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Étude probabiliste des équations de Smoluchowski ; Schéma d'Euler pour des fonctionnelles ; Amplitude du mouvement brownien avec dérive.

THÈSE

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par

Étienne TANRÉ

Composition du jury

Président et Rapporteur :	James NORRIS, Professeur à l'Université de Cambridge
Rapporteur :	Damien LAMBERTON, Professeur à l'Université de Marne la Vallée
Éxaminateurs :	Vlad BALLY, Professeur à l'Université du Maine Jean BERTOIN, Professeur à l'Université Pierre et Marie Curie Paris 6 Philippe LAURENÇOT, Chargé de recherche CNRS à l'Université Paul Sabatier Toulouse
Directeurs de thèse :	Bernard ROYNETTE, Professeur à l'Université Henri Poincaré Nancy 1 Pierre VALLOIS, Professeur à l'Université Henri Poincaré Nancy 1

Institut Élie Cartan Nancy

Mis en page avec la classe TheseCRIN.

À Muriel,
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INTRODUCTION

Mon travail s'articule autour de trois thèmes de recherche en probabilité :

A- les phénomènes de coagulation régis par les équations de Smoluchowski,
B- une méthode d'approximation de diffusions irrégulières par le schéma d'Euler,
C- l'amplitude du mouvement brownien avec dérive.

Ces trois sujets étant indépendants, je vais les introduire séparément.

A : Phénomènes de coagulation

Cette partie traite des résultats obtenus par des méthodes probabilistes sur les solutions des équations de coagulation de Smoluchowski. Il s'agit d'équations déterministes. L'utilisation des probabilités pour étudier certaines E.D.P. est bien souvent une méthode simple et rapide : elle permet de retrouver des résultats connus et d'en montrer d'autres qui n'étaient que conjectures pour les analystes [36]. Le dernier chapitre de cette partie explicitera un algorithme de simulation des solutions de ces E.D.P. dans un cadre assez large.

Les équations de coagulation de Smoluchowski décrivent de nombreux phénomènes de coalescence aussi bien physiques que chimiques. Ils modélisent, par exemple, la formation des gouttes d'eau dans l'atmosphère, la formation des étoiles et des planètes ou encore la polymérisation en chimie. C'est dans ce dernier cadre que nous choisissons de les présenter :
Considérons un récipient dans lequel on souhaite observer une réaction de polymérisation. Pour nous, un polymère est un assemblage de monomères qui sont des particules identiques. Le polymère est caractérisé par le nombre de monomères qui le forment. La géométrie de cet assemblage n'est pas prise en compte. Lorsque des particules sont assez proches, elles peuvent se lier pour former un polymère de taille plus grande. Les hypothèses faites sur cette réaction sont :

1. Le mélange est spatialement homogène.
2. Les réactions ne font intervenir que deux polymères pour en donner un troisième.
3. La masse du polymère après la réaction est égale à la somme des masses des deux polymères qui le constituent.

La première hypothèse nous permet de décrire le comportement dans une unité de volume. La troisième nous donne une information essentielle, la conservation de la masse. Le nombre de particules est toujours suffisamment grand pour être supposé infini. La proportion du nombre de particules de taille donnée k parmi le nombre total de particules est notée $n(k,t)$ et dépend du temps t.

Notons P_j un polymère générique de longueur j. Les réactions autorisées sont celles de la forme : P_j se lie avec P_k pour donner P_{k+j}. Ce type de réaction se fait de manière proportionnelle au nombre de particules de taille j et k présentes dans le récipient à l'instant considéré. La constante de proportionnalité est notée $K(j,k)$ et appelée *noyau de coagulation*. Ce noyau K est supposé indépendant du temps et symétrique en j et k.

Ceci permet d'écrire les équations de coagulation de Smoluchowski. Pour tout $k \geq 1$:

$$\begin{cases} \dfrac{d}{dt}n(k,t) = \dfrac{1}{2}\displaystyle\sum_{j=1}^{k-1} K(j,k-j)n(j,t)n(k-j,t) - n(k,t)\sum_{j=1}^{\infty} K(j,k)n(j,t) \\ n(k,0) = n_0(k) \end{cases} \qquad (SD)$$

où n_0 représente la condition initiale.

L'évolution temporelle de $n(k,t)$ fait intervenir deux sommes. Dans la première, on tient compte de la création de particules de taille k après la coagulation de deux particules de tailles plus petites ; le facteur $\frac{1}{2}$ provient de la symétrie du problème. La seconde somme décrit la disparition de particules de taille k qui se lient avec d'autres particules pour former un polymère de taille plus grande.

(SD) est ainsi un système infini d'équations non-linéaires toutes liées. Notons que ces équations sont purement déterministes et que l'existence de solutions n'est, en général, pas évident. De nombreux travaux déterministes traitent de ce sujet. Récemment, Aldous a rassemblé les résultats obtenus au moyen d'outils probabilistes et a formulé un certain nombre de problèmes ouverts [1]

Lorsque la taille des particules n'est plus restreinte à être un multiple de la taille d'un monomère mais devient une variable réelle x, nous obtenons les équations *continues* qui généralisent naturellement (SD). Pour tout $x \in \mathbb{R}_+$:

$$\begin{cases} \dfrac{\partial}{\partial t}n(x,t) = \dfrac{1}{2}\displaystyle\int_0^x K(y,x-y)n(y,t)n(x-y,t)dy - n(x,t)\int_0^{\infty} K(x,y)n(y,t)dy \\ n(x,0) = n_0(x) \end{cases} \qquad (SC)$$

Dans le premier chapitre, un travail effectué en collaboration avec Madalina DEACONU, nous étudions des propriétés de solutions de (SD) et (SC) pour trois noyaux particuliers :

- Le noyau constant : $\forall\ (k,j) \in \mathbb{N}^* \times \mathbb{N}^*\ ,\ K(j,k) = 1$.

- Le noyau additif : $\forall\ (k,j) \in \mathbb{N}^* \times \mathbb{N}^*\ ,\ K(j,k) = j + k$.
- Le noyau multiplicatif : $\forall\ (k,j) \in \mathbb{N}^* \times \mathbb{N}^*\ ,\ K(j,k) = jk$.

Ces trois noyaux ont l'avantage de permettre des calculs explicites.

Dans le cas particulier où la condition initiale n_0 est la masse de Dirac en 1 ($n_0(k) = \delta_1(k)$), on peut obtenir les solutions exactes pour ces trois noyaux. Cette condition initiale correspond à la présence uniquement de monomères dans le récipient au début de la polymérisation. Nous retrouvons les solutions exactes grâce à une interprétation probabiliste de ces équations. Nous avons également su lier les solutions des équations de Smoluchowski à la population totale d'un processus de branchement de type Galton-Watson. Grâce à cela, nous établissons une correspondance entre les solutions des équations de coagulation de Smoluchowski avec noyau additif et celles avec noyau multiplicatif, sans restriction sur les conditions initiales. Ce résultat important permet d'obtenir les propriétés pour les deux noyaux lorsqu'elles ne sont pas aisées à prouver directement.

Nous décrivons par ailleurs le comportement asymptotique des solutions de (SC), toujours pour les noyaux constant, additif et multiplicatif. Après une bonne renormalisation, nous montrons que les solutions convergent vers une variable aléatoire, la limite dépendant peu de la condition initiale (uniquement par ses premiers moments). L'ensemble de ces résultats constitue le Chapitre 1.

Le Chapitre 2 a fait l'objet d'une publication à la suite de la conférence internationale Monte Carlo 2000. Dans ce court travail, (en collaboration avec Madalina DEACONU), nous montrons comment l'on peut exploiter notre correspondance entre les solutions liées aux noyaux additif et multiplicatif au travers des solutions à valeur mesure, données par Norris [52], [53].

Les Chapitres 3 et 4 sont le fruit d'un travail en collaboration avec Madalina DEACONU et Nicolas FOURNIER.
Tout comme dans le Chapitre 1, nous nous intéressons à une forme faible de solutions des équations de coagulation de Smoluchowski.

Une propriété *physique* très importante des solutions de (SD) et (SC) que nous étudions est qu'elles conservent la masse : pour toute solution de (SC), jusqu'à un instant T_0, on a, pour tout $t \in [0,T_0[$:

$$\int_{\mathbb{R}_+} xn(x,t)\,dx = \int_{\mathbb{R}_+} xn(x,0)\,dx.$$

De même, pour toute solution de (SD), il existe un instant T_0 tel que pour tout $t \in [0,T_0[$:

$$\sum_{k\geq 1} kn(k,t) = \sum_{k\geq 1} kn(k,0).$$

De ce fait, la quantité

$$Q_t(dx) = \sum_{k\geq 1} k\, n(k,t)\delta_k(dx) \quad \text{resp.} \quad Q_t(dx) = x\, n(x,t)\, dx$$

est une mesure de probabilité sur $\mathbb{R}_+$.

Cette notation permet de traiter ensemble les problèmes (SC) et (SD) de la manière suivante : nous dirons que la famille de probabilités $(Q_t)_{t\in[0,T_0[}$ est solution faible de l'équation (MS) si pour toute fonction test $\varphi \in C_0^1(\mathbb{R}_+)$ et pour tout $t \in [0,T_0[$, on a :

$$\int_0^\infty \varphi(x) Q_t(dx) = \int_0^\infty \varphi(x) Q_0(dx)$$
$$+ \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} [\varphi(x+y) - \varphi(x)]\, \frac{K(x,y)}{y} Q_s(dy) Q_s(dx) ds.$$

Nous réussissons ainsi à établir un lien entre les équations de Smoluchowski et les processus vérifiant l'équation différentielle stochastique à sauts, dirigée par un processus de Poisson :

$$X_t = X_0 + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{z \leq \frac{K(X_{s-},\tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds,d\alpha,dz). \qquad (EDS)$$

Le processus X vit sur l'espace de probabilité $(\Omega,\mathcal{F},\mathbb{P})$ alors que $\tilde{X}$ est un processus sur l'espace de probabilité auxiliaire $[0,1]$ muni de la mesure (de probabilité) de Lebesgue.

On cherche les solutions de (EDS) telles que les processus X et $\tilde{X}$ aient même loi. La quantité $N(\omega,dt,d\alpha,dz)$ est une mesure de Poisson sur $[0,T_0[\times[0,1] \times \mathbb{R}_+$ de mesure d'intensité $dt\, d\alpha\, dz$ et est indépendante de X_0.

Le lien entre les solutions de cette équation et les équations de coagulation de Smoluchowski est le suivant : toute solution $(X,\tilde{X})$ de cette équation est telle que les marginales temporelles Q_t de $X_.$ sont solutions de (MS) :

$$Q_t = \mathcal{L}(X_t).$$

Dans le Chapitre 3, l'équation (EDS) est étudiée (existence des solutions, unicité, propriétés trajectorielles). Dès à présent, nous pouvons expliquer le comportement des solutions de (EDS). Nous partons d'une variable aléatoire distribuée suivant la loi X_0. Le rôle de la mesure de Poisson est le suivant : à des instants aléatoires, nous choisissons une particule au hasard dans la solution - c'est le choix de α - et un niveau z. La particule $\tilde{X}_t(\alpha)$ se colle ou non sur la particule que l'on observe suivant la valeur du niveau z. L'indicatrice dans cette E.D.S. non linéaire est un contrôle de la coagulation en fonction du noyau.

Dans le Chapitre 4, les résultats (théoriques) du Chapitre 3 sont utilisés pour approcher les solutions des équations de Smoluchowski par un système de particules. Nous prouvons la convergence du système de particules et montrons qu'il est parfaitement simulable dès que la condition initiale l'est. Nous présentons également des résultats numériques qui valident l'étude.

De plus, un théorème de la limite centrale est démontré sous les hypothèses suivantes :

- Q_0 a un moment d'ordre 2 fini.
- Q_0 a son support inclus dans $\mathbb{N}$.
- Le noyau K est borné.

Les résultats numériques laissent à penser que le champ d'application du théorème est plus large et que chacune de ces hypothèses peut être affaiblie.

B : Approximation de diffusions irrégulières

Dans la deuxième partie de ce travail, en collaboration avec Jean-Sébastien GIET, nous étudions la validité du schéma d'Euler pour approcher des diffusions irrégulières.

La quantité que l'on souhaite estimer est :

$$\mathbb{E} \int_0^T f(X_s) ds$$

lorsque X est une diffusion régulière, f une fonction quelconque (mesurable et bornée) et T un temps fixe déterministe.

L'apport des processus stochastiques dans la résolution des E.D.P. est bien connu. Commençons par rappeler l'exemple de l'équation de la chaleur :

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial t}(t,x) & = & \frac{1}{2}\Delta u(t,x) \\ u(0,x) & = & u_0(x), \end{array} \right.$$

où u_0 est une fonction continue. La solution s'exprime sous la forme :

$$u(t,x) = \mathbb{E}\left\{u_0\left(x + B_t\right)\right\}$$

où B est un mouvement brownien issu de 0.

Lorsque l'on considère une diffusion X solution de l'équation :

$$X_t = X_0 + \int_0^t \sigma(X_s)\, dB_s + \int_0^t b(X_s)\, ds.$$

La fonction $u(t,x) = \mathbb{E}\left\{u_0(X_t^x)\right\}$ est solution de l'E.D.P. (équation de la chaleur généralisée) :

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &= \mathcal{A}u(t,x) \\ u(0,x) &= u_0(x) \end{cases}$$

Le lien entre la diffusion X et l'opérateur $\mathcal{A}$ intervenant dans l'E.D.P. est :

$$\mathcal{A}g(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial g}{\partial x_i}(x).$$

Donnons ici un exemple d'E.D.P. pour laquelle la solution fait intervenir une fonctionnelle de la totalité de la trajectoire d'un processus stochastique : le problème de Cauchy avec potentiel r et Lagrangien g, qui est une extension de l'équation de la chaleur :

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) &= \mathcal{A}u(t,x) - r(x)u(t,x) + g(x) \\ u(0,x) &= u_0(x) \end{cases}$$

où les fonctions r et g sont continues ; r est de plus supposée minorée.

La formule de Feynman-Kac permet d'exprimer la solution du problème de Cauchy sous la forme suivante :

$$\begin{aligned} u(t,x) \;=\; \mathbb{E}\Bigg\{ & u_0(X_t^x)\exp\left(-\int_0^t r(X_s^x)\,ds\right) \\ & + \int_0^t g(X_\theta^x)\exp\left(-\int_0^\theta r(X_s^x)\,ds\right)\,d\theta\Bigg\}. \end{aligned}$$

L'approximation des solutions de l'équation de la chaleur généralisée grâce à son interprétation probabiliste a déjà été étudiée. Par exemple, Talay et Tubaro [61], Bally et Talay [6], [5] donnent un développement de l'erreur commise en approchant $\mathbb{E}\left\{f(X_T)\right\}$ par $\mathbb{E}\left\{f(\bar{X}_T)\right\}$ où $\bar{X}$ désigne l'approximation de X obtenue par le schéma d'Euler.

T est un temps fixé, X est une diffusion régulière et f est une fonction régulière dans [61], seulement mesurable bornée dans [6] et [5].

Précisons maintenant le cadre dans lequel notre travail est effectué :

- X est solution de l'équation :

$$X_t = X_0 + \int_0^t \sigma(X_s)\,dB_s + \int_0^t b(X_s)\,ds.$$

- σ et b sont des fonctions C^∞.
- σ' , σ'' , b', b'' sont des fonctions bornées.

- La diffusion est uniformément elliptique, on note $a = \sigma\sigma^t$:

$$\exists \eta \in \mathbb{R}^* \text{ t.q. } \forall \xi \neq 0 \ , \ \xi^t a(x) \xi \geq \eta ||\xi||^2$$

- f est une fonction mesurable et bornée.

Nous pouvons formuler différemment ce problème en augmentant la dimension de la diffusion :

$$\begin{cases} dX_t = \sigma(X_t)\, dB_t + b(X_t)\, dt \\ dY_t = f(X_t)\, dt \end{cases} \qquad (EDS_2)$$

Ainsi écrit, le but de ce travail est d'approcher $\mathbb{E}(Y_T)$, Y étant ici une diffusion *irrégulière.*

Les résultats connus pour la méthode de discrétisation par le schéma d'Euler s'appliquent si la fonction f est elle-même régulière. La diffusion (X_t, Y_t) l'est alors aussi et l'approximation de $\mathbb{E}(Y_T)$ par le schéma d'Euler converge vers $\mathbb{E}(Y_T)$ à une vitesse d'ordre $\frac{1}{n}$ où $\frac{T}{n}$ est le pas de discrétisation.

Ici, nous étendons ce résultat pour f mesurable bornée quelconque :

$$\left| \mathbb{E}\left(\int_0^T f(X_s)\, ds \right) - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} f\left(\bar{X}^{\frac{T}{n}} \left(\frac{iT}{n} \right) \right) \right| \leq \frac{K}{n};$$

$\bar{X}^{\frac{T}{n}}$ désigne l'approximation de X obtenue par le schéma d'Euler de pas $\frac{T}{n}$.

Pour estimer $\mathbb{E} \displaystyle\int_0^T f(X_s)\, ds$, on rencontre 3 types de problèmes :

- L'espérance est approchée par une méthode de Monte-Carlo classique : on effectue un grand nombre de réalisations et on en prend la moyenne. Nous ne développons pas davantage cette partie dont les résultats sont connus.
- La diffusion n'est connue qu'en tant que solution de l'E.D.S. et nous sommes donc amenés à l'approcher. Nous avons privilégié l'approximation par le schéma d'Euler assez facile à mettre en oeuvre.
- La totalité de la trajectoire de l'approximation par le schéma d'Euler n'est pas connue or nous devons en calculer l'intégrale sur $[0,T]$. Nous remplaçons cette intégrale par les sommes de Riemann associées.

C'est ainsi que nous approchons $\mathbb{E}\left(\int_0^T f(X_s)\, ds\right)$ par $\frac{1}{n}\sum_{i=1}^n \mathbb{E} f\left(\bar{X}^{\frac{T}{n}}\left(\frac{iT}{n}\right)\right)$.

Notons ici que lorsque la fonction f est régulière, nous avons vu que le problème se ramène à estimer $\mathbb{E}(Y_T)$ pour $(X_., Y_.)$ solution de (EDS_2). Dans ce cas, l'approximation donnée par le schéma d'Euler est précisément : $\frac{1}{n}\sum_{i=1}^n \mathbb{E} f\left(\bar{X}^{\frac{T}{n}}\left(\frac{iT}{n}\right)\right)$.

Nous avons découpé l'erreur commise en deux parts : l'une vient des sommes de Riemann, l'autre du schéma d'Euler.

$$\left|\mathbb{E}\left(\int_0^T f(X_s)\,ds\right)-\frac{1}{n}\sum_{i=1}^n \mathbb{E}f\left(\bar{X}^{\frac{T}{n}}\left(\frac{i}{n}\right)\right)\right| \le$$

$$\left|\mathbb{E}\left(\int_0^T f(X_s)\,ds\right)-\frac{1}{n}\sum_{i=1}^n \mathbb{E}f\left(X_{\frac{i}{n}}\right)\right|$$

$$+\left|\frac{1}{n}\sum_{i=1}^n \mathbb{E}f\left(X_{\frac{i}{n}}\right)-\frac{1}{n}\sum_{i=1}^n \mathbb{E}f\left(\bar{X}^{\frac{T}{n}}\left(\frac{i}{n}\right)\right)\right|.$$

Pour majorer la première erreur, nous utilisons essentiellement la régularité de la diffusion X et la propriété pour f d'être bornée.

Pour majorer la seconde partie, la technique utilisée repose sur une idée de Kurtz et Protter consistant à introduire la famille d'approximations de X $(\bar{X}_r^\delta(t), 0 \le t \le T)$ définie ainsi :

- δ désigne le pas de discrétisation.
- $\bar{X}_r^\delta(t) = \bar{X}^\delta(t)$ pour $t \in [0,r]$.
- $\bar{X}_r^\delta(t) = X^{r,\bar{X}_r^\delta(r)}(t)$ pour $t \in [r,T]$.

La diffusion $\bar{X}_r^\delta$ est donnée par le schéma d'Euler jusqu'à l'instant r puis est gouvernée par les coefficients σ et b entre r et T.

Ainsi $\bar{X}_0^\delta(.) = X_.$ et $\bar{X}_T^\delta(.) = \bar{X}^\delta(.)$.

La différence $\varphi(\bar{X}^\delta(.)) - \varphi(X.)$ peut donc être vue comme l'intégrale :

$$\int_0^T \frac{\partial}{\partial r}\varphi(\bar{X}_r^\delta(.))\,dr$$

pour tout fonctionnelle φ de la trajectoire.

