Algebraic and Differential Invariants

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Contents

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1 Scaling symmetries

Dimensional analysis

Prey-predator model [Murray, Mathematical Biology (2002)]

$$\begin{cases} \dot{n} &= \left(\left(1-\frac{n}{k_1}\right) r - k_2 \frac{p}{n+e}\right) n, \\ \dot{p} &= s\left(1-h\frac{p}{n}\right) p. \end{cases}$$

r, s, e, h, k_1, k_2 parameters.

$$\begin{cases} \dot{\mathfrak{n}} &= \left(1-\frac{\mathfrak{n}}{\mathfrak{k}} - \mathfrak{h}\frac{\mathfrak{p}}{\mathfrak{n}+1}\right) \mathfrak{n}, \\ \dot{\mathfrak{p}} &= \mathfrak{s}\left(1-\frac{\mathfrak{p}}{\mathfrak{n}}\right) \mathfrak{p}. \end{cases}$$

$\mathfrak{s}, \mathfrak{h}, \mathfrak{k}$ parameters

$$\begin{array}{lllllllll} t & \to & \lambda^{-1} t, & r & \to & \lambda r, & h & \to & \nu h, \\ n & \to & \mu n, & s & \to & \lambda s, & k_1 & \to & \mu k_1, \\ p & \to & \mu\nu^{-1} p, & e & \to & \mu e, & k_2 & \to & \lambda\nu k_2 \end{array}$$

$$\mathfrak{t} = r\,t,\;\; \mathfrak{n} = \frac{n}{e},\;\; \mathfrak{p} = \frac{h\,p}{e},\;\; \mathfrak{s} = \frac{s}{r},\;\; \mathfrak{h} = \frac{k_2}{rh}, \mathfrak{k} = \frac{k_1}{e}.$$

Symmetry reduction [Mansfield 01]

- Determine the scaling symmetry
- Compute a generating set of monomial invariants
- Rewrite the system in terms of those

It's all linear algebra in the case of scalings!

Recipe for scaling reduction [H., Labahn 2012]

- Fill the matrix A determining the scaling

$$\begin{array}{lllllllll} t & \to & \lambda^{-1} t, & r & \to & \lambda r, & h & \to & \nu h, \\ n & \to & \mu n, & s & \to & \lambda s, & k_1 & \to & \mu k_1, \\ p & \to & \mu\nu^{-1} p, & e & \to & \mu e, & k_2 & \to & \lambda\nu k_2 \end{array}$$

	s	r	e	h	k_1	k_2	n	p	t
ν	0	0	0	1	0	1	0	-1	0
μ	0	0	1	0	1	0	1	1	0
λ	1	1	0	0	0	1	0	0	-1

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$$

- Hermite Normal Form : there exists $V \in \mathbb{Z}^{n\times n}$ unimodular

$$A\,V = (H, 0)$$

$$A\begin{pmatrix} V_i & V_n \end{pmatrix} = \begin{pmatrix} H & 0 \end{pmatrix} \quad V^{-1} = \begin{pmatrix} W_u \\ W_d \end{pmatrix}$$

$$V = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- Invariants:

$$\begin{pmatrix} s & r & e & h & k_1 & k_2 & n & p & t \end{pmatrix}^{V_n} = \begin{pmatrix} \frac{s}{r} & \frac{k_1}{e} & \frac{k_2}{r\,h} & \frac{n}{e} & \frac{h\,p}{e} & r\,t \end{pmatrix}$$

$$\begin{pmatrix} s & r & e & h & k_1 & k_2 & n & p & t \end{pmatrix}^{V_n} = \begin{pmatrix} \mathfrak{s} & \mathfrak{k} & \mathfrak{h} & \mathfrak{n} & \mathfrak{p} & \mathfrak{t} \end{pmatrix}$$

- Rewriting:

