invariant subspaces are tangent to the corresponding manifolds at the origin and so the linearisations of both $\hat{\mathbf{z}}^s$ & $\hat{\mathbf{z}}^u$ are zero at the origin.

The classical techniques described in the previous section may also be applied to (3.1). The proofs are straightforward adaptations of the arguments in section 6, however, and so we merely state the corresponding results. Confirming numerical results will be given in subsection 3.1.

The Liapunov-Perron approach [21,32] looks for a solution of (3.1) with boundary conditions

$$\hat{P}^s\mathbf{u}_0 = \boldsymbol{\xi} \qquad \text{and} \qquad \lim_{n\to\infty} \hat{P}^u\mathbf{u}_n = \mathbf{0}, \tag{3.3}$$

which forces $\lim_{n\to\infty} \mathbf{u}_n = \mathbf{0}$ and so $\hat{\mathbf{z}}^s(\boldsymbol{\xi})$ is given by $\hat{P}^u\mathbf{u}_0$. By using the equivalent equation

$$\mathbf{u}_n - L\mathbf{u}_{n-1} = \mathbf{g}(\mathbf{u}_{n-1}),$$

where $\mathbf{g} \equiv \mathbf{f} - L$, we may re-write (3.1/3) in the form

$$\begin{aligned} \mathbf{w}_n - L\mathbf{w}_{n-1} &= \hat{P}^s\mathbf{g}(\mathbf{w}_{n-1} + \mathbf{y}_{n-1}) \quad & \mathbf{w}_0 = \boldsymbol{\xi} \\ \mathbf{y}_n - L\mathbf{y}_{n-1} &= \hat{P}^u\mathbf{g}(\mathbf{w}_{n-1} + \mathbf{y}_{n-1}) \quad & \lim_{n\to\infty} \mathbf{y}_n = \mathbf{0}, \end{aligned} \tag{3.4}$$

for $\boldsymbol{\xi} \in \hat{\mathcal{E}}^s_\varepsilon$, and set up the approximate Newton method

$$\begin{aligned} \mathbf{w}_n^{(k)} - L\mathbf{w}_{n-1}^{(k)} &= \hat{P}^s \mathbf{g}(\mathbf{w}_{n-1}^{(k-1)} + \mathbf{y}_{n-1}^{(k-1)}) \quad \mathbf{w}_0^{(k)} = \boldsymbol{\xi} \\ \mathbf{y}_n^{(k)} - L\mathbf{y}_{n-1}^{(k)} &= \hat{P}^u \mathbf{g}(\mathbf{w}_{n-1}^{(k-1)} + \mathbf{y}_{n-1}^{(k-1)}) \quad \lim_{n\to\infty} \mathbf{y}_n^{(k)} = \mathbf{0} \end{aligned} \tag{3.5}$$

with starting value $\{\mathbf{u}_n^{(0)}\}_{n=0}^\infty = \{\mathbf{0}, \ldots\}$. For ε sufficiently small, a solution of (3.4) is a trajectory $\{\mathbf{u}_n\}_{n=0}^\infty$ in $\hat{\mathcal{W}}^s_{loc}$ starting from $\mathbf{u}_0 = \boldsymbol{\xi} + \hat{\mathbf{z}}^s(\boldsymbol{\xi})$ and iteration (3.5) converges to it at an $O(||\boldsymbol{\xi}||^\gamma)$ rate. The traditional way of describing this algorithm is to invoke the discrete variation-of-constants formula and write (3.4) as

$$\begin{aligned} \mathbf{w}_n &= L^n \boldsymbol{\xi} + \sum_{j=0}^{n-1} L^{n-1-j} \hat{P}^s \mathbf{g}(\mathbf{w}_j + \mathbf{y}_j) \\ \mathbf{y}_n &= -\sum_{j=n}^{\infty} L^{n-1-j} \hat{P}^u \mathbf{g}(\mathbf{w}_j + \mathbf{y}_j) \end{aligned}$$

and (3.5) as

$$\begin{aligned} \mathbf{w}_n^{(k)} &= L^n \boldsymbol{\xi} + \sum_{j=0}^{n-1} L^{n-1-j} \hat{P}^s \mathbf{g}(\mathbf{w}_j^{(k-1)} + \mathbf{y}_j^{(k-1)}) \\ \mathbf{y}_n^{(k)} &= -\sum_{j=n}^{\infty} L^{n-1-j} \hat{P}^u \mathbf{g}(\mathbf{w}_j^{(k-1)} + \mathbf{y}_j^{(k-1)}) \end{aligned}$$

or

$$\mathbf{u}_n^{(k)} = L^n \boldsymbol{\xi} + \sum_{j=0}^{n-1} L^{n-1-j} \hat{P}^s \mathbf{g}(\mathbf{u}_j^{(k-1)}) - \sum_{j=n}^{\infty} L^{n-1-j} \hat{P}^u \mathbf{g}(\mathbf{u}_j^{(k-1)}).$$

[Here, of course, we are only employing negative powers of L when this operator is regarded as restricted to $\hat{\mathcal{E}}^u$.] If $\{\check{\mathbf{u}}_n\}_{n=0}^N$ is the solution of (3.1) with truncated boundary conditions

$$\hat{P}^s \mathbf{u}_0 = \boldsymbol{\xi} \qquad \text{and} \qquad \hat{P}^u \mathbf{u}_N = \mathbf{0}, \tag{3.6}$$

then the error created is

$$||\mathbf{u}_0 - \check{\mathbf{u}}_0|| = O(||\boldsymbol{\xi}||^{1+\gamma}[\hat{\alpha}^{1+\gamma}\hat{\beta}]^N),$$

where

(3.7)
- $|\mu| < \hat{\alpha} < 1$ for all eigenvalues μ of L with modulus less than 1,
- $|\mu|^{-1} < \hat{\beta} < 1$ for all eigenvalues μ of L with modulus greater than 1.

We may obtain an algorithm with $O(||\boldsymbol{\xi}||^{2\gamma})$ convergence rate by writing the unknowns/equations of (3.1) in the order

$$\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n, \ldots \; : \; \ldots, \mathbf{y}_n, \ldots, \mathbf{y}_1, \mathbf{y}_0$$

and

$$\begin{aligned} \mathbf{w}_n &= \hat{P}^s \mathbf{f}(\mathbf{w}_{n-1} + \mathbf{y}_{n-1}) \quad n = 1, 2, \ldots \\ \mathbf{y}_{n+1} &= \hat{P}^u \mathbf{f}(\mathbf{w}_n + \mathbf{y}_n) \quad n = \ldots, 1, 0 \end{aligned}$$

and then applying the nonlinear Gauss-Seidel method. The iteration is therefore

$$\begin{aligned} \mathbf{w}_n^{(k)} &= \hat{P}^s \mathbf{f}(\mathbf{w}_{n-1}^{(k)} + \mathbf{y}_{n-1}^{(k-1)}) \qquad \mathbf{w}_0^{(k)} = \boldsymbol{\xi} \\ \mathbf{y}_{n+1}^{(k)} &= \hat{P}^u \mathbf{f}(\mathbf{w}_n^{(k)} + \mathbf{y}_n^{(k)}) \qquad \lim_{n\to\infty} \mathbf{y}_n^{(k)} = \mathbf{0} \end{aligned} \tag{3.8}$$

with starting value $\{\mathbf{y}_n^{(0)}\}_{n=0}^\infty = \{\mathbf{0}, \ldots\}$.

A more recent algorithm for computing a trajectory in the stable manifold, which does not have an analogue for continuous dynamical systems, is described in [15]. This assumes that $\mathbf{f}$ is a local diffeomorphism and is based explicitly on the conditions for $\{\mathbf{u}_n\}_{n=0}^{\infty}$ to be a solution of (3.1), i.e.

$$\mathbf{u}_n = \mathbf{f}(\mathbf{u}_{n-1}) \quad \text{and} \quad \mathbf{u}_n = \mathbf{f}^{-1}(\mathbf{u}_{n+1}).$$

Hence a trajectory $\{\mathbf{u}_n\}_{n=0}^{\infty}$ in the stable manifold, with $\hat{P}^s\mathbf{u}_0 = \boldsymbol{\xi} \in \hat{\mathcal{E}}^s_{\varepsilon}$, must satisfy

$$\mathbf{u}_n = \hat{P}^s\mathbf{f}(\mathbf{u}_{n-1}) + \hat{P}^u\mathbf{f}^{-1}(\mathbf{u}_{n+1}) \quad n = 1, 2, \ldots \tag{3.9}$$

with $\mathbf{u}_0 = \boldsymbol{\xi} + \hat{P}^u\mathbf{f}^{-1}(\mathbf{u}_1)$. If the unknowns are ordered

$$\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_n, \ldots,$$

this leads to the nonlinear Gauss-Seidel scheme

$$\mathbf{u}_n^{(k)} = \hat{P}^s\mathbf{f}(\mathbf{u}_{n-1}^{(k)}) + \hat{P}^u\mathbf{f}^{-1}(\mathbf{u}_{n+1}^{(k-1)}) \quad n = 1, 2, \ldots \tag{3.10}$$

with $\mathbf{u}_0^{(k)} = \boldsymbol{\xi} + \hat{P}^u\mathbf{f}^{-1}(\mathbf{u}_1^{(k-1)})$, starting from $\{\mathbf{u}_n^{(0)}\}_{n=0}^{\infty} = \{\mathbf{0}, \ldots\}$. In [15] it is shown that $\lim_{k\to\infty} \mathbf{u}_0^{(k)} = \boldsymbol{\xi} + \hat{\mathbf{z}}^s(\boldsymbol{\xi})$ with rate of convergence $\max\{\hat{\alpha}, \hat{\beta}\} + O(\|\boldsymbol{\xi}\|^{\gamma})$. We shall refer to (3.10) as the original HOV method. Now we want to relate (3.10) to the nonlinear Gauss-Seidel method applied to (3.1) when the unknowns/equations are taken in the order

$$\mathbf{y}_0, \mathbf{w}_1, \mathbf{y}_1, \mathbf{w}_2, \mathbf{y}_3, \ldots,$$

which leads to the system

$$\left.\begin{aligned} \mathbf{y}_n &= \hat{P}^u\mathbf{f}(\mathbf{w}_{n-1} + \mathbf{y}_{n-1}) \\ \mathbf{w}_n &= \hat{P}^s\mathbf{f}(\mathbf{w}_{n-1} + \mathbf{y}_{n-1}) \end{aligned}\right\} \quad n = 1, 2, \ldots$$

and the iteration

$$\left.\begin{aligned} \mathbf{y}_n^{(k-1)} &= \hat{P}^u\mathbf{f}(\mathbf{w}_{n-1}^{(k)} + \mathbf{y}_{n-1}^{(k)}) \\ \mathbf{w}_n^{(k)} &= \hat{P}^s\mathbf{f}(\mathbf{w}_{n-1}^{(k)} + \mathbf{y}_{n-1}^{(k)}) \end{aligned}\right\} \quad n = 1, 2, \ldots,$$

which we shall call the 'HOV' method. On the other hand, when the original HOV method is written in terms of $\mathbf{w}$ and $\mathbf{y}$ we obtain the system

$$\left.\begin{aligned} \mathbf{y}_{n-1} &= \hat{P}^u\mathbf{f}^{-1}(\mathbf{w}_n + \mathbf{y}_n) \\ \mathbf{w}_n &= \hat{P}^s\mathbf{f}(\mathbf{w}_{n-1} + \mathbf{y}_{n-1}) \end{aligned}\right\} \quad n = 1, 2, \ldots$$

with the iteration

$$\left.\begin{aligned} \mathbf{y}_{n-1}^{(k)} &= \hat{P}^u\mathbf{f}^{-1}(\mathbf{w}_n^{(k-1)} + \mathbf{y}_n^{(k-1)}) \\ \mathbf{w}_n^{(k)} &= \hat{P}^s\mathbf{f}(\mathbf{w}_{n-1}^{(k)} + \mathbf{y}_{n-1}^{(k)}) \end{aligned}\right\} \quad n = 1, 2, \ldots.$$

These, of course, are not the same iterative method. If, however, the $\mathbf{y}$ equation is replaced by one step of modified Newton iteration, as recommended in [15], then in both cases we obtain

$$\mathbf{y}_{n-1}^{(k)} = \mathbf{y}_{n-1}^{(k-1)} + L^{-1}\left\{\mathbf{y}_n^{(k-1)} - \hat{P}^u\mathbf{f}(\mathbf{w}_{n-1}^{(k)} + \mathbf{y}_{n-1}^{(k-1)})\right\}. \tag{†}$$

In section 6.6 it is proved that the 'HOV' method converges at an $\hat{\alpha}\hat{\beta} + O(\|\boldsymbol{\xi}\|^{\gamma})$ rate, and it is easily seen that this result also applies to the original HOV method and the simplification based on (†). [As a final practical comment, note that when these methods are truncated, i.e. only $\mathbf{w}_n, \mathbf{y}_n \quad n = 0, \ldots, N$ are computed, then $\mathbf{y}_N^{(k)}$ is always set to zero.]

The Hadamard graph transform method [28,32], on the other hand, computes a sequence of graphs $\hat{\mathbf{z}}^{(k)} : \hat{\mathcal{E}}^s_{\varepsilon} \mapsto \hat{\mathcal{E}}^u$ with $\lim_{k\to\infty} \hat{\mathbf{z}}^{(k)} = \hat{\mathbf{z}}^s$. It relies on the attractivity of $\hat{\mathcal{W}}^s_{loc}$ as $n \to -\infty$ in (3.1). Thus given $\hat{\mathbf{z}}^{(k)}$, and we start from $\hat{\mathbf{z}}^{(0)} \equiv \mathbf{0}$, $\hat{\mathbf{z}}^{(k+1)}$ is computed by, for each $\boldsymbol{\xi} \in \hat{\mathcal{E}}^s_\varepsilon$, solving (3.1) with boundary conditions

$$\hat{P}^s \mathbf{u}_0 = \boldsymbol{\xi} \qquad \text{and} \qquad \hat{P}^u \mathbf{u}_{\tilde{N}} = \hat{\mathbf{z}}^{(k)}(\hat{P}^s \mathbf{u}_{\tilde{N}})$$

and setting $\hat{\mathbf{z}}^{(k+1)}(\boldsymbol{\xi}) = \hat{P}^u \mathbf{u}_0$. Any $\tilde{N} > 0$ can be used but obviously convergence is more rapid for larger $\tilde{N}$, with $\tilde{N} = \infty$ giving $\hat{\mathbf{z}}^{(1)} = \hat{\mathbf{z}}^s$. Note that $\hat{\mathbf{z}}^{(k)}(\boldsymbol{\xi})$ computed with $\tilde{N}$ is the same as $\hat{\mathbf{z}}^{(1)}(\boldsymbol{\xi})$ computed with $k\tilde{N}$, which is also identical with the approximation obtained from the Liapunov-Perron approach with truncated boundary condition $\hat{P}^u \mathbf{u}_N = \mathbf{0}$, where $N = k\tilde{N}$. This then indicates the rate of convergence of the Hadamard method is

$$|||\hat{\mathbf{z}}^s - \hat{\mathbf{z}}^{(k)}||| = O([\hat{\alpha}^{1+\gamma}\hat{\beta}]^{\tilde{N}}),$$

where $|||.|||$ is defined in (2.8). Note that each iteration still involves the solution of a nonlinear boundary value recurrence relation, although now of finite length $\tilde{N}$, and so an inner iteration is necessary. We choose this inner iteration to consist of r steps of (3.8), i.e.

$$\text{(3.11)} \qquad \begin{array}{lll} \text{a)} & \mathbf{w}_n^{(\ell)} = \hat{P}^s \mathbf{f}(\mathbf{w}_{n-1}^{(\ell)} + \mathbf{y}_{n-1}^{(\ell-1)}) & \mathbf{w}_0^{(\ell)} = \boldsymbol{\xi} \\ \text{b)} & \mathbf{y}_{n+1}^{(\ell)} = \hat{P}^u \mathbf{f}(\mathbf{w}_n^{(\ell)} + \mathbf{y}_n^{(\ell)}) & y_{\tilde{N}}^{(\ell)} = \hat{\mathbf{z}}^{(k)}(\mathbf{w}_{\tilde{N}}^{(\ell)}) \end{array}$$

so that $\lim_{\ell\to\infty} \mathbf{y}_0^{(\ell)} = \hat{\mathbf{z}}^{(k+1)}(\boldsymbol{\xi})$, but we use the natural starting value

$$\mathbf{w}_n^{(1)} = \hat{P}^s \mathbf{f}(\mathbf{w}_{n-1}^{(1)} + \hat{\mathbf{z}}^{(k)}(\mathbf{w}_{n-1}^{(1)})) \qquad \mathbf{w}_0^{(1)} = \boldsymbol{\xi}.$$

The Perron algorithm again corresponds to taking $r = 1$ and linearising (3.11b) to obtain

$$\hat{\mathbf{z}}^{(k+1)}(\boldsymbol{\xi}) = L^{-\tilde{N}} \hat{\mathbf{z}}^{(k)}(\mathbf{w}_{\tilde{N}}) - \sum_{j=0}^{\tilde{N}-1} L^{-1-j} \hat{P}^u \mathbf{g}(\mathbf{w}_j + \hat{\mathbf{z}}^{(k)}(\mathbf{w}_j))$$

with

$$\mathbf{w}_n = \hat{P}^s \mathbf{f}(\mathbf{w}_{n-1} + \hat{\mathbf{z}}^{(k)}(\mathbf{w}_{n-1})) \qquad \mathbf{w}_0 = \boldsymbol{\xi},$$

or

$$\hat{\mathbf{z}}^{(k+1)}(\boldsymbol{\xi}) \equiv \mathbf{y}_0 = -\sum_{j=0}^{\infty} L^{-1-j} \hat{P}^u \mathbf{g}(\mathbf{w}_j + \hat{\mathbf{z}}^{(k)}(\mathbf{w}_j))$$

if $\tilde{N} = \infty$. The rate of convergence is

$$|||\hat{\mathbf{z}}^s - \hat{\mathbf{z}}^{(k+1)}||| = O([\hat{\alpha}^{1+\gamma}\hat{\beta}]^{\tilde{N}} + \varepsilon^\gamma)|||\hat{\mathbf{z}}^s - \hat{\mathbf{z}}^{(k)}|||.$$

The nonlinear Perron method applies one full iteration of (3.11), i.e. $\hat{\mathbf{z}}^{(k+1)}(\boldsymbol{\xi}) = \mathbf{y}_0$ where

$$\begin{array}{lll} \text{a)} & \mathbf{w}_n = \hat{P}^s \mathbf{f}(\mathbf{w}_{n-1} + \hat{\mathbf{z}}^{(k)}(\mathbf{w}_{n-1})) & \mathbf{w}_0 = \boldsymbol{\xi} \\ \text{b)} & \mathbf{y}_{n+1} = \hat{P}^u \mathbf{f}(\mathbf{w}_n + \mathbf{y}_n) & y_{\tilde{N}} = \hat{\mathbf{z}}^{(k)}(\mathbf{w}_{\tilde{N}}). \end{array}$$

This has the convergence rate

$$|||\hat{\mathbf{z}}^s - \hat{\mathbf{z}}^{(k+1)}||| = O([\hat{\alpha}^{1+\gamma}\hat{\beta}]^{\tilde{N}} + \varepsilon^{2\gamma})|||\hat{\mathbf{z}}^s - \hat{\mathbf{z}}^{(k)}|||.$$

3.1 Results when computing a point on the Manifold

In this section we shall assume that a discrete dynamical system (3.1) has been obtained by discretising (1.1). In this case the convergence of $\hat{\mathcal{W}}^{s/u}_{loc}$ to $\mathcal{W}^{s/u}_{loc}$ has been dealt with by several authors, e.g. [4,32]. Here, however, we wish to consider the various algorithms introduced above and link them with appropriate discretisations of (1.1). (We shall only be concerned with one-step formulae, multi-step formulae are also analysed in [4].) As we have seen, it is natural to march forwards when solving for the stable components

$\mathbf{w}$ and backwards when solving for the unstable components $\mathbf{y}$. This is distorted by the notation of (3.1), which gives the impression that only *explicit* methods are being discussed. In this sub-section, therefore, we shall first illustrate the previous algorithms on the trapezoidal rule applied to (1.1), this being an exemplar for the class of one-step symmetric methods [30] with no bias in either direction. Hence, using the truncated boundary conditions (3.6), we need to solve the set of nonlinear equations

$$\mathbf{u}_n - \frac{h}{2}\mathbf{F}(\mathbf{u}_n) = \mathbf{u}_{n-1} + \frac{h}{2}\mathbf{F}(\mathbf{u}_{n-1}) \qquad n = 1, \ldots, N \tag{3.12}$$

with $P^s\mathbf{u}_0 = \boldsymbol{\xi}$ and $P^u\mathbf{u}_N = \mathbf{0}$. [Note that for the trapezoidal rule, $L \equiv (I - \frac{h}{2}A)^{-1}(I + \frac{h}{2}A)$ and so $\hat{\mathcal{E}}^{s/u} \equiv \mathcal{E}^{s/u}$ and $\hat{P}^{s/u} \equiv P^{s/u}$.]