C : Amplitude du mouvement brownien avec dérive

Cette dernière partie de ma thèse a fait l'objet d'une collaboration avec Pierre VALLOIS, un de mes directeurs de thèse. Le mouvement brownien avec dérive constante est un objet qui apparaît naturellement dans un certain nombres de problèmes comme, par exemple, en mathématiques financières. En effet, le logarithme

du cours d'un actif qui suit le modèle de Black et Scholes est précisément un mouvement brownien avec dérive.
L'étude des fluctuations des cours dans ce modèle amène à s'intéresser aux amplitudes de ce processus.
On définit l'amplitude A_t par :

$$A_t = \sup_{s\in[0,t]} \{B_s + \delta s\} - \inf_{s\in[0,t]} \{B_s + \delta s\}$$

Dans ce travail, nous donnons une description des trajectoires du mouvement brownien avec dérive en le « décomposant » suivant ses extremums successifs. La quantité $B_t + \delta t$ est presque sûrement minorée (resp. majorée) si $\delta > 0$ (resp. $\delta < 0$). On note $-S_1$ le minimum (resp. S_1 le maximum) absolu de $B_t+\delta t$ sur une trajectoire et ρ_1 le dernier instant où il est atteint. Nous retrouvons au moyen d'une technique de grossissement de filtration un résultat de Williams [66] qui décrit :

- La loi de S_1.
- Conditionnellement à $S_1 = x_1$,
  - la loi de $(B_t + \delta t \ , \ 0 \leq t \leq \rho_1)$,
  - la loi de $(B_t + \delta t \ , \ t \geq \rho_1)$.

Notre technique permet de poursuivre cette décomposition : sur $[0,\rho_1]$, $B_t + \delta t$ est presque sûrement majoré (resp. minoré). On note S_2 l'extremum correspondant et ρ_2 le dernier instant où il est atteint. Nous obtenons alors (toujours conditionnellement à $S_1 = x_1$) :

- La loi de S_2.
- Conditionnellement à $S_2 = x_2$,
  - la loi de $(B_t + \delta t \ , \ 0 \leq t \leq \rho_2)$,
  - la loi de $(B_t + \delta t \ , \ \rho_2 \leq t \leq \rho_1)$.

Et nous poursuivons cette décomposition.

Nous décrivons complètement les trajectoires à l'aide des lois des extremums successifs, du mouvement brownien avec dérive et de la diffusion $Z^{(\delta)}$, solution positive de l'équation différentielle stochastique suivante :

$$Z^{(\delta)}(t) = B_t + \delta \int_0^t \coth\left(\delta Z^{(\delta)}(s)\right) \, ds.$$

En particulier nous obtenons la loi des n-uples $(A_1,A_2,\cdots,A_n)$ où $A_k = S_k + S_{k+1}$.

Proposition $(A_1,\cdots,A_n)$ *a pour densité :*

$$\delta^n \left[\prod_{k=1}^{n-1} \frac{e^{(-1)^k \delta a_k}}{\mathrm{sh}(\delta a_k)}\right] \frac{e^{(-1)^n 2\delta a_n} - 1 + (-1)^{n+1} 2\delta a_n}{\mathrm{sh}^2\, \delta a_n} \mathbb{1}_{\{0\leq a_n\leq\cdots\leq a_1\}}$$

Nous obtenons également une représentation plus probabiliste de ses variables aléatoires :

Proposition

$$\left(\psi(2\delta A_1),\frac{\psi(-2\delta A_2)}{\psi(-2\delta A_1)},\cdots,\frac{\psi((-1)^{n+1}2\delta A_n)}{\psi((-1)^{n+1}2\delta A_{n-1})}\right) = (U_1,U_2,\cdots,U_n)$$

où $U_1,U_2,\cdots,U_n$ *sont des variables aléatoires indépendantes uniformes sur* $[0,1]$*, et*

$$\psi(x) = \frac{e^x - 1 - x}{e^x - 1}.$$

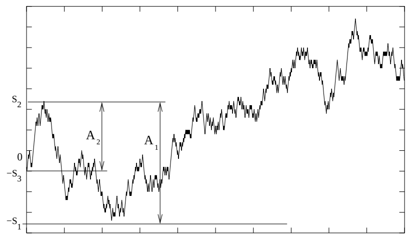

FIG. 1 – $B_t + \delta t$

Première partie

Phénomènes de coagulation

1

Smoluchowski's Coagulation Equation : Probabilistic Interpretation of Solutions for Constant, Additive and Multiplicative Kernels

Une grande partie de ce chapitre a fait l'objet d'une publication à Annali della Scuola Normale Superiore di Pisa, Serie IV, Vol. XXIX (3) : 549-580, 2000.

Abstract

This paper is devoted to the study of the Smoluchowski's coagulation equation, discrete and continuous version, for the case of constant, additive and multiplicative kernels. Even though, for the discrete case the results stated in this work are not new, our approach allows the simplification of existing proofs. For the continuous case we obtain new results : a connection between the solutions of the additive and multiplicative cases and renormalisation theorems which show that after a convenient scaling, the solution converges to a limit which depends on the initial condition only through its moments of order 1, 2 and 3.

Key Words : Smoluchowski's coagulation equation, branching processes, stochastic processes, partial differential equations, Laplace transform.

AMS 2000 subject classification : 60J80, 44A10

1.1 Introduction

1.1.1 Paper's Plan

The aim of this paper is to provide a probabilistic representation for some solutions of the Smoluchowski's coagulation equation.
In the introduction we give the heuristic motivation of this problem. Afterwards we furnish a survey of some results that we can find in the literature on the Smoluchowski's equation, without any intention of being exhaustive, but by pointing out the results obtained by using probabilistic methods. For a more detailed survey we refer to Aldous ([1]).
The two following parts discuss three particular kernels : constant, additive and multiplicative. This choice for the kernel allows the development of the computation.
The second part recalls briefly some known results on the discrete case for this three particular kernels. We obtain the explicit form of these solutions via branching processes. The results we give on this part are not new but our approach allows the simplification of existing proofs.
The last part deals with the continuous case. We express here our main results : a transformation which connects the solution of the additive case with the one of the multiplicative case (theorems 1.3.9 and 1.3.11), and some renormalisation theorems (theorems 1.3.6, 1.3.20 and 1.3.24). These last theorems insure the convergence of the solution to a limit, which depends weakly on the initial condition.

1.1.2 Heuristic Motivation of the Problem

The Smoluchowski's coagulation equation models various kind of phenomena as for example : in chemistry (polymerisation), in physics (aggregation of colloidal particles), in astrophysics (formation of stars and planets), in engineering (behaviour of fuel mixtures in engines), in genetics, in random graphs theory etc.
In order to fix the ideas, we present the appearance of this equation, in polymerisation.
For $k \in \mathbb{N}^*$, let P_k denote a polymer of mass k, that is a set of k identical particles (monomers). As time advances, the polymers evolve and, if they are sufficiently close, there is some chance that they merge into a single polymer whose mass equals the sum of the two polymers' masses which take part in this binary reaction. By convention, we admit only binary reactions. This phenomenon is called coalescence and we write formally

$$P_k + P_j \longrightarrow P_{k+j},$$

for the coalescence of a polymer of mass k with a polymer of mass j.
Let $n(k,t)$ denote the average number of polymers of mass k per unit volume, at time t. The expression $k\,n(k,t)$ denotes the part of mass consisting on polymers of

length k, per unit volume.
It is thus natural to consider that the coalescence phenomenon $(P_k + P_j \longrightarrow P_{k+j})$, is proportional to $n(k,t)\, n(j,t)$ with a proportionality constant $K(k,j)$, called coalescence kernel.
In the sequel we employ for the discrete case letters i, j, k ... while for the continuous case we use x, y, z .... Furthermore, throughout this paper, time t is always continuous, discrete and continuous refer to polymers' masses.
Hereafter (discrete and continuous case), the coagulation kernel K will satisfy the following hypothesis :

(H_1) K is positive i.e., $K : (\mathbb{N}^*)^2$ or $(\mathbb{R}_+)^2 \to \mathbb{R}_+$,
and
(H_2) K is symmetric i.e., $K(i,j) = K(j,i)$.

The Smoluchowski's coagulation equation, in the discrete case, is the equation on $n(k,t)$, for $k \in \mathbb{N}^*$. It is usually written on the following form

$$\begin{cases} \dfrac{d}{dt}n(k,t) = \dfrac{1}{2}\displaystyle\sum_{j=1}^{k-1} K(j,k-j)n(j,t)n(k-j,t) - n(k,t)\sum_{j=1}^{\infty} K(j,k)n(j,t) \\ n(k,0) = n_0(k),\ k \geq 1. \end{cases} \tag{SD}$$

This system describes a non linear evolution equation of infinite dimension, with initial condition $(n_0(k))_{k\geq 1}$. Due to the presence of the infinite series, (SD) is not a classical initial value problem for a system of non linear ordinary differential equations, and even the existence of a local solution is not guaranteed by the theory of ordinary differential equations. According to the form of the coalescence kernel K and the one of the initial condition, we obtain or not solutions for this system. In the first line of (SD), the first term on the right hand side describes the creation of polymers of mass k by coagulation of polymers of mass j and $k-j$. The coefficient $\frac{1}{2}$ is due to the fact that K is symmetric. The second term corresponds to the depletion of polymers of mass k after coalescence with other polymers.
The notion of solution will be given in the following section (definitions 1.1.1, 1.1.2 for the discrete case and definitions 1.1.3, 1.1.4 for the continuous case).
The continuous analogue of the equation (SD) can be written naturally

$$\begin{cases} \dfrac{\partial}{\partial t}n(x,t) = \dfrac{1}{2}\displaystyle\int_0^x K(y,x-y)n(y,t)n(x-y,t)dy - n(x,t)\int_0^{\infty} K(x,y)n(y,t)dy \\ n(x,0) = n_0(x). \end{cases} \tag{SC}$$

1.1.3 Some Known Results

The references, while numerous, are not intended to be complete, except that we have sought to represent the major direction of research from a probabilistic point of view. In this study on the Smoluchowski's coagulation equation we consider three kernels : constant, additive and multiplicative.

1.1.3.1 Analysis of the Discrete Case

Let us consider the discrete case. Introduce first of all the notion of weak solution for the system (SD) (we can find this definition for example in Laurençot ([43])).

Definition 1.1.1 *Let $T \in (0,\infty]$ and $(n_0(k))_{k\geq 1}$ be a sequence of positive real numbers. We call weak solution of the system (SD) on $[0,T)$, a sequence of nonnegative continuous functions such that, for all $t \in (0,T)$ and $k \geq 1$:*

(a) $n(k,.) \in \mathcal{C}([0,T])$ *and* $\displaystyle\sum_{j=1}^{\infty} K(j,k)\, n(j,t) \in L^1(0,t)$,

and
(b)

$$n(k,t) = n_0(k) + \int_0^t \Big(\frac{1}{2}\sum_{j=1}^{k-1} K(j,k-j)n(j,s)n(k-j,s) - n(k,s)\sum_{j=1}^{\infty} K(j,k)n(j,s)\Big)ds.$$

For our study we shall consider rather the notion of strong solution.

Definition 1.1.2 *Let $T \in (0,\infty]$ and $(n_0(k))_{k\geq 1}$ be a sequence of positive real numbers. We call strong solution of (SD) on $[0,T)$, a sequence of positive functions such that, for all $t \in (0,T)$ and $k \geq 1$: the derivative of $n(k,t)$ with respect to t exists, $\displaystyle\sum_{j=1}^{\infty} K(j,k)\, n(j,t) < \infty$ and (SD) is verified.*

A global solution is a solution with $T = \infty$. The solution will be called local in the opposite case.
Hereafter, we denote by C a constant whose value changes from line to line.
Let us make the following remark. Formal computations give

$$\frac{d}{dt}\sum_{k=1}^{\infty} k\, n(k,t) \;=\; 0. \tag{1.1}$$

Therefore it is natural to ask if (1.1) is preserved on time. From a physical point of view the equality (1.1) is equivalent to the conservation of the mass for the system (SD). As far as (1.1) is true we have

$$\sum_{k=1}^{\infty} k\, n(k,t) \;=\; \sum_{k=1}^{\infty} k\, n(k,0). \tag{1.2}$$

The first moment for which (1.2) is not longer valid is called gelification time and is defined by

$$T_{gel} = \inf\{t \geq 0;\, \sum_{k=1}^{\infty} k\, n(k,t) < \sum_{k=1}^{\infty} k\, n(k,0)\}. \tag{1.3}$$

In polymerisation this time corresponds to the appearance of an infinite polymer called gel (and is equivalent to mass removal). From a physical point of view the gelation phenomenon might be interpreted as follows: the process of formation of large polymers takes place at sufficiently large rate so that a part of monomers is transfered to larger and larger polymers, and eventually gives rise to a huge polymer called gel. This gel can be regarded as a polymer formed by an infinite number of monomers so it is not accounted into (SD).
Let us introduce also, the space of real positive sequences defined by

$$E = \{x = (x_i)_{i \geq 1},\, x_i \geq 0,\, \sum_{i=1}^{\infty} i\, x_i < \infty\}. \tag{1.4}$$

Constant Kernels
In 1916 Smoluchowski ([56]) studied (SD) for the constant kernel and gave an explicit solution. More generally, Melzak ([51]) proved that, for

$$K(i,j) \leq C, \quad \text{for} \quad i,j \geq 1 \quad \text{and} \quad C > 0, \tag{1.5}$$

one can obtain existence and uniqueness of a global solution for (SD), for any initial condition in E.

Additive Kernels
Ball and Carr ([4]) were interested on kernels satisfying

$$K(i,j) \leq C(i+j), \quad i,j \geq 1. \tag{1.6}$$

They got the existence of a global solution for (SD), which conserves the mass, for any initial condition from E. In order to get uniqueness of this solution we have either to impose

$$K(i,j) \leq C\sqrt{ij}, \tag{1.7}$$

or to restrict the class of initial conditions to E. This last situation was treated by Heilmann ([33]). For kernels satisfying the inequality (1.6) and with initial condition $(n(k,0))_{k\geq 1}$, from E, such that

$$\sum_{k=1}^{\infty} k^2 n(k,0) < \infty, \tag{1.8}$$

Heilmann ([33]) proved the existence and uniqueness of a global solution for (SD), which satisfies for all $T \in [0, +\infty)$

$$\sup_{t\in[0,T]} \sum_{k=1}^{\infty} k^2 n(k,t) < \infty. \tag{1.9}$$

Previously, the case $K(i,j) = C(i+j)$ was considered by Golovin ([28]).
For "additive" kernels K of the form

$$K(i,j) = i^{\alpha} + j^{\alpha} \tag{1.10}$$

with $\alpha > 1$, Carr and da Costa ([8]) proved that there is no solution (even local) of (SD). This result holds for any initial condition in E, different from the 0 function.

Multiplicative Kernels
The case of the kernel $K(i,j) = i\,j$ under the initial condition $n(k,0) = \delta_1(k)$, for all $k \in \mathbb{N}^*$, was treated by McLeod ([48]). This initial condition insures that, at $t = 0$, there are only monomers. McLeod deduced the existence and uniqueness of a local solution for all $t \in [0,1)$ verifying

$$\sup_{s\in(0,t]} \sum_{k=1}^{\infty} k^2 n(k,s) < \infty, \qquad \forall t \in (0,1). \tag{1.11}$$

For the same kernel and under the same initial condition, Kokholm ([40]) proved the existence of an unique global solution for (SD), satisfying for all $T \in [0,+\infty)$

$$\sup_{t\in[0,T]} \sum_{k=1}^{\infty} kn(k,t) < \infty. \tag{1.12}$$

This solution is given by

$$n(k,t) = \begin{cases} \dfrac{k^{k-3}}{(k-1)!} t^{k-1} e^{-kt} & \text{for} \quad t \in [0,1] \\ \dfrac{k^{k-3}}{(k-1)!} \dfrac{e^{-t}}{t} & \text{for} \quad t \in [1,+\infty). \end{cases}$$

Leyvraz and Tschudi ([45]), obtained a more general result. More precisely, for

$$K(i,j) \geq Cij, \qquad \forall i,j \geq 1, \tag{1.13}$$

any solution of (SD), when it exists, cannot satisfy the mass conservation for all t. Furthermore, for the initial condition $n(k,0) = \delta_1(k)$ they constructed a solution for which :

$$\sum_{k=1}^{\infty} k\,n(k,t) = \begin{cases} 1 & \text{for} \quad t \leq 1 \\ \dfrac{1}{t} & \text{for} \quad t \geq 1. \end{cases}$$

Obviously, in this particular case $T_{gel} = 1$. Their result explains also the local solution obtained previously by McLeod ([48]).
Furthermore, Leyvraz ([44]) treated the case $K(i,j) = i^\alpha\, j^\alpha$, and proved that for all $\alpha \in (\frac{1}{2}, 1]$, there exists an initial condition such that $T_{gel} = 0$ (that is the total mass decreases from the beginning).
Several works in the literature were concentrated on kernels which combine the additive, constant and multiplicative kernels, of the following form :

$$K(i,j) = A + B(i+j) + Cij, \tag{1.14}$$

where A, B and C are given constants.
For these kernels, the existence of a solution for the Smoluchowski's coagulation equation was obtained by using various techniques : Flory ([21], [22], [23]) and Spouge ([58], [59]), from a combinatorial point of view, Gordon ([29]) and Spouge ([57]), by using branching processes.

1.1.3.2 Analysis of the Continuous Case

Let us introduce first the notion of weak solution in the continuous case.

Definition 1.1.3 *Let $T \in (0,\infty]$ and $(n_0(x))_{x\geq 0}$ be a positive real function. We call weak solution of (SC) on $[0,T)$, a set of continuous and positive functions satisfying, for all $t \in (0,T)$ and $x \geq 0$:*

(a) $n(x,.) \in \mathcal{C}([0,T])$ *and* $\int_0^\infty K(x,y)\, n(y,t)dy \in L^1(0,t)$
and
(b)

$$n(x,t) = n_0(x) + \int_0^t \Big(\frac{1}{2}\int_0^x K(y,x-y)n(y,s)n(x-y,s)dy - n(x,s)\int_0^\infty K(x,y)n(y,s)dy\Big)ds.$$

The notion of strong solution becomes then:

Definition 1.1.4 *Let $T \in (0,\infty]$ and $(n(x,0))_{x\geq 0}$ be a real positive function. We call strong solution of (SC) on $[0,T)$, a set of positive and continuous functions such that, for all $t \in (0,T)$ and $x \geq 0$: the derivative with respect to t of $n(x,t)$ exists, $\int_0^\infty K(x,y)\, n(y,t)dy < \infty$ and (SC) is verified.*

For the additive, constant and multiplicative kernels, in the continuous case, the uniqueness is far of being obvious. We can obtain it for some special kernels, as for example, those satisfying ([1])

$$K(x,y) \leq C(1 + x + y), \tag{1.15}$$

More precisely, if the condition (C_t)

$$\int_0^\infty n(x,t)dx < \infty, \int_0^\infty xn(x,t)dx = 1 \text{ and } \int_0^\infty x^2 n(x,t)dx < \infty, \tag{C_t}$$

is satisfied for $t = 0$, then the solution of the continuous Smoluchowski's coagulation equation (SC), exists and is unique. Furthermore (C_t) is valid for all $t \in [0,\infty)$.
The work of Drake ([12]) and Aldous ([1]) gave examples of kernels K which have appeared in physics and chemistry. The model initially proposed by Smoluchowski ([56]) in 1916 had a kernel of the form

$$K(x,y) = C(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{-\frac{1}{3}} + y^{-\frac{1}{3}}), \tag{1.16}$$

and corresponds to a coagulation controled by the Brownian diffusion.
Ernst, Ziff and Hendricks ([16]) gave in their paper the construction of a solution

for (SC) after the gelification time, for kernels admitting a finite gelification time, of the form

$$K(x,y) = (xy)^{\alpha}, \text{ with } \frac{1}{2} < \alpha \leq 1. \tag{1.17}$$

For kernels of the form

$$K(x,y) = A + B(x+y) + Cxy, \tag{1.18}$$

Spouge ([60]) studied the "critical" time

$$t_c = \inf\{t; \int_0^{\infty} x^2\, n(x,t)dx < \infty\}, \tag{1.19}$$

and proved that it corresponds to a critical branching process. Times $t > t_c$ correspond to super-critical branching processes.

1.1.3.3 Probabilistic Approach for the Smoluchowski's Coagulation Equation

Many authors have treated the Smoluchowski's coagulation equation by using probabilistic methods.
In his paper, Aldous ([1]) made a survey of the present situation for (SD) and (SC) from a probabilistic point of view. He brought also to the fore some open problems which can be solved by using probabilistic methods.
Lang and Nguyen ([42]) used the propagation of chaos method in order to prove the convergence of an infinite particles system, directed by 3 dimensional Brownian Motions, to the initial model of coagulation proposed by Smoluchowski ([56]).
Recently, Jeon ([36]) approached the solution of a more general equation than (SD), in that, we have also the fragmentation of polymers, by a sequence of finite Markov chains. Jeon gave a general result. More precisely, if we have

$$K(i,j) \geq (ij)^{\alpha}, \text{ with } \alpha \in \left(\frac{1}{2},1\right), \tag{1.20}$$

and furthermore

$$\lim_{i+j\to\infty} \frac{K(i,j)}{ij} = 0, \tag{1.21}$$

then we have gelification in finite time ($T_{gel} < \infty$), for a large class of initial conditions. This result was generalised to the continuous situation by Norris ([52]).
The equation for the kernel $K(i,j) = ij$ can be connected to studies on coagulation via random graphs and forests ([1], [17]).
Another interesting direction in the present study of the Smoluchowski's coagulation equation is the approximation of the solution by using Monte Carlo methods ([32], [3]).