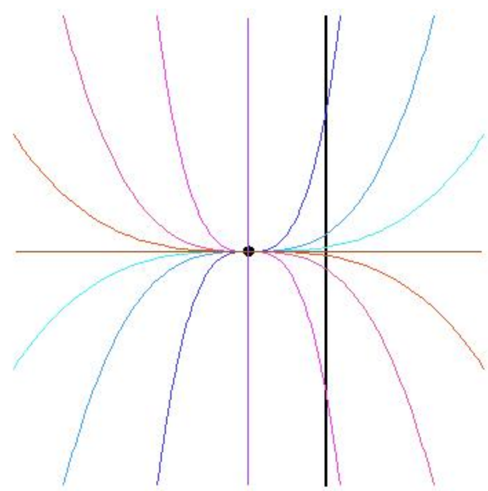
$$\begin{pmatrix} s & r & e & h & k_1 & k_2 & n & p & t \end{pmatrix} \to \begin{pmatrix} \mathfrak{s} & \mathfrak{k} & \mathfrak{h} & \mathfrak{n} & \mathfrak{p} & \mathfrak{t} \end{pmatrix}^{W_d}$$

$$s \to \mathfrak{s}, r \to 1, e \to 1, h \to 1, k_1 \to \mathfrak{k}, k_2 \to \mathfrak{h}, n \to \mathfrak{n}, p \to \mathfrak{p}, t \to \mathfrak{t}$$

A geometric picture

$$A = \begin{pmatrix} 1 & 3 \end{pmatrix}$$

$$A\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$



Scaling: $X = \lambda\,x,\ Y = \lambda^3\,y$

Invariant: $r = \frac{y}{x^3}$

Rewriting: $x \to 1,\ y \to r$

Symmetry reduction $\simeq$ Projection on the cross-section along the orbits

2 Rational and algebraic invariants

Rational Action of an algebraic group $\mathcal{G}$ on $\mathcal{M} = \mathbb{K}^n$

$\mathcal{G} \subset \mathbb{R}^l$ or $\mathbb{C}^l$ an algebraic variety

$G \subset \mathbb{K}[\lambda_1, \ldots, \lambda_l]$ its ideal

$\mathbb{K} \subset \mathbb{C}$

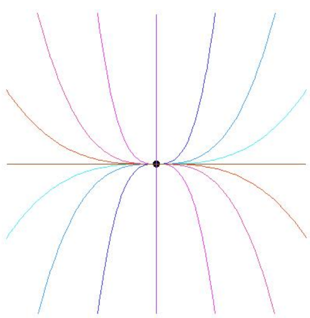
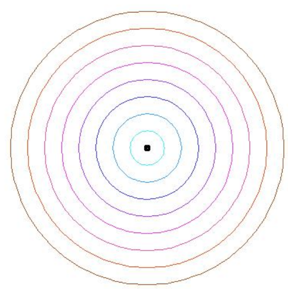
$$\begin{array}{rrcl} g: & \mathcal{G} \times \mathcal{M} & \to & \mathcal{M} \\ & (\lambda, z) & \mapsto & \lambda \star z \end{array} \qquad (\lambda \cdot \mu) \star z = \lambda \star (\mu \star z)$$

$$\lambda \star z = \left(\frac{g_1(\lambda, z)}{h(\lambda, z)}, \ldots, \frac{g_n(\lambda, z)}{h(\lambda, z)} \right)$$

$h, g_1, \ldots, g_n \in \mathbb{K}[\lambda_1, \ldots, \lambda_l, z_1, \ldots, z_n]$

Orbit of $z \in \mathcal{M}$: $\mathcal{O}_z = \{ \lambda \star z \mid \lambda \in \mathcal{G} \}$

Rational Actions on $\mathbb{R}^2$

$\mathcal{G}$	$\mathbb{R}^*$	$SO(2)$
G	$(\lambda \mu - 1)$	$(\lambda^2 + \mu^2 - 1)$
$\lambda \star z$	$\begin{pmatrix} \lambda x \\ \lambda^3 y \end{pmatrix}$	$\begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$
Orbits		