The classical Liapunov-Perron method is just an approximate Newton method for (3.12), i.e.

$$(I - \frac{h}{2}A)\mathbf{u}_n^{(k)} - (I + \frac{h}{2}A)\mathbf{u}_{n-1}^{(k)} = \frac{h}{2}\left\{\mathbf{G}(\mathbf{u}_n^{(k-1)}) + \mathbf{G}(\mathbf{u}_{n-1}^{(k-1)})\right\}.$$

Since $\mathcal{E}^{s/u}$ are invariant under A, and because of the boundary conditions, these can be solved forwards for the stable components and backwards for the unstable components, i.e.

$$\mathbf{w}_n^{(k)} = (I - \frac{h}{2}A)^{-1}\left[(I + \frac{h}{2}A)\mathbf{w}_{n-1}^{(k)} + \frac{h}{2}P^s\left\{\mathbf{G}(\mathbf{w}_n^{(k-1)} + \mathbf{y}_n^{(k-1)}) + \mathbf{G}(\mathbf{w}_{n-1}^{(k-1)} + \mathbf{y}_{n-1}^{(k-1)})\right\}\right] \tag{3.13a}$$

for $n = 1, \ldots, N$ with $\mathbf{w}_0^{(k)} = \boldsymbol{\xi}$ and

$$\mathbf{y}_n^{(k)} = (I + \frac{h}{2}A)^{-1}\left[(I - \frac{h}{2}A)\mathbf{y}_{n+1}^{(k)} - \frac{h}{2}P^u\left\{\mathbf{G}(\mathbf{w}_n^{(k-1)} + \mathbf{y}_n^{(k-1)}) + \mathbf{G}(\mathbf{w}_{n+1}^{(k-1)} + \mathbf{y}_{n+1}^{(k-1)})\right\}\right] \tag{3.13b}$$

for $n = N-1, \ldots, 0$ with $\mathbf{y}_N^{(k)} = \mathbf{0}$.

For the sake of completeness, and also to illustrate how decoupling ideas even arise here, we mention that the full Newton method for (3.12) is of course possible; i.e.

$$\begin{aligned}\left(I - \frac{h}{2}\mathbf{F}'(\mathbf{u}_n^{(k-1)})\right)\boldsymbol{\delta}\mathbf{u}_n^{(k)} - \left(I + \frac{h}{2}\mathbf{F}'(\mathbf{u}_{n-1}^{(k-1)})\right)\boldsymbol{\delta}\mathbf{u}_{n-1}^{(k)}& \\ = \mathbf{u}_{n-1}^{(k-1)} - \mathbf{u}_n^{(k-1)} + \frac{h}{2}\left\{\mathbf{F}(\mathbf{u}_n^{(k-1)}) + \mathbf{F}(\mathbf{u}_{n-1}^{(k-1)})\right\}&\end{aligned}$$

with $\mathbf{u}_n^{(k)} \equiv \mathbf{u}_n^{(k-1)} + \boldsymbol{\delta}\mathbf{u}_n^{(k)}$ $\quad n = 0, \ldots, N$, which leads to block bidiagonal systems of the form

$$\begin{pmatrix} -C_1 & B_1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -C_N & B_N \end{pmatrix}\begin{pmatrix}\mathbf{x}_0 \\ \vdots \\ \vdots \\ \mathbf{x}_N\end{pmatrix} = \begin{pmatrix}\mathbf{b}_1 \\ \vdots \\ \vdots \\ \mathbf{b}_N\end{pmatrix} \tag{3.14}$$

with $P^s\mathbf{x}_0 = \mathbf{0}$ and $P^u\mathbf{x}_N = \mathbf{0}$. From our viewpoint, the most natural solution strategy for (3.14), which links up with our decoupling ideas, is the *stable compactification* method of [2 § 7.2.4]. Since this reference describes the technique in detail, and emphasises the importance of decoupling, we merely state it in our notation.

- Let Q_0 be an $m \times m$ orthogonal matrix whose last p^u columns span the unstable space $\mathcal{E}^u$.
- Set
$$\left.\begin{aligned}\tilde{Q}_i^T[C_iQ_{i-1}] = L_i \\ [\tilde{Q}_i^TB_i]Q_i = R_i\end{aligned}\right\} \quad i = 1, \ldots, N;$$
where $\tilde{Q}_i$ is the orthogonal matrix which performs the first p^u stages of orthogonal factorisation *from the right* on C_iQ_{i-1}, i.e. the last p^u columns of L_i have zeroes above the principal diagonal, and Q_i is

the orthogonal matrix which performs the first p^s stages of orthogonal factorisation on $\tilde{Q}_i B_i$, i.e. the first p^s rows of R_i have zeroes to the right of the principal diagonal. This gives the factorisation

$$\begin{pmatrix} -C_1 & B_1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -C_N & B_N \end{pmatrix} = \tilde{\mathcal{D}}^T \begin{pmatrix} -L_1 & R_1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -L_N & R_N \end{pmatrix} \mathcal{D}^T,$$

where $\tilde{\mathcal{D}}$ and $\mathcal{D}$ are the block diagonal orthogonal matrices

$$\tilde{\mathcal{D}} \equiv \begin{pmatrix} \tilde{Q}_1 & & \\ & \ddots & \\ & & \ddots & \\ & & & \tilde{Q}_N \end{pmatrix} \quad \text{and} \quad \mathcal{D} \equiv \begin{pmatrix} Q_0 & & \\ & \ddots & \\ & & \ddots & \\ & & & Q_N \end{pmatrix}.$$

- Set $\hat{\mathbf{b}}_i = \tilde{Q}_i \mathbf{b}_i \quad i = 1, \ldots, N$ and denote $Q_i^T \mathbf{x}_i$ by $\hat{\mathbf{x}}_i$ for $i = 0, \ldots, N$, so that the first p^s components of $\hat{\mathbf{x}}_0$ must be zero.
- Solve the system

$$\begin{pmatrix} -L_1 & R_1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & -L_N & R_N \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}}_0 \\ \vdots \\ \vdots \\ \hat{\mathbf{x}}_N \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{b}}_1 \\ \vdots \\ \vdots \\ \hat{\mathbf{b}}_N \end{pmatrix}$$

  forward for the first p^s components of $\hat{\mathbf{x}}_1, \ldots, \hat{\mathbf{x}}_N$ in turn. The last p^u components of $\hat{\mathbf{x}}_N$ are then chosen so that $P^u Q_N \hat{\mathbf{x}}_N = \mathbf{0}$ and the above system solved backwards to obtain the last p^u components of $\hat{\mathbf{x}}_{N-1}, \ldots, \hat{\mathbf{x}}_0$ in turn.

Since the quadratic convergence of Newton's method is well-known, we do not consider if further. Note however that this iteration is considerably more expensive than the Liapunov-Perron algorithm.

Now we look at Gauss-Seidel methods, which are motivated by the fact that $\mathbf{F} \approx A$ and so the stable/unstable components approximately decouple in (3.12). Thus, if the unknowns are ordered

$$\mathbf{w}_1, \ldots, \mathbf{w}_N, \mathbf{y}_{N-1}, \ldots, \mathbf{y}_0$$

then the nonlinear Gauss-Seidel method [25] solves each of

$$\text{(3.15a)} \qquad \mathbf{w}_n^{(k)} - \frac{h}{2} P^s \mathbf{F}(\mathbf{w}_n^{(k)} + \mathbf{y}_n^{(k-1)}) = \mathbf{w}_{n-1}^{(k)} + \frac{h}{2} P^s \mathbf{F}(\mathbf{w}_{n-1}^{(k)} + \mathbf{y}_{n-1}^{(k-1)}) \qquad n = 1, \ldots, N$$

for $\mathbf{w}_n^{(k)}$, with $\mathbf{w}_0^{(k)} = \boldsymbol{\xi}$, and solves each of

$$\text{(3.15b)} \qquad \mathbf{y}_n^{(k)} + \frac{h}{2} P^u \mathbf{F}(\mathbf{w}_n^{(k)} + \mathbf{y}_n^{(k)}) = \mathbf{y}_{n+1}^{(k)} - \frac{h}{2} P^u \mathbf{F}(\mathbf{w}_{n+1}^{(k)} + \mathbf{y}_{n+1}^{(k)}) \qquad n = N-1, \ldots, 0$$

for $\mathbf{y}_n^{(k)}$, with $\mathbf{y}_N^{(k)} = \mathbf{0}$. The convergence of this method is governed by the convergence of the standard Gauss-Seidel method on the linearisation of (3.12) at the solution [25]. This matrix is *two-cyclic* and so Gauss-Seidel converges at an $O(\|\boldsymbol{\xi}\|^{2\gamma})$ rate, i.e. twice as fast as Jacobi [13], and even the SOR theory (for complex eigenvalues of the Jacobi iteration matrix in general [33]) applies, but it would be impractical to make use of this. It is usually regarded as inefficient to solve the subproblems in (3.15) exactly and the Gauss-Seidel Newton [25] method is used, i.e. one or more steps of Newton's method is applied to each equation. For us, however, it is more efficient to just apply the Gauss-Seidel approximate Newton method. If r-steps of this are applied then $\mathbf{w}_n^{(k)} = \mathbf{v}^{(r)}$, where

$$\mathbf{v}^{(\ell)} = (I - \frac{h}{2} A)^{-1} \left\{ \frac{h}{2} P^s \mathbf{G}(\mathbf{v}^{(\ell-1)} + \mathbf{y}_n^{(k-1)}) + \mathbf{d}_n^{(k)} \right\}$$

with $\mathbf{d}_n^{(k)} \equiv \mathbf{w}_{n-1}^{(k)} + \frac{h}{2} P^s \mathbf{F}(\mathbf{w}_{n-1}^{(k)} + \mathbf{y}_{n-1}^{(k-1)})$ and $\mathbf{v}^{(0)} = \mathbf{w}_n^{(k-1)}$; and $\mathbf{y}_n^{(k)} = \mathbf{v}^{(r)}$, where

$$\mathbf{v}^{(\ell)} = (I + \frac{h}{2}A)^{-1} \left\{ \mathbf{d}_n^{(k)} - \frac{h}{2} P^u \mathbf{G}(\mathbf{w}_n^{(k)} + \mathbf{v}^{(\ell-1)}) \right\}$$

with $\mathbf{d}_n^{(k)} \equiv \mathbf{y}_{n+1}^{(k)} - \frac{h}{2} P^u \mathbf{F}(\mathbf{w}_{n+1}^{(k)} + \mathbf{y}_{n+1}^{(k)})$ and $\mathbf{v}^{(0)} = \mathbf{y}_n^{(k-1)}$. With $r = 1$ we may explicitly write

$$\mathbf{w}_n^{(k)} = (I - \frac{h}{2}A)^{-1} \left[\mathbf{w}_{n-1}^{(k)} + \frac{h}{2} P^s \left\{ \mathbf{G}(\mathbf{w}_n^{(k)} + \mathbf{y}_n^{(k-1)}) + \mathbf{F}(\mathbf{w}_{n-1}^{(k)} + \mathbf{y}_{n-1}^{(k-1)}) \right\} \right] \qquad n = 1, \ldots, N$$

with $\mathbf{w}_0^{(k)} = \boldsymbol{\xi}$ and

$$\mathbf{y}_n^{(k)} = (I + \frac{h}{2}A)^{-1} \left[\mathbf{y}_{n+1}^{(k)} - \frac{h}{2} P^u \left\{ \mathbf{G}(\mathbf{w}_n^{(k)} + \mathbf{y}_n^{(k)}) + \mathbf{F}(\mathbf{w}_{n+1}^{(k)} + \mathbf{y}_{n+1}^{(k)}) \right\} \right] \qquad n = N-1, \ldots, 0$$

with $\mathbf{y}_N^{(k)} = \mathbf{0}$. Comparing this with the classical Liapunov-Perron method (3.13), we see that the above method makes use of new iterates more quickly. To retain the $O(||\boldsymbol{\xi}||^{2\gamma})$ rate of the full nonlinear Gauss-Seidel method, however, we expect to have to take $r = 2$, since each approximate Newton step will give an $O(||\boldsymbol{\xi}||^{\gamma})$ improvement.

The other possibility is to order the unknowns

$$\mathbf{y}_0, \mathbf{w}_1, \mathbf{y}_1, \ldots, \mathbf{w}_{N-1}, \mathbf{y}_{N-1}, \mathbf{w}_N,$$

and the equations

$$\left.\begin{aligned} \mathbf{y}_{n-1} + \frac{h}{2} P^u \mathbf{F}(\mathbf{w}_{n-1} + \mathbf{y}_{n-1}) &= \mathbf{y}_n - \frac{h}{2} P^u \mathbf{F}(\mathbf{w}_n + \mathbf{y}_n) \\ \mathbf{w}_n - \frac{h}{2} P^s \mathbf{F}(\mathbf{w}_n + \mathbf{y}_n) &= \mathbf{w}_{n-1} + \frac{h}{2} P^s \mathbf{F}(\mathbf{w}_{n-1} + \mathbf{y}_{n-1}) \end{aligned}\right\} \quad n = 1, \ldots, N$$

which is the 'HOV' choice. The nonlinear Gauss-Seidel method is then

$$\left.\begin{aligned} \mathbf{y}_{n-1}^{(k)} + \frac{h}{2} P^u \mathbf{F}(\mathbf{w}_{n-1}^{(k)} + \mathbf{y}_{n-1}^{(k)}) &= \mathbf{y}_n^{(k-1)} - \frac{h}{2} P^u \mathbf{F}(\mathbf{w}_n^{(k-1)} + \mathbf{y}_n^{(k-1)}) \\ \mathbf{w}_n^{(k)} - \frac{h}{2} P^s \mathbf{F}(\mathbf{w}_n^{(k)} + \mathbf{y}_n^{(k-1)}) &= \mathbf{w}_{n-1}^{(k)} + \frac{h}{2} P^s \mathbf{F}(\mathbf{w}_{n-1}^{(k)} + \mathbf{y}_{n-1}^{(k)}) \end{aligned}\right\} \quad n = 1, \ldots, N. \tag{3.16}$$

[Note that, as commented on earlier, this is slightly different from the original HOV method which would take $\mathbf{y}_{n-1}$ to be $P^u\mathbf{v}$, where

$$\mathbf{v} + \frac{h}{2}\mathbf{F}(\mathbf{v}) = \mathbf{w}_n + \mathbf{y}_n - \frac{h}{2}\mathbf{F}(\mathbf{w}_n + \mathbf{y}_n)$$

and $\mathbf{w}_n$ to be $P^s\mathbf{v}$, where

$$\mathbf{v} - \frac{h}{2}\mathbf{F}(\mathbf{v}) = \mathbf{w}_{n-1} + \mathbf{y}_{n-1} + \frac{h}{2}\mathbf{F}(\mathbf{w}_{n-1} + \mathbf{y}_{n-1}).]$$

In practice we would again replace (3.16) by a Gauss-Seidel r-step approximate Newton method and, in the case of $r = 1$, end up with the same scheme from both (3.16) and the original HOV equations. The important fact, however, is that, as stated earlier, the convergence rate of these methods depends on $\hat{\alpha}, \hat{\beta}$ for $L \equiv (I - \frac{h}{2}A)^{-1}(I + \frac{h}{2}A)$. Since these are $1 \pm O(h)$, convergence will be slow and will deteriorate as $h \to 0$. Hence the conclusion is that this ordering of the unknowns is inappropriate, since the convergence rate is based on only *one* step of the discrete problem rather than a complete sweep.

Now we illustrate the above algorithms on two simple well-known examples and compare their rates of convergence.

Example 1 The Lorenz equations

$$\begin{aligned}\dot{x} &= a(y-x)\\ \dot{y} &= bx - y - xz\\ \dot{z} &= xy - cz\end{aligned}$$

We use the parameter values $(a, b, c) = (10, 14, \frac{8}{3})$ for which the stationary solution at the origin is hyperbolic. The Jacobian matrix there has real eigenvalues

$$-c, \quad -\frac{1}{2}\left[(a+1) \pm \sqrt{(a+1)^2 + 4a(b-1)}\right],$$

two of which are negative. The trapezoidal rule was applied with

$$\boldsymbol{\xi} = \delta(\boldsymbol{\phi}_1 + \boldsymbol{\phi}_2)$$

(where $\boldsymbol{\phi}_i$ are the normalised eigenvectors corresponding to the two negative eigenvalues) and various values of δ, h and N.

We first fixed $\delta = 4$ and $h = 0.01$, and investigated the effect of truncating the infinite interval. Table 1 shows the absolute error [ERR] between $\hat{z}^s(\boldsymbol{\xi})$, using the value for $N = 1000$ as an approximation to $N = \infty$, and the computed value for smaller N.

N	10	20	30	40	50
ERR	2.85(−2)	2.09(−3)	1.46(−4)	9.87(−6)	6.50(−7)
RATE	35.59	30.86	29.44	28.81	28.49

N	60	70	80	90	100
ERR	4.20(−8)	2.67(−9)	1.68(−10)	1.05(−11)	6.48(−13)
RATE	28.31	28.20	28.13	28.05	28.06

Table 1

The second line of the table shows RATE = $(\log ERR)/(Nh)$ which, because of the absence of a z^2 term in the Lorenz equations, we expect to approach the sum of the moduli of the eigenvalues, i.e. ≈ 27.98.

Table 2 illustrates the $O(h^2)$ convergence rate. For $\delta = 4$ and various values of h, we list the discretisation error $||\mathbf{z}^s(\boldsymbol{\xi}) - \hat{\mathbf{z}}^s(\boldsymbol{\xi})||$ and the number of steps N required for the truncation error to reach this value.

h	.8(−2)	.4(−2)	.2(−2)	.1(−2)	.5(−3)	.25(−3)
ERR	2.82(−4)	7.04(−5)	1.76(−5)	4.40(−6)	1.10(−6)	2.75(−7)
N	35	80	180	405	915	2030

Table 2

Table 3 compares three different methods by showing first the convergence rate, calculated in the usual way $\lim_{k\to\infty} ||\mathrm{e}^{(k)}||^{\frac{1}{k}}$, and secondly the number of iterations required to reach machine accuracy 10^{-15}. We vary δ while keeping h and N fixed at $.25 \times 10^{-3}$ and 2500 respectively.

δ	16	8	4	2
L–P	.67	.29	.14	.67(−1)
G–S[1]	.46	.11	.39(−1)	.14(−1)
G–S[2]	.37	.45(−1)	.76(−2)	.15(−2)

Table 3a

δ	16	8	4	2
L–P	84	28	18	13
G–S[1]	42	16	11	8
G–S[2]	34	12	8	6

Table 3b

The terminology is almost self-explanatory: L–P refers to iteration (3.13) and G–S[r] to (3.15) with r inner approximate Newton steps. The number of outer iterations required did not change when more inner iterations were applied.

To confirm that the convergence rate of the 'HOV' method deteriorates with h, we show results for fixed $\delta = 4$ and let h and N vary according to the Table 2. The convergence rate (C–RATE) is calculated as in Table 3a and the number of iterations (ITS) as in Table 3b.

h/N	.008/35	.004/80	.002/180	.001/405
C–RATE	.74	.86	.93	.97
ITS	111	227	463	943

Table 4

These results were obtained with one approximate Newton inner iteration. No significant change was observed if more inner iterations, or the original HOV method, were used.

Example 2 Chua's electronic circuit

$$\begin{aligned}\dot{x} =& a(y + \frac{1}{6}(x - x^3))\\ \dot{y} =& x - y + z\\ \dot{z} =& -by\end{aligned}$$

We use the parameter values $(a, b) = (4, 5)$ for which the stationary solution at the origin is again hyperbolic. The Jacobian matrix there has one positive eigenvalue and a pair of complex conjugate eigenvalues with negative real part, i.e. 1.32 and $-0.83 \pm 1.36i$. The trapezoidal rule was applied with

$$\boldsymbol{\xi} = \delta(\boldsymbol{\phi}_R + \boldsymbol{\phi}_I)$$

(where $\boldsymbol{\phi}_R$ and $\boldsymbol{\phi}_I$ are the real and imaginary parts of the complex eigenvector).

The following results correspond to Table 1 for the Lorenz equations, but with $\delta = .4$ and $h = 0.01$.

N	100	200	300	400	500	600
ERR	4.06(−3)	6.39(−7)	1.77(−6)	1.39(−8)	4.80(−10)	1.79(−11)
RATE	5.50	7.13	4.41	4.52	4.31	4.12

Table 5

Since there are no quadratic terms in Chua's equations, we expect the second line of the table to approach $3 \times 0.83 + 1.32 \approx 3.81$.