1.2 Connection Between the Discrete Smoluchowski's Coagulation Equation (SD) and Branching Processes

Let us consider the discrete Smoluchowski's coagulation equation (SD). We are going to study three coagulation kernels K: constant, additive and multiplicative. The initial condition that we consider for these three cases is $n(k,0) = \delta_1(k)$. It corresponds to an initial configuration in which there are only monomers. In the sequel we shall use the notion of solution in the sense of strong solution, given in the definition 1.1.2.
For each one of these cases we shall emphasise a connection between the solution $n(k,t)$ of (SD) and a branching process. This allows to obtain explicit solutions (corollaries 1.2.3, 1.2.5 and 1.2.7). The choice of this initial condition is essential and it corresponds, for the branching process, to the fact that it has one ancestor.
On one hand, our results are not new, we can find them for example in Aldous ([1]). On the other hand, our approach is new and consists in using generating functions, which allow the simplification of existing proofs. The goal of this section is to exhibit clearly the breadth and importance of branching processes in the study of the discrete Smoluchowski's coagulation equation.
A formal calculation allows the following remark.

Lemma 1.2.1 *Before the gelification time ($t < T_{gel}$), we have for (SD)*

$$\frac{d}{dt}\sum_{k=1}^{\infty} k\, n(k,t) = 0. \tag{1.22}$$

By re-normalising the initial condition, we can always suppose that

$$\sum_{k=1}^{\infty} k\, n(k,t) = 1. \tag{1.23}$$

1.2.1 Discrete Coagulation Equation with $K(i,j) = i\,j$

For this particular case the equation (SD) can be written

$$\begin{cases} \dfrac{d}{dt}n(k,t) = \dfrac{1}{2}\displaystyle\sum_{j=1}^{k-1} j(k-j)n(j,t)n(k-j,t) - kn(k,t)\sum_{j=1}^{\infty} jn(j,t) \\ n(k,0) = \delta_1(k),\ k \geq 1. \end{cases} \tag{$SD*$}$$

The choice of this initial condition corresponds to consider that at $t = 0$ we have only monomers. It is essential for the results which follow. We have $T_{gel} = 1$ (cf.

Leyvraz and Tschudi ([45])), and we shall consider only $t \leq T_{gel}$. By using lemma 1.2.1, we have the conservation of mass

$$\sum_{k=1}^{\infty} k\, n(k,t) = \sum_{k=1}^{\infty} k\, n(k,0) = 1, \quad \forall t \leq T_{gel} = 1. \tag{1.24}$$

Let us denote by

$$p(k,t) = k\, n(k,t), \tag{1.25}$$

such that $(p(k,t))_{k\geq 1}$ form a probability distribution on $\mathbb{N}^*$. The generating function of this probability,

$$G(t,s) = \sum_{k=1}^{\infty} p(k,t)\, s^k, \quad |s| \leq 1, \tag{1.26}$$

verifies

Proposition 1.2.2 *The function G, given by (1.26), is the unique solution of the non linear partial differential equation*

$$\begin{cases} \dfrac{\partial G}{\partial t}(t,s) = \dfrac{s}{2}\dfrac{\partial G^2}{\partial s}(t,s) - s\dfrac{\partial G}{\partial s}(t,s) \\ G(t,1) = 1 \\ G(0,s) = s, \end{cases} \tag{1.27}$$

for all $t \leq 1 = T_{gel}$ and $|s| \leq 1$.

The proof of this proposition is obvious in view of the equation $(SD*)$, on $n(k,t)$. The solution of the equation (1.27) is also the generating function of the total population of a particular branching process. Indeed, let us consider a branching process with one ancestor and offspring distribution a Poisson distribution of parameter t. Denote by X_t its total population. We consider the situation $X_t < \infty$ which is equivalent with $t \leq 1$ (by using classical properties of branching processes). The generating function of X_t satisfies the equation (1.27). We remark that on one hand, from a probabilistic point of view, $t = 1$ is a critical time in that, for $t > 1$ the probability that the total population of the branching process be finite is strictly less than 1 (that is $P\{X_t < \infty\} < 1$). On the other hand, for the Smoluchowski's coagulation equation $(SD*)$, this time is the gelification time. It becomes thus natural to consider for both cases $t \leq 1$.

The equality (via uniqueness), of these two generating functions allows us to express the solution of $(SD*)$

Corollary 1.2.3 *For $t \leq 1$, the solution of the equation $(SD*)$, is given by:*

$$n(k,t) = \frac{1}{k^2}\frac{(kt)^{k-1}}{(k-1)!}\, e^{-kt}, \ \forall k \in \mathbb{N}^*. \tag{1.28}$$

Proof This result follows immediately from a remarkable property of branching processes, which connects the law of the total population to the law of the offspring (Athreya and Ney ([2])). More precisely, if X_t denotes the total population of the previous branching process, we have

$$P(X_t = k) = \frac{1}{k} P(Y_t^1 + ... + Y_t^k = k - 1),$$

where Y_t^i are i.i.d. random variables having a Poisson distribution of parameter t. Thus, we have obtained for the total population X_t, a Borel distribution of parameter t. □

The previous corollary implicitly gives an existence and uniqueness result for the solution of the equation $(SD*)$. We observe that, the connection between the solution of $(SD*)$ and the branching process with one ancestor and offspring distribution a Poisson distribution of parameter t, denoted by $\mathcal{P}(t)$, allows to find the explicit form for the solution of $(SD*)$, by using a generating function argument.

1.2.2 Discrete Coagulation Equation with $K(i,j) = i + j$

The form of the equation (SD) becomes in this case

$$\begin{cases} \dfrac{d}{dt} n(k,t) = \dfrac{k}{2} \displaystyle\sum_{j=1}^{k-1} n(j,t) n(k-j,t) - k\, n(k,t) \sum_{j=1}^{\infty} n(j,t) - n(k,t) \\ n(k,0) = \delta_1(k),\ k \geq 1. \end{cases} \qquad (SD+)$$

We have now $T_{gel} = \infty$ and the solution will be defined on the real positive line. By summing over k in $(SD+)$ we deduce

$$\sum_{k=1}^{\infty} n(k,t) = e^{-t}. \qquad (1.29)$$

This leads us to choose the normalisation

$$p(k,t) = e^t n(k,t). \qquad (1.30)$$

The generating function associated with $(p(k,t))_{k \geq 1}$

$$G(t,s) = \sum_{k=1}^{\infty} p(k,t) s^k, \quad |s| \leq 1, \qquad (1.31)$$

satisfies

Proposition 1.2.4 *The function $G(t,s)$, given by* (1.31)*, is the unique solution of the following partial differential equation:*

$$\begin{cases} \dfrac{\partial G}{\partial t}(t,s) = s\, e^{-t} (G(t,s) - 1) \dfrac{\partial G}{\partial s}(t,s) \\ G(t,1) = 1 \\ G(0,s) = s, \end{cases} \qquad (1.32)$$

for all $t \in [0,+\infty)$ *and* $|s| \leq 1$.

We don't prove this proposition. It follows easily from the equation $(SD+)$. Let us construct a branching process for which the generating function of its total population is G. In order to do this, consider a branching process with one ancestor and offspring distribution a Poisson distribution of parameter $1-e^{-t}, t \geq 0$. The generating function of its total population X_t, is the unique solution of (1.32). Classical results on branching processes allow to express the distribution of X_t. We can deduce thus the solution of the equation $(SD+)$:

Corollary 1.2.5 *The solution* $n(k,t)$ *of the equation* $(SD+)$, *is given by*

$$n(k,t) = \frac{1}{k}\frac{(k(1-e^{-t}))^{k-1}}{(k-1)!}e^{-t}\,e^{-k(1-e^{-t})},\ k \geq 1,\ t \geq 0. \tag{1.33}$$

Let us also remark that the previous corollary furnishes an existence and uniqueness result for the solution of $(SD+)$.

1.2.3 Discrete Coagulation Equation with $K(i,j) = 1$

The equation (SD) can be written in this case

$$\begin{cases} \dfrac{d}{dt}n(k,t) = \dfrac{1}{2}\displaystyle\sum_{j=1}^{k-1} n(j,t)n(k-j,t) - n(k,t)\sum_{j=1}^{\infty} n(j,t) \\ n(k,0) = \delta_1(k),\ k \geq 1. \end{cases} \tag{SD1}$$

By summing over k in $(SD1)$ we deduce

$$\sum_{k=1}^{\infty} n(k,t) = \frac{2}{t+2}.$$

This leads us to consider

$$p(k,t) = \left(1+\frac{t}{2}\right)n(k,t), \tag{1.34}$$

such that $(p(k,t))_{k\geq 1}$ form a probability distribution. The generating function associated with this probability:

$$G(t,s) = \sum_{k=1}^{\infty} p(k,t)s^k, \quad |s| \leq 1, \tag{1.35}$$

satisfies the following equation:

Proposition 1.2.6 *The function G, given in (1.35), is the unique solution of the partial differential equation*

$$\begin{cases} (2+t)\dfrac{\partial G}{\partial t}(t,s) = G(t,s)(G(t,s)-1) \\ G(t,1) = 1 \\ G(0,s) = s, \end{cases} \tag{1.36}$$

for all $t \in [0,+\infty)$ and $|s| \leq 1$.

Again we observe that the proof of this result is obvious in view of the equation ($SD1$). Associate with G a branching process. While we are on the subject, consider a branching process with one ancestor and offspring distribution a Bernoulli distribution of parameter $p_t = \dfrac{t}{2+t}$. Denote by X_t its total population. The generating function of X_t is solution of (1.36) and, by uniqueness, we are able to express $n(k,t)$ with respect to $P(X_t = k)$. This remark gives us the form of the solution of the Smoluchowski's coagulation equation for $K(i,j) = 1$.

Corollary 1.2.7 *The solution $n(k,t)$ of the equation ($SD1$), is given by*

$$n(k,t) = \left(1 + \frac{t}{2}\right)^{-2} \left(\frac{t}{t+1}\right)^{k-1}, \; k \geq 1, \; t \geq 0. \tag{1.37}$$

Remark 1.2.8 In the three preceding sections, the choice of the initial condition $n(k,0) = \delta_1(k)$ is essential in order to be able to use properties of branching processes with one ancestor.

Let us denote by $\mathcal{P}(\lambda)$ a Poisson distribution of parameter λ and by $\mathcal{B}(\lambda)$ a Bernoulli distribution of parameter λ. We use the following scheme to summarise the results for the discrete case

$K(k,j)$	$n(k,t)$	T_{gel}	Offspring distribution
1	$\left(1+\frac{t}{2}\right)^{-2}\left(\frac{t}{t+1}\right)^{k-1}$	∞	$\mathcal{B}\left(\frac{t}{2+t}\right)$
$k+j$	$\frac{1}{k}\frac{(k(1-e^{-t}))^{k-1}}{(k-1)!}e^{-t}\,e^{-k(1-e^{-t})}$	∞	$\mathcal{P}(1-e^{-t})$
kj	$\frac{1}{k^2}\frac{(kt)^{k-1}}{(k-1)!}\,e^{-kt}$	1	$\mathcal{P}(t)$

Remark 1.2.9 We have presented in this section a simple manner which allows to obtain a probabilistic representation of solutions for the equation (SD), by making use of the PDEs verified by a generating function associated with (SD).
Our approach gives a construction for each fixed time t. We refer to the Aldous' paper ([1]) for a dynamic construction in t, via processes for the constant, additive and multiplicative coalescence (see section 3, constructions 5, 6 and 8 of his paper). Jeon ([36]) gives also an interesting construction for a general coagulation-fragmentation equation as a limit of a sequence of finite state Markov chains.

1.3 Continuous Smoluchowski's Coagulation Equation (SC)

We shall now be interested on the Smoluchowski's coagulation equation (SC), for which the mass is a continuous parameter. First of all we present some general results for (SC), which are independent from the kernel K. Part of these results (proposition 1.3.1), are taken from Aldous ([1]).
Afterwards, as for the discrete case we shall consider the constant, additive and multiplicative kernels.
The aim of this part is to present some new results, more precisely: a duality result connecting the solutions of the multiplicative case with those of the additive case (theorems 1.3.9 and 1.3.11) and also some renormalisation theorems (theorems 1.3.6, 1.3.20 and 1.3.24), which insure the convergence of the solution, for a large class of initial conditions, to a limit, which depends weakly on the initial condition.

1.3.1 General Results for (SC)

Hereafter we shall use for solution the notion of strong solution introduced in definition 1.1.4.

Proposition 1.3.1 *Let K be a symmetric and positive kernel and $n(x,t)$ be the solution of (SC). Denote, for $i \in \mathbb{N}$*

$$\phi_i(t) = \int_0^\infty x^i n(x,t)\, dx.$$

When these expressions are well defined, ϕ_0 is non-increasing, ϕ_1 is constant and ϕ_2 is increasing. Furthermore

$$\phi_0'(t) = -\frac{1}{2}\int_0^\infty \int_0^\infty K(x,y)n(x,t)n(y,t)\, dy\, dx. \tag{1.38}$$

These results can be found for example in Dubovskii ([13]). We shall usually consider, for normalisation reasons

$$\phi_1(0) = \phi_1(t) = 1. \tag{1.39}$$

Let us denote

$$p(x,t) = xn(x,t),\ x \geq 0,\ t > 0. \tag{1.40}$$

By using (1.39), $(p(x,t))_x$ is the probability density of a positive random variable. We have the following characterisation:

Proposition 1.3.2 *Let X_t be a positive random variable of density $p(x,t) = xn(x,t)$ and $\tilde{X}_t$ a random variable independent from X_t and with same distribution. Then $n(x,t)$ is solution of (SC), if and only if*

$$\frac{d}{dt}E(f(X_t)) = E\left(\frac{f(X_t + \tilde{X}_t) - f(X_t)}{\tilde{X}_t} K(X_t,\tilde{X}_t)\right), \tag{1.41}$$

for all smooth functions f.

Remark 1.3.3 It suffices for example to consider f derivable, with compact support.

Proof Let us evaluate

$$\begin{aligned}
\frac{d}{dt}E(f(X_t)) &= \int_0^\infty f(x)\frac{\partial p}{\partial t}(x,t)\,dx \\
&= \int_0^\infty \frac{x}{2}f(x)\int_0^x n(y,t)n(x-y,t)K(y,x-y)\,dy\,dx \\
&\quad - \int_0^\infty f(x)\,x\,n(x,t)\int_0^\infty K(x,y)n(y,t)\,dy\,dx.
\end{aligned}$$

By using the variable change

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x-y \\ y \end{pmatrix}$$

in the first term on the right hand, we deduce

$$\begin{aligned}
\frac{d}{dt}E(f(X_t)) &= \int_0^\infty\int_0^\infty \frac{(x+y)f(x+y)}{2}K(x,y)n(x,t)n(y,t)\,dy\,dx \\
&\quad - \int_0^\infty\int_0^\infty xf(x)K(x,y)n(x,t)n(y,t)\,dy\,dx \\
&= \int_0^\infty\int_0^\infty \left(\frac{f(x+y)-f(x)}{y}\right)K(x,y)p(x,t)p(y,t)\,dy\,dx.
\end{aligned}$$

For the last equality we used the symmetry of K. This ends the proof of the proposition. □

1.3.2 Continuous Coagulation Equation with Constant Kernel

For the constant kernel, a time normalisation is more interesting than the mass normalisation that we have announced in the general case. For this case we shall use this normalisation.
Let us first remark that the result in (1.15) applies in particular for constant and additive kernels, once the initial condition satisfies (C_0). We can deduce thus an existence and uniqueness result for the solution of $(SC1)$ and $(SC+)$.
The equation (SC) becomes, for $K(x,y) = 1$

$$\begin{cases} \dfrac{\partial}{\partial t} n(x,t) = \dfrac{1}{2}\displaystyle\int_0^x n(y,t)n(x-y,t)\,dy - n(x,t)\int_0^\infty n(y,t)\,dy \\ n(x,0) = n_0(x). \end{cases} \tag{SC1}$$

For the same kernel, the differential equation (1.38), can be written

$$\phi_0'(t) = -\frac{\phi_0^2(t)}{2}. \tag{1.42}$$

We obtain

$$\phi_0(t) = \frac{2}{t+\alpha}$$

where $\alpha > 0$ is given by

$$\alpha = \frac{2}{\phi_0(0)}.$$

Let us denote by

$$p(x,t) = \frac{t+\alpha}{2} n(x,t), \tag{1.43}$$

such that $(p(x,t))_{x\geq 0}$ is the probability distribution of a positive random variable. There is a similar result to the proposition 1.3.2 and it writes, for this particular case

Proposition 1.3.4 *Let X_t denote a random variable with probability density $p(x,t)$ given by* (1.43)*, and $\tilde{X}_t$ an independent copy of X_t. Then, $n(x,t)$ is solution of $(SC1)$ if and only if, for any smooth function f we have:*

$$\frac{d}{dt}E(f(X_t)) = \frac{1}{t+\alpha}(E(f(X_t+\tilde{X}_t)) - E(f(X_t))). \tag{1.44}$$

Proof We proceed as for the proof of the proposition 1.3.2. Let us evaluate

$$\begin{aligned}
\frac{d}{dt}E[f(X_t)] &= \\
&= \frac{1}{2}\int_0^\infty f(x)n(x,t)\,dx + \frac{t+\alpha}{2}\int_0^\infty \frac{f(x)}{2}\left(\int_0^x n(y,t)n(x-y,t)\,dy\right)dx \\
&\quad -\frac{t+\alpha}{2}\int_0^\infty f(x)n(x,t)\,dx\int_0^\infty n(y,t)\,dy \\
&= -\frac{1}{2}\int_0^\infty f(x)n(x,t)\,dx + \frac{t+\alpha}{4}\int_0^\infty\int_0^\infty f(x+y)n(y,t)n(x,t)\,dy\,dx \\
&= \frac{1}{t+\alpha}\int_0^\infty\int_0^\infty f(x+y)p(y,t)p(x,t)\,dy\,dx - \frac{1}{t+\alpha}\int_0^\infty f(x)p(x,t)\,dx
\end{aligned}$$

This achieves the proof. □

This result allows us to describe entirely, for any initial condition, the solution of the Smoluchowski's coagulation equation with constant kernel, $(SC1)$.

Theorem 1.3.5 *Let T_t be a random variable of geometric law, with parameter $\frac{t}{t+\alpha}$ i.e.*

$$\forall k \geq 1,\ P(T_t = k) = \frac{\alpha}{t+\alpha}\left(\frac{t}{t+\alpha}\right)^{k-1}. \tag{1.45}$$

Let $(Y_i)_{i\geq 1}$ denote a sequence of i.i.d. random variables having same law as X_0, of probability density $p(x,0)$, and independent from T_t.
Define

$$X_t = \sum_{i=1}^{T_t} Y_i,$$

and denote by $p(x,t)$ the distribution of the random variable X_t. Then

$$n(x,t) = \frac{2}{t+\alpha}p(x,t),$$

is the solution of the Smoluchowski's coagulation equation corresponding to the constant kernel $(SC1)$.

Proof We prove this result by using Laplace transforms. Let us apply the equation (1.44) with $f_\lambda(x) = e^{-\lambda x}$. Denote by $\psi(\lambda,t) = E(e^{-\lambda X_t})$, the Laplace transform of X_t and by $g(\lambda) = \psi(\lambda,0)$, the Laplace transform of the initial condition. We deduce

$$\left\{\begin{array}{l}\frac{\partial}{\partial t}\psi(\lambda,t) = \frac{1}{t+\alpha}(\psi^2(\lambda,t) - \psi(\lambda,t)) \\ \psi(\lambda,0) = g(\lambda).\end{array}\right. \tag{1.46}$$

After integration we get

$$\frac{1}{(t+\alpha)\psi(\lambda,t)} = \frac{1}{t+\alpha} + d(\lambda), \tag{1.47}$$

where $d(\lambda)$ is a function depending only on λ. We obtain so the form of the Laplace transform

$$\psi(\lambda,t) = \frac{1}{(t+\alpha)d(\lambda)+1} \tag{1.48}$$

under the initial condition

$$\psi(\lambda,0) = g(\lambda) = \frac{1}{\alpha d(\lambda)+1}. \tag{1.49}$$

Let us also remark that the equation (1.47) insures that

$$(t+\alpha)\, d(\lambda) + 1 \neq 0.$$

From (1.49) we deduce the expression of d

$$d(\lambda) = \left(\frac{1}{g(\lambda)} - 1\right)\frac{1}{\alpha}. \tag{1.50}$$

By reporting in (1.48) the form of d found in (1.50), we obtain for ψ the following formula :

$$\psi(\lambda,t) = \frac{\alpha g(\lambda)}{(t+\alpha)(1-g(\lambda)+\frac{\alpha}{t+\alpha}g(\lambda))} = \frac{\alpha g(\lambda)}{(t+\alpha)(1-\frac{t}{t+\alpha}g(\lambda))}. \tag{1.51}$$

As $g(\lambda) < 1$, we can expand in integer series this expression and obtain

$$\psi(\lambda,t) = \frac{\alpha g(\lambda)}{t+\alpha}\sum_{k\geq 1}\left(\frac{t}{t+\alpha}\right)^{k-1}(g(\lambda))^{k-1}. \tag{1.52}$$

$g(\lambda)$ being a Laplace transform it is not difficult to verify that $\psi(\lambda,t)$ is also a Laplace transform (we can use the structure of ψ in terms of convolution). We have thus construct a solution of (1.46) ; more precisely, the equation (1.44) is satisfied for all exponential functions $f_\lambda(x) = e^{-\lambda x}$. This family is sufficiently large in order that (1.44) be verified by all smooth functions. This ends the proof of the theorem 1.3.5, because the equation (1.44) is nothing else that the integral version of the Smoluchowski's coagulation equation ($SC1$). □

Let us focus now on the asymptotic behaviour of the random variables introduced in the theorem 1.3.5.