Field of Rational Invariants $\mathbb{K}(z)^G$

Rational invariant: $\dfrac{p}{q} \in \mathbb{K}(z)$ $\qquad \frac{p(\lambda \star z)}{q(\lambda \star z)} = \frac{p(z)}{q(z)}$

Field of rational invariants: $\mathbb{K}(z)^G$ $\qquad \mathbb{K}(z)^G = \mathbb{K}(r_1, \ldots, r_\kappa)$

Separation: $z' \in \mathcal{O}_z \;\Leftrightarrow\; r_1(z') = r_1(z), \ldots, r_\kappa(z') = r_\kappa(z)$
for $z, z' \in \mathcal{M} \setminus \mathcal{W}$

Algorithm **[H., Kogan; JSC 07]**

In : G, $\lambda \star z$, P

Out : $\{r_1, \ldots, r_\kappa\} \subset \mathbb{K}(z)^G$

$$\longrightarrow_Q : \begin{array}{ccc} \mathbb{K}(z)^G & \to & \mathbb{K}(y_1, \ldots, y_\kappa) \\ r & \mapsto & R \end{array} \qquad r = R(r_1, \ldots, r_\kappa)$$

So : $\mathbb{K}(z)^G = \mathbb{K}(r_1, \ldots, r_\kappa)$

Tool: Compute the Gröbner basis Q of $(P(Z) + G(\lambda) + (Z - \lambda \star z)) \cap \mathbb{K}(z)[Z]$
A zero dimensional ideal.

Previous: [Rosenlicht 56], [Vinberg & Popov 89], [Müller-Quade & Beth 99]

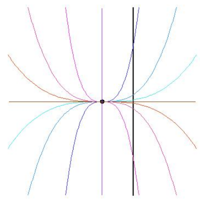
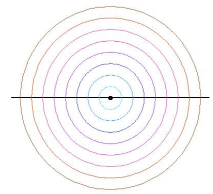
Cross-section of degree e
A variety $\mathcal{P}$ that intersects generic orbits in e simple points.

P defines a cross-section $\mathcal{P}$ of degree e:

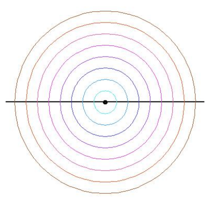
- $P \subset \mathbb{K}[Z]$ prime ideal of complementary dimension to $\mathcal{O}_z$
- $I = (P + G + (Z - \lambda \star z)) \cap \mathbb{K}(z)[Z]$
  radical and zero-dimensional
- $\dim_{\mathbb{K}(z)} \mathbb{K}(z)[Z]/I = e$

Take $P = (a_{i1}Z_1 + \ldots + a_{in}Z_n - b_i,\ 1 \leq i \leq r_o)$ or simply $P = \left(Z_{i_1} - b_1,\ \ldots,\ Z_{i_{r_o}} - b_{r_o}\right)$

Examples

$\mathcal{G}$	$\mathbb{K}^*$	$SO(2)$
$\lambda \star z$	$\begin{pmatrix} \lambda\, x \\ \lambda^3 y \end{pmatrix}$	$\begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$
		
O $P+G$	$(X-\lambda\, x,\, Y-\lambda^3 y)$ $(X-1)+(\lambda\mu-1)$	$(X-\lambda\, x+\mu\, y, Y-\mu\, x-\lambda\, y)$ $(X)+(\lambda^2+\mu^2-1)$
Q	$X-1,\ Y-\frac{y}{x^3}$	$X,\ Y^2-(x^2+y^2)$
$\mathbb{K}(z)^G$	$r=\frac{y}{x^3}$	$r=x^2+y^2$
$\rightarrow$	$x\rightarrow 1,\ y\rightarrow r$	$x\rightarrow 0,\ y^2\rightarrow r$

Algebraic invariants [H. & Kogan, FoCM 07]