Table 6 illustrates the $O(h^2)$ convergence rate. For $\delta = .4$ and various values of h, we list the discretisation error $||\mathbf{z}^s(\boldsymbol{\xi}) - \hat{\mathbf{z}}^s(\boldsymbol{\xi})||$ and the number of steps N required for the truncation error to reach this value.

h	.16	.8(−1)	.4(−1)	.2(−1)	.1(−1)	.5(−2)
ERR	5.39(−4)	1.32(−5)	3.29(−5)	8.22(−6)	2.05(−6)	5.13(−7)
N	30	60	120	235	470	940

Table 6

The next two sets of results use the same key as Table 3. Here we take $h = .5\times 10^{-2}$ and $N = 1000$.

δ	.64	.32	.16	.08
L–P	.70	.93(−1)	.18(−1)	.37(−2)
G–S[1]	.51	.24(−1)	.26(−2)	.34(−3)
G–S[2]	.33	.21(−2)	.20(−4)	.12(−6)

Table 7a

δ	.64	.32	.16	.08
L–P	96	15	9	7
G–S[1]	51	10	6	5
G–S[2]	32	6	4	3

Table 7b

Again the number of iterations did not significantly alter if more inner iterations were performed.

The following table shows the convergence rate and number of iterations for the 'HOV' method with fixed $\delta = .4$ and varying h and N, as in Table 6.

h/N	.16/30	.08/60	.04/120	.02/235	.01/470
C–RATE	.52	.71	.84	.92	.96
ITS	53	99	191	372	726

Table 8

Again, these results were obtained with one approximate Newton inner iteration. No significant change was observed if more inner iterations, or the original HOV method, were used.

It is clear from the above considerations that we will always be marching forwards to compute new stable components and backwards to compute new unstable components. Thus, if numerical stability requirements permit, an obvious alternative to discretising with a symmetric implicit method, like the trapezoidal rule, is to use the same explicit method for both types of components, but backwards in the unstable case. Using the Euler method merely as an illustration, we have

$$\begin{aligned} \mathbf{w}_n &= \mathbf{w}_{n-1} + hP^s\mathbf{F}(\mathbf{w}_{n-1}+\mathbf{y}_{n-1}) \\ \mathbf{y}_n &= \mathbf{y}_{n+1} + hP^u\mathbf{F}(\mathbf{w}_{n+1}+\mathbf{y}_{n+1}). \end{aligned}$$

There is now no practical objection to using the nonlinear Gauss-Seidel method itself, since it is totally explicit and does not require any equation solving! For the stable/unstable ordering we solve

$$\begin{aligned} \mathbf{w}_n &= \mathbf{w}_{n-1} + hP^s\mathbf{F}(\mathbf{w}_{n-1}+\mathbf{y}_{n-1}) \qquad && n = 1,\dots,N-1 \\ \mathbf{y}_n &= \mathbf{y}_{n+1} + hP^u\mathbf{F}(\mathbf{w}_{n+1}+\mathbf{y}_{n+1}) \qquad && n = N-1,\dots,0, \end{aligned}$$

with $\mathbf{w}_0 = \boldsymbol{\xi}$ and $\mathbf{y}_N = \mathbf{0}$, which may be regarded as a discretisation of the pair of equations in (2.6). [On the other hand, we do not consider the 'HOV' method

$$\left.\begin{aligned} \mathbf{y}_{n-1} &= \mathbf{y}_n + hP^u\mathbf{F}(\mathbf{w}_n+\mathbf{y}_n) \\ \mathbf{w}_n &= \mathbf{w}_{n-1} + hP^s\mathbf{F}(\mathbf{w}_{n-1}+\mathbf{y}_{n-1}) \end{aligned}\right. \qquad n = 1,\dots,N$$

again, since it does not correspond to (2.6) and, as we have seen above, converges more slowly as h is decreased.] Using this approach on the above examples, but with the improved Euler method

$$\mathbf{v}_n = \mathbf{v}_{n-1} + \frac{h}{2}\left\{\mathbf{F}(\mathbf{v}_{n-1}) + \mathbf{F}(\mathbf{v}_{n-1} + h\mathbf{F}(\mathbf{v}_{n-1}))\right\}$$

rather than the Euler method [note that for higher order methods we would need to carry a natural interpolant for the auxiliary unknown], gives

δ	16	8	4	2
C–RATE	.37	.45(−1)	.72(−2)	.11(−2)
ITS	34	11	7	6

Table 9

for the Lorenz equations, employing the same values of h and N as in Table 3 which therefore should be compared, and

δ	.64	.32	.16	.08
C–RATE	.33	.20(−2)	.20(−4)	.12(−6)
ITS	32	6	4	3

Table 10

for Chua's equation, which corresponds to Table 7.

3.2 Results when computing the whole Manifold

We have described the Hadamard and Perron methods for a general discrete dynamical system earlier in this section. In order to obtain a practical algorithm, however, we must replace the computation of $\hat{\mathbf{z}}^s$ by that of a finite-dimensional approximation $\hat{\mathbf{Z}}^s$. Thus one possibility is to place a grid on $\hat{\mathcal{E}}^s_\varepsilon$ and define $\hat{\mathbf{Z}}^s$ to be the continuous piecewise-linear mapping satisfying (3.1) with boundary conditions

$$\mathbf{u}_0 = \boldsymbol{\xi}_i + \hat{\mathbf{Z}}^s(\boldsymbol{\xi}_i) \text{ and } \hat{P}^u\mathbf{u}_{\tilde{N}} = \hat{\mathbf{Z}}^s(\hat{P}^s\mathbf{u}_{\tilde{N}})$$

for all grid points $\boldsymbol{\xi}_i$: i.e. we are imposing the collocation conditions that each grid point value of the manifold remains on the manifold after $\tilde{N}$ steps of the discrete dynamical system. Another possibility, which is the one we use in the examples below, is to choose $\hat{\mathbf{Z}}^s$ to be a tensor-product of polynomials and collocate at the Chebyshev points $\boldsymbol{\xi}_i$. The three algorithms for computing $\hat{\mathbf{Z}}^s$ iteratively are thus:-

- Hadamard

  $\hat{\mathbf{Z}}^{(k+1)}(\boldsymbol{\xi}_i) = \hat{P}^u\mathbf{u}_0$ where

  $$\mathbf{u}_n = \mathbf{f}(\mathbf{u}_{n-1}) \qquad n = 1,\ldots,\tilde{N}$$

  with $\hat{P}^s\mathbf{u}_0 = \boldsymbol{\xi}_i$ and $\hat{P}^u\mathbf{u}_{\tilde{N}} = \hat{\mathbf{Z}}^{(k)}(\hat{P}^s\mathbf{u}_{\tilde{N}})$;
- Perron

  $\hat{\mathbf{Z}}^{(k+1)}(\boldsymbol{\xi}_i) = \mathbf{y}_0$ where

  $$\begin{aligned}\mathbf{w}_n =& \hat{P}^s\mathbf{f}(\mathbf{w}_{n-1} + \hat{\mathbf{Z}}^{(k)}(\mathbf{w}_{n-1})) \qquad n = 1,\ldots,\tilde{N}\\ \mathbf{y}_n =& L^{-1}\mathbf{y}_{n+1} - L^{-1}\hat{P}^u\mathbf{g}(\mathbf{w}_n + \hat{\mathbf{Z}}^{(k)}(\mathbf{w}_n)) \qquad n = \tilde{N}-1,\ldots,0\end{aligned}$$

  with $\mathbf{w}_0 = \boldsymbol{\xi}_i$ and $\mathbf{y}_{\tilde{N}} = \hat{\mathbf{Z}}^{(k)}(\mathbf{w}_{\tilde{N}})$;
- Nonlinear Perron

  $\hat{\mathbf{Z}}^{(k+1)}(\boldsymbol{\xi}_i) = \mathbf{y}_0$ where

  $$\begin{aligned}\mathbf{w}_n =& \hat{P}^s\mathbf{f}(\mathbf{w}_{n-1} + \hat{\mathbf{Z}}^{(k)}(\mathbf{w}_{n-1})) \qquad n = 1,\ldots,\tilde{N}\\ \mathbf{y}_{n+1} =& \hat{P}^u\mathbf{f}(\mathbf{w}_n + \mathbf{y}_n) \qquad n = \tilde{N}-1,\ldots,0\end{aligned}$$

  with $\mathbf{w}_0 = \boldsymbol{\xi}_i$ and $\mathbf{y}_{\tilde{N}} = \hat{\mathbf{Z}}^{(k)}(\mathbf{w}_{\tilde{N}})$.

If, however, we are applying these algorithms to a discretisation of (1.1), then, in the light of the results at the end of the previous subsection, it seems a good idea to use a pair of forward/backward explicit discretisations for the Hadamard and nonlinear Perron methods. One would also prefer to use an explicit discretisation for the stable component in Perron's method, but the equation for the unstable component is

linear and so an implicit choice here does not involve additional work. Hence, using the improved Euler and trapezoidal methods as examples, the three algorithms become:-
$\hat{\mathbf{Z}}^{(k+1)}(\boldsymbol{\xi}_i) = \mathbf{y}_0$ where

- Hadamard

$$\mathbf{w}_n = \mathbf{w}_{n-1} + \frac{h}{2} P^s \{\mathbf{F}(\mathbf{w}_{n-1} + \mathbf{y}_{n-1}) + \mathbf{F}(\bar{\mathbf{w}}_n + \mathbf{y}_n)\} \qquad n = 1, \ldots, \tilde{N}$$
$$\text{where } \bar{\mathbf{w}}_n = \mathbf{w}_{n-1} + hP^s\mathbf{F}(\mathbf{w}_{n-1} + \mathbf{y}_{n-1})$$
$$\mathbf{y}_n = \mathbf{y}_{n+1} - \frac{h}{2} P^u \{\mathbf{F}(\mathbf{w}_{n+1} + \mathbf{y}_{n+1}) + \mathbf{F}(\mathbf{w}_n + \bar{\mathbf{y}}_n)\} \qquad n = \tilde{N} - 1, \ldots, 0$$
$$\text{where } \bar{\mathbf{y}}_n = \mathbf{y}_{n+1} - hP^u\mathbf{F}(\mathbf{w}_{n+1} + \mathbf{y}_{n+1})$$

- Perron

$$\mathbf{w}_n = \mathbf{w}_{n-1} + \frac{h}{2} P^s \left\{\mathbf{F}(\mathbf{w}_{n-1} + \hat{\mathbf{Z}}^{(k)}(\mathbf{w}_{n-1})) + \mathbf{F}(\bar{\mathbf{w}}_n + \hat{\mathbf{Z}}^{(k)}(\bar{\mathbf{w}}_n))\right\} \qquad n = 1, \ldots, \tilde{N}$$
$$\text{where } \bar{\mathbf{w}}_n = \mathbf{w}_{n-1} + hP^s\mathbf{F}(\mathbf{w}_{n-1} + \hat{\mathbf{Z}}^{(k)}(\mathbf{w}_{n-1}))$$
$$\mathbf{y}_n + \frac{h}{2} A\mathbf{y}_n = \mathbf{y}_{n+1} - \frac{h}{2} A\mathbf{y}_{n+1} - \mathbf{d}_n \qquad n = \tilde{N} - 1, \ldots, 0$$
$$\text{where } \mathbf{d}_n = \tfrac{h}{2} P^u \left\{\mathbf{G}(\mathbf{w}_{n+1} + \hat{\mathbf{Z}}^{(k)}(\mathbf{w}_{n+1})) + \mathbf{G}(\mathbf{w}_n + \hat{\mathbf{Z}}^{(k)}(\mathbf{w}_n))\right\}$$

- Nonlinear Perron

$$\mathbf{w}_n = \mathbf{w}_{n-1} + \frac{h}{2} P^s \left\{\mathbf{F}(\mathbf{w}_{n-1} + \hat{\mathbf{Z}}^{(k)}(\mathbf{w}_{n-1})) + \mathbf{F}(\bar{\mathbf{w}}_n + \hat{\mathbf{Z}}^{(k)}(\bar{\mathbf{w}}_n))\right\} \qquad n = 1, \ldots, \tilde{N}$$
$$\text{where } \bar{\mathbf{w}}_n = \mathbf{w}_{n-1} + hP^s\mathbf{F}(\mathbf{w}_{n-1} + \hat{\mathbf{Z}}^{(k)}(\mathbf{w}_{n-1}))$$
$$\mathbf{y}_n = \mathbf{y}_{n+1} - \frac{h}{2} P^u \{\mathbf{F}(\mathbf{w}_{n+1} + \mathbf{y}_{n+1}) + \mathbf{F}(\mathbf{w}_n + \bar{\mathbf{y}}_n)\} \qquad n = \tilde{N} - 1, \ldots, 0$$
$$\text{where } \bar{\mathbf{y}}_n = \mathbf{y}_{n+1} - hP^u\mathbf{F}(\mathbf{w}_{n+1} + \mathbf{y}_{n+1})$$

with $\mathbf{w}_0 = \boldsymbol{\xi}_i$ and $\mathbf{y}_{\tilde{N}} = \hat{\mathbf{Z}}^{(k)}(\mathbf{w}_{\tilde{N}})$.

Now we shall show how these algorithms perform on the Lorenz example and, with a slight abuse of notation, we denote the domain of our graphs by

$$\mathcal{E}^s_\varepsilon \equiv \delta_1\boldsymbol{\phi}_1 + \delta_2\boldsymbol{\phi}_2 \qquad |\delta_i| \leq \varepsilon \quad i = 1, 2.$$

Our first set of tables shows the discretisation error

$$\max_i |\hat{Z}^s(\boldsymbol{\xi}_i) - z^s(\boldsymbol{\xi}_i)|,$$

caused by an improved Euler pair with steplength h and a tensor product of polynomials of degree d with fixed $\varepsilon = 4$.

$h/\tilde{N}$	$3.2(-2)/4$	$1.6(-2)/8$	$.8(-2)/16$	$.4(-2)/32$	$.2(-2)/64$
$d = 2$	$3.1(-2)$	$7.1(-3)$	$1.8(-3)$	$6.0(-4)$	$3.0(-4)$

Table 11a

$h/\tilde{N}$	$6.4(-3)/10$	$3.2(-3)/20$	$1.6(-3)/40$	$.8(-3)/80$	$.4(-3)/160$
$d = 3$	$1.2(-3)$	$3.0(-4)$	$8.7(-5)$	$3.5(-5)$	$2.2(-5)$

Table 11b

$h/\tilde{N}$	$3.2(-3)/20$	$1.6(-3)/40$	$.8(-3)/80$	$.4(-3)/160$	$.2(-3)/320$
$d = 4$	$3.0(-4)$	$7.6(-5)$	$2.0(-5)$	$6.7(-6)$	$3.2(-6)$

Table 11c

As expected, the errors decrease by a factor of 4 until the effect of the polynomial collocation is felt.

Our next set of results shows the dependence of the Hadamard method on $\tilde{N}$, with $\varepsilon = 4$, $d = 6$ and $h = 0.01$ all fixed.

$\tilde{N}$	1	2	5	10	20	40
C–RATE	7.6(−1)	5.7(−1)	2.5(−1)	6.1(−2)	3.7(−3)	1.4(−5)
RATE*	28.0	28.0	28.0	27.9	27.9	27.9
ITS	119	61	26	14	8	4

Table 12

The first row is the convergence rate based on

$$\lim_{k\to\infty} \left\{ |e^{(k)}|^{1/k} \right\},$$

where

$$|e^{(k)}| \equiv \max_i |\hat{Z}^s(\boldsymbol{\xi}_i) - \hat{Z}^{(k)}(\boldsymbol{\xi}_i)|;$$

the second row is $\frac{\log \text{C–RATE}}{\tilde{N}h}$, which may be compared with the sum of the moduli of eigenvalues in Table 1; while the third row is the number of iterations required to achieve machine accuracy.

We now move on to Perron's method and its dependence on $\tilde{N}$.

$\tilde{N}$	1	2	5	10	20	40
C–RATE	7.6(−1)	5.7(−1)	2.5(−1)	7.7(−2)	3.6(−2)	3.6(−2)
ITS	119	61	26	15	11	11

Table 13

$\tilde{N}$	1	2	5	10	20	40
C–RATE	7.6(−1)	5.7(−1)	2.5(−1)	6.2(−2)	5.2(−3)	2.7(−3)
ITS	119	61	25	13	7	6

Table 14

Tables 13/14 give convergence results for the standard Perron method and the nonlinear version respectively, again with $\varepsilon = 4$, $d = 6$ and $h = 0.01$ all fixed. Note that, for small $\tilde{N}$, the performance is close to that in Table 12, while for larger $\tilde{N}$ the convergence rate appears to be settling down. We examine this more closely in Table 15 by fixing $\tilde{N} = 50$ and varying ε.

ε	16	8	4	2
C–RATE1	2.3(−1)	7.6(−2)	3.6(−2)	1.5(−2)
C–RATE2	1.4(−1)	1.5(−2)	2.8(−3)	4.4(−4)

Table 15

The convergence rate of the nonlinear Perron method in the second row is clearly superior to the convergence rate of the standard method in the first row.

4. Direct Approximation of a point on the Manifold

To determine a particular point $\boldsymbol{\xi} + \mathbf{z}^s(\boldsymbol{\xi})$ in the stable manifold, we return to the basic equation

$$\dot{\mathbf{u}} = \mathbf{F}(\mathbf{u}) \tag{4.1}$$

with boundary conditions

$$P^s\mathbf{u}(0) = \boldsymbol{\xi} \quad \text{and} \quad \lim_{t\to\infty} P^u\mathbf{u}(t) = \mathbf{0}. \tag{4.2}$$

Instead of following the indirect approach and creating a discrete dynamical system, we simply wish to solve the infinite interval boundary value problem (4.1-2) in order to determine $\mathbf{z}^s(\boldsymbol{\xi}) \equiv P^u\mathbf{u}(0)$. This problem is closely related to that of computing homo-/hetero-clinic orbits, which has recently been considered in several papers [5,10,20,22,23].

The most obvious approach is to truncate (4.1) to a finite interval $(0,T)$ and use the boundary condition

$$P^u\mathbf{u}(T) = \mathbf{0}.$$

Then, by mapping to the standard interval $(0,1)$ through $\tau = \frac{t}{T}$, we may use a standard BVP package, such as COLSYS [3], to solve

$$\dot{\mathbf{u}} = T\mathbf{F}(\mathbf{u}) \tag{4.3}$$

with boundary conditions

$$P^s\mathbf{u}(0) = \boldsymbol{\xi} \quad \text{and} \quad P^u\mathbf{u}(1) = \mathbf{0}. \tag{4.4}$$

Here T is a known parameter, which must be chosen in the light of the truncation error introduced, cf. (2.5). These equations may be compared with those in [5] for connecting orbits. Alternatively, as suggested in [23], we may let T be an unknown parameter, which is defined by insisting that the required solution of (4.1) passes through the ε-ball in $\mathcal{E}^s$ when $t = T$; i.e. our boundary conditions are

$$P^s\mathbf{u}(0) = \boldsymbol{\xi},\ P^u\mathbf{u}(T) = \mathbf{0} \quad \text{and} \quad ||P^s\mathbf{u}(T)|| = \varepsilon.$$

Thus, again mapping to the standard interval $(0,1)$ through $\tau = \frac{t}{T}$, we arrive at

$$\dot{\mathbf{u}} = T\mathbf{F}(\mathbf{u}) \tag{4.5}$$

with boundary conditions

$$P^s\mathbf{u}(0) = \boldsymbol{\xi},\ P^u\mathbf{u}(1) = \mathbf{0} \quad \text{and} \quad ||P^s\mathbf{u}(1)|| = \varepsilon. \tag{4.6}$$

The system (4.5-6) may be solved by a standard BVP package, such as COLPAR [3], which allows unknown parameters. Here ε is a (small) parameter whose value must be chosen [23].