Theorem 1.3.6 *Consider the notations of theorem 1.3.5. For any random variable X_0 with probability density $p(x,0) = \frac{\alpha}{2}\, n(x,0)$, we have, independently of the initial condition and for all fixed t*

$$a\, X_{\frac{t}{a}} \xrightarrow[a\to 0]{(d)} R_{\frac{t}{2}},$$

where R_t is the square of a two-dimensional Bessel process starting from the origin.

Proof We shall prove this convergence by using Laplace transforms. Let us remark first that, if the initial condition has a first order moment, we can write

$$g(\lambda) = E(e^{-\lambda X_0}) = 1 - E(X_0) + o(\lambda).$$

We obtain, by using (1.51)

$$\begin{aligned}\psi(\lambda a, \frac{t}{a}) &= \frac{\alpha g(\lambda a)}{\frac{t}{a} + \alpha - \frac{t}{a} g(\lambda a)} \\ &= \frac{\alpha(1 + o(1))}{\frac{t}{a} + \alpha - \frac{t}{a}(1 - \lambda a \frac{\alpha}{2} + o(a))} \\ &= \frac{1 + o(1)}{1 + \frac{\lambda t}{2} + o(1)}.\end{aligned}$$

By letting a goes to zero we obtain the result of the theorem. □

Remark 1.3.7 R_t "corresponds" to the solution of ($SC1$) with initial condition a Dirac mass, δ_0. Consequently, for any initial condition, the scaled solution of the continuous Smoluchowski's coagulation equation, with constant kernel, converges to the solution of ($SC1$), with initial condition the Dirac mass.

By applying known results on the density of the square of a Bessel process, we deduce an exact solution for the Smoluchowski's coagulation equation ($SC1$)

$$n(x,t) = \frac{4}{t^2} \exp\left(-\frac{2x}{t}\right). \tag{1.53}$$

1.3.3 Additive and Multiplicative Kernels for the Continuous Coagulation Equation

We shall prove in this section that the solutions of the additive and multiplicative kernels are connected. We make first some remarks that simplify these equations. We will suppose in what follows, that

$$\phi_1(0) = \int_0^\infty x n(x,0)\, dx = 1. \tag{1.54}$$

In the sequel we will denote by $\overset{+}{n}(x,t)$ (respectively $\overset{*}{n}(x,t)$), a solution for the Smoluchowski's coagulation equation with additive kernel (respectively multiplicative).

Remark 1.3.8 For the additive kernel $K(x,y) = x + y$, the equation (1.38) becomes

$$\phi_0'(t) = -\phi_0(t) \text{ and so } \phi_0(t) = \beta e^{-t},$$

where

$$\beta = \phi_0(0) = \int_0^\infty \overset{+}{n}(x,0)\, dx.$$

The equation (SC) can be written in this case

$$\begin{cases} \dfrac{\partial \overset{+}{n}}{\partial t}(x,t) &= \dfrac{x}{2}\displaystyle\int_0^x \overset{+}{n}(y,t)\overset{+}{n}(x-y,t)\,dy - x\overset{+}{n}(x,t)\int_0^\infty \overset{+}{n}(y,t)\,dy - \overset{+}{n}(x,t) \\ &= \dfrac{x}{2}\displaystyle\int_0^x \overset{+}{n}(y,t)\overset{+}{n}(x-y,t)\,dy - x\overset{+}{n}(x,t)\beta e^{-t} - \overset{+}{n}(x,t) \\ \overset{+}{n}(x,0) &= \overset{+}{n}_0(x). \end{cases} \tag{SC+}$$

Let us also write the equation (SC) for the multiplicative kernel.

$$\begin{cases} \dfrac{\partial \overset{*}{n}}{\partial t}(x,t) &= \dfrac{1}{2}\displaystyle\int_0^x y(x-y)\overset{*}{n}(y,t)\overset{*}{n}(x-y,t)\,dy - x\overset{*}{n}(x,t)\int_0^\infty y\overset{*}{n}(y,t)\,dy \\ &= \dfrac{1}{2}\displaystyle\int_0^x y(x-y)\overset{*}{n}(y,t)\overset{*}{n}(x-y,t)\,dy - x\overset{*}{n}(x,t) \\ \overset{*}{n}(x,0) &= \overset{*}{n}_0(x). \end{cases} \tag{SC*}$$

We can now express the connection between the solutions of the equations $(SC+)$ and $(SC*)$.

Theorem 1.3.9 *Let $\overset{+}{n}(x,t)$ denote a solution of the Smoluchowski's coagulation equation with additive kernel $(SC+)$, and initial condition $\overset{+}{n}_0(x)$. Then $\overset{*}{n}(x,t)$ is a solution for the equation with multiplicative kernel $(SC*)$, where we have denoted by*

$$\overset{*}{n}(x,t) = \frac{1}{T-t}\frac{1}{x}\overset{+}{n}\left(x, -\,ln(1-\frac{t}{T})\right), \forall t < T, \tag{1.55}$$

and $T = \displaystyle\int_0^\infty \overset{+}{n}_0(y)\,dy$.

Proof We remark first that the initial condition (1.54), that we usually impose, is really satisfied for $\overset{*}{n}$. We have

$$\begin{aligned} \int_0^\infty x\overset{*}{n}(x,0)\,dx &= \frac{1}{T}\int_0^\infty \overset{+}{n}(x,0)\,dx \\ &= 1. \end{aligned}$$

We prove now that $\overset{*}{n}$ satisfies the Smoluchowski's equation $(SC*)$. We have

$$\frac{\partial}{\partial t}\overset{*}{n}(x,t) = \frac{1}{(T-t)^2}\frac{1}{x}\overset{+}{n}(x,-\,ln(1-\frac{t}{T})) + \frac{1}{(T-t)^2}\frac{1}{x}\frac{\partial}{\partial t}\overset{+}{n}(x,-\,ln(1-\frac{t}{T}))$$

$$\begin{aligned} &= \frac{1}{(T-t)^2}\frac{1}{x}\overset{+}{n}(x,-\,ln(1-\frac{t}{T})) \\ &\quad +\frac{1}{2}\frac{1}{(T-t)^2}\frac{1}{x}\int_0^x x\overset{+}{n}(y,-\,ln(1-\frac{t}{T}))\overset{+}{n}(x-y,-\,ln(1-\frac{t}{T}))\,dy \\ &\quad -\frac{1}{(T-t)^2}\frac{1}{x}\overset{+}{n}(x,-\,ln(1-\frac{t}{T})) \\ &\quad -\frac{1}{(T-t)^2}\overset{+}{n}(x,-\,ln(1-\frac{t}{T}))\int_0^\infty \overset{+}{n}(y,-\,ln(1-\frac{t}{T}))\,dy, \end{aligned}$$

because $\overset{+}{n}$ is solution of $(SC+)$. We conclude that

$$\frac{\partial}{\partial t}\overset{*}{n}(x,t) = \frac{1}{2}\int_0^x y(x-y)\overset{*}{n}(y,t)\overset{*}{n}(x-y,t)\,dy - x\overset{*}{n}(x,t).$$

Here we have used that for the additive solution

$$\int_0^\infty \overset{+}{n}(y,t)\,dy = Te^{-t} \text{ with } T = \int_0^\infty \overset{+}{n}(x,0)\,dx.$$

This ends the proof of the theorem 1.3.9. □

Remark 1.3.10 The transformation in theorem 1.3.9 emphasises a finite time existence interval for the solution in the multiplicative case. We find thus already known results for this kernel ($T_{gel} < \infty$, Norris ([52]))

We have also the inverse transformation.

Theorem 1.3.11 *If $\overset{*}{n}(x,t)$ is a solution of the Smoluchowski's coagulation equation with multiplicative kernel $(SC*)$ and initial condition $\overset{*}{n}_0(x)$, then $\overset{+}{n}(x,t)$ is a solution of the Smoluchowski's equation with additive kernel $(SC+)$, where we have denoted by:*

$$\overset{+}{n}(x,t) = x\,T\,e^{-t}\overset{*}{n}(x,T(1-e^{-t}))$$

and $T = \left(\int_0^\infty x^2\,\overset{*}{n}_0(x)\,dx\right)^{-1}$.

Proof The proof of this theorem is similar to the one of the theorem 1.3.9. The choice of T insures that (1.54) is satisfied, i.e.

$$\int_0^\infty x\overset{+}{n}(x,0) = 1.$$

□

Remark 1.3.12 These transformations (theorems 1.3.9 and 1.3.11), apply also to the discrete case. It suffices to replace integrals by sums.

1.3.3.1 Existence of Solutions for $(SC+)$ and $(SC*)$

The aim of this part is to prove existence of solutions for $(SC+)$ and $(SC*)$ by employing probabilistic methods and the preceding transformations. We shall first recall an existence result which can be found for example in Dubovskii ([13]). Consider the equation $(SC*)$. We have

Theorem 1.3.13 *For any initial condition $\overset{*}{n}(x,0)$, $x \geq 0$ which satisfies $\int_0^\infty x\overset{*}{n}(x,0)\,dx = 1$ and $0 < \int_0^\infty x^2\overset{*}{n}(x,0)\,dx < \infty$, there exists an unique solution of $(SC*)$ defined for $t \in [0,T)$ where*

$$T = \left(\int_0^\infty x^2\overset{*}{n}(x,0)\,dx\right)^{-1}.$$

Proof The aim of what follows is to prove this theorem by using probabilistic techniques. In order to obtain this result, we apply the proposition 1.3.2 on this particular case. We deduce that

$$\frac{d}{dt}E(f(X_t)) = E((f(X_t + \tilde{X}_t) - f(X_t))X_t), \tag{1.56}$$

where X_t has density $x\overset{*}{n}(x,t)$ and $\tilde{X}_t$ is an independent copy of the random variable X_t. We apply the equation (1.56) to functions of the form $f_\lambda(x) = e^{-\lambda x}$. Denote by $\psi(\lambda,t) = E\left(e^{-\lambda X_t}\right)$ the Laplace transform of X_t. ψ satisfies the non linear hyperbolic partial differential equation:

$$\left\{\begin{array}{l} \dfrac{\partial\psi}{\partial t} = (1-\psi)\dfrac{\partial\psi}{\partial\lambda} \\ \psi(\lambda,0) = g(\lambda). \end{array}\right. \tag{1.57}$$

We will construct a solution for this equation under the initial condition

$$\psi(\lambda,0) = \int_0^\infty e^{-\lambda x}x\overset{*}{n}(x,0)\,dx$$

and prove that it conserves, for all t, the property of being a Laplace transform once that $g(\lambda) = \psi(\lambda,0)$ is a Laplace transform.
Let us denote, for $t \in \left[0, -\dfrac{1}{g'(0)}\right[$

$$\psi(\lambda_0 + (g(\lambda_0) - 1)t,t) = g(\lambda_0). \tag{1.58}$$

Remark 1.3.14 Similar techniques, based on Laplace transform, have been used by Ernst, Ziff and Hendricks ([16]) for the multiplicative kernel, in order to find the expression of $\int_0^\infty x\overset{*}{n}(x,t)dx$ beyond the gelification time.

Proposition 1.3.15 *The function ψ defined by (1.58) is a solution of the partial differential equation (1.57).*

Proof This proposition doesn't really need a proof. By construction the result is true. Indeed, we constructed this solution by using the level sets of the equation (1.57). □

We want to prove that $\psi(\lambda,t)$, solution of (1.57), is a Laplace transform. We shall use the Karamata theorem (cf. ([18]), p. 439). In order to apply it we need some auxiliary results (lemmas 1.3.16 and 1.3.17).

Lemma 1.3.16 *The function ψ, solution of (1.57), is completely monotone, that is*

$$\forall k \geq 0,\ (-1)^k \frac{\partial^k \psi}{\partial \lambda^k} \geq 0. \tag{1.59}$$

Proof The proof will be done by recurrence on k: the result is true for $k = 0$, by construction of ψ.
Let us denote by $f_k(t) = \dfrac{\partial^k \psi}{\partial \lambda^k}(\lambda(t),t)$ where $\lambda(t) = \lambda_0 + (g(\lambda_0) - 1)t$.
We suppose that:

$$\forall j \in [\![1,k-1]\!],\ (-1)^j f_j(t) \geq 0. \tag{1.60}$$

We get then:

$$f_k'(t) = \lambda'(t)\frac{\partial^{k+1}}{\partial \lambda^{k+1}}\psi(\lambda(t),t) + \frac{\partial^{k+1}}{\partial \lambda^k\, \partial t}\psi(\lambda(t),t). \tag{1.61}$$

By taking the derivative in the equation (1.57), k times with respect to λ, we obtain

$$\frac{\partial^{k+1}\psi}{\partial \lambda^k\, \partial t}(\lambda,t) = (1-\psi(\lambda,t))\frac{\partial^{k+1}}{\partial \lambda^{k+1}}\psi(\lambda,t) - \sum_{j=1}^{k}\binom{k}{j}\frac{\partial^j}{\partial \lambda^j}\psi(\lambda,t)\frac{\partial^{k-j+1}}{\partial \lambda^{k-j+1}}\psi(\lambda,t). \tag{1.62}$$

This gives, by using (1.61) and recalling that we are on a level set $(1 - \psi(\lambda(t),t) + \lambda'(t) = 0)$

$$\begin{aligned} f_k'(t) &= -(k+1)\frac{\partial}{\partial \lambda}\psi(\lambda(t),t)f_k(t) - \sum_{j=2}^{k-1}\binom{k}{j}\frac{\partial^j}{\partial \lambda^j}\psi(\lambda(t),t)\frac{\partial^{k-j+1}}{\partial \lambda^{k-j+1}}\psi(\lambda(t),t) \\ &= -(k+1)f_1(t)f_k(t) - \sum_{j=2}^{k-1}\binom{k}{j}f_j(t)f_{k+1-j}(t). \end{aligned} \tag{1.63}$$

We remark now that each term in the previous sum has same sign as $(-1)^{k+1}$, by the recurrence hypothesis (1.60). In conclusion, we have a first order partial differential equation, that is:

$$u'(t) = a(t)u(t) + b(t) \tag{1.64}$$

with $u(0)$ and $b(t)$ of same sign.
In this case, the sign of $u(t)$ is conserved, for all t in the definition domain of the function u, solution of the equation (1.64). □

Furthermore:

Lemma 1.3.17 *For all* $t \in \left[0, -\frac{1}{g'(0)}\right[$, *we have*

$$\lim_{\lambda \to 0} \psi(\lambda,t) = 1.$$

Proof We calculate the ordinate of the intersection between the level set and the axis $\lambda = 0$. Call $t_0(\lambda_0)$ the intersection point of the level set passing through the point $(\lambda_0,0)$, with this axis. We have

$$t_0(\lambda_0) = \frac{\lambda_0}{1 - g(\lambda_0)}.$$

By taking derivatives in this expression we get

$$t_0'(\lambda_0) = \frac{1 - g(\lambda_0) - \lambda_0 g'(\lambda_0)}{(1 - g(\lambda_0))^2}.$$

This last expression is obviously positive ($g(\lambda)$ is a Laplace transform). Thus $t_0(\lambda_0)$ is increasing (see the picture) and equals $-\frac{1}{g'(0)}$ in 0. We deduce that, for all $t \in \left[0, -\frac{1}{g'(0)}\right[$,

$$\lim_{\lambda \to 0} \psi(\lambda,t) = 1.$$

□

The results of lemmas 1.3.16 and 1.3.17 prove (by the Karamata theorem, ([18]), p. 439) the following proposition:

Proposition 1.3.18 *For all* $t \in \left[0, -\frac{1}{g'(0)}\right[$, ψ *is the Laplace transform of a probability density.*

By using the resolution of the partial differential equation satisfied by the Laplace transform, we have proved that the equation (1.56) is true for all exponential functions $f_\lambda(x) = e^{-\lambda x}$. This family forms a large family of functions and the equation

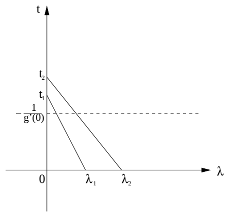

FIG. 1.1 – *Behaviour of* $t_0(\lambda_0)$

will be satisfied by all smooth functions. On the other hand this equation is the integral form of the Smoluchowski's coagulation equation $(SC*)$. This ends the proof of the theorem 1.3.13. $\square$

Let us note now the consequence of theorems 1.3.13 and 1.3.11.

Corollary 1.3.19 *For any initial condition* $\overset{+}{n}(x,0)$, $x \geq 0$ *satisfying*

$$\int_0^\infty x\overset{+}{n}(x,0)\,dx = 1,$$

there exists an unique solution of $(SC+)$ *defined for all* $t \in [0,+\infty[$.

1.3.3.2 Convergence of Solutions

In the last part we state the renormalisation theorems. More precisely, we shall see now that the solutions of $(SC+)$ have the same asymptotic behaviour under initial conditions not too restrictive. Let us state this theorem :

Theorem 1.3.20 *For* $\overset{+}{n}(x,t)$ *solution of the Smoluchowski's coagulation equation with additive kernel* $(SC+)$*, let us denote by* $\overset{+}{p}(x,t) = x\,\overset{+}{n}(x,t)$.
We suppose that $m_k(0) = \displaystyle\int_0^\infty x^k\overset{+}{p}(x,0)\,dx \leq C^k\frac{(2k)!}{k!}$. *We have then, for fix* t *:*

$$a^2 X_{\log\frac{t}{a}} \xrightarrow[a\to 0]{(d)} R_{A_1 t^2}, \tag{1.65}$$

where R_t *denotes the square of the one-dimensional Bessel process started at the origin and* $A_1 = m_1(0) = \displaystyle\int_0^\infty x\overset{+}{p}(x,0)\,dx = \int_0^\infty x^2\overset{+}{n}(x,0)\,dx$.

Remark 1.3.21 The condition of this theorem on the initial condition is satisfied by all random variables whose moments are dominated by the moments of the square of a Gaussian random variable. This is true for example for random variables having compact support.

Proof In order to make the proof we need some auxiliary results and remarks. Proposition 1.3.2 gives in this particular case, the following result : for X_t a random variable with probability density $p(x,t)$ and f a smooth function, we have

$$\frac{d}{dt}E(f(X_t)) = E\left(\frac{X_t + \tilde{X}_t}{\tilde{X}_t}(f(X_t + \tilde{X}_t) - f(X_t))\right), \tag{1.66}$$

where $\tilde{X}_t$ denotes an independent copy of the random variable X_t.
To get existence we used functions f of the form $e^{-\lambda x}$. Another way to treat the problem is to find the moments of X_t, by using a recurrence formula.
Let us apply formula (1.66) for power functions $f_k(x) = x^k$. Denote by

$$m_k(t) = \int_0^{\infty} x^k \overset{+}{p}(x,t)\, dx. \tag{1.67}$$

For $f_1(x) = x$, (1.66) writes

$$m_1'(t) = 2m_1(t),$$

that is

$$m_1(t) = A_1 e^{2t} \text{ where } A_1 = m_1(0).$$

Furthermore, by using (1.66) for $f_k(x) = x^k$, we obtain

$$m_k'(t) = (k+1)m_k(t) + \sum_{j=1}^{k-1} \binom{k+1}{j} m_j(t) m_{k-j}(t). \tag{1.68}$$

Simple computation for first order moments gives

$$\begin{aligned} m_1(t) &= A_1 e^{2t}, & A_1 &= m_1(0), \\ m_2(t) &= 3{A_1}^2 e^{4t} + A_2 e^{3t}, & A_2 &= m_2(0) - 3A_1^2, \\ m_3(t) &= 15{A_1}^3 e^{6t} + 10A_1A_2e^{5t} + A_3e^{4t}, & A_3 &= m_3(0) - 15A_1^3 - 10A_1A_2. \end{aligned}$$

These results and formula (1.68) allow us to obtain the general form of m_k.