Allowing *algebraic invariants* we could have the simpler rewrite rules

$$x \rightarrow 0, \quad y \rightarrow \sqrt{x^2+y^2}$$

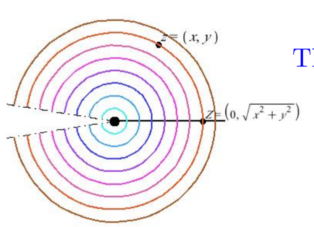
$$\begin{array}{rcl} x & \rightarrow & 0 \\ y^2 & \rightarrow & x^2+y^2 \end{array}$$

$(X, Y^2-(x^2+y^2))$

$(P+G+(Z-\lambda\star z))\cap\mathbb{R}(z)^{\mathcal{G}}[Z]=(Q)$

$$\mathfrak{z}=\left(0,\ \pm\sqrt{x^2+y^2}\right)$$

has e roots $\mathfrak{z}=(\mathfrak{z}_1,\ldots,\mathfrak{z}_n)$ in $\overline{\mathbb{R}(z)^{\mathcal{G}}}$.



Thm: In a neighborhood $\mathcal{U}$ of a point on $\mathcal{P}$, $\exists\ \mathfrak{z}$ smooth; $p_1(\mathfrak{z})=0,\ldots,p_{r_o}(\mathfrak{z})=0$

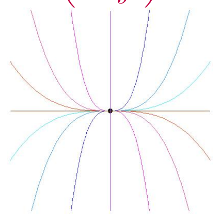
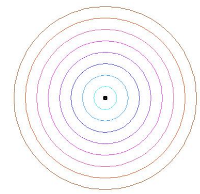
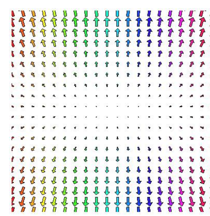
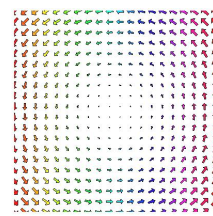
If $f:\mathcal{U}\rightarrow\mathbb{R}$ is a local invariant

$$f\left(z_1,\ldots,z_n\right)=f\left(\mathfrak{z}_1,\ldots,\mathfrak{z}_n\right).$$

$$\begin{array}{rcl} x & \rightarrow & 0 \\ y & \rightarrow & \sqrt{x^2+y^2} \end{array}$$

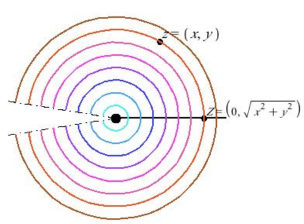
Geometric construction [Fels & Olver 99]

$\mathrm{V}_1, \ldots, \mathrm{V}_r$ the infinitesimal generators of the action

$\mathcal{G}$	$\mathbb{R}^*$	$SO(2)$
$\lambda \star z$	$\begin{pmatrix} \lambda x \\ \lambda^3 y \end{pmatrix}$	$\begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$
		
		
V	$x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y}$	$y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$

$f : \mathcal{U} \to \mathbb{R}$ smooth

f is a local invariant $\Leftrightarrow$ $\mathrm{V}_1(f) = 0, \ldots, \mathrm{V}_r(f) = 0$



Invariantization: $\iota f : \mathcal{U} \to \mathbb{R}$
the only local invariant with $\iota f_{|\mathcal{P}} = f_{|\mathcal{P}}$

$$\iota f(z) = f(Z) \text{ where } \{Z\} = \mathcal{O}_z \cap \mathcal{P}$$

Normalized invariants

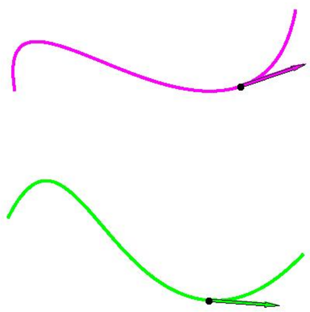
$(\iota z_1, \ldots, \iota z_n) = (\mathfrak{z}_1, \ldots, \mathfrak{z}_n)$