In [22] it was suggested that arclength parametrisation, rather than time, would often be a good choice for periodic and connecting orbits. Supporting numerical results have also been given in [20]. The required solution of (4.1-2) may also be parametrised by arclength in phase-space and, after mapping to the standard interval $(0,1)$, the resulting equations are

$$\dot{\mathbf{u}} = L\frac{\mathbf{F}(\mathbf{u})}{||\mathbf{F}(\mathbf{u})||} \tag{4.7}$$

with boundary conditions

$$P^s\mathbf{u}(0) = \boldsymbol{\xi} \quad \text{and} \quad \mathbf{u}(1) = \mathbf{0}, \tag{4.8}$$

where L is the unknown length of the solution curve in $\Re^m$. This is a singular [16,17,18] boundary value problem and, in [22], it was suggested that the natural collocation approach is either to use Gauss-Lobatto points on all subintervals or to use Gauss-Legendre points on all subintervals apart from the last, where Gauss-Radau is appropriate. This allows one to replace the collocation of (4.7) at $\sigma = 1$, which does not make sense, with the limiting equations

$$P^u\dot{\mathbf{u}}(1) = \mathbf{0} \quad \text{and} \quad ||\dot{\mathbf{u}}(1)|| = L.$$

Thus, as with connecting orbits, one ends up with an equal number of equations and unknowns. The smoothness of $\mathbf{u}(\sigma)$ as $\sigma \to 1$ is governed by the eigenvalues of A with negative real part (besides the smoothness of $\mathbf{F}$ of course). If these are $\lambda_j^R + i\lambda_j^I$ with

$$\lambda_1^R \geq \lambda_2^R \geq \cdots$$

then

$$\mathbf{u}(\sigma) = \mathbf{c}_1(1-\sigma) + \mathbf{c}_2(1-\sigma)^{\lambda_1^R/\lambda_2^R} + \cdots$$

Thus the smoothness decreases as λ_2^R approaches λ_1^R and, in the limit, when $\lambda_1^R + i\lambda_1^I$ is defective or forms part of a complex conjugate pair, even $\dot{\mathbf{u}}(1)$ can fail to exist. Numerical results for connecting orbits in this case (cf. Šilnikov bifurcation [29]) shows that arclength has difficulties [20]. A (partial) solution is to mimic the idea of (4.5-6) and solve

$$\dot{\mathbf{u}} = \tilde{L}\frac{\mathbf{F}(\mathbf{u})}{||\mathbf{F}(\mathbf{u})||} \tag{4.9}$$

with boundary conditions

$$P^s\mathbf{u}(0) = \boldsymbol{\xi},\ P^u\mathbf{u}(1) = \mathbf{0} \quad \text{and} \quad ||\mathbf{u}(1)|| = \varepsilon. \tag{4.10}$$

Here ε is a known constant and $\tilde{L}$ is an unknown parameter which approximates the length of the solution curve in $\Re^m$. These equations may be solved by any BVP package that allows unknown parameters. Of course, there is also an extra error created by this approximate boundary condition at $\sigma = 1$.

Since we may assume that the eigenvalues of A are known, there is also an alternative explicit exponential change-of-variable available, apart from the implicit arclength parametrisation. Thus if we choose

$$0 > -\mu > \lambda_1^R$$

and set

$$\tilde{s} = \exp -\mu t,$$

then (4.1-2) becomes

$$\dot{\mathbf{u}} = -\frac{\mathbf{F}(\mathbf{u})}{\mu\tilde{s}} \tag{4.11}$$

with boundary conditions

$$\mathbf{u}(0) = \mathbf{0} \quad \text{and} \quad P^s\mathbf{u}(1) = \boldsymbol{\xi}. \tag{4.12}$$

This is again a singular boundary value problem, this time at $\tilde{s} = 0$, and one should use Gauss-Radau collocation on the first subinterval (with the condition

$$P^u\dot{\mathbf{u}}(0) = \mathbf{0}$$

replacing the collocation at $\tilde{s} = 0$) and Gauss-Legendre collocation on all other subintervals. The smoothness and decay of $\mathbf{u}$ as $\tilde{s} \to 0$ is governed by the ratio of $-\mu$ and λ_1^R, i.e. if we choose μ so that $-p\mu > \lambda_1^R$, then the first p derivatives of $\mathbf{u}$ are zero at $\tilde{s} = 0$.

The following table of errors was produced for the 3-point Gauss-Lobatto scheme ($O(h^4)$) applied to Chua's equation. The value being approximated is $P^u\mathbf{u}(1)$, with $P^s\mathbf{u}(1) = 0.5(\boldsymbol{\phi}_R + \boldsymbol{\phi}_I)$.

$N \equiv h^{-1}$	$\mu = 0.28$	$\mu = 0.82$
4	.25(−2)	.63(−4)
8	.24(−3)	.19(−4)
16	.18(−4)	.10(−5)
32	.12(−5)	.60(−7)
64	.75(−7)	.45(−8)
128	.44(−8)	.22(−9)

Table 16

Setting $\mu = 0.28$ forces the solution to have three zero derivatives at $\tilde{s} = 0$ and this is sufficient for $O(h^4)$ convergence, as verified by the results. We do not, however, claim that this value is optimal since the second column of results, while still $O(h^4)$, is obviously superior. Note that, for $\mu = 0.82$, $\mathbf{u}$ is only just differentiable at $\tilde{s} = 0$. An analysis of superconvergence for collocation methods applied to singular problems, however, has yet to be given, as is remarked upon in [3,p.485].

5. Direct Approximation of the whole Manifold

As we have seen, current algorithms for approximating $\mathcal{W}^s_{loc}$ are all based on first discretising the differential equation (1.1). In contrast, we prefer to view the problem geometrically and only discretise an equation for the invariant manifold itself. Such an equation is obtained by noticing that, if $\mathcal{M}$ is a r-dimensional C^1 sub-manifold of $\Re^m$, then $\mathcal{M}$ is invariant for (1.1) iff

$$\mathbf{F}(\mathbf{x}) \in T_x\mathcal{M} \qquad \forall \mathbf{x} \in \mathcal{M}, \tag{5.1}$$

where $T_x\mathcal{M}$ is the r-dimensional tangent space of $\mathcal{M}$ at $\mathbf{x}$. This defining condition has been used to compute other types of invariant manifold in [23,24], and it provides $m-r$ equations at each point of $\mathcal{M}$. The missing r equations may be regarded as a choice of parametrisation, which must be chosen. Thus, for $\mathcal{W}^s_{loc}$, it is natural to parametrise by $\mathcal{E}^s$ so that

$$T_\xi \mathcal{W}^s_{loc} = \{\bar{\boldsymbol{\xi}} + B^s(\boldsymbol{\xi})\bar{\boldsymbol{\xi}} : \bar{\boldsymbol{\xi}} \in \mathcal{E}^s\},$$

where $B^s(\boldsymbol{\xi})$ is the linearisation of $\mathbf{z}^s$ at $\boldsymbol{\xi}$. Since $\mathbf{x} \in T_\xi \mathcal{W}^s_{loc}$ iff $P^u\mathbf{x} = B^s(\boldsymbol{\xi})P^s\mathbf{x}$, our equation for $\mathbf{z}^s$ becomes

$$P^u\mathbf{F}(\boldsymbol{\xi} + \mathbf{z}^s(\boldsymbol{\xi})) - B^s(\boldsymbol{\xi})P^s\mathbf{F}(\boldsymbol{\xi} + \mathbf{z}^s(\boldsymbol{\xi})) = \mathbf{0}. \tag{5.2}$$

To make further progress we must introduce bases $\mathbf{e}^s_i \quad i = 1, \ldots, p^s$ and $\mathbf{e}^u_i \quad i = 1, \ldots, p^u$ for $\mathcal{E}^s$ and $\mathcal{E}^u$ respectively. Then (5.1) can be written

$$\sum_{j=1}^{p^s} \frac{\partial z^s_k}{\partial w_j} P^s_j \mathbf{F}(\textstyle\sum_{i=1}^{p^s} w_i \mathbf{e}^s_i + \sum_{i=1}^{p^u} z^s_i(\mathbf{w})\mathbf{e}^u_i) = P^u_k \mathbf{F}(\sum_{i=1}^{p^s} w_i \mathbf{e}^s_i + \sum_{i=1}^{p^u} z^s_i(\mathbf{w})\mathbf{e}^u_i) \tag{5.3}$$

for $k = 1, \ldots, p^u$, where

$$\mathbf{z}^s(\textstyle\sum_{i=1}^{p^s} w_i \mathbf{e}^s_i) = \displaystyle\sum_{i=1}^{p^u} z^s_i(\mathbf{w})\mathbf{e}^u_i$$

and P^s_j & P^u_k are defined by

$$\mathbf{x} = \textstyle\sum_{i=1}^{p^s} \alpha_i \mathbf{e}^s_i + \sum_{i=1}^{p^u} \beta_i \mathbf{e}^u_i \quad \Rightarrow \quad P^s_j \mathbf{x} = \alpha_j \text{ and } P^u_k \mathbf{x} = \beta_k.$$

Equations (5.3) form a quasi-linear hyperbolic system with the same principal part [8] and these are the equations that we discretise and solve in order to obtain approximations for $\mathcal{W}^s_{loc}$. Note that the characteristic differential equations just lead back to (1.1), i.e.

$$\begin{aligned} \dot{w}_j &= P^s_j \mathbf{F}(\textstyle\sum_{i=1}^{p^s} w_i \mathbf{e}^s_i + \sum_{i=1}^{p^u} y_i(\mathbf{w})\mathbf{e}^u_i) \quad j = 1, \ldots, p^s \\ \dot{y}_k &= P^u_k \mathbf{F}(\textstyle\sum_{i=1}^{p^s} w_i \mathbf{e}^s_i + \sum_{i=1}^{p^u} y_i(\mathbf{w})\mathbf{e}^u_i) \quad k = 1, \ldots, p^u. \end{aligned}$$

Hence a characteristic method applied to (5.3) is equivalent to discretising (1.1) explicitly, which is not our aim. It is possible, however, to use an alternative computational scheme on the p.d.e. (5.3). The unusual feature here, of course, is that the characteristics all emanate from the origin, naturally because the manifold is developing from this point.

It must be admitted that converting to a p.d.e. formulation means that it is only practical to consider low dimensional manifolds; however these are an important class. In the simplest case of one-dimensional manifolds, i.e. for (5.3) with $p^s = 1$, we can write

$$\frac{dz_k^s}{dw}(w) = \begin{cases} \frac{P_k^u \mathbf{F}(w\mathbf{e}^s + \sum_{i=1}^{m-1} z_i^s(w)\mathbf{e}_i^u)}{P_1^s \mathbf{F}(w\mathbf{e}^s + \sum_{i=1}^{m-1} z_i^s(w)\mathbf{e}_i^u)} & w \neq 0 \\ 0 & w = 0 \end{cases}$$

with initial condition $z_k^s(0) = 0$ for $k = 1, \ldots, m-1$. [This is the reverse of the standard procedure of replacing m non-autonomous equations with a system of $m+1$ autonomous equations, i.e. by introducing a parametrisation we replace m autonomous equations with a non-autonomous system of $m-1$.] A natural alternative parametrisation in the one-dimensional case is arc-length, and by going back to the fundamental equation (5.1) we see that in this case our equation to solve is

$$\frac{d\mathbf{u}}{ds}(s) = \begin{cases} \frac{-\mathbf{F}(\mathbf{u})}{\|\mathbf{F}(\mathbf{u})\|} & s \neq 0 \\ \mathbf{e}^s & s = 0 \end{cases}$$

with initial condition $\mathbf{u}(0) = \mathbf{0}$. In this paper, however, we shall restrict ourselves to two-dimensional manifolds, and develop our algorithm for parametrisation by the invariant subspace in subsection 5.1. In this case too there is a natural alternative parametrisation, i.e. geodesic polar co-ordinates, and we conclude by considering this in subsection 5.2.

5.1 Stable subspace parametrisation

We consider here parametrisation by the stable subspace, and simplify (5.3) for the case $p^s = 2$. Hence we seek a solution $\mathbf{y} : \Re^2 \mapsto \Re^{m-2}$ of the equation

$$[\nabla y_k]^{\mathrm{T}} \mathbf{F}^s(\mathbf{w}, \mathbf{y}) = F_k^u(\mathbf{w}, \mathbf{y}) \qquad k = 1, \ldots, m-2 \tag{5.4}$$

where $\mathbf{F}^s : \Re^2 \times \Re^{m-2} \mapsto \Re^2$ is defined by

$$F_i^s(\mathbf{w}, \mathbf{y}) = P_i^s \mathbf{F}(\textstyle\sum_{k=1}^2 w_k \mathbf{e}_k^s + \sum_{k=1}^{m-2} y_k \mathbf{e}_k^u)$$

and $\mathbf{F}^u : \Re^2 \times \Re^{m-2} \mapsto \Re^{m-2}$ by

$$F_i^u(\mathbf{w}, \mathbf{y}) = P_i^u \mathbf{F}(\textstyle\sum_{k=1}^2 w_k \mathbf{e}_k^s + \sum_{k=1}^{m-2} y_k \mathbf{e}_k^u).$$

Since the characteristics of (5.4) emanate from the origin, it is natural to impose polar co-ordinates on $\mathcal{E}^s$ and use r as the 'time-like' independent variable. Thus, applying the co-ordinate change $w_1 = r\cos\theta$ and $w_2 = r\sin\theta$ to (5.4), we arrive at

$$\{\cos\theta F_1^s(r,\theta,\mathbf{y}) + \sin\theta F_2^s(r,\theta,\mathbf{y})\}\frac{\partial y_k}{\partial r} + \{\cos\theta F_2^s(r,\theta,\mathbf{y}) - \sin\theta F_1^s(r,\theta,\mathbf{y})\}\frac{1}{r}\frac{\partial y_k}{\partial \theta} = F_k^u(r,\theta,\mathbf{y})$$

for $k = 1, \ldots, m-2$. In order to be able to march forward in r, the characteristics must be transversal to the circles of constant radius, i.e. $\cos\theta F_1^s(r,\theta,\mathbf{y}) + \sin\theta F_2^s(r,\theta,\mathbf{y}) \neq 0$. By looking at the dominant term, this will be true near the origin iff A^s is definite, where A^s is the 2×2 matrix defined by $A_{ij}^s = P_i^s A \mathbf{e}_j^s$, and this condition depends on the basis $\{\mathbf{e}_1^s, \mathbf{e}_2^s\}$ chosen for $\mathcal{E}^s$. Hence, we choose this basis, depending on the two stable eigenvalues, in the following way.

- In the real non-defective case we use the eigenvector basis, so that $A^s = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ where the stable eigenvalues are $\lambda_{1,2}$.
- In the complex-conjugate case we use the real & imaginary parts of the eigenvectors as a basis, so that $A^s = \left(\begin{smallmatrix} \lambda_R & -\lambda_I \\ \lambda_I & \lambda_R \end{smallmatrix}\right)$ where the stable eigenvalues are $\lambda_R \pm i\lambda_I$.
- In the real defective case we use a generalised eigenvector basis, chosen so that $A^s = \left(\begin{smallmatrix} \lambda & \kappa \\ 0 & \lambda \end{smallmatrix}\right)$ with $|\kappa| < 2|\lambda|$.

[This is equivalent to using the orthogonal basis $\{\frac{\mathbf{e}_1^s+\mathbf{e}_2^s}{\sqrt{2(1+\mathbf{e}_1^s.\mathbf{e}_2^s)}}, \frac{\mathbf{e}_1^s-\mathbf{e}_2^s}{\sqrt{2(1-\mathbf{e}_1^s.\mathbf{e}_2^s)}}\}$, i.e. the interior and exterior bisectors, and then using ellipsoidal co-ordinates $w_1 = \sqrt{2(1+\mathbf{e}_1^s.\mathbf{e}_2^s} r\cos\theta$, $w_2 = \sqrt{2(1-\mathbf{e}_1^s.\mathbf{e}_2^s} r\sin\theta$.] Then we can march in r using

$$\frac{\partial y_k}{\partial r} = \frac{F_k^u(r,\theta,\mathbf{y}) - \{\cos\theta F_2^s(r,\theta,\mathbf{y}) - \sin\theta F_1^s(r,\theta,\mathbf{y})\}\frac{1}{r}\frac{\partial y_k}{\partial\theta}}{\cos\theta F_1^s(r,\theta,\mathbf{y}) + \sin\theta F_2^s(r,\theta,\mathbf{y})}$$

for $k = 1,\ldots,m-2$.

We note that, in the case $m = 3$, the above formula can be simplified by utilising vector products: i.e.

$$\{\cos\theta\hat{F}_1(r,\theta,y) + \sin\theta\hat{F}_2(r,\theta,y)\}\frac{\partial y}{\partial r} + \{\cos\theta\hat{F}_2(r,\theta,y) - \sin\theta\hat{F}_1(r,\theta,y)\}\frac{1}{r}\frac{\partial y}{\partial\theta} = \hat{F}_3(r,\theta,y)$$

where $\hat{F}_i(r,\theta,y) = \hat{\mathbf{e}}_i^{\mathrm{T}}\mathbf{F}(r\cos\theta\ \mathbf{e}_1^s + r\sin\theta\ \mathbf{e}_2^s + y\mathbf{e}^u)$ and $\hat{\mathbf{e}}_1 = \mathbf{e}_2^s\times\mathbf{e}^u, \hat{\mathbf{e}}_2 = \mathbf{e}^u\times\mathbf{e}_1^s, \hat{\mathbf{e}}_3 = \mathbf{e}_1^s\times\mathbf{e}_2^s$.

It is now time to discretise our equations. Of course there are many ways of doing this but, because our solutions will usually be smooth and also because of the periodicity of θ, we have chosen a spectral collocation method [7] in a method-of-lines approach. Thus we seek

$$Y_{ik}(r) \equiv y_k(r,\theta_i) \qquad \theta_i = \frac{2\pi i}{2N} \quad i = 0,\ldots,2N-1$$

as the solution of a set of O.D.E.'s and use the *spectral relationship* [7]

$$\frac{df}{d\theta}(\theta_i) \approx [T\mathbf{f}]_i \qquad i = 0,\ldots,2N-1,$$

where $\mathbf{f} \equiv \{f(\theta_0),\ldots,f(\theta_{2N-1})\}^{\mathrm{T}}$ and T is the $2N\times 2N$ matrix

$$T_{ij} = \begin{cases} \frac{1}{2}(-1)^{i+j}\cot\frac{\theta_i-\theta_j}{2} & i\neq j \\ 0 & i = j \end{cases}$$

for $i,j = 0,\ldots,2N-1$. This leads to the O.D.E. system

$$\begin{aligned} &\{\cos\theta F_1^s(r,\theta_i,\textstyle\sum_{j=1}^{m-2}Y_{ij}\mathbf{e}_j^u) + \sin\theta F_2^s(r,\theta_i,\sum_{j=1}^{m-2}Y_{ij}\mathbf{e}_j^u)\}\dfrac{dY_{ik}}{dr} \\ &\quad + \{\cos\theta F_2^s(r,\theta_i,\textstyle\sum_{j=1}^{m-2}Y_{ij}\mathbf{e}_j^u) - \sin\theta F_1^s(r,\theta_i,\sum_{j=1}^{m-2}Y_{ij}\mathbf{e}_j^u)\}\dfrac{1}{r}\displaystyle\sum_{j=0}^{2N-1} T_{ij}Y_{jk} \\ &\quad = F_k^u(r,\theta_i,\textstyle\sum_{j=1}^{m-2}Y_{ij}\mathbf{e}_j^u) \end{aligned}$$

for $k = 1,\ldots,m-2$ and $i = 0,\ldots,2N-1$ with initial condition $Y_{ik} = 0$. This final system can be solved by any standard package, noting that $\frac{dY_{ik}}{dr}(0) = 0$, and we display below the graph produced for the Lorenz equations by the Mathematica routine NDSolve, with $N = 8$ and integrating forward to $r = 1$.

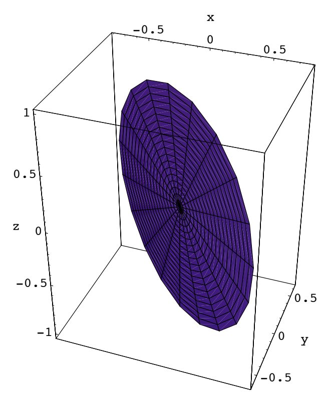

Fig. 1.

5.2 Geodesic polar co-ordinates

As an alternative parametrisation of a two-dimensional stable manifold, we consider geodesic polar co-ordinates [31]. In this case points in the stable subspace are denoted by

$$r\cos\theta\ \mathbf{e}_1 + r\sin\theta\ \mathbf{e}_2, \tag{5.5}$$

where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is an orthonormal basis for $\mathcal{E}^s$. Then $\mathcal{W}^s_{loc}$ is sought as the range of a function $\mathbf{v}(r,\theta)$, where $\mathbf{v} : \Re^+ \times S^1 \mapsto \Re^m$, which satisfies

$$\begin{aligned} &\text{i)} && \mathbf{F}(\mathbf{v}(r,\theta)) \in \left\{ \frac{\partial \mathbf{v}}{\partial r}(r,\theta), \frac{\partial \mathbf{v}}{\partial \theta}(r,\theta) \right\} \\ &\text{ii)} && \left(\frac{\partial \mathbf{v}}{\partial r}(r,\theta) \right)^T \frac{\partial \mathbf{v}}{\partial \theta}(r,\theta) = 0 \\ &\text{iii)} && \|\frac{\partial \mathbf{v}}{\partial r}(r,\theta)\| = 1 \end{aligned} \tag{5.6}$$

with $\frac{\partial \mathbf{v}}{\partial r}(0,\theta) = \cos\theta\ \mathbf{e}_1 + \sin\theta\ \mathbf{e}_2$. (Thus (5.5) is the first-order term in r for $\mathbf{v}$.) The first part of (5.6) is just the fundamental invariance condition (5.1), while the latter two force an orthogonal co-ordinate system for which the gridlines in the r-direction measure arclength. From such conditions it may be deduced that these gridlines are geodesics on the manifold [31].