Lemma 1.3.22 *For all integer k, $m_k(\log t)$ is a polynomial on t of degree $2k$, with valuation greater or equal to $k+1$ and dominating coefficient : $\frac{(2k)!}{2^k k!}A_1^k$, where $A_1 = m_1(0)$.*

Proof We have already test this result for $k = 1$ (and even for $k = 0$). Suppose it valid until $k-1$ and solve the equation (1.68) by using the method of constant variation. We obtain

$$m_k(t) = e^{(k+1)t}\Gamma(t), \tag{1.69}$$

where $\Gamma(t)$ is such that :

$$e^{(k+1)t}\Gamma'(t) = \sum_{j=1}^{k-1} \binom{k+1}{j} m_j(t) m_{k-j}(t). \tag{1.70}$$

By using the recurrence hypothesis, $\Gamma'(\log t)$ is a polynomial of degree $k-1$ and dominating coefficient $\frac{{A_1}^k}{2^k} \sum_{j=1}^{k-1} \binom{k+1}{j} \frac{(2j)!(2k-2j)!}{j!(k-j)!}$.

We can prove also the following result (see lemma 1.4.1, given in the appendix) :

$$\sum_{j=1}^{k-1} \binom{k+1}{j} \frac{(2j)!(2k-2j)!}{j!(k-j)!} = (k-1)\frac{(2k)!}{k!}.$$

We deduce that $\Gamma(\log t)$ is a polynomial on t of degree $k-1$ and dominating coefficient $\frac{(2k)!}{2^k k!} A_1^k$. This ends the proof. □

Let us write down the Laplace transform of the random variable X_t.

$$\psi(\lambda,t) = \sum_{k\geq 0} \frac{(-1)^k \lambda^k}{k!} m_k(t). \tag{1.71}$$

Evaluate this function at $(\lambda a^2, \log \frac{t}{a})$.

$$\psi(\lambda a^2, \log \frac{t}{a}) = \sum_{k\geq 0} \frac{(-1)^k \lambda^k a^{2k}}{k!} m_k(\log \frac{t}{a}). \tag{1.72}$$

By using the result in lemma 1.3.22 we can remark that, each term of this series satisfies

$$\lim_{a\to 0} \left((\lambda a^2)^k m_k(\log \frac{t}{a}) \right) = {A_1}^k \frac{(2k)!}{2^k k!} \lambda^k t^{2k}. \tag{1.73}$$

We shall prove that we can consider the limit as a goes to 0 in the sum of (1.72). In order to obtain this convergence, we need the following result :

Lemma 1.3.23 *For any initial condition of the Smoluchowski's coagulation equation (SC+), such that $m_k(0) \leq C^k \frac{(2k)!}{k!}$ for fixed C and for all k, we have*

$$m_k(t) \leq C^k \frac{(2k)!}{k!} e^{2kt}, \quad \forall t \geq 0.$$

Proof We shall use the same method as the one used in order to obtain the form of m_k. We get from (1.69) and (1.70)

$$\Gamma'(t) \leq \frac{C^k}{k-1} \frac{(2k)!}{k!} e^{(k-1)t}. \tag{1.74}$$

We have also

$$\Gamma(t) = \Gamma(0) + \int_0^t \Gamma'(s)ds,$$

which yields

$$\Gamma(t) \leq C^k \frac{(2k)!}{k!} e^{(k-1)t} + \Gamma(0) - \frac{(2k)!}{k!} C^k,$$

and $\Gamma(0) = m_k(0)$. The result is then proved. □

Lemma 1.3.23 insures the normal convergence, with respect to a, of the series appearing in the expression of $\psi(\lambda a^2, \log \frac{t}{a})$ (see (1.72)), as soon as $\frac{4\lambda t^2}{C} \leq 1$. We can now conclude the proof of the theorem 1.3.20.

By using the previous normal convergence, we easily deduce

$$\lim_{a \to 0} \psi(\lambda a^2, \log \frac{t}{a}) = \sum_{k \geq 0} \frac{(2k)!}{(k!)^2} \frac{(-1)^k {A_1}^k \lambda^k t^{2k}}{2^k} = \frac{1}{\sqrt{1 + 2A_1 \lambda t^2}}. \tag{1.75}$$

This last limit is also the Laplace transform of the square of a one-dimensional Bessel process starting from the origin. □

As in the previous section, this convergence allows us to find a particular solution of $(SC+)$:

$$n_1(x,t) = \frac{1}{\sqrt{2\pi}} e^{-t} x^{-\frac{3}{2}} e^{-\frac{x}{2} e^{-2t}}.$$

We can obtain other solutions by scaling, more precisely, consider

$$n_{A_1}(x,t) = \frac{1}{A_1^2} n_1 \left(\frac{x}{A_1}, t \right)$$

so

$$n_{A_1}(x,t) = \frac{1}{\sqrt{A_1}} \frac{1}{\sqrt{2\pi}} e^{-t} x^{-\frac{3}{2}} e^{-\frac{x}{2A_1} e^{-2t}}$$

is a solution of $(SC+)$.
Let us also express the similar result for the solutions of $(SC*)$. By using theorem 1.3.9 we have

Theorem 1.3.24 *Let $\overset{*}{n}(x,t)$ be a solution of the Smoluchowski's coagulation equation with multiplicative kernel $(SC*)$. Let*

$$\overset{*}{p}(x,t) = x^2 \, (T - t) \overset{*}{n}(x,t),$$

and denote by Y_t a random variable with probability density $\overset{*}{p}(x,t)$.
We suppose also that $m_k(0) = \int_0^\infty x^k \overset{*}{p}(x,0)\, dx \leq C^k \frac{(2k)!}{k!}$, where

$$T = \left(\int_0^\infty x^2 \overset{*}{n}(x,0)\, dx \right)^{-1}.$$

Then, by fixing t, we have:

$$a^2 Y_{T(1-\frac{t}{a})} \xrightarrow[a\to 0]{(d)} R_{A_1 t^2}, \tag{1.76}$$

where

$$A_1 = \int_0^\infty x\overset{*}{p}(x,0)dx = T\int_0^\infty x^3 \overset{*}{n}(x,0)\,dx$$

and R_t is the square of a one-dimensional Bessel process, starting at the origin.

We conclude our study by making two remarks:

Remark 1.3.25 We mention that Van Dongen and Ernst ([65]) analysed the long time behaviour in the discrete case. Part of their approach is rather intuitive. See also the section 2.4 of Aldous' paper ([1]) for a presentation on self similarity results for the solution of Smoluchowski's coagulation equation. We emphasise that our previous theorem proves that, any given solution converges, after employing a good scaling, to a self-similar solution of the equation. At our knowledge this result, stated on the previous form, is new.

Remark 1.3.26 The preceding normalisation theorems are true for fix t. It could be interesting and probably very difficult to obtain them in terms of processes.

Let us write in the following schema the solutions we got for the continuous case (by using Laplace transforms):

$K(x,y)$	$n(x,t)$	t
1	$\frac{4}{t^2}\exp\left(\frac{-2x}{t}\right)$	$0 < t < \infty$
$x+y$	$\frac{1}{\sqrt{2\pi}}e^{-t}\exp\left(-e^{-2t}\frac{x}{2}\right)\frac{1}{x^{\frac{3}{2}}}$	$-\infty < t < \infty$
xy	$\frac{1}{\sqrt{2\pi}}\frac{1}{x^{\frac{5}{2}}}\exp\left(-\frac{t^2 x}{2}\right)$	$-\infty < t < 0$

1.4 Appendix

We shall prove now the result we used in the proof of lemma 1.3.22.

Lemma 1.4.1 *For any integer $k \geq 2$, we have the following identity :*

$$\sum_{j=1}^{k-1} \binom{k+1}{j} \frac{(2j)!(2k-2j)!}{j!(k-j)!} = (k-1)\frac{(2k)!}{k!}. \tag{1.77}$$

Proof
This result can be obtained by recurrence. We present here a probabilistic proof, communicated by P. Chassaing.
The equality (1.4.1) can be written in the equivalent form

$$\binom{2k+1}{k} = \sum_{j=0}^{k} \binom{2j}{j} \frac{(2k-2j)!}{(k-j)!(k+1-j)!}. \tag{1.78}$$

Let us consider equiprobable paths $\omega = (S_0(\omega), \cdots, S_{2k+1}(\omega))$ such that $S_0 = 0$, $S_{n+1} = S_n \pm 1$. There are $\binom{2k+1}{k}$ such paths of length $2k+1$ having $S_{2k+1}(\omega) = 1$ (k descents in $2k+1$ steps). For each path ω such that $S_{2k+1}(\omega) = 1$, we note $T = \max\{0 \leq n \leq 2k / S_n(\omega) = 0\}$ the last instant where the path is 0 (this place exists and T is even). The number of paths for which $T = 2j$, equals the product of the number of paths which go from 0 to 0 in $2j$ steps (they are $\binom{2j}{j}$), with the number of paths having length $2k-2j+1$ which go from 0 to 1, without any 0 value (they are as much as Bernoulli excursions of length $2(k-j+1)$, i.e. C_{k-j} where C_n denotes the n^{th} Catalan's number).

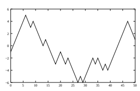

FIG. 1.2 – *Path description*

Let us recall the value for the Catalan's number :

$$C_n = \frac{(2n)!}{n!(n+1)!}.$$

Doing the sum over all possible values of j yields the identity (1.78). This ends the proof of the lemma. □

2

A generalization of the Connection Between the Additive and Multiplicative Solutions for the Smoluchowski's Coagulation Equation

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Abstract This study will reformulate part of our previous results ([10]) obtained for the Smoluchowski's coagulation equation for the additive, multiplicative and constant kernels for fixed time, in terms of measures. In order to obtain this, we use a construction introduced by Norris ([52]).
Key Words : Smoluchowski's coagulation equation, stochastic processes, partial differential equations.
AMS 2000 subject classification : 60J80, 44A10

2.1 Introduction

Let us consider the coagulation phenomenon described as follows. Consider a large system of particles and allow only the coagulation of particles in pairs. For physical reasons, the rate at which pairs of particles aggregate depends on some positive parameter associated with each particles, such as mass or size.
We can write down the continuous Smoluchowski's coagulation equation in the fol-

lowing form :

$$\begin{cases} \dfrac{\partial}{\partial t} n(x,t) = \dfrac{1}{2} \displaystyle\int_0^x K(y,x-y) n(y,t) n(x-y,t) dy \\ \qquad\qquad\qquad\qquad -n(x,t) \displaystyle\int_0^\infty K(x,y) n(y,t) dy \\ n(x,0) = n_0(x). \end{cases} \tag{SC}$$

where $K : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a symmetric kernel and $n(x,t)$ denotes the average number of particles with mass $[x, x + dx]$ per unit volume.
In the equation (SC) the first term on the right corresponds to the creation of particles of mass x while the second stands for the disappearance of particles of mass x after coagulation with other particles.

2.1.1 Norris results and construction

We shall present here the construction and some of results in Norris ([52]).
Let $K : (\mathbb{R}_+^*)^2 \to \mathbb{R}_+$ be a symmetric measurable function and $\mathcal{M}$ the space of signed Radon measures on $\mathbb{R}_+^*$. Denote by $\mathcal{M}^+$ the set of non negative measures on $\mathcal{M}$. Let $\mu \in \mathcal{M}^+$ be such that, for every compact set $B \subseteq \mathbb{R}_+^*$:

$$\int_B \int_0^\infty K(x,y) \mu(dx) \mu(dy) < \infty.$$

For such μ we consider the following weak form of the Smoluchowski's coagulation equation :

$$\mu_t = \mu_0 + \int_0^t L(\mu_s) ds, \tag{SCW}$$

where $L(\mu) \in \mathcal{M}$ is defined by the relation

$$< f, L(\mu) > = \frac{1}{2} \int_0^\infty \int_0^\infty (f(x+y) - f(x) - f(y)) K(x,y) \mu(dx) \mu(dy), \tag{2.1}$$

for all bounded measurable functions f of compact support.

Definition 2.1.1 *We call* <u>*local solution of (SCW)*</u> *any map* $t \mapsto \mu_t : [0,T) \mapsto \mathcal{M}^+$ *such that :*
(i) $\displaystyle\int_0^\infty x \mathbb{1}_{\{x \le 1\}} \mu_0(dx) < \infty$
(ii) For all $B \subseteq \mathbb{R}_+^*$ *compact, the map* $t \mapsto \mu_t(B) : [0,T) \mapsto [0,\infty)$ *is measurable.*
(iii) For all $t < T$ *and* $B \subseteq \mathbb{R}_+^*$ *compact*

$$\int_0^t \int_B \int_0^\infty K(x,y) \mu_s(dx) \mu_s(dy) ds < \infty.$$

(iv) For all measurable function f *with compact support and for* $f(x) = x\mathbb{1}_{\{x \le 1\}}$ *and for all* $t < T$

$$\int_0^\infty f(x) \mu_t(dx) = \int_0^\infty f(x) \mu_0(dx)$$

$$+\frac{1}{2}\int_0^t ds \int_0^\infty \int_0^\infty (f(x+y)-f(x)-f(y))K(x,y)\mu_s(dx)\mu_s(dy).$$

For $T=\infty$ we have a <u>*solution*</u>.
We make also the following hypothesis on K

$$K(x,y) \leq \varphi(x)\varphi(y), \quad \forall x,y \in \mathbb{R}_+^*, \tag{2.2}$$

where $\varphi : \mathbb{R}_+^* \mapsto \mathbb{R}_+^*$ is a continuous sub-linear function (that is $\varphi(\lambda x) \leq \lambda\varphi(x)$, for all $x \in \mathbb{R}_+^*$ and $\lambda \geq 1$).
Further hypothesis

$$\int_0^\infty \varphi(x)\mu_0(dx) < \infty. \tag{2.3}$$

Any local solution such that

$$\int_0^t \int_0^\infty \varphi^2(x)\mu_s(dx)ds < \infty, \tag{2.4}$$

is called <u>*strong solution*</u>.
Under assumptions (2.2) and (2.3), Norris ([52]) proved that, starting from μ_0, we have uniqueness of a strong solution.
If $\varphi(x) \geq \varepsilon x$ for all x and some $\varepsilon > 0$, then any strong solution is conservative.
If $\int_0^\infty \varphi^2(x)\mu_0(dx) < \infty$ then there exists a unique maximal strong solution $(\mu_t)_{t \leq \xi(\mu_0)}$ where $\xi(\mu_0) \geq \left(\int_0^\infty \varphi^2(x)\mu_0(dx)\right)^{-1}$.
Furthermore, if φ^2 is sub-linear or $K(x,y) \leq \varphi(x)+\varphi(y)$, for all $x, y \in \mathbb{R}_+^*$ then $\xi(\mu_0) = \infty$.

Markov process associated with (SCW)
We shall introduce now a particle system connected to the Smoluchowski's coagulation equation.
Let X_0 be a finite, integer valued measure on $\mathbb{R}_+^*$

$$X_0 = \sum_{i=1}^m \delta_{x_i},$$

for some $x_1, \ldots, x_m \in \mathbb{R}_+^*$. X_0 is a system of m particles labelled by their masses.
We construct a Markov process $(X_t)_{t\geq 0}$ of finite, integer valued measures on $\mathbb{R}_+^*$ as follows: for each pair $i<j$ take T_{ij} an independent exponential random variable of parameter $K(x_i,x_j)$ and define

$$T = \min_{i<j} T_{ij}.$$

Set $X_t = X_0$ for $t < T$ and, if $T = T_{ij}$ set

$$X_T = X_0 + \delta_{x_i+x_j} - \delta_{x_i} - \delta_{x_j}$$

and then begin the construction afresh from X_T.
In the process described above each pair of particles $\{x_i, x_j\}$ coalesces at a rate $K(x_i, x_j)$ to form a new particle $x_i + x_j$. We call $(X_t)_{t\geq 0}$ a *stochastic coalescent with coagulation kernel K*.
Let d be a metric on $\mathcal{M}$ such that
(a) $d(\mu_n,\mu) \mapsto 0$ if and only if $\int_0^\infty f(x)\mu_n(dx) \mapsto \int_0^\infty f(x)\mu(dx)$, for all bounded continuous functions $f : \mathbb{R}_+^* \mapsto \mathbb{R}$.
(b) $d(\mu,\mu') \leq \|\mu - \mu'\|$, for all $\mu,\mu' \in \mathcal{M}$.
When we consider f of bounded support we obtain a weaker topology, also metrizable, and we denote by d_0 some compatible metric with $d_0 \leq d$.
We have the following weak convergence result :

Theorem 2.1.2 *(see Norris ([52]), theorem 4.1)*
Let $K : \mathbb{R}_+^* \times \mathbb{R}_+^* \mapsto [0,\infty)$ be a symmetric continuous function and μ_0 a measure on $\mathbb{R}_+^*$. Assume that for $\varphi : \mathbb{R}_+^* \mapsto \mathbb{R}_+^*$ continuous and sublinear we have

$$K(x,y) \leq \varphi(x)\varphi(y), \quad \forall x,y \in \mathbb{R}_+^*$$

$$\lim_{(x,y)\to(\infty,\infty)} \frac{K(x,y)}{\varphi(x)\varphi(y)} = 0.$$

Assume also that $\int_0^\infty \varphi(x)\mu_0(dx) < \infty$.
Let $(X_t^n)_{t\geq 0}$ be a sequence of stochastic coalescents with coagulation kernel K. Set

$$\tilde{X}_t^n := \frac{1}{n} X_{\frac{t}{n}}^n$$

and suppose that

$$\lim_{n\to\infty} d_0(\varphi \tilde{X}_0^n, \varphi\mu_0) = 0$$

and that, for some constant $C < \infty$ and for all n

$$\int_0^\infty \varphi(x) d\tilde{X}_0^n \leq C.$$

Then the sequence of laws of $\varphi\tilde{X}^n$ on $D([0,\infty),(\mathcal{M},d_0))$ is tight. Moreover, for any weak limit point φX, almost surely, $(X_t)_{t\geq 0}$ is a solution of Smoluchowski's coagulation equation (SCW). In particular, this equation has at least one solution.

2.2 Connection between the additive and multiplicative case in terms of measures

Let us denote by $(SCW+)$ and $(SCW*)$ the weak form for the Smoluchowski's coagulation equation for $K(x,y) = x+y$ and $K(x,y) = xy$ respectively. The previous results from Norris ([52]) apply with $\varphi(x) = x$. We express in the following theorem the connection between the additive and the multiplicative kernels.

Theorem 2.2.1 *Let X_t be solution for the multiplicative equation with initial condition X_0. Then Y_t defined as*

$$dY_t(x) = xTe^{-t}dX_{T(1-e^{-t})}(x) \tag{2.5}$$

is solution for the additive equation (SCW+), where $T = \left(\int_0^\infty x^2 dX_0(x)\right)^{-1}$.