Prop: $\iota f(z_1, \ldots, z_n) = f\,(\iota z_1, \ldots, \iota z_n)$

Symmetry reduction:
$(z_1, \ldots, z_n) \to (\iota z_1, \ldots, \iota z_n)$ subject to $p_1(\iota z) = 0, \ldots, p_{r_o}(\iota z) = 0$

3 Differential Invariants

Classical differential invariants



$E(2)$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$\lambda^2 + \mu^2 = 1$

$$Y_X = \frac{\mu + \lambda y_x}{\lambda - \mu\, y_x} \qquad Y_{XX} = \frac{y_{xx}}{(\lambda - \mu\, y_x)^3}$$

Curvature: $\sigma = \sqrt{\frac{y_{xx}^2}{(1+y_x^2)^3}}$ a differential invariant

Invariant derivation: $\dfrac{d}{ds} = \dfrac{1}{\sqrt{1+y_x^2}} \dfrac{d}{dx}$

Jets and Action Prolongation

$\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ $\qquad g^{(0)} : \mathcal{G} \times \mathrm{J}^0 \to \mathrm{J}^0$ $\qquad \mathrm{V}_1^0, \ldots, \mathrm{V}_r^0$

$(x_1, \ldots, x_m)$ coordinates on $\mathcal{X} \rightsquigarrow$ independent variables

$(u_1, \ldots, u_n)$ coordinates on $\mathcal{U} \rightsquigarrow$ dependent variables

$\mathrm{J}^k = \mathcal{X} \times \mathcal{U}^{(k)}$ $\qquad g^{(k)} : \mathcal{G} \times \mathrm{J}^k \to \mathrm{J}^k$ $\qquad \mathrm{V}_1^k, \ldots, \mathrm{V}_r^k$

additional coordinates $u_\alpha = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}$, $|\alpha| \leq k$

$\rightsquigarrow$ the derivatives of u w.r.t x up to order k

$$\mathrm{D}_i = \frac{\partial}{\partial x_i} + \sum_\alpha u_{\alpha+\epsilon_i} \frac{\partial}{\partial u_\alpha}$$

[DifferentialGeometry]

Differential Invariants

$\mathrm{J}^0 = \mathcal{X} \times \mathcal{U}$ $\qquad g^{(0)} : \mathcal{G} \times \mathrm{J}^0 \to \mathrm{J}^0$ $\qquad \mathrm{V}_1^0, \ldots, \mathrm{V}_r^0$

$\mathrm{J}^k = \mathcal{X} \times \mathcal{U}^{(k)}$ $\qquad g^{(k)} : \mathcal{G} \times \mathrm{J}^k \to \mathrm{J}^k$ $\qquad \mathrm{V}_1^k, \ldots, \mathrm{V}_r^k$

$f : \mathrm{J}^k \to \mathbb{R}$ differential invariant of order k if $\mathrm{V}^k(f) = 0$.

Given a cross-section $\mathcal{P}^k$ on J^k, we define the normalized invariants of order k

$$\mathcal{I}^k = \{\bar{\imath} x_1, \ldots, \bar{\imath} x_m\} \cup \{\bar{\imath} u_\alpha \,|\, |\alpha| \leq k\}$$

Invariant derivations [Fels & Olver 99]

$\mathcal{D} : \mathcal{F}(\mathrm{J}^k) \to \mathcal{F}(\mathrm{J}^{k+1})$ s.t $\mathcal{D} \circ \mathrm{V} = \mathrm{V} \circ \mathcal{D}$

$f : \mathrm{J}^k \to \mathbb{R}$ a differential invariant
$\Rightarrow$ $\mathcal{D}(f)$ a differential invariant of order $k+1$.

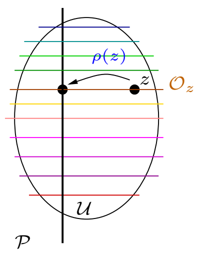
The dimensions r_k of orbits on J^k stabilizes:
$r_0 \leq r_1 \leq \ldots \leq r_s = r_{s+1} = \ldots = r$.