An equation for $\frac{\partial \mathbf{v}}{\partial r}$ may easily be obtained from (5.6), i.e.

$$\frac{\partial \mathbf{v}}{\partial r} = \frac{\frac{\frac{\partial \mathbf{v}}{\partial \theta}^T \mathbf{F}(\mathbf{v})}{\frac{\partial \mathbf{v}}{\partial \theta}^T \frac{\partial \mathbf{v}}{\partial \theta}} \frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{F}(\mathbf{v})}{\|\frac{\frac{\partial \mathbf{v}}{\partial \theta}^T \mathbf{F}(\mathbf{v})}{\frac{\partial \mathbf{v}}{\partial \theta}^T \frac{\partial \mathbf{v}}{\partial \theta}} \frac{\partial \mathbf{v}}{\partial \theta} - \mathbf{F}(\mathbf{v})\|}. \tag{5.7}$$

Unless A^s is normal, however, we cannot use (5.7) to march forward in r because the denominator will pass through zero, as with parametrisation by $\mathcal{E}^s$. This difficulty can be circumvented, of course, by proceeding as in the previous subsection and choosing new co-ordinates. Such a solution, however, which is equivalent to using a different inner-product in (5.6), destroys the canonical nature of our parametrisation and so we prefer the following method.

If $\{\mathbf{e}^s_1, \mathbf{e}^s_2\}$ is a (in general non-orthogonal) basis of unit vectors for $\mathcal{E}^s$, which is chosen as in the previous subsection to ensure that A^s is definite, then the interior and exterior bisectors

$$\begin{aligned} \mathbf{e}_1 &= \frac{\mathbf{e}^s_1 + \mathbf{e}^s_2}{\sqrt{2(1 + \mathbf{e}^s_1 . \mathbf{e}^s_2)}} \\ \mathbf{e}_2 &= \frac{\mathbf{e}^s_2 - \mathbf{e}^s_1}{\sqrt{2(1 - \mathbf{e}^s_1 . \mathbf{e}^s_2)}} \end{aligned}$$

form a particular orthonormal basis for $\mathcal{E}^s$. If points in the stable space are represented by polar co-ordinates (ρ, ϕ) and (r, θ) with respect to these two bases, i.e.

$$\rho\cos\phi\ \mathbf{e}^s_1 + \rho\sin\phi\ \mathbf{e}^s_2 \qquad \text{and} \qquad r\cos\theta\ \mathbf{e}_1 + r\sin\theta\ \mathbf{e}_2,$$

this defines a diffeomorphism between (ρ, ϕ) and (r, θ) given by

$$\begin{aligned} r &= \rho\sqrt{1 + \mathbf{e}^s_1 . \mathbf{e}^s_2 \sin 2\phi} \\ \tan\theta &= \frac{\sqrt{1 - \mathbf{e}^s_1 . \mathbf{e}^s_2}\ (\sin\phi - \cos\phi)}{\sqrt{1 + \mathbf{e}^s_1 . \mathbf{e}^s_2}\ (\sin\phi + \cos\phi)} \end{aligned} \tag{5.8a}$$

and

$$\begin{aligned} \rho &= r\left\{ \frac{\cos^2\theta}{1 + \mathbf{e}^s_1 . \mathbf{e}^s_2} + \frac{\sin^2\theta}{1 - \mathbf{e}^s_1 . \mathbf{e}^s_2} \right\} \\ \tan\phi &= \frac{\sqrt{1 - \mathbf{e}^s_1 . \mathbf{e}^s_2}\cos\theta + \sqrt{1 + \mathbf{e}^s_1 . \mathbf{e}^s_2}\sin\theta}{\sqrt{1 - \mathbf{e}^s_1 . \mathbf{e}^s_2}\cos\theta - \sqrt{1 + \mathbf{e}^s_1 . \mathbf{e}^s_2}\sin\theta}. \end{aligned} \tag{5.8b}$$

Now, if we seek the stable manifold in the form $\tilde{\mathbf{v}}(\rho,\phi)$, equations (5.6) transform to

$$(5.9)\qquad \begin{aligned} &\text{i)} && \mathbf{F}(\tilde{\mathbf{v}}(\rho,\phi)) \in \left\{\frac{\partial\tilde{\mathbf{v}}}{\partial\rho}(\rho,\phi), \frac{\partial\tilde{\mathbf{v}}}{\partial\phi}(\rho,\phi)\right\} \\ &\text{ii)} && \left(\frac{\partial\tilde{\mathbf{v}}}{\partial\rho}(\rho,\phi)\right)^T \frac{\partial\tilde{\mathbf{v}}}{\partial\phi}(\rho,\phi) = \rho\mathbf{e}_1^s.\mathbf{e}_2^s\cos 2\phi \\ &\text{iii)} && \left(\frac{\partial\tilde{\mathbf{v}}}{\partial\rho}(\rho,\phi)\right)^T \frac{\partial\tilde{\mathbf{v}}}{\partial\rho}(\rho,\phi) = 1+\mathbf{e}_1^s.\mathbf{e}_2^s\sin 2\phi. \end{aligned}$$

Thus, since

$$\tilde{\mathbf{v}}(\rho,\phi) \approx \rho\cos\phi\ \mathbf{e}_1^s + \rho\sin\phi\ \mathbf{e}_2^s$$

to first order in ρ, we must have $\frac{\partial\tilde{\mathbf{v}}}{\partial\phi}$ and $\mathbf{F}(\tilde{\mathbf{v}})$ linearly independent in a neighbourhood of $\rho = 0$ and so (5.9) defines $\frac{\partial\tilde{\mathbf{v}}}{\partial\rho}$. Consequently, we can use (5.9) to march forward in ρ, with

$$\frac{\partial\tilde{\mathbf{v}}}{\partial\rho}(0,\phi) = \cos\phi\ \mathbf{e}_1^s + \sin\phi\ \mathbf{e}_2^s.$$

Then, having computed $\mathcal{W}_{loc}^s$ in the form $\tilde{\mathbf{v}}(\rho,\phi)$, we can use (5.8) to transform back to $\mathbf{v}(r,\theta)$. Equation (5.9) needs to be discretised, of course, and, as in the previous subsection, we have used a spectral collocation method in ϕ. The graph below was again produced for the Lorenz equations by the Mathematica routine NDSolve, with the same parameters as above.

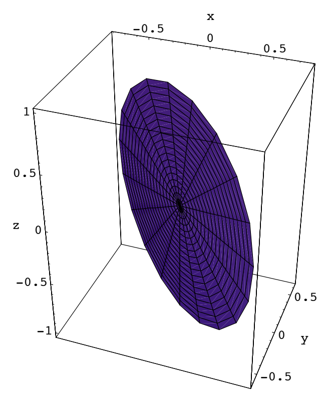

Fig. 2.

6. Appendix - proofs of convergence results

In subsections 6.1-3 we shall use the following assumptions.

- Norm on $\Re^m$

$$||\mathbf{x}||_\star \equiv \max\{||P^s\mathbf{x}||, ||P^u\mathbf{x}||\}$$

- Nonlinearity

  $\exists\ \gamma > 0$ such that

$$||\mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{y})||_\star \le K(r)\max\{||\mathbf{x}||_\star^\gamma, ||\mathbf{y}||_\star^\gamma\}||\mathbf{x}-\mathbf{y}||_\star$$

  $\forall ||\mathbf{x}||_\star, ||\mathbf{y}||_\star \le r$ where $K(r)$ is a non-decreasing function of r.

- Dynamics

  $\exists\ \alpha > 0, \beta > 0, C \ge 1$ such that

$$||e^{At}P^s|| \le Ce^{-\alpha t} \quad ||e^{-At}P^u|| \le Ce^{-\beta t} \qquad \forall t \ge 0$$

6.1 Liapunov-Perron Method

We restrict attention to $\boldsymbol{\xi} \in \mathcal{E}^s_\varepsilon$ with ε satisfying

$$\varepsilon^\gamma K(2C\varepsilon)[2C]^{1+\gamma} \leq \gamma\alpha,$$

and consider the mapping $\mathbf{u} \mapsto \mathbf{v}$ defined by

$$\mathbf{v}(t) = e^{At}\boldsymbol{\xi} + \int_0^t e^{A(t-s)} P^s \mathbf{G}(\mathbf{u}(s))\, ds - \int_t^\infty e^{A(t-s)} P^u \mathbf{G}(\mathbf{u}(s))\, ds. \tag{6.1}$$

a) The set $||\mathbf{u}(t) - e^{At}\boldsymbol{\xi}||_\star \leq C||\boldsymbol{\xi}||e^{-\alpha t}$ is mapped to itself by (6.1) since $||P^s\mathbf{u}(t)|| \leq 2C||\boldsymbol{\xi}||e^{-\alpha t}$ and $||P^u\mathbf{u}(t)|| \leq C||\boldsymbol{\xi}||e^{-\alpha t}$ imply

$$\begin{aligned} || \int_0^t e^{A(t-s)} P^s \mathbf{G}(\mathbf{u}(s))\, ds|| &\leq CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^{1+\gamma} \int_0^t e^{-\alpha(t-s)} e^{-(1+\gamma)\alpha s}\, ds \\ &\leq CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^{1+\gamma} \frac{e^{-\alpha t}}{\gamma\alpha} \leq C||\boldsymbol{\xi}||e^{-\alpha t} \end{aligned}$$

and

$$\begin{aligned} || \int_t^\infty e^{A(t-s)} P^u \mathbf{G}(\mathbf{u}(s))\, ds|| &\leq CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^{1+\gamma} \int_t^\infty e^{\beta(t-s)} e^{-(1+\gamma)\alpha s}\, ds \\ &\leq CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^{1+\gamma} \frac{e^{-(1+\gamma)\alpha t}}{(1+\gamma)\alpha} \leq C||\boldsymbol{\xi}||e^{-\alpha t}. \end{aligned}$$

b) The norm $|||\mathbf{u}||| \equiv \max_{t\geq 0}\{||\mathbf{u}(t)||_\star\}$ sets up (6.1) as a contraction on the set in a) since

$$\begin{aligned} &|| \int_0^t e^{A(t-s)} P^s \{\mathbf{G}(\mathbf{u}_1(s)) - \mathbf{G}(\mathbf{u}_2(s))\}\, ds|| \\ &\qquad \leq CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^\gamma \int_0^t e^{-\alpha(t-s)} e^{-\gamma\alpha s} ||\mathbf{u}_1(s) - \mathbf{u}_2(s)||_\star\, ds \\ &\qquad \leq CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^\gamma \frac{1}{\gamma\alpha} |||\mathbf{u}_1 - \mathbf{u}_2||| \leq \frac{1}{2} |||\mathbf{u}_1 - \mathbf{u}_2||| \end{aligned}$$

and

$$\begin{aligned} &|| \int_t^\infty e^{A(t-s)} P^u \{\mathbf{G}(\mathbf{u}_1(s)) - \mathbf{G}(\mathbf{u}_2(s))\}\, ds|| \\ &\qquad \leq CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^\gamma \int_t^\infty e^{\beta(t-s)} e^{-\gamma\alpha s} ||\mathbf{u}_1(s) - \mathbf{u}_2(s)||_\star\, ds \\ &\qquad \leq CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^\gamma \frac{1}{\gamma\alpha} |||\mathbf{u}_1 - \mathbf{u}_2||| \leq \frac{1}{2} |||\mathbf{u}_1 - \mathbf{u}_2|||. \end{aligned}$$

Hence the actual fixed point of (6.1) will satisfy

- $||P^s\mathbf{u}(t) - e^{At}\boldsymbol{\xi}|| = O(e^{-\alpha t}||\boldsymbol{\xi}||^{1+\gamma})$
- $||P^u\mathbf{u}(t)|| = O(e^{-\alpha(1+\gamma)t}||\boldsymbol{\xi}||^{1+\gamma})$
- $||P^u\mathbf{u}(T)|| = O(e^{-\alpha(1+\gamma)T}||\boldsymbol{\xi}||^{1+\gamma})$.

6.2 Truncated Solution

We again restrict attention to $\boldsymbol{\xi} \in \mathcal{E}^s_\varepsilon$ with ε satisfying

$$\varepsilon^\gamma K(2C\varepsilon)[2C]^{1+\gamma} \leq \gamma\alpha,$$

and now consider the mapping $\mathbf{u} \mapsto \mathbf{v}$ defined by

$$\mathbf{v}(t) = e^{At}\boldsymbol{\xi} + \int_0^t e^{A(t-s)} P^s \mathbf{G}(\mathbf{u}(s))\, ds - \int_t^T e^{A(t-s)} P^u \mathbf{G}(\mathbf{u}(s))\, ds. \tag{6.2}$$

The same contraction argument as in subsection 6.1, but now over $[0,T]$, shows the existence of a fixed point $\check{\mathbf{u}}(t)$ of (6.2) with analogous properties. Then the difference satisfies

$$\begin{aligned}\mathbf{u}(t) - \check{\mathbf{u}}(t) = e^{-A(T-t)}\boldsymbol{\eta} + \int_0^t e^{A(t-s)} P^s \{\mathbf{G}(\mathbf{u}(s)) - \mathbf{G}(\check{\mathbf{u}}(s))\}\, ds \\ - \int_t^T e^{A(t-s)} P^u \{\mathbf{G}(\mathbf{u}(s)) - \mathbf{G}(\check{\mathbf{u}}(s))\}\, ds,\end{aligned}$$

where $\boldsymbol{\eta} \equiv P^u\{\mathbf{u}(T) - \check{\mathbf{u}}(T)\} = P^u \mathbf{u}(T) = O(e^{-\alpha(1+\gamma)T}||\boldsymbol{\xi}||^{1+\gamma})$. Since we have

$$\begin{aligned}&|| \int_0^t e^{A(t-s)} P^s \{\mathbf{G}(\mathbf{u}(s)) - \mathbf{G}(\check{\mathbf{u}}(s))\}\, ds|| \\ &\qquad \le CK(2C\epsilon)[2C||\boldsymbol{\xi}||]^\gamma \int_0^t e^{-\alpha(t-s)} e^{-\gamma\alpha s} ||\mathbf{u}(s) - \check{\mathbf{u}}(s)||_\star \, ds \\ &\qquad \le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^\gamma \int_0^t e^{-\alpha(t-s)} e^{-\gamma\alpha s} e^{-\beta(T-s)} \{e^{\beta(T-s)} ||\mathbf{u}(s) - \check{\mathbf{u}}(s)||_\star\}\, ds \\ &\qquad \le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^\gamma \frac{e^{-\beta(T-t)}}{\gamma\alpha} \max_{s\in[0,T]} \left\{e^{\beta(T-s)} ||\mathbf{u}(s) - \check{\mathbf{u}}(s)||_\star\right\} \\ &\qquad \le \frac{1}{2} e^{-\beta(T-t)} \max_{s\in[0,T]} \left\{e^{\beta(T-s)} ||\mathbf{u}(s) - \check{\mathbf{u}}(s)||_\star\right\}\end{aligned}$$

and

$$\begin{aligned}&|| \int_t^T e^{A(t-s)} P^u \{\mathbf{G}(\mathbf{u}(s)) - \mathbf{G}(\check{\mathbf{u}}(s))\}\, ds|| \\ &\qquad \le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^\gamma \int_t^T e^{\beta(t-s)} e^{-\gamma\alpha s} ||\mathbf{u}(s) - \check{\mathbf{u}}(s)||_\star \, ds \\ &\qquad \le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^\gamma \int_t^T e^{\beta(t-s)} e^{-\gamma\alpha s} e^{-\beta(T-s)} \{e^{\beta(T-s)} ||\mathbf{u}(s) - \check{\mathbf{u}}(s)||_\star\}\, ds \\ &\qquad \le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^\gamma \frac{e^{-\beta(T-t)}}{\gamma\alpha} \max_{s\in[0,T]} \{e^{\beta(T-s)} ||\mathbf{u}(s) - \check{\mathbf{u}}(s)||_\star\} \\ &\qquad \le \frac{1}{2} e^{-\beta(T-t)} \max_{s\in[0,T]} \left\{e^{\beta(T-s)} ||\mathbf{u}(s) - \check{\mathbf{u}}(s)||_\star\right\},\end{aligned}$$

it follows that

$$\max_{t\in[0,T]} \left\{e^{\beta(T-t)} ||\mathbf{u}(t) - \check{\mathbf{u}}(t)||_\star\right\} \le \max_{t\in[0,T]} \left\{e^{\beta(T-t)} ||e^{-A(T-t)}\boldsymbol{\eta}||\right\} + \frac{1}{2} \max_{t\in[0,T]} \left\{e^{\beta(T-t)} ||\mathbf{u}(t) - \check{\mathbf{u}}(t)||_\star\right\}$$

and so

$$\begin{aligned}||\mathbf{u}(0) - \check{\mathbf{u}}(0)||_\star &\le 2C||\boldsymbol{\eta}||e^{-\beta T} \\ &= O(e^{-(\beta+\alpha(1+\gamma))T}||\boldsymbol{\xi}||^{1+\gamma}).\end{aligned}$$

6.3 Gauss-Seidel

We again restrict attention to $\boldsymbol{\xi} \in \mathcal{E}^s_\varepsilon$ with ε satisfying

$$\varepsilon^\gamma K(2C\varepsilon)[2C]^{1+\gamma} \le \gamma\alpha,$$

and now consider the mappings $\mathbf{w} \mapsto \mathbf{v}$ and $\mathbf{y} \mapsto \tilde{\mathbf{v}}$ defined by

$$\mathbf{v}(t) = e^{At}\boldsymbol{\xi} + \int_0^t e^{A(t-s)} P^s \mathbf{G}(\mathbf{w}(s) + \mathbf{y}(s))\, ds \tag{6.3}$$

$$\tilde{\mathbf{v}}(t) = -\int_t^\infty e^{A(t-s)} P^u \mathbf{G}(\mathbf{w}(s) + \mathbf{y}(s))\, ds. \tag{6.4}$$

a) We consider (6.3) under the assumption $||\mathbf{y}(t)|| \le C||\boldsymbol{\xi}||e^{-\alpha t}$.

i) The set $||\mathbf{w}(t) - e^{At}\boldsymbol{\xi}|| \le C||\boldsymbol{\xi}||e^{-\alpha t}$ is mapped to itself by (6.3) since

$$\begin{aligned} ||\int_0^t e^{A(t-s)} P^s \mathbf{G}(\mathbf{w}(s) + \mathbf{y}(s))\, ds|| &\le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^{1+\gamma} \int_0^t e^{-\alpha(t-s)} e^{-(1+\gamma)\alpha s}\, ds \\ &\le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^{1+\gamma} \frac{e^{-\alpha t}}{\gamma\alpha} \le C||\boldsymbol{\xi}||e^{-\alpha t}. \end{aligned}$$

ii) With the norm $|||\mathbf{w}||| \equiv \max_{t\ge 0}\{||\mathbf{w}(t)||\}$, (6.3) is a contraction on the set in i) because

$$\begin{aligned} &||\int_0^t e^{A(t-s)} P^s \{\mathbf{G}(\mathbf{w}_1(s){+}\mathbf{y}(s)) - \mathbf{G}(\mathbf{w}_2(s) + \mathbf{y}(s))\}\, ds|| \\ &\qquad \le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^{\gamma} \int_0^t e^{-\alpha(t-s)} e^{-\gamma\alpha s} ||\mathbf{w}_1(s) - \mathbf{w}_2(s)||\, ds \\ &\qquad \le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^{\gamma} \frac{1}{\gamma\alpha} |||\mathbf{w}_1 - \mathbf{w}_2||| \le \frac{1}{2}|||\mathbf{w}_1 - \mathbf{w}_2|||. \end{aligned}$$

b) We consider (6.4) under the assumption $||\mathbf{w}(t) - e^{At}\boldsymbol{\xi}|| \le C||\boldsymbol{\xi}||e^{-\alpha t}$.

i) The set $||\mathbf{y}(t)|| \le C||\boldsymbol{\xi}||e^{-(1+\gamma)\alpha t}$ is mapped to itself by (6.4) since

$$\begin{aligned} ||\int_t^\infty e^{A(t-s)} P^u \mathbf{G}(\mathbf{w}(s) + \mathbf{y}(s))\, ds|| &\le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^{1+\gamma} \int_t^\infty e^{\beta(t-s)} e^{-(1+\gamma)\alpha s}\, ds \\ &\le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^{1+\gamma} \frac{e^{-(1+\gamma)\alpha t}}{(1+\gamma)\alpha} \le C||\boldsymbol{\xi}||e^{-(1+\gamma)\alpha t}. \end{aligned}$$

ii) With the norm $|||\mathbf{y}||| \equiv \max_{t\ge 0}\{||\mathbf{y}(t)||\}$, (6.4) is a contraction on the set in i) because

$$\begin{aligned} &||\int_t^\infty e^{A(t-s)} P^u \{\mathbf{G}(\mathbf{w}(s){+}\mathbf{y}_1(s)) - \mathbf{G}(\mathbf{w}(s) + \mathbf{y}_2(s))\}\, ds|| \\ &\qquad \le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^{\gamma} \int_t^\infty e^{\beta(t-s)} e^{-\gamma\alpha s} ||\mathbf{y}_1(s) - \mathbf{y}_2(s)||\, ds \\ &\qquad \le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^{\gamma} \frac{1}{\gamma\alpha} |||\mathbf{y}_1 - \mathbf{y}_2||| \le \frac{1}{2}|||\mathbf{y}_1 - \mathbf{y}_2|||. \end{aligned}$$

c) The improvement in $\mathbf{w}$, compared to $\mathbf{y}$, gained by applying (6.3) is described by

$$\mathbf{w}(t) - P^s\mathbf{u}(t) = \int_0^t e^{A(t-s)} P^s \{\mathbf{G}(\mathbf{w}(s) + \mathbf{y}(s)) - \mathbf{G}(\mathbf{u}(s))\}\, ds.$$