Proof We remark first of all that X_t and Y_t are obtained as weak limit points of the corresponding coalescents. As X_t is solution for $K(x,y) = xy$ we have, by using definition 2.1, that for any function f with compact support

$$\begin{aligned} \int_0^\infty f(x)dX_t(x) &= \int_0^\infty f(x)dX_0(x) \\ &+\frac{1}{2}\int_0^t ds \int_0^\infty \int_0^\infty (f(x+y) - f(x) - f(y))xydX_s(x)dX_s(y). \end{aligned} \tag{2.6}$$

We shall prove that we can write a similar formula for a function of the form $g(t,x) = \alpha(t)f(x)$. In order to make rigorous the calculus to follow, we have to prove that we can use time-dependent test functions. We can justify this by using a result due to Norris (see proposition 2.3. in [52]).
First remark that, for any smooth functions $\alpha(t), \beta(t)$ we have, after integration by parts

$$\alpha(t)\int_0^t \beta(s)ds = \int_0^t \alpha(s)\beta(s)ds + \int_0^t \alpha'(s)\int_0^s \beta(u)duds. \tag{2.7}$$

Let us also denote, in order to simplify notations

$$\Gamma_f(s) = \int_0^\infty \int_0^\infty (f(x+y) - f(x) - f(y))xydX_s(x)dX_s(y). \tag{2.8}$$

With this notation equation (2.6) writes

$$\int_0^\infty f(x)dX_t(x) = \int_0^\infty f(x)dX_0(x) + \frac{1}{2}\int_0^t \Gamma_f(s)ds. \tag{2.9}$$

Multiplying equality (2.9) by $\alpha(t)$ and using (2.7) one gets

$$\begin{aligned}
&\alpha(t)\int_0^\infty f(x)dX_t(x)\\
&=\alpha(t)\int_0^\infty f(x)dX_0(x)+\frac{1}{2}\alpha(t)\int_0^t \Gamma_f(s)ds\\
&=\alpha(t)\int_0^\infty f(x)dX_0(x)+\frac{1}{2}\int_0^t \alpha(s)\Gamma_f(s)ds+\frac{1}{2}\int_0^t \alpha'(s)\int_0^s \Gamma_f(u)duds\\
&=\alpha(t)\int_0^\infty f(x)dX_0(x)+\frac{1}{2}\int_0^t \alpha(s)\Gamma_f(s)ds\\
&\quad+\int_0^t \alpha'(s)ds\Big[\int_0^\infty f(x)dX_s(x)-\int_0^\infty f(x)dX_0(x)\Big]\\
&=\alpha(t)\int_0^\infty f(x)dX_0(x)+\frac{1}{2}\int_0^t \alpha(s)\Gamma_f(s)ds\\
&\quad+\int_0^t \alpha'(s)ds\int_0^\infty f(x)dX_s(x)-\int_0^\infty f(x)dX_0(x)(\alpha(t)-\alpha(0))\\
&=\alpha(0)\int_0^\infty f(x)dX_0(x)+\frac{1}{2}\int_0^t \alpha(s)\Gamma_f(s)ds+\int_0^t \alpha'(s)ds\int_0^\infty f(x)dX_s(x).
\end{aligned} \tag{2.10}$$

Recall that $g(t,x)=\alpha(t)f(x)$ so the last equality writes

$$\begin{aligned}
\int_0^\infty g(t,x)dX_t(x)&=\int_0^\infty g(0,x)dX_0(x)\\
&+\frac{1}{2}\int_0^t ds\int_0^\infty\int_0^\infty (g(s,x+y)-g(s,x)-g(s,y))xydX_s(x)dX_s(y)\\
&+\int_0^t ds\int_0^\infty \frac{\partial g}{\partial s}(s,x)dX_s(x).
\end{aligned} \tag{2.11}$$

Apply first the equality (2.11) for $g(t,x)=(T-t)f(x)$. This yields

$$\begin{aligned}
\int_0^\infty (T-t)f(x)dX_t(x)&=\int_0^\infty Tf(x)dX_0(x)\\
&+\frac{1}{2}\int_0^t (T-s)ds\int_0^\infty\int_0^\infty (f(x+y)-f(x)-f(y))xydX_s(x)dX_s(y)\\
&-\int_0^t ds\int_0^\infty f(x)dX_s(x).
\end{aligned} \tag{2.12}$$

Replacing t by $T(1-e^{-t})$ in (2.12) we get

$$\int_0^\infty Te^{-t}f(x)dX_{T(1-e^{-t})}(x) = \int_0^\infty Tf(x)dX_0(x)$$
$$+\frac{1}{2}\int_0^{T(1-e^{-t})}(T-s)ds\int_0^\infty\int_0^\infty (f(x+y)-f(x)-f(y))xydX_s(x)dX_s(y)$$
$$-\int_0^{T(1-e^{-t})}ds\int_0^\infty f(x)dX_s(x). \tag{2.13}$$

Make in the last two terms on the right side of (2.13) the change of variable $s = T(1-e^{-u})$. This gives $0 \le u \le t$ and

$$\int_0^\infty Te^{-t}f(x)dX_{T(1-e^{-t})}(x) = \int_0^\infty Tf(x)dX_0(x)$$
$$+\frac{T^2}{2}\int_0^t e^{-2u}du\int_0^\infty\int_0^\infty (f(x+y)-f(x)-f(y))xydX_{T(1-e^{-u})}(x)dX_{T(1-e^{-u})}(y)$$
$$-\int_0^t Te^{-u}du\int_0^\infty f(x)dX_{T(1-e^{-u})}(x). \tag{2.14}$$

Further, apply (2.14) to $f(x) = xh(x)$.

$$\int_0^\infty Te^{-t}xh(x)dX_{T(1-e^{-t})}(x) = \int_0^\infty Txh(x)dX_0(x)$$
$$+\frac{T^2}{2}\int_0^t e^{-2u}du\int_0^\infty\int_0^\infty [(x+y)h(x+y)-xh(x)-yh(y)]xydX_{T(1-e^{-u})}(x)dX_{T(1-e^{-u})}(y)$$
$$-\int_0^t Te^{-u}du\int_0^\infty xh(x)dX_{T(1-e^{-u})}(x). \tag{2.15}$$

Replace now $dY_t(x) = xTe^{-t}dX_{T(1-e^{-t})}(x)$ (as asked in the theorem) to obtain

$$\int_0^\infty h(x)dY_t(x) = \int_0^\infty h(x)dY_0(x)$$
$$+\frac{1}{2}\int_0^t ds \int_0^\infty \int_0^\infty (h(x+y) - h(x) - h(y))(x+y)dY_s(x)dY_s(y)$$
$$+\frac{1}{2}\int_0^t ds \int_0^\infty \int_0^\infty (xh(y) + yh(x))dY_s(x)dY_s(y) - \int_0^t ds \int_0^\infty h(x)dY_s(x) \quad (2.16)$$
$$= \int_0^\infty h(x)dY_0(x) + \int_0^t ds \int_0^\infty h(x)dY_s(x)\Big(\int_0^\infty xdY_s(x) - 1\Big)$$
$$+\frac{1}{2}\int_0^t ds \int_0^\infty \int_0^\infty (h(x+y) - h(x) - h(y))(x+y)dY_s(x)dY_s(y).$$

Once we have proved that

$$\int_0^\infty xdY_t(x) = 1, \ \forall t < T \quad (2.17)$$

we can conclude from (2.16) that

$$\begin{aligned}\int_0^\infty h(x)dY_t(x) &= \int_0^\infty h(x)dY_0(x) \\ &+\frac{1}{2}\int_0^t ds \int_0^\infty \int_0^\infty (h(x+y) - h(x) - h(y))(x+y)dY_s(x)dY_s(y),\end{aligned} \quad (2.18)$$

which shows that Y_t defined by using X_t is a solution for the additive kernel. Let us prove that (2.17) holds. Evaluate (2.6) for $f(x) = x^2$

$$\int_0^\infty x^2dX_t(x) = \int_0^\infty x^2dX_0(x) + \int_0^t \Big(\int_0^\infty x^2dX_s(x)\Big)^2 ds. \quad (2.19)$$

Recalling that $T = \Big(\int_0^\infty x^2dX_0(x)\Big)^{-1}$, we get

$$m_2^*(t) := \int_0^\infty x^2dX_t(x) = \frac{1}{T-t}, \quad \forall t < T. \quad (2.20)$$

Introducing (2.20) into (2.17) yields

$$\begin{aligned}\int_0^\infty xdY_t(x) &= Te^{-t}\int_0^\infty x^2dX_{T(1-e^{-t})}(x) \\ &= m_2^*(T(1-e^{-t}))Te^{-t} \\ &= 1.\end{aligned} \quad (2.21)$$

This ends the proof of the theorem 2.2.1.
We can refer to the paper of Norris ([53]) for much more on the multiplicative kernel.
□

There is a similar result once we know the additive solution. More precisely

Theorem 2.2.2 *Let* Y_t *be solution of the additive equation with initial condition* Y_0*. Then* X_t *given by*

$$dX_t(x) = \frac{1}{1-T}\frac{1}{x}dY_{-\ln(1-\frac{t}{T})}(x), \quad \forall t < T. \tag{2.22}$$

is solution for the multiplicative equation $(SCW*)$*, where* $T = \int_0^\infty dY_0(x)$.

3

A pure jump Markov process associated with Smoluchowski's coagulation equation

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Abstract

The Smoluchowski coagulation equation models the evolution of the density $n(x,t)$ of the particles of size (or mass) x at the instant $t \geq 0$ for a system in which a coalescence phenomenon occurs. Two versions of this equation exist : the case of discrete sizes (when $x \in \mathbb{N}^*$) and the case of continuous sizes (when $x \in \mathbb{R}^*$).

The aim of the present paper is to construct a stochastic process, whose law is the solution of the Smoluchowski's coagulation equation. This approach is at our knowledge the first in this direction, in that, for the first time the solution of Smoluchowski's coagulation equation is obtained as the law of a stochastic process.

We first introduce a modified equation, dealing with the evolution of the repartition $Q_t(dx)$ of the mass in the system. The advantage we take on this is that we can do an unified study for both continuous and discrete models.

The integro-partial-differential equation satisfied by $\{Q_t\}_{t\geq 0}$ can be interpreted as the evolution equation of the time marginals of a Markov pure jump process. At this end we introduce a nonlinear Poisson driven stochastic differential equation related to the Smoluchowski equation in the following way : if X_t satisfies this stochastic equation, then the law of X_t satisfies the modified Smoluchowski equation. Existence, uniqueness, and pathwise behaviour of a solution to this S.D.E. are studied.

Key words: Smoluchowski's coagulation equations, nonlinear stochastic differential equations, Poisson measures.

MSC 2000: 60H30, 60K35, 60J75.

3.1 Introduction

Smoluchowski's coagulation equation governs various phenomenons as for example : polymerisation, aggregation of colloidal particles, formation of stars and planets, behaviour of fuel mixtures in engines etc.
We describe this equation as modelling the polymerisation phenomenon.
For $k \in \mathbb{N}^*$, let P_k denote a polymer of mass k, that is a set of k identical particles (monomers). As time advances, the polymers evolve and, if they are sufficiently close, there is some chance that they merge into a single polymer whose mass equals the sum of the two polymers' masses which take part in this binary reaction. We admit here only binary reactions.
Denote by $n(k,t)$ the average number of polymers of mass k per unit volume, at time t so $k\,n(k,t)$ stands for the part of mass consisting on polymers of length k, per unit volume. The coalescence phenomenon of a polymer of mass k with a polymer of mass j, can be written formally as $P_k + P_j \longrightarrow P_{k+j}$, and is proportional to $n(k,t)\,n(j,t)$ with a proportionality constant $K(k,j)$, called coalescence kernel.
Throughout this paper, time t is always continuous, discrete and continuous refer to polymers' masses.
Hereafter (discrete and continuous case), the coagulation kernel K will satisfy the following hypothesis : K is positive (i.e., $K : (\mathbb{N}^*)^2$ or $(\mathbb{R}_+)^2 \to \mathbb{R}_+$) and symmetric (i.e., $K(i,j) = K(j,i)$).

The Smoluchowski coagulation equation, in the discrete case, is the equation on $n(k,t)$, for $k \in \mathbb{N}^*$. It writes :

$$\begin{cases} \dfrac{d}{dt}n(k,t) = \dfrac{1}{2}\displaystyle\sum_{j=1}^{k-1} K(j,k-j)n(j,t)n(k-j,t) \\ \qquad\qquad\qquad -n(k,t)\displaystyle\sum_{j=1}^{\infty} K(j,k)n(j,t) \\ n(k,0) = n_0(k). \end{cases} \qquad (SD)$$

This system describes a non linear evolution equation of infinite dimension, with initial condition $(n_0(k))_{k\geq 1}$. In the first line of (SD), the first term on the right hand side describes the creation of polymers of mass k by coagulation of polymers of mass j and $k-j$. The coefficient $\frac{1}{2}$ is due to the fact that K is symmetric. The second term corresponds to the depletion of polymers of mass k after coalescence with other polymers.
The continuous analog of the equation (SD) can be written naturally :

$$\begin{cases} \dfrac{\partial}{\partial t}n(x,t) = \dfrac{1}{2}\displaystyle\int_0^x K(y,x-y)n(y,t)n(x-y,t)dy \\ \qquad\qquad\qquad -n(x,t)\displaystyle\int_0^{\infty} K(x,y)n(y,t)dy \\ n(x,0) = n_0(x) \end{cases} \qquad (SC)$$

for all $x \in \mathbb{R}_+$.

We present briefly some recent results on the existence and uniqueness for the Smoluchowski coagulation equation, obtained by employing a probabilistic approach. These results furnish answers to some phenomenon that seems to be accepted as granted by the physicists.
A detailed survey on the present situation of the research on this equation is provided in Aldous [1].
In the discrete case few situations allow to conclude to the existence and uniqueness of the solution of (SD). If the kernel K satisfies :

$$K(i,j) \leq C(i+j), \quad i,j \geq 1. \tag{3.1}$$

and the initial condition is such that $\sum_{k=1}^{\infty} kn(k,0) < \infty$ then existence (see Ball and Carr [4]) and uniqueness (see Heilmann [33]) are known.
Jeon [36] approached the solution of a more general equation than (SD), in that, we have also the fragmentation of polymers, by a sequence of finite Markov chains. Jeon gave a general result about the gelification time T_{gel}, i.e. the first instant when particles of infinite mass appear. More precisely, if we have

$$K(i,j) \geq (ij)^{\alpha}, \text{ with } \alpha \in \left]\frac{1}{2},1\right[\tag{3.2}$$

and furthermore

$$\lim_{i+j\to\infty} \frac{K(i,j)}{ij} = 0 \tag{3.3}$$

then we have gelification in finite time ($T_{gel} < \infty$), for a large class of initial conditions.
For the continuous case, Aldous [1], states hypotheses which insure existence and uniqueness of the solution of the Smoluchowski coagulation equation. More precisely existence and uniqueness hold for (SC) if the kernel K satisfies, for all x and y in $\mathbb{R}_+$:

$$K(x,y) \leq C(1+x+y) \tag{3.4}$$

and the initial condition is such that

$$\int_0^{\infty} n(x,0)dx < \infty \text{ and } \int_0^{\infty} x^2 n(x,0)dx < \infty \tag{3.5}$$

and furthermore we have conservation of mass

$$\int_0^{\infty} xn(x,0)dx = 1. \tag{3.6}$$

Our conditions will be less restrictive because we don't need to impose

$$\int_0^{\infty} n(x,0)dx < \infty. \tag{3.7}$$

Norris [52], [53] obtains recently some new results on (SC) by generalising to the continuous case the results in Jeon [36]. Norris has two kinds of hypotheses :
either $\dfrac{K(x,y)}{xy} \to 0$ as $(x,y) \to \infty$
or $K(x,y) \leq \varphi(x)\varphi(y)$ where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is sub-linear and $\int_0^\infty \varphi^2(x)n(0,x)dx < \infty$.
Under one of this conditions he proved existence and uniqueness of the solution of (SC). The last hypothesis includes the case where $K(x,y)$ explodes as $x \mapsto 0$ or $y \mapsto 0$ and allows also to obtain uniqueness in some cases when the initial condition has no second, or even first, moment. These results are also interesting because one needs no local regularity on K.
Norris constructs a sequence of stochastic processes which converges to a deterministic limit which is a solution of (SC).
Deaconu and Tanré [10] furnish a probabilistic interpretation for the additive, multiplicative and constant kernels in both discrete and continuous cases. They find a duality between the additive and multiplicative solutions which permits to obtain the results for one of this solutions by those on the other one. They have also presented a « long time behaviour » for the solution.

Our approach of (SC) or (SD) is new and purely stochastic. We construct a pure jump stochastic process $(X_t)_{t\geq 0}$ whose law is the solution of the Smoluchowski coagulation equation in the following sense : in the discrete case, $P[X_t = k] = kn(k,t)$ for all $t \geq 0$ and all $k \in \mathbb{N}^*$, while in the continuous case, $P[X_t \in dx] = xn(x,t)dx$ for all $t \geq 0$. For each ω, $X_t(\omega)$ may be seen as the evolution of the size of one « mean » particle in the system.
The jump process satisfies a non linear Poisson driven stochastic differential equation.
This approach is strongly inspired by probabilistic works on the Boltzmann equation. The Boltzmann equation deals with the distribution of the speeds in a gas, and can be related to the Smoluchowski equation for two reasons : first, it concerns the evolution of the « density of particles of speed v at the instant t », while the Smoluchowski equation deals with the « density of particles of mass x at the instant t ».
Second, the phenomenon is discontinuous : in each case, a particle moves instantly from a mass x (or a speed v) to a new mass x' (or a speed v') after a coagulation (or a collision).

Let us first mention Tanaka [63], who first introduced a non linear jump process in order to study the Boltzmann equation of Maxwell molecules. Many new results about this equation has been obtained thanks to this approach, as regularity, positivity, and numerical approximation : Desvillettes, Graham and Méléard [11], [31], have initiated a new way for studying non linear P.D.E.s, by using recent probabilistic tools, such as Malliavin Calculus and propagation of chaos.
Very recently, Tanaka's approach has been extended, in Fournier, Méléard [25], [26], to the case of non Maxwell molecules, which is technically much less easy. We follow essentially here the approach of [25]. We will transpose it here in order to give a

probabilistic interpretation to the Smoluchowski equations.
The main fact that makes the Maxwell molecules easy is that the rate of collisions of a particle does not depend on its speed, which is not the case for non Maxwell molecules. In the Smoluchowski equation, the « rate of coagulation » of a particle depends on its size.
We get rid of this problem by using a sort of « reject » procedure: as in [25], there is, in our stochastic equation, an indicator function which allows to control the rate of coagulation.

Let us finally describe the plan of the present paper.
In Section 3.2, we introduce our notations, we state a modified Smoluchowski equation (MS) which allows to study equations (SC) and (SD) together. This equation (MS) describes the evolution of the repartition $Q_t(dx)$ (either discrete or continuous) of the sizes: for each t, Q_t is a probability measure on $\mathbb{R}_+^*$. Afterwards we relate (MS) to a nonlinear martingale problem (MP): for Q a solution to (MP), its time marginals Q_t satisfy the modified Smoluchowski equation (MS). We finally exhibit a non linear Poisson driven stochastic differential equation (SDE), which gives a pathwise representation of (MP). If X_t satisfies (SDE), then its law is a solution to (MP). Notice that X_t can be seen as the evolution of a particle chosen randomly in the system, which coagulates randomly with other particles also chosen randomly. In other words, X_t is the evolution of the mass of a « mean particle ». In Section 3.3, we state and prove an existence result for (SDE), under quite general assumptions. The pathwise properties of the solution to (SDE) are briefly discussed in Section 3.4. Section 3.5 deals with uniqueness results for (SDE). In Section 3.6, we study the case where $K(x,y) = xy$. This case drives to very simple computations, and the results we obtain are of course easy and satisfactory. The last section contains an appendix which includes some useful classical results.

In the sequel A and B stand for universal constants whose values may change from line to line.

A forthcoming paper will present a stochastic particle system associated with the process constructed in the present paper, which will permit to solve numerically the Smoluchowski's coagulation equation.

3.2 Framework

Our probabilistic approach is based on the following remark: there is conservation of mass in (SC) and (SD). This means in the discrete case that a solution $(n(k,t))_{t\geq 0, k\in\mathbb{N}^*}$ of (SD) will satisfy until a time $T_0 \leq \infty$,

$$\text{for all } t \in [0,T_0[, \quad \sum_{k\geq 1} k\, n(k,t) = 1 \tag{3.8}$$

and in the continuous one that a solution $(n(x,t))_{t\geq 0, x\in\mathbb{R}_+^*}$ of (SC) will satisfy until a time $T_0 \leq \infty$,

$$\text{for all } t \in [0,T_0[, \quad \int_0^\infty x\, n(x,t)dx = 1. \tag{3.9}$$

Thus, either in the discrete or continuous case, the quantity

$$Q_t(dx) = \sum_{k\geq 1} k\, n(k,t)\delta_k(dx) \quad \text{or} \quad Q_t(dx) = x\, n(x,t)\, dx \tag{3.10}$$

(where δ_k denotes the Dirac mass at k) is a probability measure on $\mathbb{R}_+$ for all $t \in [0,T_0[$

Let us first define the weak solution for (SC) (or (SD)).

Definition 3.2.1 *We say that $n(x,t)_{\{x,t\geq 0\}}$ is a weak solution of (SC) on $[0,T_0[$ if for all test function $\varphi \in C_0^1(\mathbb{R}_+)$ and all $t \in [0,T_0[$ we have*

$$\begin{aligned}\int_{\mathbb{R}_+} \varphi(x)n(x,t)\, dx &= \int_{\mathbb{R}_+} \varphi(x)n(x,0)\, dx \\ &+ \int_0^t ds \int_{\mathbb{R}_+} \varphi(x)\Big[\frac{1}{2}\int_0^x n(x-y,s)n(y,s)K(x-y,y)dy \\ &\qquad - \int_{\mathbb{R}_+} n(x,s)n(y,s)K(x,y)dy\Big]dx.\end{aligned} \tag{3.11}$$

For any t, $Q_t(dx)$ can be seen as the repartition of the mass of the particles at instant t. This leads us to define a modified Smoluchowski equation. We begin with some notations.

Notation 3.2.2 *1. We denote by $\mathcal{P}_1$ the set of probability measures Q on $\mathbb{R}_+$ such that*

$$Q\left(]0,\infty[\right) = 1 \quad ; \quad \int_{\mathbb{R}_+} xQ(dx) < \infty. \tag{3.12}$$

2. Let $Q_0 \in \mathcal{P}_1$. We denote by

$$H_{Q_0} = \overline{\left\{\sum_{i=1}^n x_i \,;\, x_i \in \text{Supp}\, Q_0 \,,\, n \in \mathbb{N}^*\right\}}^{\mathbb{R}_+}. \tag{3.13}$$

Notice that H_{Q_0} is a closed subset of $\mathbb{R}_+$ which contains the support of Q_0.

Since Q_0 is the repartition of the sizes of the particles in the initial system, H_{Q_0} simply represents the smallest closed subset of $\mathbb{R}_+$ in which the sizes of the particles will always take their values.