$\mathrm{V}_1^s, \ldots, \mathrm{V}_r^s$ of rank r

$\mathrm{V}(P) = (\mathrm{V}_i(p_j))$ full rank r

$P = (p_1, \ldots, p_r)$ a cross-section on J^{s+k}

$\rho : \mathrm{J}^k \to \mathcal{G}$ equivariant $\rho(\lambda \star z) = \rho(z) \cdot \lambda^{-1}$



Invariant derivations:

$$\begin{pmatrix} \mathcal{D}_1 \\ \vdots \\ \mathcal{D}_m \end{pmatrix} = \rho^* \left(\mathrm{D}_i(\lambda \star x_j)\right)_{ij} \begin{pmatrix} \mathrm{D}_1 \\ \vdots \\ \mathrm{D}_m \end{pmatrix}$$

[Fels & Olver 99]

$\mathcal{D}_i(\bar{\iota} f) = \bar{\iota}(\mathrm{D}_i(f)) - K_{ia}\, \bar{\iota}(\mathrm{V}_a(f))$

$K = \bar{\iota}\left(\mathrm{D}(P)\, \mathrm{V}(P)^{-1}\right)$ $\quad D(P) = (\mathrm{D}_i(p_j))_{ij}$

$$\mathcal{D}_i \mathcal{D}_j - \mathcal{D}_j \mathcal{D}_i = \sum_{k=1}^{m} \Lambda_{ijk}\, \mathcal{D}_k$$

$$\Lambda_{ijk} = \sum_{c=1}^{r} K_{ic}\, \bar{\iota}(\mathrm{D}_j(\xi_{ck})) - K_{jc}\, \bar{\iota}(\mathrm{D}_i(\xi_{ck}))$$

Finite Generation [Fels & Olver 99] [Olver 07], [H. 09] [H. 12]

$$\bar{\iota} u_{\alpha+\epsilon_i} = \mathcal{D}_i(\bar{\iota} u_\alpha) + K_{ia}\, \bar{\iota}\,(\mathrm{V}_a(u_\alpha)) \qquad K = \bar{\iota}\left(\mathrm{D}(P)\,\mathrm{V}(P)^{-1}\right)$$

Any differential invariant can be contructively written in terms of:

- the normalized invariants of order $s+1$

$$\mathcal{I}^{s+1} = \{\bar{\iota} x_1, \ldots, \bar{\iota} x_m\} \cup \{\bar{\iota} u_\alpha \,|\, |\alpha| \leq s+1\}$$

$$\#\mathcal{I}^{s+1} = m + n \begin{pmatrix} s+1+m \\ m \end{pmatrix}$$

- the *edge invariants*, when the cross-section is of minimal order

$$\mathcal{E} = \{\bar{\iota}(\mathcal{D}_i(p_a))\} \cup \mathcal{I}^0$$

$$\#\mathcal{E} \leq n + m(r+1)$$

- the *Maurer-Cartan invariants* $\quad \mathcal{K} = \{K_{ia}\} \cup \mathcal{I}^0$

$$\#\mathcal{K} \leq n+m(r+1)$$

and their derivatives w.r.t. $\mathcal{D}_1, \ldots, \mathcal{D}_m$

Syzygies for Normalized Invariants [H. 09]

A subset S of the following relationships

$$p_1(\bar{\iota} x, \bar{\iota} u_\alpha) = 0, \,\ldots,\, p_r(\bar{\iota} x, \bar{\iota} u_\alpha) = 0$$