Hence

$$\begin{aligned} ||\mathbf{w}(t) - P^s\mathbf{u}(t)|| &\le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^{\gamma} \int_0^t e^{-\alpha(t-s)} e^{-\gamma\alpha s} ||\mathbf{w}(s) + \mathbf{y}(s) - \mathbf{u}(s)||_\star\, ds \\ &\le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^{\gamma} \frac{1}{\gamma\alpha} \max\{|||\mathbf{w} - P^s\mathbf{u}|||, |||\mathbf{y} - P^u\mathbf{u}|||\} \end{aligned}$$

shows that

$$|||\mathbf{w} - P^s\mathbf{u}||| \leq CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^\gamma \frac{1}{\gamma\alpha} |||\mathbf{y} - P^u\mathbf{u}|||.$$

Thus applying (6.3) gives an $O(||\boldsymbol{\xi}||^\gamma)$ improvement.

d) The improvement in $\mathbf{y}$, compared to $\mathbf{w}$, gained by applying (6.4) is described by

$$\mathbf{y}(t) - P^u\mathbf{u}(t) = -\int_t^\infty e^{A(t-s)} P^u \{\mathbf{G}(\mathbf{w}(s) + \mathbf{y}(s)) - \mathbf{G}(\mathbf{u}(s))\}\, ds.$$

Hence

$$\begin{aligned} ||\mathbf{y}(t) - P^u\mathbf{u}(t)|| &\leq CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^\gamma \int_t^\infty e^{\beta(t-s)} e^{-\gamma\alpha s} ||\mathbf{w}(s) + \mathbf{y}(s) - \mathbf{u}(s)||_\star\, ds \\ &\leq CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^\gamma \frac{1}{\gamma\alpha} \max\{|||\mathbf{w} - P^s\mathbf{u}|||, |||\mathbf{y} - P^u\mathbf{u}|||\} \end{aligned}$$

shows that

$$|||\mathbf{y} - P^u\mathbf{u}||| \leq CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^\gamma \frac{1}{\gamma\alpha} |||\mathbf{w} - P^s\mathbf{u}|||.$$

Thus applying (6.4) gives an $O(||\boldsymbol{\xi}||^\gamma)$ improvement. [In addition, we also see that

$$\max_{s\geq t}\{||\mathbf{y}(s) - P^u\mathbf{u}(s)||\} \leq CK(2C\varepsilon)[2C||\boldsymbol{\xi}||]^\gamma \frac{e^{-\gamma\alpha t}}{\gamma\alpha} \max_{s\geq t}\{||\mathbf{w}(s) - P^s\mathbf{u}(s)||\},$$

which corresponds with the $O(e^{-(1+\gamma)\alpha t})$ decay for $\mathbf{y}$ and $O(e^{-\alpha t})$ decay for $\mathbf{w}$.]

Therefore we can conclude that each step of the Gauss-Seidel method results in an $O(||\boldsymbol{\xi}||^{2\gamma})$ improvement.

6.4 Linear Perron Method

In analysing the Perron methods, we need to be more careful in our choice of norm on $\Re^m$. Thus we employ norms $||.||_s$ for $\mathcal{E}^s$ and $||.||_u$ for $\mathcal{E}^u$ such that

$$||e^{A\tilde{T}}||_s < 1, \quad ||e^{-A\tilde{T}}||_u < 1,$$

and then make the usual assumptions

- Norms
$$||\mathbf{x}||_\star \equiv \max\{||P^s\mathbf{x}||_s, ||P^u\mathbf{x}||_u\}$$
- Nonlinearity

  $\exists\ \gamma > 0$ such that
$$||\mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{y})||_\star \leq K(r) \max\{||\mathbf{x}||_\star^\gamma, ||\mathbf{y}||_\star^\gamma\} ||\mathbf{x} - \mathbf{y}||_\star$$
  $\forall ||\mathbf{x}||_\star, ||\mathbf{y}||_\star \leq r$ where $K(r)$ is a non-decreasing function of r.
- Dynamics

  $\exists\ \alpha > 0, \beta > 0, C \geq 1$ such that
$$||e^{At}||_s \leq Ce^{-\alpha t} \quad ||e^{-At}||_u \leq Ce^{-\beta t} \qquad \forall t \geq 0.$$

We must also restrict ε so that, with

$$M_\varepsilon \equiv \frac{CK(2C\varepsilon)[2C]^{1+\gamma}}{\gamma\alpha},$$

we have

$$\theta_\varepsilon^{\mathcal{L}} \equiv ||e^{-A\tilde{T}}||_u \left\{ ||e^{A\tilde{T}}||_s + 2\varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde{T}} \right\}^{1+\gamma} + \varepsilon^\gamma M_\varepsilon [2C]^\gamma < 1.$$

In particular, this means that

$$||e^{-A\tilde{T}}||_u \left\{||e^{A\tilde{T}}||_s + \varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde{T}}\right\}^{1+\gamma} < 1$$

and, defining

$$C_\varepsilon \equiv \frac{M_\varepsilon}{1 - ||e^{-A\tilde{T}}||_u \left\{||e^{A\tilde{T}}||_s + \varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde{T}}\right\}^{1+\gamma}},$$

that

- $\varepsilon^\gamma M_\varepsilon < 1$
- $\varepsilon^\gamma C_\varepsilon [2C]^\gamma < 1,$

which we shall use repeatedly. Finally, we shall also require the set of functions defined by

(6.5)
$$\begin{gathered} \mathbf{z} : \mathcal{E}^s_\varepsilon \mapsto \mathcal{E}^u \qquad \mathbf{z}(\mathbf{0}) = \mathbf{0} \\ ||\mathbf{z}(\boldsymbol{\xi}_1) - \mathbf{z}(\boldsymbol{\xi}_2)||_u \le C_\varepsilon \max\{||\boldsymbol{\xi}_1||^\gamma_s, ||\boldsymbol{\xi}_1||^\gamma_s\}||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2||_s, \end{gathered}$$

where $\mathcal{E}^s_\varepsilon$ is now defined in terms of $||.||_s$, and the norm

(6.6)
$$|||\mathbf{z}||| \equiv \max_{\boldsymbol{\xi}\neq\mathbf{0}} \left\{\frac{||\mathbf{z}(\boldsymbol{\xi})||_u}{||\boldsymbol{\xi}||^{1+\gamma}_s}\right\}.$$

a) We first consider the $\mathbf{w}$ equation and define the mapping $\mathbf{w} \mapsto \mathbf{v}$ by

(6.7)
$$\mathbf{v}(t) = e^{At}\boldsymbol{\xi} + \int_0^t e^{A(t-s)} P^s \mathbf{G}(\mathbf{w}(s) + \mathbf{z}(\mathbf{w}(s)))\, ds.$$

i) The set $||\mathbf{w}(t) - e^{At}\boldsymbol{\xi}||_s \le C||\boldsymbol{\xi}||_s e^{-\alpha t}$ is mapped to itself by (6.7) since

$$\begin{aligned} || \int_0^t e^{A(t-s)} P^s \mathbf{G}(\mathbf{w}(s) + \mathbf{z}(\mathbf{w}(s)))\, ds||_s &\le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||_s]^{1+\gamma} \int_0^t e^{-\alpha(t-s)} e^{-(1+\gamma)\alpha s}\, ds \\ &\le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||_s]^{1+\gamma} \frac{e^{-\alpha t}}{\gamma\alpha} \\ &\le C||\boldsymbol{\xi}||_s e^{-\alpha t}. \end{aligned}$$

ii) With the norm $|||\mathbf{w}||| \equiv \max_{0\le t\le\tilde{T}}\{||\mathbf{w}(t)||_s\}$, (6.7) is a contraction on the set in i) since

$$\begin{aligned} &|| \int_0^t e^{A(t-s)} P^s \{\mathbf{G}(\mathbf{w}_1(s) + \mathbf{z}(\mathbf{w}_1(s))) - \mathbf{G}(\mathbf{w}_2(s) + \mathbf{z}(\mathbf{w}_2(s)))\}\, ds||_s \\ &\qquad \le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||_s]^\gamma \int_0^t e^{-\alpha(t-s)} e^{-\gamma\alpha s} ||\mathbf{w}_1(s) - \mathbf{w}_2(s)||_s\, ds \\ &\qquad \le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||_s]^\gamma \frac{1}{\gamma\alpha} |||\mathbf{w}_1 - \mathbf{w}_2||| \\ &\qquad \le \frac{1}{2} |||\mathbf{w}_1 - \mathbf{w}_2|||. \end{aligned}$$

Hence (6.7) has a fixed point $\mathbf{w}$, which also satisfies

$$\begin{aligned} ||\mathbf{w}(\tilde{T}) - e^{A\tilde{T}}\boldsymbol{\xi}||_s &\le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||_s]^{1+\gamma} \frac{e^{-\alpha\tilde{T}}}{\gamma\alpha} \\ &\le \varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde{T}} ||\boldsymbol{\xi}||_s. \end{aligned}$$

b) The $\mathbf{y}$ equation is linear and so we may just write down the solution

$$\mathbf{y}(t) = e^{-A(\tilde{T}-t)}\mathbf{z}(\mathbf{w}(\tilde{T})) - \int_t^{\tilde{T}} e^{A(t-s)} P^u \mathbf{G}(\mathbf{w}(s) + \mathbf{z}(\mathbf{w}(s)))\, ds.$$

c) Now we define the mapping $\mathbf{z} \mapsto \mathcal{K}\mathbf{z}$ by

$$[\mathcal{K}\mathbf{z}](\boldsymbol{\xi}) = e^{-A\tilde{T}}\mathbf{z}(\mathbf{w}(\tilde{T})) - \int_0^{\tilde{T}} e^{-As} P^u \mathbf{G}(\mathbf{w}(s) + \mathbf{z}(\mathbf{w}(s)))\, ds,$$

where $\mathbf{w}$ is the fixed point of (6.7), and prove that $\mathcal{K}$ maps to itself the set of functions $\mathbf{z}$ defined by (6.5). Consequently, we must consider $[\mathcal{K}\mathbf{z}](\boldsymbol{\xi}_1) - [\mathcal{K}\mathbf{z}](\boldsymbol{\xi}_2)$ and deal with

$$\mathbf{w}_1(t) - \mathbf{w}_2(t) = e^{At}(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) + \int_0^t e^{A(t-s)} P^s \{\mathbf{G}(\mathbf{w}_1(s) + \mathbf{z}(\mathbf{w}_1(s))) - \mathbf{G}(\mathbf{w}_2(s) + \mathbf{z}(\mathbf{w}_2(s)))\}\, ds$$

and

$$\begin{aligned}\mathbf{y}_1(0) - \mathbf{y}_2(0) = e^{-A\tilde{T}} &\left\{\mathbf{z}(\mathbf{w}_1(\tilde{T})) - \mathbf{z}(\mathbf{w}_2(\tilde{T}))\right\} \\ &- \int_0^{\tilde{T}} e^{-As} P^u \left\{\mathbf{G}(\mathbf{w}_1(s) + \mathbf{z}(\mathbf{w}_1(s))) - \mathbf{G}(\mathbf{w}_2(s) + \mathbf{z}(\mathbf{w}_2(s)))\right\}\, ds.\end{aligned}$$

i) In order to bound $||\mathbf{w}_1(s) - \mathbf{w}_2(s)||_s$, we write

$$\begin{aligned}||\int_0^t & e^{A(t-s)} P^s \{\mathbf{G}(\mathbf{w}_1(s) + \mathbf{z}(\mathbf{w}_1(s))) - \mathbf{G}(\mathbf{w}_2(s) + \mathbf{z}(\mathbf{w}_2(s)))\}\, ds||_s \\ &\leq CK(2C\varepsilon)[2C\varepsilon]^\gamma \int_0^t e^{-\alpha(t-s)} e^{-\gamma\alpha s} ||\mathbf{w}_1(s) - \mathbf{w}_2(s)||_s\, ds \\ &\leq CK(2C\varepsilon)[2C\varepsilon]^\gamma \frac{e^{-\alpha t}}{\gamma\alpha} \max_{0 \leq s \leq t} \{e^{\alpha s} ||\mathbf{w}_1(s) - \mathbf{w}_2(s)||_s\} \\ &\leq \frac{1}{2} e^{-\alpha t} \max_{0 \leq s \leq t} \{e^{\alpha s} ||\mathbf{w}_1(s) - \mathbf{w}_2(s)||_s\}.\end{aligned}$$

and hence obtain

$$\begin{aligned}\max_{0 \leq s \leq \tilde{T}} \{e^{\alpha s} ||\mathbf{w}_1(s) - \mathbf{w}_2(s)||_s\} &\leq 2 \max_{0 \leq s \leq \tilde{T}} \{e^{\alpha s} ||e^{As}(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)||_s\} \\ &\leq 2C||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2||_s\end{aligned}$$

and thus

$$||\mathbf{w}_1(t) - \mathbf{w}_2(t)||_s \leq 2Ce^{-\alpha t}||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2||_s.$$

This also gives

$$\begin{aligned}||\mathbf{w}_1(\tilde{T}) - \mathbf{w}_2(\tilde{T}) &- e^{A\tilde{T}}(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)||_s \\ &\leq CK(2C\varepsilon)[2C\varepsilon]^\gamma \frac{e^{-\alpha\tilde{T}}}{\gamma\alpha} 2C||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2||_s \\ &\leq \varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde{T}}||\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2||_s.\end{aligned}$$

ii) For the $\mathbf{y}$ equation we write

$$
\begin{aligned}
&\|\mathbf{y}_1(0)-\mathbf{y}_2(0)\|_u \\
&\qquad \le \|e^{-A\tilde{T}}\left\{\mathbf{z}(\mathbf{w}_1(\tilde{T}))-\mathbf{z}(\mathbf{w}_2(\tilde{T}))\right\}\|_u \\
&\qquad\quad + \|\int_0^{\tilde{T}} e^{-As}P^u\left\{\mathbf{G}(\mathbf{w}_1(s)+\mathbf{z}(\mathbf{w}_1(s)))-\mathbf{G}(\mathbf{w}_2(s)+\mathbf{z}(\mathbf{w}_2(s)))\right\}\,ds\|_u \\
&\qquad \le \|e^{-A\tilde{T}}\|_u C_\varepsilon \max\{\|\mathbf{w}_1(\tilde{T})\|_s^\gamma,\|\mathbf{w}_2(\tilde{T})\|_s^\gamma\}\|\mathbf{w}_1(\tilde{T})-\mathbf{w}_2(\tilde{T})\|_s \\
&\qquad\quad + CK(2C\varepsilon)[2C]^\gamma \max\{\|\boldsymbol{\xi}_1\|_s^\gamma,\|\boldsymbol{\xi}_2\|_s^\gamma\}\int_0^{\tilde{T}} e^{-\beta s}e^{-\gamma\alpha s}\|\mathbf{w}_1(s)-\mathbf{w}_2(s)\|_s\,ds \\
&\qquad \le C_\varepsilon\|e^{-A\tilde{T}}\|_u\left[\|e^{A\tilde{T}}\|_s+\varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde{T}}\right]^{1+\gamma}\max\{\|\boldsymbol{\xi}_1\|_s^\gamma,\|\boldsymbol{\xi}_2\|_s^\gamma\}\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|_s \\
&\qquad\quad + M_\varepsilon\max\{\|\boldsymbol{\xi}_1\|_s^\gamma,\|\boldsymbol{\xi}_2\|_s^\gamma\}\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|_s \\
&\qquad = C_\varepsilon\max\{\|\boldsymbol{\xi}_1\|_s^\gamma,\|\boldsymbol{\xi}_2\|_s^\gamma\}\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|_s
\end{aligned}
$$

and hence

$$\|\mathcal{K}\mathbf{z}(\boldsymbol{\xi}_1)-\mathcal{K}\mathbf{z}(\boldsymbol{\xi}_2)\|_u \equiv \|\mathbf{y}_1(0)-\mathbf{y}_2(0)\|_u \le C_\varepsilon\max\{\|\boldsymbol{\xi}_1\|_s^\gamma,\|\boldsymbol{\xi}_2\|_s^\gamma\}\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|_s.$$

d) Finally we prove that $\mathcal{K}$ is a contraction on the set (6.5) endowed with the norm (6.6). Thus we write, for $i=1,2$,

$$[\mathcal{K}\mathbf{z}_i](\boldsymbol{\xi}) = e^{-A\tilde{T}}\mathbf{z}_i(\mathbf{w}_i(\tilde{T})) - \int_0^{\tilde{T}} e^{-As}P^u\mathbf{G}(\mathbf{w}_i(s)+\mathbf{z}_i(\mathbf{w}_i(s)))\,ds,$$

where

$$\mathbf{w}_i(t) = e^{At}\boldsymbol{\xi} + \int_0^t e^{A(t-s)}P^s\mathbf{G}(\mathbf{w}_i(s)+\mathbf{z}_i(\mathbf{w}_i(s)))\,ds,$$

and consider $\mathcal{K}\mathbf{z}_1(\boldsymbol{\xi})-\mathcal{K}\mathbf{z}_2(\boldsymbol{\xi})$, which means that we must deal with

$$\mathbf{w}_1(t)-\mathbf{w}_2(t) = \int_0^t e^{A(t-s)}P^s\{\mathbf{G}(\mathbf{w}_1(s)+\mathbf{z}_1(\mathbf{w}_1(s)))-\mathbf{G}(\mathbf{w}_2(s)+\mathbf{z}_2(\mathbf{w}_2(s)))\}\,ds$$

and

$$
\begin{aligned}
\mathbf{y}_1(0)-\mathbf{y}_2(0) = e^{-A\tilde{T}}&\left\{\mathbf{z}_1(\mathbf{w}_1(\tilde{T}))-\mathbf{z}_2(\mathbf{w}_2(\tilde{T}))\right\} \\
&- \int_0^{\tilde{T}} e^{-As}P^u\left\{\mathbf{G}(\mathbf{w}_1(s)+\mathbf{z}_1(\mathbf{w}_1(s)))-\mathbf{G}(\mathbf{w}_2(s)+\mathbf{z}_2(\mathbf{w}_2(s)))\right\}\,ds.
\end{aligned}
$$

i) In order to bound $\|\mathbf{w}_1(t)-\mathbf{w}_2(t)\|_s$, we write

$$
\begin{aligned}
&\|\int_0^t e^{A(t-s)}P^s\{\mathbf{G}(\mathbf{w}_1(s)+\mathbf{z}_1(\mathbf{w}_1(s)))-\mathbf{G}(\mathbf{w}_2(s)+\mathbf{z}_2(\mathbf{w}_2(s)))\}\,ds\|_s \\
&\quad \le CK(2C\varepsilon)[2C\|\boldsymbol{\xi}\|_s]^\gamma\int_0^t e^{-\alpha(t-s)}e^{-\gamma\alpha s}\|\mathbf{w}_1(s)-\mathbf{w}_2(s) \\
&\qquad\qquad + \mathbf{z}_1(\mathbf{w}_1(s))-\mathbf{z}_2(\mathbf{w}_2(s))\|_\star\,ds \\
&\quad \le CK(2C\varepsilon)[2C\|\boldsymbol{\xi}\|_s]^\gamma\int_0^t e^{-\alpha(t-s)}e^{-\gamma\alpha s}\,\{\|\mathbf{w}_1(s)-\mathbf{w}_2(s)\|_s \\
&\qquad\qquad + \|\mathbf{z}_1(\mathbf{w}_2(s))-\mathbf{z}_2(\mathbf{w}_2(s))\|_u\}\,ds \\
&\quad \le CK(2C\varepsilon)[2C\|\boldsymbol{\xi}\|_s]^\gamma e^{-\alpha t}\left\{\frac{1}{\gamma\alpha}\max_{0\le s\le t}\{e^{\alpha s}\|\mathbf{w}_1(s)-\mathbf{w}_2(s)\|_s\}\right. \\
&\qquad\qquad \left. + |||\mathbf{z}_1-\mathbf{z}_2|||[2C\|\boldsymbol{\xi}\|_s]^{1+\gamma}\frac{1}{2\gamma\alpha}\right\} \\
&\quad \le \frac{1}{2}\max_{0\le s\le t}\{e^{\alpha s}\|\mathbf{w}_1(s)-\mathbf{w}_2(s)\|_s\} + \frac{1}{2\gamma\alpha}CK(2C\varepsilon)[2C\|\boldsymbol{\xi}\|_s]^{1+2\gamma}e^{-\alpha t}|||\mathbf{z}_1-\mathbf{z}_2|||.
\end{aligned}
$$