Definition 3.2.3 *Let Q_0 be a probability measure on $\mathbb{R}_+$ belonging to $\mathcal{P}_1$ and let $T_0 \leq \infty$. We will say that $(Q_t(dx))_{t\in[0,T_0[}$ is a weak solution to (MS) on $[0,T_0[$ with*

initial condition Q_0 *if:*
for all $t \in [0,T_0[$, Supp $Q_t \subset H_{Q_0}$ *and* $Q_t \in \mathcal{P}_1$, *and for all* $\varphi \in C_0^1(\mathbb{R}_+)$ *and all* $t \in [0,T_0[$,

$$\int_0^\infty \varphi(x)Q_t(dx) = \int_0^\infty \varphi(x)Q_0(dx)$$
$$+ \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} [\varphi(x+y) - \varphi(x)] \frac{K(x,y)}{y} Q_s(dy)Q_s(dx)ds. \qquad (3.14)$$

It is obvious the procedure to pass from (SC) to (MS). It suffices to multiply by $\varphi(x)$ and integrate over $\mathbb{R}_+$. This definition allows us to consider together both discrete and continuous cases. To make this assertion clear, let us state the following result:

Proposition 3.2.4 *Let* $(Q_t(dx))_{t\in[0,T_0[}$ *be a weak solution to* (MS), *with initial condition* $Q_0 \in \mathcal{P}_1$, *for some* $T_0 \leq \infty$.

1. *If* Supp $Q_0 \subset \mathbb{N}^*$, *then clearly* $H_{Q_0} \subset \mathbb{N}^*$. *Thus for all* $t \in [0,T_0[$, Supp $Q_t \subset \mathbb{N}^*$, *and we can write* Q_t *as:*
$$Q_t(dx) = \sum_{k\geq 1} \alpha_k(t)\, \delta_k(dx) \quad \textit{where} \quad \alpha_k(t) = Q_t(\{k\}). \qquad (3.15)$$
Then, the function $n(k,t) = \alpha_k(t)/k$ *is a solution to* (SD) *on* $[0,T_0[$, *with initial condition* $n_0(k) = \alpha_k(0)/k$
2. *Assume now that for all* $t \in [0,T_0[$, *the probability measure* Q_t *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}_+$. *We can thus write* $Q_0(dx) = f_0(x)\, dx$ *and for any* $t \in]0,T_0[$, $Q_t(dx) = f(x,t)\, dx$. *Then* $n(x,t) = f(x,t)/x$ *is a weak solution to* (SC) *on* $[0,T_0[$, *with initial condition* $n_0(x) = f_0(x)/x$.
3. *Other cases, as mixed cases, are contained in* (MS).

Notice that the assumption $Q_0 \in \mathcal{P}_1$ simply means that the initial condition to the Smoluchowski equation admits a moment of order 2: in the discrete case this writes $\sum_k k^2 n_0(k) < \infty$, while in the continuous case we have, $\int x^2 n_0(x)dx < \infty$.

Proof 1. Since $Q_t(dx) = \sum_{k\geq 1} \alpha_k(t)\, \delta_k(dx)$, with $\alpha_k(t) = k\, n(k,t)$, is a weak solution to (MS), we may apply (3.14) with $\varphi_k(x) \in C_c^1(\mathbb{R}_+^*)$ such that for some $k \geq 1$

$$\varphi_k(x) = \begin{cases} 0 \text{ if } x \notin \left[k - \frac{1}{2}, k + \frac{1}{2}\right] \\ \\ \dfrac{1}{k} \text{ if } x = k. \end{cases} \qquad (3.16)$$

We obtain:

$$\frac{\alpha_k(t)}{k} = \frac{\alpha_k(0)}{k} + \int_0^t \frac{1}{k} \sum_{i\geq 1} \alpha_i(s) \sum_{j\geq 1} \alpha_j(s) \left[\mathbb{1}_{\{i+j=k\}} - \mathbb{1}_{\{i=k\}}\right] \frac{K(i,j)}{j} ds \qquad (3.17)$$

and thus

$$\begin{aligned} n(k,t) &= n_0(k) + \int_0^t \Big[\sum_{i=1}^{k-1} \alpha_i(s) n(k-i,s) \frac{K(i,k-i)}{k} \\ &\qquad - \sum_{j\geq 1} n(k,s)n(j,s)K(k,j) \Big] ds \\ &= n_0(k) + \int_0^t \Big[\frac{1}{2} \sum_{i=1}^{k-1} n(i,s)n(k-i,s)K(i,k-i) \\ &\qquad - \sum_{j\geq 1} n(k,s)n(j,s)K(k,j) \Big] ds \end{aligned} \tag{3.18}$$

where the last equality comes from the facts that $\alpha_i(s) = in(i,s)$ and $K(i,j)$ is a symmetric kernel.

2. We now assume that $Q_t(dx) = f(x,t)\,dx$ for all $t \in [0,T_0[$; let $\varphi \in C_c^1(\mathbb{R}_+^*)$. Let $\psi(x) = \varphi(x)/x$. Applying (3.14) to ψ, we obtain $Q_0(dx) = f(x,0)dx = xn_0(x)dx$ and :

$$\begin{aligned} &\int_{\mathbb{R}_+} \varphi(x) n(x,t)\,dx = \int_{\mathbb{R}_+} \varphi(x) n_0(x)\,dx \\ &\quad + \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{\varphi(x+y)}{x+y} \frac{K(x,y)}{y} f(x,s) f(y,s)\,dxdyds \\ &\quad - \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{\varphi(x)}{x} \frac{K(x,y)}{y} f(x,s) f(y,s)\,dxdyds. \end{aligned} \tag{3.19}$$

Using a symmetry argument, we obtain :

$$\begin{aligned} I &= \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \frac{\varphi(x+y)}{x+y} \frac{K(x,y)}{y} f(x,s) f(y,s)\,dxdyds \\ &= \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \varphi(x+y) K(x,y) f(x,s) f(y,s) \\ &\qquad \left(\frac{1}{y(x+y)} + \frac{1}{x(x+y)} \right) dxdyds \\ &= \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \varphi(x+y) K(x,y) n(x,s) n(y,s) dxdyds. \end{aligned} \tag{3.20}$$

Using the substitution $x' = x + y$, $y' = y$, we obtain :

$$I = \frac{1}{2} \int_0^t \int_{\mathbb{R}_+} dx' \int_0^{x'} dy' \varphi(y') K(x'-y',y') n(x'-y',s) n(y',s) ds. \tag{3.21}$$

We have proved that for any $\varphi \in C_c^1(\mathbb{R}_+^*)$,

$$\begin{aligned} \int_{\mathbb{R}_+} \varphi(x) n(x,t)\, dx = & \int_{\mathbb{R}_+} \varphi(x) n(x,0)\, dx \\ & + \int_0^t ds \int_{\mathbb{R}_+} \varphi(x) \Big[\frac{1}{2} \int_0^x n(x-y,s) n(y,s) K(x-y,y) dy \\ & \qquad - \int_{\mathbb{R}_+} n(x,s) n(y,s) K(x,y) dy \Big]\, dx \end{aligned} \tag{3.22}$$

which is the definition of a weak solution for (SC). This ends the proof. □

Equation (MS) has to be understood as the evolution equation of the time marginals of a pure jump Markov process. In order to exploit this remark, we will associate to (MS) a martingale problem. We begin with some notations.

Notation 3.2.5 *Let $T_0 \leq \infty$ and $Q_0 \in \mathcal{P}_1$ be fixed. We denote by $\mathbb{D}^{\uparrow}([0,T_0[,\ H_{Q_0})$ the set of positive increasing càdlàg functions from $[0,T_0[$ to H_{Q_0}. We denote by $\mathcal{P}_1^{\uparrow}([0,T_0[,\ H_{Q_0})$ the set of probability measures Q on $\mathbb{D}^{\uparrow}([0,T_0[,\ H_{Q_0})$ such that*

$$Q\left(\left\{x \in \mathbb{D}^{\uparrow}([0,T_0[,\ H_{Q_0})\ ;\ x(0) > 0\right\}\right) = 1 \tag{3.23}$$

and for all $t < T_0$,

$$\int_{x \in \mathbb{D}^{\uparrow}([0,T_0[,\ H_{Q_0})} x(t) Q(dx) = \int_{x \in \mathbb{D}^{\uparrow}([0,T_0[,\ H_{Q_0})} \left(\sup_{s \in [0,t]} x(s) \right) Q(dx) < \infty. \tag{3.24}$$

The last equality comes naturally from the fact that $x(t)$ is increasing.

Definition 3.2.6 *Let $T_0 \leq \infty$, and $Q_0 \in \mathcal{P}_1$ be fixed. Consider $Q \in \mathcal{P}_1^{\uparrow}([0,T_0[,\ H_{Q_0})$. Let Z be the canonical process of $\mathbb{D}^{\uparrow}([0,T_0[,\ H_{Q_0})$. We say that Q is a solution to the martingale problem (MP) on $[0,T_0[$ if for all $\varphi \in C_b^1(\mathbb{R}_+)$ and $t \in [0,T_0[$,*

$$\varphi(Z_t) - \varphi(Z_0) - \int_0^t \int_{\mathbb{R}_+} [\varphi(Z_s + y) - \varphi(Z_s)] \frac{K(Z_s,y)}{y} Q_s(dy) ds \tag{3.25}$$

is a $Q-$martingale, where Q_s denotes the law of Z_s under Q.

Taking expectations in (3.25), we obtain the following remark :

Remark 3.2.7 *Let Q be a solution to the martingale problem (MP) on $[0,T_0[$. For $t \in [0,T_0[$, let Q_t be its time marginal. Then $(Q_t)_{t\in[0,T_0[}$ is a weak solution of (MS) with initial condition Q_0.*

We are now seeking for a pathwise representation of the martingale problem (MP). To this aim, let us introduce some notations. The main ideas of the following notations and definition are taken from Tanaka [63].

Notation 3.2.8
1. *We consider two probability spaces : $(\Omega,\mathcal{F},\mathbb{P})$ is an abstract space and $([0,1],\mathcal{B}[0,1],d\alpha)$ is an auxiliary space (here, $d\alpha$ denotes the Lebesgue measure). In order to avoid confusion, the expectation on $[0,1]$ will be denoted E_α, the laws $\mathcal{L}_\alpha$, the processes will be said to be α-processes, etc.*
2. *Let $T_0 \le \infty$ and $Q_0 \in \mathcal{P}_1$ be fixed. An increasing positive càdlàg process $(X_t(\omega))_{t\in[0,T_0[}$ is said to belong to $L_1^{T_0,\uparrow}(\ H_{Q_0})$ if its law belongs to $\mathcal{P}_1^{\uparrow}([0,T_0[,\ H_{Q_0})$. In the same way, an increasing positive càdlàg α-process $(\tilde{X}_t(x))_{t\in[0,T_0[}$ is said to belong to $L_1^{T_0,\uparrow}(\ H_{Q_0})$-$\alpha$ if its α-law belongs to $\mathcal{P}_1^{\uparrow}([0,T_0[,\ H_{Q_0})$.*

Definition 3.2.9 *Let $T_0 \le \infty$ and $Q_0 \in \mathcal{P}_1$ be fixed. We say that $(X_0,X,\tilde{X},N)$ is a solution to the problem (SDE) on $[0,T_0[$ if :*

1. *$X_0 : \Omega \to \mathbb{R}_+$ is a random variable whose law is Q_0.*
2. *$X_t(\omega) : [0,T_0[\times\Omega \to \mathbb{R}_+$ is a $L_1^{T_0,\uparrow}(\ H_{Q_0})$-process*
3. *$\tilde{X}_t(\alpha) : [0,T_0[\times[0,1] \to \mathbb{R}_+$ is a $L_1^{T_0,\uparrow}(\ H_{Q_0})$-$\alpha$-process.*
4. *$N(\omega,dt,d\alpha,dz)$ is a Poisson measure on $[0,T_0[\times[0,1]\times\mathbb{R}_+$ with intensity measure $dt\,d\alpha\,dz$ and is independent of X_0.*
5. *X and $\tilde{X}$ have the same law on their respective probability spaces : $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ (this equality holds in $\mathcal{P}_1^{\uparrow}([0,T_0[,\ H_{Q_0})$).*
6. *Finally, the following S.D.E. is satisfied on $[0,T_0[$:*

$$X_t = X_0 + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{z \le \frac{K(X_{s-},\tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds,d\alpha,dz). \tag{3.26}$$

The motivation of this definition is the following :

Proposition 3.2.10 *Let $(X_0,X,\tilde{X},N)$ be a solution to (SDE) on $[0,T_0[$. Then the law $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ satisfies the martingale problem (MP) on $[0,T_0[$ with initial condition $Q_0 = \mathcal{L}(X_0)$. Hence $\{\mathcal{L}(X_t)\}_{t\in[0,T_0[}$ is a solution to the modified Smoluchowski equation (MS) with initial condition Q_0.*

Before proving rigorously this result, we explain its main intuition : why is it natural to choose (X_t) satisfying (SDE), in order to obtain a stochastic process whose law satisfies the modified Smoluchowski equation (MS)?

We wish that the law Q_t of X_t describes the evolution of the repartition particles' masses in the system. A natural way to do this is to choose one particle randomly, and to use a random (but natural) coagulation dynamic. Thus, X_t should be understood as the evolution of the size of a sort of « mean » particle. Of course, X_0 has to follow the initial distribution Q_0. Then, at some random instants, which are typically Poissonian instants (for Markovian reasons), coalescence phenomenons occur. Let τ be one of these instants. We choose another particle, randomly, and we denote by $\tilde{X}_\tau(\alpha)$ its size. Then we describe the coagulation as $X_\tau = X_{\tau-} + \tilde{X}_\tau(\alpha)$. The indicator function in (3.26) allows to control the frequency of the coagulations. Thus, from a time-evolution point of view, $X_.$ mimics randomly the evolution of the size of one particle, thus its law is given by the (deterministic) « true » repartition of the sizes in the system at the instant t, which is exactly the solution of (MS). For fixed t, X_t may be understood as a random variable representing the following experience : we choose randomly one particle in the system, at the instant t, (according to an « uniform law »), and we denote by X_t its size.

Let us now prove Proposition 3.2.10.

Proof Let φ be a $C^1_c(\mathbb{R}_+)$ function. Then for all $t \in [0,T_0[$,

$$\begin{aligned}\varphi(X_t) &= \varphi(X_0) + \sum_{s\le t} [\varphi(X_s) - \varphi(X_{s-})] \qquad (3.27)\\ &= \varphi(X_0) + \int_0^t \int_0^1 \int_0^\infty \left[\varphi\left(X_{s-} + \tilde{X}_{s-}(\alpha)\mathbb{1}_{\left\{z \le \frac{K(X_{s-},\tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}}\right) - \varphi(X_{s-})\right] \\ &\qquad N(ds,d\alpha,dz)\\ &= \varphi(X_0) + \int_0^t \int_0^1 \int_0^\infty \left[\varphi\left(X_{s-} + \tilde{X}_{s-}(\alpha)\right) - \varphi(X_{s-})\right] \mathbb{1}_{\left\{z \le \frac{K(X_{s-},\tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} \\ &\qquad N(ds,d\alpha,dz).\end{aligned}$$

Hence :

$$\begin{aligned}M_t^\varphi &= \varphi(X_t) - \varphi(X_0) \qquad (3.28)\\ &- \int_0^t \int_0^1 \int_0^\infty \left[\varphi\left(X_s + \tilde{X}_s(\alpha)\right) - \varphi(X_s)\right] \mathbb{1}_{\left\{z \le \frac{K(X_s,\tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)}\right\}} dz\, d\alpha\, ds\end{aligned}$$

can be written as a stochastic integral with respect to the compensated Poisson measure, and thus is a martingale. But

$$\begin{aligned} M_t^{\varphi} &= \varphi(X_t) - \varphi(X_0) \qquad (3.29) \\ &\quad - \int_0^t E_\alpha \left[\Big(\varphi(X_s + \tilde{X}_s(\alpha)) - \varphi(X_s) \Big) \frac{K(X_s,\tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)} \right] ds \\ &= \varphi(X_t) - \varphi(X_0) - \int_0^t \int_{\mathbb{R}_+} [\varphi(X_s + y) - \varphi(y)] \frac{K(X_s,y)}{y} Q_s(dy) ds \end{aligned}$$

where $Q_s = \mathcal{L}_\alpha(\tilde{X}_s) = \mathcal{L}(X_s)$. We have proved that $\mathcal{L}(X)$ satisfies (MP) on $[0,T_0[$. □

Let us now state hypotheses which will allow to prove existence results for (SDE). In the sequel, we will always suppose that the coagulation kernel K satisfies the following hypothesis :

(H_β) : The initial condition Q_0 belongs to $\mathcal{P}_1$. The symmetric kernel $K : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ is locally Lipschitz continuous on $(\ H_{Q_0})^2$, and satisfies, for some constant $C < \infty$,

$$K(x,y) \leq C(1 + x + y + x^\beta y^\beta). \qquad (3.30)$$

Two different cases will appear according with $\beta = 1/2$ or $\beta = 1$. We will always prove the results for the case $\beta = 1$ the other one being similar and easier to treat. Let us also remark that all results for $\beta = 1/2$ apply also for $0 \leq \beta \leq 1/2$ and similarly the ones for $\beta = 1$ are true for $1/2 < \beta \leq 1$.
Notice that in the discrete case, H_{Q_0} is contained in $\mathbb{N}^*$, so that we don't need the local Lipschitz continuity condition.

3.3 Existence results for (SDE)

The aim of this section is to prove the following result.

Theorem 3.3.1 *Let $Q_0 \in \mathcal{P}_1$ satisfy $\int x^2 Q_0(dx) < \infty$. Assume (H_β).*

1. *If $\beta = 1/2$ then there exists a solution $(X_0,X,\tilde{X},N)$ to (SDE), on $[0,T_0[$, where $T_0 = \infty$.*
2. *If $\beta = 1$ then there exists a solution $(X_0,X,\tilde{X},N)$ to (SDE), on $[0,T_0[$, where $T_0 = 1/C(1 + E(X_0))$.*

Remark 3.3.2 *From now on we state that under (H_β) if $\beta = 1/2$ then $T_0 = \infty$ and if $\beta = 1$ we put $T_0 = 1/C(1 + E(X_0))$.*

Thus, under (H_β), if $\beta = 1/2$ we obtain an existence result on $[0,\infty[$. This is not the case if $\beta = 1$, but this is not a limitation. It is classical that for $\beta = 1$, there may be a gelification time : for example, in the discrete case, Jeon has proved in [36] that if $K(x,y) \geq x^\beta y^\beta$ for some $\beta \in]1/2,1[$, then a solution $n(k,t)$ to (SD) will satisfy

$$T_{gel} = \inf \left\{ t \geq 0 \; ; \; \sum_{k\geq 1} k^2 n(k,t) = \infty \right\} < \infty \tag{3.31}$$

which writes, with our notations,

$$T_{gel} = \inf \left\{ t \geq 0 \; ; \; E(X_t) = \infty \right\} < \infty. \tag{3.32}$$

It is thus clear that an existence result on $[0,\infty[$ can not be proved under the general assumption (H_β) for $\beta = 1$.
Finally, notice that for $\beta = 1$, $T_0 = 1/C(1 + E(X_0))$ is not the exact gelification time, except if $K(x,y) = C(1 + x + y + xy)$: since we only assume an upper bound on K, we are able only to prove an existence result for (SDE) on $[0,T_0[$, for some $T_0 \leq T_{gel}$. We however will give exact gelification times corresponding to a class of coagulation kernels for which explicit computations are easy. In such cases, our existence result will easily extend to $[0,T_{gel}[$.

Technically, Theorem 3.3.1 is not easy to prove, because the coefficients of (SDE) are not globally Lipschitz continuous. Due to the nonlinearity, a direct construction is difficult. Thus, in a first proposition, we prove a result, which combined with Proposition 3.2.10 shows that the existence (resp. uniqueness in law) for (SDE) is equivalent to existence (resp. uniqueness) for (MP). It will thus be sufficient to prove an existence result for (MP).
Next, we use a cutoff procedure, which transforms the coefficients of our equation globally Lipschitz continuous : we obtain the existence of a solution X^ε to a cutoff equation $(SDE)_\varepsilon$. Then we prove tightness and uniform integrability results, which allow to prove that the family $\mathcal{L}(X^\varepsilon)$ has limiting points, and that these limiting points satisfy (MP).

As said previously, we begin with a proposition, which, combined with Proposition 3.2.10, shows a sort of equivalence between (MP) and (SDE).