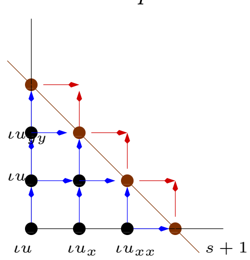
$$\mathcal{D}_i(\bar{\iota} x_j) = \delta_{ij} - K_{ia}\bar{\iota}\,(\mathrm{V}(x_j)),$$

$$\mathcal{D}_i(\bar{\iota} u_\alpha) = \bar{\iota} u_{\alpha+\epsilon_i} - K_{ia}\bar{\iota}\,(\mathrm{V}(u_\alpha)),\ |\alpha| \leq s$$

$$\mathcal{D}_i(\bar{\iota} u_\alpha) - \mathcal{D}_j(\bar{\iota} u_\beta) = K_{ja}\bar{\iota}\,(\mathrm{V}(u_\beta)) - K_{ia}\bar{\iota}\,(\mathrm{V}(u_\alpha)),$$

$$\alpha + \epsilon_i = \beta + \epsilon_j,\ |\alpha| = |\beta| = s+1.$$

form a *complete set of differential syzygies.*



$$\bar{\iota} u_{\alpha+\epsilon_i} = \mathcal{D}_i(\bar{\iota} u_\alpha) + K_{ia}\, \bar{\iota}\,(\mathrm{V}_a(u_\alpha))$$

Maurer-Cartan inv. & Serret-Frenet [Mansfield & van der Kamp 06]

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 & 0 \\ -\kappa(s) & 0 & \tau(s) & 0 \\ 0 & -\tau(s) & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \\ \mathbf{x} \end{pmatrix}$$

$$\mathcal{E}_3 = \left\{ \left(\begin{array}{cc} R & 0 \\ a^t & 1 \end{array} \right) \middle| \, R \in SO(3),\, a \in \mathbb{R}^3 \right\}$$

$$\kappa \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \tau \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \mathbf{0} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + 0\dots$$

$$K = \begin{pmatrix} \bar{\iota}x_{ss}^2 & -\frac{\bar{\iota}x_{sss}^3}{\bar{\iota}x_{ss}^2} & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \kappa & \tau & 1 & 0 & 0 \end{pmatrix}$$

Syzygies on M-C Invariants [H. 2012]

$$\tilde{P}\,(K_{ia}) = 0$$

$$\mathcal{D}_i(K_{jc}) - \mathcal{D}_j(K_{ic}) = \sum_{1 \le a < b \le r} C_{abc}(K_{ia}K_{jb} - K_{ja}K_{ib}) + \sum_{k=1}^{m} \Lambda_{ijk}\, K_{kc} = 0$$

where

$$[\mathrm{v}_i, \mathrm{v}_j] = \sum_{k=1}^{r} C_{ijk}\mathrm{v}_k \qquad [\mathcal{D}_i, \mathcal{D}_j] = \sum_{k=1}^{m} \Lambda_{ijk}\, \mathcal{D}_k.$$

4 Generalized Differential Algebra

Differential Polynomial Rings

$$\mathbb{F} = \mathbb{Q}(x,y)$$

$$\partial_1 = \frac{\partial}{\partial x},\ \partial_2 = \frac{\partial}{\partial y}$$

$$\mathcal{Y} = \{\phi, \psi\}$$

$$\mathbb{F}\,[\![\phi,\psi]\!] = \mathbb{F}[\phi, \phi_x, \phi_y, \dots, \psi \dots]$$

$$\phi_{xxy} \rightsquigarrow \phi_{x^2y} \rightsquigarrow \phi_{(2,1)}$$

$$\frac{\partial}{\partial x}(\phi_{xxy}) = \phi_{xxxy} \rightsquigarrow \partial_1\left(\phi_{(2,1)}\right) = \phi_{(3,1)}$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}$$