and hence obtain

$$\max_{0\le s\le \tilde T}\{e^{\alpha s}||\mathbf{w}_1(s)-\mathbf{w}_2(s)||_s\}\le 2CK(2C\varepsilon)[2C||\boldsymbol{\xi}||_s]^{1+2\gamma}\frac{1}{2\gamma\alpha}|||\mathbf{z}_1-\mathbf{z}_2|||$$
$$= M_\varepsilon[2C]^\gamma||\boldsymbol{\xi}||_s^{1+2\gamma}|||\mathbf{z}_1-\mathbf{z}_2|||$$

and thus

$$||\mathbf{w}_1(t)-\mathbf{w}_2(t)||_s\le M_\varepsilon[2C]^\gamma||\boldsymbol{\xi}||_s^{1+2\gamma}e^{-\alpha t}|||\mathbf{z}_1-\mathbf{z}_2|||.$$

ii) Bounding $||\mathbf{y}_1(0)-\mathbf{y}_2(0)||_u$ then gives

$$\begin{aligned}
&||\mathbf{y}_1(0)-\mathbf{y}_2(0)||_u\\
&\le ||e^{-A\tilde T}\left\{\mathbf{z}_1(\mathbf{w}_2(\tilde T))-\mathbf{z}_2(\mathbf{w}_2(\tilde T))\right\}||_u+||e^{-A\tilde T}\left\{\mathbf{z}_1(\mathbf{w}_1(\tilde T))-\mathbf{z}_1(\mathbf{w}_2(\tilde T))\right\}||_u\\
&\quad +||\int_0^{\tilde T}e^{-As}P^u\left\{\mathbf{G}(\mathbf{w}_1(s)+\mathbf{z}_1(\mathbf{w}_1(s)))-\mathbf{G}(\mathbf{w}_2(s)+\mathbf{z}_2(\mathbf{w}_2(s)))\right\}ds||_u\\
&\le ||e^{-A\tilde T}||_u||\mathbf{w}_2(\tilde T)||_s^{1+\gamma}|||\mathbf{z}_1-\mathbf{z}_2|||\\
&\quad +||e^{-A\tilde T}||_u C_\varepsilon\max\{||\mathbf{w}_1(\tilde T)||_s^\gamma,||\mathbf{w}_2(\tilde T)||_s^\gamma\}||\mathbf{w}_1(\tilde T)-\mathbf{w}_2(\tilde T)||_s\\
&\quad +CK(2C\varepsilon)[2C||\boldsymbol{\xi}||_s]^\gamma\int_0^{\tilde T}e^{-\beta s}e^{-\gamma\alpha s}\{||\mathbf{w}_1(s)-\mathbf{w}_2(s)||_s\\
&\qquad\qquad +||\mathbf{z}_1(\mathbf{w}_2(s))-\mathbf{z}_2(\mathbf{w}_2(s))||_u\}\,ds\\
&\le ||e^{-A\tilde T}||_u\left[||e^{A\tilde T}||_s+\varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde T}\right]^{1+\gamma}||\boldsymbol{\xi}||_s^{1+\gamma}|||\mathbf{z}_1-\mathbf{z}_2|||\\
&\quad +C_\varepsilon||e^{-A\tilde T}||_u\left[||e^{A\tilde T}||_s+\varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde T}\right]^\gamma||\boldsymbol{\xi}||_s^\gamma e^{-\alpha\tilde T}M_\varepsilon[2C]^\gamma||\boldsymbol{\xi}||_s^{1+2\gamma}|||\mathbf{z}_1-\mathbf{z}_2|||\\
&\quad +CK(2C\varepsilon)[2C||\boldsymbol{\xi}||_s]^\gamma\frac{1}{\gamma\alpha}M_\varepsilon[2C]^\gamma||\boldsymbol{\xi}||_s^{1+2\gamma}|||\mathbf{z}_1-\mathbf{z}_2|||\\
&\quad +CK(2C\varepsilon)[2C||\boldsymbol{\xi}||_s]^\gamma\frac{1}{2\gamma\alpha}[2C||\boldsymbol{\xi}||_s]^{1+\gamma}|||\mathbf{z}_1-\mathbf{z}_2|||\\
&\le ||e^{-A\tilde T}||_u\left[||e^{A\tilde T}||_s+\varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde T}\right]^{1+\gamma}||\boldsymbol{\xi}||_s^{1+\gamma}|||\mathbf{z}_1-\mathbf{z}_2|||\\
&\quad +||e^{-A\tilde T}||_u\left[||e^{A\tilde T}||_s+\varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde T}\right]^\gamma\varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde T}||\boldsymbol{\xi}||_s^{1+\gamma}|||\mathbf{z}_1-\mathbf{z}_2|||\\
&\quad +\frac{1}{2}\varepsilon^\gamma M_\varepsilon[2C]^\gamma||\boldsymbol{\xi}||_s^{1+\gamma}|||\mathbf{z}_1-\mathbf{z}_2|||+\frac{1}{2}\varepsilon^\gamma M_\varepsilon[2C]^\gamma||\boldsymbol{\xi}||_s^{1+\gamma}|||\mathbf{z}_1-\mathbf{z}_2|||\\
&\le \theta_\varepsilon^{\mathcal{L}}||\boldsymbol{\xi}||_s^{1+\gamma}|||\mathbf{z}_1-\mathbf{z}_2|||.
\end{aligned}$$

Hence we have established the contraction

$$|||\mathcal{K}\mathbf{z}_1-\mathcal{K}\mathbf{z}_2|||\le\theta_\varepsilon^{\mathcal{L}}|||\mathbf{z}_1-\mathbf{z}_2|||.$$

Thus, replacing $\mathbf{z}_1$ by $\mathbf{z}^{(k)}$ and $\mathbf{z}_2$ by $\mathbf{z}^s$ and using the definition of $\theta_\varepsilon^{\mathcal{L}}$, gives the convergence result

$$|||\mathbf{z}^{(k+1)}-\mathbf{z}^s|||\le\left(||e^{-A\tilde T}||_u||e^{A\tilde T}||_s^{1+\gamma}+O(\varepsilon^\gamma)\right)|||\mathbf{z}^{(k)}-\mathbf{z}^s|||.$$

6.5 Nonlinear Perron Method

We require the same assumptions on norms, nonlinearity and dynamics as at the beginning of subsection 6.4. and we must also restrict ε so that, with

$$M_\varepsilon\equiv\frac{CK(2C\varepsilon)[2C]^{1+\gamma}}{\gamma\alpha},$$

we have

$$\left\{||e^{-A\tilde{T}}||_u + \varepsilon^\gamma M_\varepsilon e^{-\beta\tilde{T}}\right\}\left\{||e^{A\tilde{T}}||_s + 2\varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde{T}}\right\}^{1+\gamma} + \varepsilon^\gamma M_\epsilon[2C]^\gamma < 1.$$

In particular, this means that

$$||e^{-A\tilde{T}}||_u \left\{||e^{A\tilde{T}}||_s + \varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde{T}}\right\}^{1+\gamma} < 1$$

and, defining

$$C_\varepsilon \equiv \frac{M_\varepsilon}{1 - ||e^{-A\tilde{T}}||_u \left\{||e^{A\tilde{T}}||_s + \varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde{T}}\right\}^{1+\gamma}},$$

that

- $\varepsilon^\gamma M_\varepsilon \leq 1$
- $\varepsilon^\gamma C_\varepsilon[2C]^\gamma \leq 1$
- $\varepsilon^\gamma\{CC_\varepsilon[2C]^{1+\gamma} + M_\varepsilon\} \leq 2C,$

which we shall use repeatedly, and that

$$\theta_\varepsilon^{\mathcal{N}} \equiv \left\{||e^{-A\tilde{T}}||_u + \varepsilon^\gamma M_\varepsilon e^{-\beta\tilde{T}}\right\}\left\{||e^{A\tilde{T}}||_s + 2\varepsilon^\gamma M_\varepsilon^2 e^{-\alpha\tilde{T}}\right\}^{1+\gamma} + \varepsilon^{2\gamma} M_\varepsilon[2C]^\gamma < 1.$$

Finally, we shall employ again the set of functions (6.5) and norm (6.6).

a) Exactly as for the linear Perron method, we can show that

$$\mathbf{w}(t) = e^{At}\boldsymbol{\xi} + \int_0^t e^{A(t-s)}P^s\mathbf{G}(\mathbf{w}(s) + \mathbf{z}(\mathbf{w}(s)))\,ds \tag{6.8}$$

has a solution satisfying

$$||\mathbf{w}(t) - e^{At}\boldsymbol{\xi}||_s \leq C||\boldsymbol{\xi}||_s e^{-\alpha t}$$

and

$$||\mathbf{w}(\tilde{T}) - e^{A\tilde{T}}\boldsymbol{\xi}||_s \leq \varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde{T}}||\boldsymbol{\xi}||_s.$$

b) Now we consider the mapping $\mathbf{y} \mapsto \mathbf{v}$, defined by

$$\mathbf{v}(t) = e^{-A(\tilde{T}-t)}\mathbf{z}(\mathbf{w}(\tilde{T})) - \int_t^{\tilde{T}} e^{A(t-s)}P^u\mathbf{G}(\mathbf{w}(s) + \mathbf{y}(s))\,ds, \tag{6.9}$$

with $\mathbf{w}$ satisfying the bounds in a) above and $\mathbf{z}$ a member of the set (6.5).

i) The set $||\mathbf{y}(t)||_u \leq \{CC_\varepsilon[2C]^{1+\gamma} + M_\varepsilon\}||\boldsymbol{\xi}||_s^{1+\gamma}e^{-\alpha(1+\gamma)t}$ is mapped to itself by (6.9) since

$$\begin{aligned}||e^{-A(\tilde{T}-t)}\mathbf{z}(\mathbf{w}(\tilde{T}))||_u &\leq Ce^{-\beta(\tilde{T}-t)}C_\varepsilon[2Ce^{-\alpha\tilde{T}}]^{1+\gamma}||\boldsymbol{\xi}||_s^{1+\gamma} \\ &\leq CC_\varepsilon[2C]^{1+\gamma}||\boldsymbol{\xi}||_s^{1+\gamma}e^{-\alpha(1+\gamma)t}\end{aligned}$$

and

$$\begin{aligned}||\int_t^{\tilde{T}} e^{A(t-s)}P^u\mathbf{G}(\mathbf{w}(s) + \mathbf{y}(s))\,ds||_s &\leq CK(2C\varepsilon)[2C||\boldsymbol{\xi}||_s]^{1+\gamma}\int_t^{\tilde{T}} e^{\beta(t-s)}e^{-(1+\gamma)\alpha s}\,ds \\ &\leq \left\{CK(2C\varepsilon)[2C]^{1+\gamma}\frac{1}{\gamma\alpha}\right\}||\boldsymbol{\xi}||_s^{1+\gamma}e^{-\alpha(1+\gamma)t} \\ &= M_\varepsilon||\boldsymbol{\xi}||_s^{1+\gamma}e^{-\alpha(1+\gamma)t}.\end{aligned}$$

ii) With the norm $|||\mathbf{y}||| \equiv \max_{0\le t\le \tilde{T}}\{||\mathbf{y}(t)||_u\}$, (6.9) is a contraction on the set in i) since

$$\begin{aligned}||\int_t^{\tilde{T}} e^{A(t-s)}P^u\{\mathbf{G}(\mathbf{w}(s)+\mathbf{y}_1(s)) - \mathbf{G}(\mathbf{w}(s)+\mathbf{y}_2(s))\}\,ds||_u \\ &\le CK(2C\varepsilon)[2C||\boldsymbol{\xi}||_s]^\gamma \int_t^{\tilde{T}} e^{\beta(t-s)}e^{-\gamma\alpha s}||\mathbf{y}_1(s)-\mathbf{y}_2(s)||_u\,ds \\ &\le CK(2C\varepsilon)[2C\varepsilon]^\gamma \frac{1}{\gamma\alpha}|||\mathbf{y}_1-\mathbf{y}_2||| \\ &\le \frac{1}{2}|||\mathbf{y}_1-\mathbf{y}_2|||.\end{aligned}$$

c) Now we consider the mapping $\mathbf{z}\mapsto \mathcal{K}\mathbf{z}$ defined by $[\mathcal{K}\mathbf{z}](\boldsymbol{\xi})\equiv \mathbf{y}(0)$, where $\mathbf{y}$ is the fixed point of (6.9) and $\mathbf{w}$ the solution of (6.8), and show that $\mathcal{K}$ maps to itself the set of functions defined by (6.5). Consequently we must consider $[\mathcal{K}\mathbf{z}](\boldsymbol{\xi}_1)-[\mathcal{K}\mathbf{z}](\boldsymbol{\xi}_2)$ and deal with

$$\mathbf{w}_1(t)-\mathbf{w}_2(t) = e^{At}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2) + \int_0^t e^{A(t-s)}P^s\{\mathbf{G}(\mathbf{w}_1(s)+\mathbf{z}(\mathbf{w}_1(s))) - \mathbf{G}(\mathbf{w}_2(s)+\mathbf{z}(\mathbf{w}_2(s)))\}\,ds$$

and

$$\begin{aligned}\mathbf{y}_1(t)-\mathbf{y}_2(t) = e^{-A(\tilde{T}-t)}&\left\{\mathbf{z}(\mathbf{w}_1(\tilde{T}))-\mathbf{z}(\mathbf{w}_2(\tilde{T}))\right\} \\ &- \int_t^{\tilde{T}} e^{A(t-s)}P^u\left\{\mathbf{G}(\mathbf{w}_1(s)+\mathbf{y}_1(s)) - \mathbf{G}(\mathbf{w}_2(s)+\mathbf{y}_2(s))\right\}\,ds.\end{aligned}$$

i) Exactly as for the linear Perron method we can show that

$$||\mathbf{w}_1(t)-\mathbf{w}_2(t)||_s \le 2Ce^{-\alpha t}||\boldsymbol{\xi}_1-\boldsymbol{\xi}_2||_s.$$

and

$$||\mathbf{w}_1(\tilde{T})-\mathbf{w}_2(\tilde{T}) - e^{A\tilde{T}}(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)||_s \le \varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde{T}}||\boldsymbol{\xi}_1-\boldsymbol{\xi}_2||_s.$$

ii) To bound $||\mathbf{y}_1(0)-\mathbf{y}_2(0)||_u$ we may write

$$\begin{aligned}||\mathbf{y}_1(0)-\mathbf{y}_2(0)||_u &\le ||e^{-A\tilde{T}}\left\{\mathbf{z}(\mathbf{w}_1(\tilde{T}))-\mathbf{z}(\mathbf{w}_2(\tilde{T}))\right\}||_u \\ &\quad + ||\int_0^{\tilde{T}} e^{-As}P^u\left\{\mathbf{G}(\mathbf{w}_1(s)+\mathbf{y}_1(s)) - \mathbf{G}(\mathbf{w}_2(s)+\mathbf{y}_2(s))\right\}\,ds||_u \\ &\le ||e^{-A\tilde{T}}||_u C_\varepsilon \max\{||\mathbf{w}_1(\tilde{T})||_s^\gamma, ||\mathbf{w}_2(\tilde{T})||_s^\gamma\}||\mathbf{w}_1(\tilde{T})-\mathbf{w}_2(\tilde{T})||_s \\ &\quad + CK(2C\varepsilon)[2C]^\gamma \max\{||\boldsymbol{\xi}_1||_s^\gamma, ||\boldsymbol{\xi}_2||_s^\gamma\} \\ &\qquad \int_0^{\tilde{T}} e^{-\beta s}e^{-\gamma\alpha s}||\mathbf{w}_1(s)-\mathbf{w}_2(s)+\mathbf{y}_1(s)-\mathbf{y}_2(s)||_\star\,ds \\ &\le C_\varepsilon||e^{-A\tilde{T}}||_u\left[||e^{A\tilde{T}}||_s + \varepsilon^\gamma M_\varepsilon e^{-\alpha\tilde{T}}\right]^{1+\gamma}\max\{||\boldsymbol{\xi}_1||_s^\gamma, ||\boldsymbol{\xi}_2||_s^\gamma\}||\boldsymbol{\xi}_1-\boldsymbol{\xi}_2||_s \\ &\qquad + M_\varepsilon\max\{||\boldsymbol{\xi}_1||_s^\gamma, ||\boldsymbol{\xi}_2||_s^\gamma\}||\boldsymbol{\xi}_1-\boldsymbol{\xi}_2||_s \\ &= C_\varepsilon\max\{||\boldsymbol{\xi}_1||_s^\gamma, ||\boldsymbol{\xi}_2||_s^\gamma\}||\boldsymbol{\xi}_1-\boldsymbol{\xi}_2||_s\end{aligned}$$

<u>provided that</u> $||\mathbf{y}_1(t)-\mathbf{y}_2(t)||_u \le 2Ce^{-\alpha t}||\boldsymbol{\xi}_1-\boldsymbol{\xi}_2||_s$. This condition is justified by

$$\begin{aligned}||e^{-A(\tilde{T}-t)}\left\{\mathbf{z}(\mathbf{w}_1(\tilde{T}))-\mathbf{z}(\mathbf{w}_2(\tilde{T}))\right\}||_u &\le Ce^{-\beta(\tilde{T}-t)}C_\varepsilon[2C\varepsilon e^{-\alpha\tilde{T}}]^\gamma 2Ce^{-\alpha\tilde{T}}||\boldsymbol{\xi}_1-\boldsymbol{\xi}_2||_s \\ &\le \varepsilon^\gamma CC_\varepsilon[2C]^{1+\gamma}e^{-(1+\gamma)\alpha\tilde{T}}||\boldsymbol{\xi}_1-\boldsymbol{\xi}_2||_s\end{aligned}$$

and

$$||\int_t^{\tilde{T}} e^{A(t-s)}P^u\left\{\mathbf{G}(\mathbf{w}_1(s)+\mathbf{y}_1(s))-\mathbf{G}(\mathbf{w}_2(s)+\mathbf{y}_2(s))\right\}ds||_u$$

$$\begin{aligned}
&\le CK(2C\varepsilon)[2C\varepsilon]^\gamma\int_t^{\tilde{T}} e^{\beta(t-s)}2Ce^{-(1+\gamma)\alpha s}\,ds||\boldsymbol{\xi}_1-\boldsymbol{\xi}_2||_s\\
&\le CK(2C\varepsilon)[2C\varepsilon]^\gamma 2C\frac{1}{\gamma\alpha}e^{-\alpha(1+\gamma)t}||\boldsymbol{\xi}_1-\boldsymbol{\xi}_2||_s\\
&=\varepsilon^\gamma M_\varepsilon e^{-\alpha(1+\gamma)t}||\boldsymbol{\xi}_1-\boldsymbol{\xi}_2||_s.
\end{aligned}$$

Hence we obtain

$$||\mathcal{K}\mathbf{z}(\boldsymbol{\xi}_1)-\mathcal{K}\mathbf{z}(\boldsymbol{\xi}_2)||_u\equiv||\mathbf{y}_1(0)-\mathbf{y}_2(0)||_u\le C_\varepsilon\max\{||\boldsymbol{\xi}_1||_s^\gamma,||\boldsymbol{\xi}_2||_s^\gamma\}||\boldsymbol{\xi}_1-\boldsymbol{\xi}_2||_s.$$

d) Finally we prove that $\mathcal{K}$ is a contraction on the set (6.5) endowed with the norm (6.6). Thus we write $[\mathcal{K}\mathbf{z}_i](\boldsymbol{\xi})\equiv\mathbf{y}_i(0)$, for $i=1,2$, where

$$\mathbf{y}_i(t)=e^{-A(\tilde{T}-t)}\mathbf{z}_i(\mathbf{w}_i(\tilde{T}))-\int_t^{\tilde{T}} e^{A(t-s)}P^u\mathbf{G}(\mathbf{w}_i(s)+\mathbf{y}_i(s))\,ds$$

and

$$\mathbf{w}_i(t)=e^{At}\boldsymbol{\xi}+\int_0^t e^{A(t-s)}P^s\mathbf{G}(\mathbf{w}_i(s)+\mathbf{z}_i(\mathbf{w}_i(s)))\,ds,$$

and consider $\mathcal{K}\mathbf{z}_1(\boldsymbol{\xi})-\mathcal{K}\mathbf{z}_2(\boldsymbol{\xi})$, which means that we must deal with

$$\mathbf{w}_1(t)-\mathbf{w}_2(t)=\int_0^t e^{A(t-s)}P^s\{\mathbf{G}(\mathbf{w}_1(s)+\mathbf{z}_1(\mathbf{w}_1(s)))-\mathbf{G}(\mathbf{w}_2(s)+\mathbf{z}_2(\mathbf{w}_2(s)))\}\,ds$$

and

$$\begin{aligned}
\mathbf{y}_1(t)-\mathbf{y}_2(t)=e^{-A(\tilde{T}-t)}&\left\{\mathbf{z}_1(\mathbf{w}_1(\tilde{T}))-\mathbf{z}_2(\mathbf{w}_2(\tilde{T}))\right\}\\
&-\int_t^{\tilde{T}} e^{A(t-s)}P^u\left\{\mathbf{G}(\mathbf{w}_1(s)+\mathbf{y}_1(s))-\mathbf{G}(\mathbf{w}_2(s)+\mathbf{y}_2(s))\right\}\,ds.
\end{aligned}$$

i) We bound $||\mathbf{w}_1(t)-\mathbf{w}_2(t)||_s$ exactly as for the linear Perron method and obtain

$$||\mathbf{w}_1(t)-\mathbf{w}_2(t)||_s\le M_\varepsilon[2C]^\gamma e^{-\alpha t}||\boldsymbol{\xi}||_s^{1+2\gamma}|||\mathbf{z}_1-\mathbf{z}_2|||.$$

ii) In order to bound $||\mathbf{y}_1(0)-\mathbf{y}_2(0)||_u$, we define

$$\begin{aligned}
\boldsymbol{\eta}_i&\equiv\mathbf{z}_i(\mathbf{w}_i(\tilde{T}))\\
\mathbf{r}_i(t)&\equiv\mathbf{y}_i(t)-e^{-A(\tilde{T}-t)}\boldsymbol{\eta}_i
\end{aligned}$$

for $i=1,2$, and hence we can write

$$\begin{aligned}
\mathbf{r}_1(t)-\mathbf{r}_2(t)=-\int_t^{\tilde{T}} e^{-A(s-t)}P^u&\left\{\mathbf{G}(\mathbf{w}_1(s)+e^{-A(\tilde{T}-s)}\boldsymbol{\eta}_1+\mathbf{r}_1(s))\right.\\
&\left.-\mathbf{G}(\mathbf{w}_2(s)+e^{-A(\tilde{T}-s)}\boldsymbol{\eta}_2+\mathbf{r}_2(s))\right\}\,ds.
\end{aligned}$$

Consequently

$$\begin{aligned}||\mathbf{r}_1(t)-\mathbf{r}_2(t)||_u &\le CK(2C\varepsilon)[2C\varepsilon]^\gamma \int_t^{\tilde{T}} e^{-\beta(s-t)}e^{-\gamma\alpha s}\,\{||\mathbf{w}_1(s)-\mathbf{w}_2(s)||_s \\ &\qquad + ||e^{-A(\tilde{T}-s)}(\boldsymbol{\eta}_1-\boldsymbol{\eta}_2)||_u \\ &\qquad + ||\mathbf{r}_1(s)-\mathbf{r}_2(s)||_u\}\,ds \\ &\le \frac{CK(2C\varepsilon)[2C\varepsilon]^\gamma}{\gamma\alpha}\left\{\max_{\tilde{T}\ge s\ge t}\{||\mathbf{w}_1(s)-\mathbf{w}_2(s)||_s\} + Ce^{-\beta\tilde{T}}||\boldsymbol{\eta}_1-\boldsymbol{\eta}_2)||_u \right. \\ &\qquad \left. + \max_{\tilde{T}\ge s\ge t}\{||\mathbf{r}_1(s)-\mathbf{r}_2(s)||_u\}\right\}\end{aligned}$$

and thus, since $\frac{CK(2C\varepsilon)[2C\varepsilon]^\gamma}{\gamma\alpha} \le \frac{1}{2}$, we have

$$\max_{\tilde{T}\ge s\ge t}\{||\mathbf{r}_1(t)-\mathbf{r}_2(t)||_u\} \le \varepsilon^\gamma M_\varepsilon\left\{\max_{\tilde{T}\ge s\ge t}\{||\mathbf{w}_1(s)-\mathbf{w}_2(s)||_s\} + Ce^{-\beta\tilde{T}}||\boldsymbol{\eta}_1-\boldsymbol{\eta}_2)||_u\right\}.$$

Therefore

$$\begin{aligned}||\mathbf{y}_1(0)-\mathbf{y}_2(0)||_u &\le ||e^{-A\tilde{T}}(\boldsymbol{\eta}_1-\boldsymbol{\eta}_2)||_u + ||\mathbf{r}_1(0)-\mathbf{r}_2(0)||_u \\ &\le ||e^{-A\tilde{T}}(\boldsymbol{\eta}_1-\boldsymbol{\eta}_2)||_u + \varepsilon^\gamma M_\varepsilon Ce^{-\beta\tilde{T}}||\boldsymbol{\eta}_1-\boldsymbol{\eta}_2)||_u \\ &\qquad + \varepsilon^{2\gamma}M_\varepsilon^2[2C]^\gamma||\boldsymbol{\xi}||_s^{1+\gamma}|||\mathbf{z}_1-\mathbf{z}_2||| \\ &\le \theta_\varepsilon^{\mathcal{N}}||\boldsymbol{\xi}||_s^{1+\gamma}|||\mathbf{z}_1-\mathbf{z}_2|||.\end{aligned}$$

Then, similarly to the linear Perron method, on replacing $\mathbf{z}_1$ with $\mathbf{z}^{(k)}$ and $\mathbf{z}_2$ with $\mathbf{z}^s$, we have

$$|||\mathbf{z}^{(k+1)}-\mathbf{z}^s||| \le \left(||e^{-A\tilde{T}}||_u||e^{A\tilde{T}}||_s^{1+\gamma} + O(\varepsilon^{2\gamma})\right)|||\mathbf{z}^{(k)}-\mathbf{z}^s|||.$$

6.6 The 'HOV' Method

We must choose norms $||.||_s$ for $\hat{\mathcal{E}}^s$ and $||.||_u$ for $\hat{\mathcal{E}}^u$ so that

$$||L||_s < 1, \quad ||L^{-1}||_u < 1,$$

and make the usual assumptions

- Norm

$$||\mathbf{x}||_\star \equiv \max\{||\hat{P}^s\mathbf{x}||_s, ||\hat{P}^u\mathbf{x}||_u\}$$

- Nonlinearity

  $\exists\,\gamma > 0$ such that

$$||\mathbf{g}(\mathbf{x})-\mathbf{g}(\mathbf{y})||_\star \le K(r)\max\{||\mathbf{x}||_\star^\gamma, ||\mathbf{y}||_\star^\gamma\}||\mathbf{x}-\mathbf{y}||_\star$$

  $\forall ||\mathbf{x}||_\star, ||\mathbf{y}||_\star \le r$ where $K(r)$ is a non-decreasing function of r.

We also need to restrict the size of $||\boldsymbol{\xi}||_s$ so that the following conditions hold:

i) $\quad ||L^{-1}||_u\,(||L||_s + 2D_0) \le 1 \quad$ which implies that $\quad ||L^{-1}||_u D_0 \le \dfrac{1}{2}$

ii) $\quad ||L||_s + D_0 < 1$

iii) $\quad \theta^{\mathcal{H}} \equiv \dfrac{||L^{-1}||_u}{1-||L^{-1}||_u D_0}\left\{||L||_s + \dfrac{2D_0}{1-(||L^{-1}||_u D_0)^\gamma}\right\} < 1$

where

$$D_n \equiv K(\|\boldsymbol{\xi}\|_s)\,(\|L\|_s + 2D_0)^{n\gamma}\,\|\boldsymbol{\xi}\|_s^{\gamma}.$$

If

$$\|\mathbf{y}_n^{(0)}\|_u \le (\|L\|_s + D_0)^n\,\|\boldsymbol{\xi}\|_s \qquad n = 1, 2, \dots$$

then we can show by induction that

$$\|\mathbf{y}_n^{(k)}\|_u \le (\|L\|_s + D_0)^n\,\|\boldsymbol{\xi}\|_s \qquad n = 0, 1, \dots$$

for $k = 1, 2, \dots$.

a) If the mapping $\mathbf{y} \mapsto \mathbf{v}$ is defined by

$$\mathbf{v} = L^{-1}\mathbf{y}_1^{(k-1)} - L^{-1}\hat{P}^u\mathbf{g}(\boldsymbol{\xi} + \mathbf{y}), \tag{6.10}$$

then (6.10) maps the set $\|\mathbf{y}\|_u \le \|\boldsymbol{\xi}\|_s$ to itself since

$$\begin{aligned}\|L^{-1}\mathbf{y}_1^{(k-1)} - L^{-1}\hat{P}^u\mathbf{g}(\boldsymbol{\xi} + \mathbf{y})\|_u &\le \|L^{-1}\|_u \left\{\|\mathbf{y}_1^{(k-1)}\|_u + \|\mathbf{g}(\boldsymbol{\xi} + \mathbf{y})\|_\star\right\} \\ &\le \|L^{-1}\|_u \left\{(\|L\|_s + D_0)\,\|\boldsymbol{\xi}\|_s + K(\|\boldsymbol{\xi}\|_s)\|\boldsymbol{\xi}\|_s^{1+\gamma}\right\} \\ &\le \|L^{-1}\|_u\,(\|L\|_s + 2D_0)\,\|\boldsymbol{\xi}\|_s \le \|\boldsymbol{\xi}\|_s.\end{aligned}$$

Also, (6.10) is a contraction on this set because

$$\|L^{-1}\|_u\|\mathbf{g}(\boldsymbol{\xi} + \mathbf{y}) - \mathbf{g}(\boldsymbol{\xi} + \tilde{\mathbf{y}})\|_\star \le \|L^{-1}\|_u D_0\|\mathbf{y} - \tilde{\mathbf{y}}\|_u.$$

Hence $\mathbf{y}_0^{(k)} \le \|\boldsymbol{\xi}\|_s$.

b) From $\mathbf{w}_1^{(k)} = L\boldsymbol{\xi} + \hat{P}^s\mathbf{g}(\boldsymbol{\xi} + \mathbf{y}_0^{(k)})$ we have

$$\|\mathbf{y}_0^{(k)}\|_u \le \|\boldsymbol{\xi}\|_s \quad \Rightarrow \quad \|\mathbf{w}_1^{(k)}\|_s \le (\|L\|_s + D_0)\,\|\boldsymbol{\xi}\|_s.$$

c) If the mapping $\mathbf{y} \mapsto \mathbf{v}$ is defined by

$$\mathbf{v} = L^{-1}\mathbf{y}_{n+1}^{(k-1)} - L^{-1}\hat{P}^u\mathbf{g}(\mathbf{w}_n^{(k)} + \mathbf{y}) \tag{6.11}$$

then, with $\|\mathbf{w}_n\|_s \le (\|L\|_s + D_0)^n\,\|\boldsymbol{\xi}\|_s$, (6.11) maps the set $\|\mathbf{y}\|_u \le (\|L\|_s + D_0)^n\,\|\boldsymbol{\xi}\|_s$ to itself since

$$\begin{aligned}\|L^{-1}\left\{\mathbf{y}_{n+1}^{(k-1)} - \hat{P}^u\mathbf{g}(\mathbf{w}_n + \mathbf{y})\right\}\|_u &\le \|L^{-1}\|_u \left(\|\mathbf{y}_{n+1}^{(k-1)}\|_u + \|\mathbf{g}(\mathbf{w}_n + \mathbf{y})\|_\star\right) \\ &\le \|L^{-1}\hat{P}^u\|_u \Big\{(\|L\|_s + D_0)^{n+1}\,\|\boldsymbol{\xi}\|_s \\ &\qquad\qquad + K(\|\boldsymbol{\xi}\|_s)\,(\|L\|_s + D_0)^{n(1+\gamma)}\,\|\boldsymbol{\xi}\|_s^{1+\gamma}\Big\} \\ &\le \|L^{-1}\|_u\,(\|L\|_s + 2D_0)\,(\|L\|_s + D_0)^n\,\|\boldsymbol{\xi}\|_s \\ &\le (\|L\|_s + D_0)^n\,\|\boldsymbol{\xi}\|_s.\end{aligned}$$

Also, (6.11) is a contraction on this set because

$$\|L^{-1}\|_u\|\mathbf{g}(\mathbf{w}_n + \mathbf{y}) - \mathbf{g}(\mathbf{w}_n + \tilde{\mathbf{y}})\|_\star \le \|L^{-1}\|_u D_0\|\mathbf{y} - \tilde{\mathbf{y}}\|_u.$$

Hence $\mathbf{y}_n^{(k)} \le (\|L\|_s + D_0)^n\,\|\boldsymbol{\xi}\|_s$.

d) From $\mathbf{w}_{n+1}^{(k)} = L\mathbf{w}_n^{(k)} + \hat{P}^s\mathbf{g}(\mathbf{w}_n^{(k)} + \mathbf{y}_n^{(k)})$ we obtain, since $\|\mathbf{w}_n^{(k)}\|_s, \|\mathbf{y}_n^{(k)}\|_u \le (\|L\|_s + D_0)^n\,\|\boldsymbol{\xi}\|_s$, that

$$\begin{aligned}\|\mathbf{w}_{n+1}^{(k)}\|_s &\le \|L\|_s\|\mathbf{w}_n^{(k)}\|_s + K(\|\boldsymbol{\xi}\|_s)\,(\|L\|_s + D_0)^{n(1+\gamma)}\,\|\boldsymbol{\xi}\|_s^{1+\gamma} \\ &\le \|L\|_s\|\mathbf{w}_n\|_s + D_0\,(\|L\|_s + D_0)^n\,\|\boldsymbol{\xi}\|_s \\ &\le (\|L\|_s + D_0)^{n+1}\,\|\boldsymbol{\xi}\|_s.\end{aligned}$$

Now we can bound the error in $\{\mathbf{y}_n^{(k)}\}$ in terms of the error in $\{\mathbf{y}_n^{(k-1)}\}$, making use of the norm

$$|||\hat{P}^u\mathbf{e}||| \equiv \max_n \left\{ \frac{||\hat{P}^u\mathbf{e}_n||_u}{(||L||_s + D_0)^n} \right\}$$

where $\hat{P}^u\mathbf{e}_n^{(k)} \equiv \mathbf{y}_n^{(k)} - \hat{P}^u\mathbf{u}_n$.

a) From $\hat{P}^u\mathbf{e}_0^{(k)} = L^{-1}\hat{P}^u\mathbf{e}_1^{(k-1)} - L^{-1}\hat{P}^u\left\{\mathbf{g}(\boldsymbol{\xi} + \mathbf{y}_0^{(k)}) - \mathbf{g}(\boldsymbol{\xi} + \hat{P}^u\mathbf{u}_0)\right\}$ we obtain

$$||\hat{P}^u\mathbf{e}_0^{(k)}||_u \le ||L^{-1}||_u \left\{||\hat{P}^u\mathbf{e}_1^{(k-1)}||_u + D_0||\hat{P}^u\mathbf{e}_0^{(k)}||_u\right\}$$

and so

$$\begin{aligned} ||\hat{P}^u\mathbf{e}_0^{(k)}||_u &\le \frac{||L^{-1}||_u}{1 - ||L^{-1}||_u D_0}||\hat{P}^u\mathbf{e}_1^{(k-1)}||_u \\ &\le \frac{||L^{-1}||_u}{1 - ||L^{-1}||_u D_0}\left(||L||_s + D_0\right)|||\hat{P}^u\mathbf{e}^{(k-1)}|||. \end{aligned}$$

b) The equation $\hat{P}^s\mathbf{e}_1^{(k)} = \hat{P}^s\left\{\mathbf{g}(\boldsymbol{\xi} + \mathbf{y}_0^{(k)}) - \mathbf{g}(\boldsymbol{\xi} + \hat{P}^u\mathbf{u}_0)\right\}$ gives

$$\begin{aligned} ||\hat{P}^s\mathbf{e}_1^{(k)}||_s &\le D_0||\hat{P}^u\mathbf{e}_0^{(k)}||_u \\ &\le \frac{||L^{-1}||_u}{1 - ||L^{-1}||_u D_0}D_0||\hat{P}^u\mathbf{e}_1^{(k-1)}||_u. \end{aligned}$$

c) From $\hat{P}^u\mathbf{e}_n^{(k)} = L^{-1}\hat{P}^u\left\{\hat{P}^u\mathbf{e}_{n+1}^{(k-1)} - [\mathbf{g}(\mathbf{w}_n^{(k)} + \mathbf{y}_n^{(k)}) - \mathbf{g}(\mathbf{u}_n)]\right\}$ for $n \ge 1$, we obtain

$$||\hat{P}^u\mathbf{e}_n^{(k)}||_u \le ||L^{-1}||_u\left\{||\hat{P}^u\mathbf{e}_{n+1}^{(k-1)}||_u + D_n\left(||\hat{P}^s\mathbf{e}_n^{(k)}||_s + ||\hat{P}^u\mathbf{e}_n^{(k)}||_u\right)\right\}$$

and so

$$||\hat{P}^u\mathbf{e}_n^{(k)}||_u \le \frac{||L^{-1}||_u}{1 - ||L^{-1}||_u D_0}\left\{||\hat{P}^u\mathbf{e}_{n+1}^{(k-1)}||_u + D_1||\hat{P}^s\mathbf{e}_n^{(k)}||_s\right\}.$$

d) The equation $\hat{P}^s\mathbf{e}_{n+1}^{(k)} = L\hat{P}^s\mathbf{e}_n^{(k)} + \hat{P}^s\left\{\mathbf{g}(\mathbf{w}_n^{(k)} + \mathbf{y}_n^{(k)}) - \mathbf{g}(\mathbf{u}_n)\right\}$, for $n \ge 1$, gives

$$||\hat{P}^s\mathbf{e}_{n+1}^{(k)}||_s \le \left(||L||_s + D_0\right)||\hat{P}^s\mathbf{e}_n^{(k)}||_s + D_n||\hat{P}^u\mathbf{e}_n^{(k)}||_u.$$

Thus, combining c) & d) we obtain

$$||\hat{P}^s\mathbf{e}_{n+1}^{(k)}||_s \le \left(||L||_s + 2D_0\right)||\hat{P}^s\mathbf{e}_n^{(k)}||_s + \frac{||L^{-1}||_u}{1 - ||L^{-1}||_u D_0}D_n||\hat{P}^u\mathbf{e}_{n+1}^{(k-1)}||_u,$$

which, together with

$$||\hat{P}^s\mathbf{e}_1^{(k)}||_s \le \frac{||L^{-1}||_u}{1 - ||L^{-1}||_u D_0}D_0||\hat{P}^u\mathbf{e}_1^{(k-1)}||_u,$$

leads to

$$\begin{aligned} ||\hat{P}^s\mathbf{e}_n^{(k)}||_s &\le \frac{||L^{-1}||_u}{1 - ||L^{-1}||_u D_0}\sum_{j=1}^{n}\left(||L||_s + 2D_0\right)^{n-j}D_{j-1}||\hat{P}^u\mathbf{e}_j^{(k-1)}||_u \\ &\le \frac{||L^{-1}||_u}{1 - ||L^{-1}||_u D_0}\left(||L||_s + 2D_0\right)^n\frac{D_0}{1 - \left(||L||_s + D_0\right)^\gamma}|||\hat{P}^u\mathbf{e}^{(k-1)}|||. \end{aligned}$$

Hence, finally, using c) again gives

$$\begin{aligned}\frac{||\hat{P}^u\mathbf{e}_n^{(k)}||_u}{(||L||_s+2D_0)^n} &\leq \frac{||L^{-1}||_u}{1-||L^{-1}||_u D_0}\left\{(||L||_s+2D_0)\,|||\hat{P}^u\mathbf{e}^{(k-1)}|||+\frac{2||L^{-1}||_u D_1 D_0}{1-(||L||_s+D_0)^\gamma}|||\hat{P}^u\mathbf{e}^{(k-1)}|||\right\}\\ &\leq \frac{||L^{-1}||_u}{1-||L^{-1}||_u D_0}\left(||L||_s+\frac{2D_0}{1-(||L||_s+D_0)^\gamma}\right)|||\hat{P}^u\mathbf{e}^{(k-1)}|||,\end{aligned}$$

which means that

$$|||\hat{P}^u\mathbf{e}^{(k)}||| \leq \theta^{\mathcal{H}}\,|||\hat{P}^u\mathbf{e}^{(k-1)}|||$$

and

$$||\mathbf{y}_0^{(k)}-\hat{\mathbf{z}}^s(\boldsymbol{\xi})||_u \leq \left\{\hat{\alpha}\hat{\beta}+O(||\boldsymbol{\xi}||_s^\gamma)\right\}||\mathbf{y}_0^{(k-1)}-\hat{\mathbf{z}}^s(\boldsymbol{\xi})||_u.$$

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