$\mathbb{F}$ a field

$$\partial = \{\partial_1, \dots, \partial_m\} \text{ derivations on } \mathbb{F}$$

$$\mathcal{Y} = \{y_1, \dots, y_n\}$$

$$\mathbb{F}[\,y_\alpha \mid \alpha \in \mathbb{N}^m,\ y \in \mathcal{Y}\,] = \mathbb{F}\,[\![\mathcal{Y}]\!]$$

$$\partial_i\,(y_\alpha) = y_{\alpha+\epsilon_i}$$

$$\epsilon_i = (0, \dots, \underset{i^{\text{th}}}{1}, \dots, 0)$$

$$\partial_i\partial_j = \partial_j\partial_i$$

Derivations with nontrivial commutations

$$\mathcal{Y} = \{y_1, \dots, y_n\}$$

$$\mathcal{D} = \{\mathcal{D}_1, \dots, \mathcal{D}_m\}$$

$$\mathcal{D}_i\,\mathcal{D}_j - \mathcal{D}_j\,\mathcal{D}_i = \sum_{l=1}^{m} \Lambda_{ijl}\,\mathcal{D}_l \qquad \Lambda_{ijl} \in \mathbb{K}\,[\![\mathcal{Y}]\!]$$

$$\mathbb{K}\,[\![\mathcal{Y}]\!]?$$

Mononotone derivatives: $y_\alpha = \mathcal{D}_1^{\alpha_1} \dots \mathcal{D}_m^{\alpha_m} y$

Differential polynomial ring $\mathbb{K}\,[\![\mathcal{Y}]\!]$ with non commuting derivations [H. 05]

$$\mathcal{Y} = \{y_1, \dots, y_n\}$$

$$\mathcal{D} = \{\mathcal{D}_1, \dots, \mathcal{D}_m\}$$

$$\mathbb{K}[y_\alpha \mid \alpha \in \mathbb{N}^m,\ y \in \mathcal{Y}]$$

$$\mathcal{D}_i\,(y_\alpha) = \begin{cases} y_{\alpha+\epsilon_i} \text{ if} \alpha_1 = \dots = \alpha_{i-1} = 0 \\ \mathcal{D}_j\mathcal{D}_i\left(y_{\alpha-\epsilon_j}\right) + \sum_{l=1}^{m} c_{ijl}\,\mathcal{D}_l\left(y_{\alpha-\epsilon_j}\right) \\ \quad \text{where } j < i \text{ is s.t. } \alpha_j > 0 \\ \quad \text{while } \alpha_1 = \dots = \alpha_{j-1} = 0 \end{cases}$$

If the c_{ijl} satisfy

- $c_{ijl} = -c_{jil}$
- $\mathcal{D}_k(c_{ijl}) + \mathcal{D}_i(c_{jkl}) + \mathcal{D}_j(c_{kil}) = \sum_{\mu=1}^{m} c_{ij\mu}c_{\mu kl} + c_{jk\mu}c_{\mu il} + c_{ki\mu}c_{\mu jl}$

& there exists an *admissible ranking* $\prec$

- $|\alpha| < |\beta| \ \Rightarrow\ y_\alpha \prec y_\beta$,
- $y_\alpha \prec z_\beta \ \Rightarrow\ y_{\alpha+\gamma} \prec z_{\beta+\gamma}$,
- $\sum_{l \in \mathbb{N}_m} c_{ijl}\,\mathcal{D}_l(y_\alpha) \prec y_{\alpha+\epsilon_i+\epsilon_j}$

then $\mathcal{D}_i\mathcal{D}_j(p) - \mathcal{D}_j\mathcal{D}_i(p) = \sum_{l=1}^{m} c_{ijl}\,\mathcal{D}_l(p)$
$\forall p \in \mathbb{K}[\,y_\alpha \,|\, \alpha \in \mathbb{N}^m\,] = \mathbb{K}\,[\![\mathcal{Y}]\!]$

Results obtained with the software developed

[aida, diffalg]

Conformal & Projective Surfaces

A differential invariants for surfaces in $\mathbb{R}^3$ under the conformal/projective group can be effectively written in terms of a one (two) differential invariants of order 3/4, and their invariant (monotone) derivatives. [H. & Olver 07]

Generalized Orthogonal Group $O(3-l,l) \ltimes \mathbb{R}^3$ acting on $\mathbb{R}^3 \times \mathbb{R}$:

Differential invariants of all orders can be effectively written in terms of 3 second order differential invariants. [H. 09]