Proposition 3.3.3 *Let Q_0 belong to $\mathcal{P}_1$. Assume that $Q \in \mathcal{P}_1^\uparrow([0,T_0[, \, H_{Q_0})$ is a solution to (MP) with initial condition $Q_0 \in \mathcal{P}_1$ on $[0,T_0[$ for some $T_0 \leq \infty$. Consider any $L_1^{T_0,\uparrow}(\, H_{Q_0})$-$\alpha$-process $\tilde{X}$ such that $\mathcal{L}_\alpha(\tilde{X}) = Q$. Consider also the canonical process Z of $\mathbb{D}^\uparrow([0,T_0[, \, H_{Q_0}))$. Then there exists, on an enlarged probability space (from the canonical one), a Poisson measure $N(\omega,dt,d\alpha,dz)$, independent of Z_0 (all of this under Q), such that $(Z_0,Z,\tilde{X},N)$ is a solution to (SDE) (still under Q).*

Proof It follows from (MP), since $\mathcal{L}_\alpha(\tilde{X}_s) = Q_s$ for all s, that for any $\varphi \in C_b^1(\mathbb{R}_+)$,

$$M_t^\varphi = \varphi(Z_t) - \varphi(Z_0) \tag{3.33}$$

$$- \int_0^t \int_0^1 \left[\varphi(Z_s + \tilde{X}_s(\alpha)) - \varphi(Z_s)\right] \frac{K(Z_s,\tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)} d\alpha\, ds$$

$$= \varphi(Z_t) - \varphi(Z_0)$$

$$- \int_0^t \int_0^1 \int_0^\infty \left[\varphi\left(Z_s + \tilde{X}_s(\alpha) 1\!\!1_{\left\{z \leq \frac{K(Z_s,\tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)}\right\}}\right) - \varphi(Z_s)\right] dz\, d\alpha\, ds$$

is a martingale under Q. Applying this result with $\varphi(x) = x$, we deduce that

$$Z_t = Z_0 + M_t + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_s(\alpha) 1\!\!1_{\left\{z \leq \frac{K(Z_s,\tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)}\right\}} dz\, d\alpha\, ds \tag{3.34}$$

where M_t is a martingale (under Q). This decomposition is unique, in the sense that if $Z_t = Z_0 + L_t + F_t$, where L_t is a local martingale and F_t has bounded variations, then $L_t = M_t$ and

$$F_t = \int_0^t \int_0^1 \int_0^\infty \tilde{X}_s(\alpha) 1\!\!1_{\left\{z \leq \frac{K(Z_s,\tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)}\right\}} dz\, d\alpha\, ds \tag{3.35}$$

(see Jacod-Shiryaev [35], p 43). Hence, applying the Itô formula for jump processes (see e.g. [35]), we see that for any $\varphi \in C_b^2(\mathbb{R}_+)$ and $t \in [0,T_0[$,

$$\begin{aligned} \varphi(Z_t) &= \varphi(Z_0) + \int_0^t \varphi'(Z_{s-})\, dZ_s + \frac{1}{2} \int_0^t \varphi''(Z_s)\, dM_s^c \\ &\quad + \sum_{s \leq t} \left[\varphi(Z_{s-} + \Delta Z_s) - \varphi(Z_{s-}) - \Delta Z_s \varphi'(Z_{s-})\right] \end{aligned} \tag{3.36}$$

where M^c denotes the continuous martingale part of M. A comparison with (3.33) shows that $M^c \equiv 0$, and hence that M is a pure jump martingale.
A second comparison between (3.33) and (3.36) shows that the compensator of the jump measure $\mu = \sum_{s \leq T_0} \delta_{(s,\Delta M_s)}$ of M is the image measure of the Lebesgue measure $ds\, d\alpha\, dz$ by the map:

$$(s,\alpha,z) \to \tilde{X}_{s-}(\alpha) 1\!\!1_{\left\{z \leq \frac{K(Z_{s-},\tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}}. \tag{3.37}$$

Using a representation theorem for point processes (see El Karoui, Lepeltier [39]) we see that there exists, on an enlarged probability space, a Poisson measure $N(\omega,dt,d\alpha,dz)$, with intensity measure $dt\, d\alpha\, dz$, such that:

$$M_t = \int_0^t \int_0^1 \int_{\mathbb{R}_+} \tilde{X}_{s-}(\alpha) 1\!\!1_{\left\{z \leq \frac{K(Z_{s-},\tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} \bar{N}(ds,d\alpha,dz), \tag{3.38}$$

$\bar{N}(ds,d\alpha,dz)$ denoting the compensated Poisson measure of N, i.e.

$$\bar{N}(ds,d\alpha,dz) = N(ds,d\alpha,dz) - ds\,d\alpha\,dz. \tag{3.39}$$

We finally obtain :

$$Z_t = Z_0 + \int_0^t \int_0^1 \int_{\mathbb{R}_+} \tilde{X}_{s-}(\alpha) 1\!\!1_{\left\{z \leq \frac{K(Z_{s-},\tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds,d\alpha,dz). \tag{3.40}$$

Since we work under Q and since $\mathcal{L}_\alpha(\tilde{X}) = Q$, we deduce that $(Z_0,Z,\tilde{X},N)$ satisfies (SDE). This was our aim. □

In order to prove Theorem 3.3.1, we first consider a simpler problem with cutoff. For Q_0 in $\mathcal{P}_1$, we define a solution $(X_0,X^\varepsilon,\tilde{X}^\varepsilon,N)$ to $(SDE)_\varepsilon$ exactly in the same way as in Definition 3.2.9, but replacing (3.26) by

$$X_t^\varepsilon = X_0 + \int_0^t \int_0^1 \int_0^\infty \left(\tilde{X}_{s-}^\varepsilon(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}\right) 1\!\!1_{\left\{z \leq \frac{K(X_{s-}^\varepsilon \wedge \frac{1}{\varepsilon},\tilde{X}_{s-}^\varepsilon(\alpha)\wedge\frac{1}{\varepsilon})}{\tilde{X}_{s-}^\varepsilon(\alpha)\vee\varepsilon\wedge\frac{1}{\varepsilon}}\right\}} N(ds,d\alpha,dz) \tag{3.41}$$

with the conditions that $\mathcal{L}(X^\varepsilon) \in \mathcal{P}_1^\uparrow([0,T_0[,\ H_{Q_0})$ and that $\mathcal{L}_\alpha(\tilde{X}^\varepsilon) = \mathcal{L}(X^\varepsilon)$.

We begin with an important remark :

Remark 3.3.4 *We need that for each $\varepsilon > 0$ and for $(X_0,X^\varepsilon,\tilde{X}^\varepsilon,N)$ a solution to $(SDE)_\varepsilon$, X^ε takes its values in H_{Q_0}. Indeed, the regularity assumption (H_β) on K holds only on H_{Q_0}. Hence, in (3.41),* $x \vee \varepsilon \wedge (1/\varepsilon)$ is only a notation, *of which the rigorous definition is, for any $x \in H_{Q_0}$, any $\varepsilon > 0$,*

$$x \vee \varepsilon \wedge (1/\varepsilon) = \begin{cases} \inf\{y \in H_{Q_0}\ ;\ y \geq \varepsilon\} & \text{if} \quad 0 \leq x \leq \varepsilon \\ x & \text{if} \quad x \in [\varepsilon,1/\varepsilon] \\ \sup\{y \in H_{Q_0}\ ;\ y \leq 1/\varepsilon\} & \text{if} \quad 1/\varepsilon \leq x. \end{cases} \tag{3.42}$$

Of course, $x \wedge (1/\varepsilon)$ is defined in the same way. With these definitions, $x \vee \varepsilon \wedge (1/\varepsilon)$ and $x \wedge (1/\varepsilon)$ belong to H_{Q_0} for any $x \in H_{Q_0}$, $\varepsilon > 0$.

We now prove an existence result for $(SDE)_\varepsilon$.

Proposition 3.3.5 *Let $Q_0 \in \mathcal{P}_1$ and $\varepsilon > 0$. Assume (H_β). Let X_0 be a random variable whose law is Q_0 and N be a Poisson measure independent of X_0. Then there exists a solution $(X_0,X^\varepsilon,\tilde{X}^\varepsilon,N)$ to $(SDE)_\varepsilon$.*

Proof The proof mimics that of Tanaka, who proved in [63] a similar result in the case of a nonlinear S.D.E. related to a Boltzmann equation (and with globally Lipschitz coefficients). To this end, we introduce the following non-classical Picard approximations. First, we consider the process $X^{0,\varepsilon} \equiv X_0$, and any α-process $\tilde{X}^{0,\varepsilon}$

such that $\mathcal{L}_\alpha(\tilde{X}^{0,\varepsilon}) = \mathcal{L}(X^{0,\varepsilon})$.
Once everything is built up to n, we set

$$X_t^{n+1,\varepsilon} = X_0 + \int_0^t \int_0^1 \int_0^\infty \left(\tilde{X}_{s-}^{n,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon} \right) 1\!\!1_{\left\{ z \leq \frac{K(X_{s-}^{n,\varepsilon} \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^{n,\varepsilon}(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^{n,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}} \right\}} N(ds,d\alpha,dz) \tag{3.43}$$

and we consider any α-process $\tilde{X}^{n+1,\varepsilon}$ such that

$$\mathcal{L}_\alpha(\tilde{X}^{n+1,\varepsilon} | \tilde{X}^{0,\varepsilon}, \ldots, \tilde{X}^{n,\varepsilon}) = \mathcal{L}(X^{n+1,\varepsilon} | X^{0,\varepsilon}, \ldots, X^{n,\varepsilon}). \tag{3.44}$$

One easily checks recursively that for each n, $X^{n,\varepsilon}$ is an $L_1^{\infty,\uparrow}(\ H_{Q_0})$-process.
Let us show now that the sequence $\{X^{n,\varepsilon}\}_n$ is Cauchy in $L_1^{\infty,\uparrow}(\ H_{Q_0})$. A simple computation gives

$$\begin{aligned} X_t^{n+1,\varepsilon} - X_t^{n,\varepsilon} = & \int_0^t \int_0^1 \int_0^\infty \left(\tilde{X}_{s-}^{n,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon} - \tilde{X}_{s-}^{n-1,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon} \right) \\ & 1\!\!1_{\left\{ z \leq \frac{K(X_{s-}^{n,\varepsilon} \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^{n,\varepsilon}(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^{n,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}} \right\}} N(ds,d\alpha,dz) \\ & + \int_0^t \int_0^1 \int_0^\infty \left(\tilde{X}_{s-}^{n-1,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon} \right). \\ & \left(1\!\!1_{\left\{ z \leq \frac{K(X_{s-}^{n,\varepsilon} \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^{n,\varepsilon}(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^{n,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}} \right\}} - 1\!\!1_{\left\{ z \leq \frac{K(X_{s-}^{n-1,\varepsilon} \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^{n-1,\varepsilon}(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^{n-1,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}} \right\}} \right) N(ds,d\alpha,dz). \end{aligned} \tag{3.45}$$

Let us for example present the proof under (H_β) when $\beta = 1$. Then

$$\begin{aligned} \left| X_t^{n+1,\varepsilon} - X_t^{n,\varepsilon} \right| \leq & \int_0^t \int_0^1 \int_0^\infty \left| \tilde{X}_{s-}^{n,\varepsilon}(\alpha) - \tilde{X}_{s-}^{n-1,\varepsilon}(\alpha) \right| \\ & 1\!\!1_{\left\{ z \leq \frac{C}{\varepsilon}(1+2/\varepsilon+1/\varepsilon^2) \right\}} N(ds,d\alpha,dz) \\ & + \frac{1}{\varepsilon} \int_0^t \int_0^1 \int_0^\infty \left| 1\!\!1_{\left\{ z \leq \frac{K(X_{s-}^{n,\varepsilon} \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^{n,\varepsilon}(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^{n,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}} \right\}} - 1\!\!1_{\left\{ z \leq \frac{K(X_{s-}^{n-1,\varepsilon} \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^{n-1,\varepsilon}(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^{n-1,\varepsilon}(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}} \right\}} \right| \\ & N(ds,d\alpha,dz). \end{aligned} \tag{3.46}$$

By setting $\varphi_n(t) := E\left[\sup_{s\in[0,t]} \left|X_s^{n+1,\varepsilon} - X_s^{n,\varepsilon}\right|\right]$, we obtain, for some constant A, depending only on ε,

$$\varphi_n(t) \le A\int_0^t \varphi_{n-1}(s)\,ds \tag{3.47}$$

$$+A\int_0^t EE_\alpha \left|\frac{K(X_s^{n,\varepsilon}\wedge\frac{1}{\varepsilon},\tilde{X}_s^{n,\varepsilon}\wedge\frac{1}{\varepsilon})}{\tilde{X}_s^{n,\varepsilon}\vee\varepsilon\wedge\frac{1}{\varepsilon}} - \frac{K(X_s^{n-1,\varepsilon}\wedge\frac{1}{\varepsilon},\tilde{X}_s^{n-1,\varepsilon}\wedge\frac{1}{\varepsilon})}{\tilde{X}_s^{n-1,\varepsilon}\vee\varepsilon\wedge\frac{1}{\varepsilon}}\right| ds.$$

Since K is locally Lipschitz continuous on $(\ H_{Q_0})^2$, it is clear that the map

$$(x,y) \mapsto \frac{K\left(x\wedge\frac{1}{\varepsilon},y\wedge\frac{1}{\varepsilon}\right)}{x\vee\varepsilon\wedge\frac{1}{\varepsilon}} \tag{3.48}$$

is globally Lipschitz continuous on $(\ H_{Q_0})^2$. Hence, using the fact that $\int_0^1 |\tilde{X}^{n,\varepsilon}(\alpha) - \tilde{X}^{n-1,\varepsilon}(\alpha)|\,d\alpha \le \varphi_{n-1}(s)$, we obtain

$$\varphi_n(t) \le A\int_0^t \varphi_{n-1}(s)\,ds. \tag{3.49}$$

We conclude, thanks to the usual Picard Lemma, that there exists a $L_1^{\infty,\uparrow}(\ H_{Q_0})$-process X^ε such that, for any $T<\infty$, when n tends to infinity,

$$E\left[\sup_{t\in[0,T]} |X_t^{n,\varepsilon} - X_t^\varepsilon|\right] \longrightarrow 0. \tag{3.50}$$

By construction, the α-law of the sequence of processes $\tilde{X}^{0,\varepsilon},\dots,\tilde{X}^{n,\varepsilon},\dots$ is the same as the law of the sequence $X^{0,\varepsilon},\dots,X^{n,\varepsilon},\dots$. We thus deduce the existence of an $L_1^{\infty,\uparrow}(\ H_{Q_0})$-$\alpha$-process $\tilde{X}^\varepsilon$ such that $\mathcal{L}_\alpha(\tilde{X}^\varepsilon) = \mathcal{L}(X^\varepsilon)$, and such that for all $T<\infty$, when n tends to infinity,

$$E_\alpha\left[\sup_{t\in[0,T]} |\tilde{X}_t^{n,\varepsilon} - \tilde{X}_t^\varepsilon|\right] \longrightarrow 0. \tag{3.51}$$

Letting n go to infinity in (3.43) concludes the proof. □

We now prove the tightness of the family $\{\mathcal{L}(X^\varepsilon)\}_\varepsilon$.

Lemma 3.3.6 *Let $Q_0\in\mathcal{P}_1$. Assume (H_β). For $\beta=1/2$ set $T_0=\infty$, while for $\beta=1$ set $T_0 = 1/C\left(1+\int xQ_0(dx)\right)$, where C is the constant which appear in the hypothesis (H_β).*
Consider a family $(X_0,X^\varepsilon,\tilde{X}^\varepsilon,N)$ of solutions to $(SDE)_\varepsilon$. Then, for all $T<T_0$,

$$\sup_{\varepsilon>0} E\left[\sup_{t\in[0,T]} |X_t^\varepsilon|\right] = \sup_{\varepsilon>0} E_\alpha\left[\sup_{t\in[0,T]} |\tilde{X}_t^\varepsilon|\right] < \infty. \tag{3.52}$$

Furthermore, the family $\mathcal{L}(X^\varepsilon) = \mathcal{L}_\alpha(\tilde{X}^\varepsilon)$ of probability measures on $\mathbb{D}^\uparrow([0,T_0[,\ H_{Q_0})$ is tight, and any limiting point Q of a convergent subsequence is the law of a quasi-left continuous process (for the definition see Jacod, Shiryaev [35]). .

Proof Let us again prove the result under (H_β) for $\beta = 1$, the case of $\beta = 1/2$ being similar. We first check (3.52). Setting

$$f_\varepsilon(t) = E\left[\sup_{s\in[0,t]} |X_s^\varepsilon|\right] \tag{3.53}$$

it is immediate, since the processes are positive and increasing and since for each ε, $\mathcal{L}_\alpha(\tilde{X}^\varepsilon) = \mathcal{L}(X^\varepsilon)$, that :

$$f_\varepsilon(t) = E\left[X_t^\varepsilon\right] = E_\alpha\left[\tilde{X}_t^\varepsilon\right] \tag{3.54}$$

A simple computation, using (3.41), yields that

$$f_\varepsilon(t) = E(X_0) + \int_0^t EE_\alpha\left[K\left(X_s^\varepsilon \wedge \frac{1}{\varepsilon}, \tilde{X}_s^\varepsilon \wedge \frac{1}{\varepsilon}\right)\right] ds. \tag{3.55}$$

But since we are under (H_β) with $\beta = 1$, it is clear that

$$EE_\alpha\left[K\left(X_s^\varepsilon \wedge \frac{1}{\varepsilon}, \tilde{X}_s^\varepsilon \wedge \frac{1}{\varepsilon}\right)\right] \leq C\left(1 + 2f_\varepsilon(s) + f_\varepsilon^2(s)\right) = C\left(1 + f_\varepsilon(s)\right)^2. \tag{3.56}$$

Lemma 3.7.3 of the appendix, applied to the function $g_\varepsilon = 1 + f_\varepsilon$, which is clearly continuous thanks to (3.55) allows to conclude that for any $t < T_0 = 1/C(1+E(X_0))$,

$$f_\varepsilon(t) \leq \frac{1 + E(X_0)}{1 - t/T_0} - 1 \tag{3.57}$$

from which (3.52) is straightforward.

In order to obtain the tightness of the family $\{\mathcal{L}(X^\varepsilon)\}_\varepsilon$, we use the Aldous criterion, which is recalled in the appendix (Theorem 3.7.1).
We just have to check (for example) that for all $T < T_0$ fixed, there exists a constant A_T such that for all $\delta > 0$, all couple of stopping times S and S' satisfying *a.s.* $0 \leq S \leq S' \leq (S + \delta) \wedge T$, and all ε,

$$E|X_{S'}^\varepsilon - X_S^\varepsilon| \leq A_T\,\delta \tag{3.58}$$

the constant A_T being independent of ε, δ, S and S'. This is not hard. Indeed,

$$|X_{S'}^\varepsilon - X_S^\varepsilon| = \int_S^{S'} \left(\tilde{X}_{s-}^\varepsilon(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}\right) \mathbb{1}_{\left\{z \leq \frac{K(X_{s-}^\varepsilon \wedge \frac{1}{\varepsilon}, \tilde{X}_{s-}^\varepsilon(\alpha) \wedge \frac{1}{\varepsilon})}{\tilde{X}_{s-}^\varepsilon(\alpha) \vee \varepsilon \wedge \frac{1}{\varepsilon}}\right\}} N(ds, d\alpha, dz). \tag{3.59}$$

Hence

$$\begin{aligned} E\left[|X_{S'}^\varepsilon - X_S^\varepsilon|\right] &= EE_\alpha\left[\int_S^{S'} K(X_s^\varepsilon, \tilde{X}_s^\varepsilon(\alpha))ds\right] \\ &\leq \delta \sup_{s\in[0,T]} EE_\alpha\left[K(X_s^\varepsilon, \tilde{X}_s^\varepsilon)\right]. \end{aligned} \tag{3.60}$$

But thanks to (H_β) for $\beta = 1$ and to (3.52) (since $T < T_0$),

$$\begin{aligned}\sup_{s\in[0,T]} EE_\alpha\left[K(X_s^\varepsilon,\tilde{X}_s^\varepsilon)\right] &\leq C \sup_{s\in[0,T]} EE_\alpha\left[1 + X_s^\varepsilon + \tilde{X}_s^\varepsilon + X_s^\varepsilon \tilde{X}_s^\varepsilon\right] \\ &\leq A_T \qquad (3.61)\end{aligned}$$

which concludes the proof. □

To prove that any limiting point Q of $\mathcal{L}(X^\varepsilon)$ satisfies (MP), we will also need a property of uniform integrability, which will be obtained in the next lemma.

Lemma 3.3.7 *Assume that $Q_0 \in \mathcal{P}_1$, and that $\int x^2 Q_0(dx) < \infty$. Assume (H_β), and following the value of β consider the associated T_0. Consider a family $(X_0,X^\varepsilon,\tilde{X}^\varepsilon,N)$ of solutions to $(SDE)_\varepsilon$. Then for all $T < T_0$ fixed,*

$$\sup_{\varepsilon>0} E\left[\sup_{t\in[0,T]} |X_t^\varepsilon|^2\right] < \infty. \qquad (3.62)$$

Proof For $k \in \mathbb{N}^*$, we define

$$g_k^\varepsilon(t) = E\left[\sup_{t\in[0,T]} |X_t^\varepsilon|^k\right] = E\left[(X_T^\varepsilon)^k\right]. \qquad (3.63)$$

For all $t < T_0$,

$$\begin{aligned}(X_t^\varepsilon)^2 &= (X_0)^2 + \sum_{s\leq t}\left(\left(X_{s-}^\varepsilon + \Delta X_s^\varepsilon\right)^2 - \left(X_{s-}^\varepsilon\right)^2\right) \qquad (3.64)\\ &= (X_0)^2 + \sum_{s\leq t}\left(2X_{s-}^\varepsilon \Delta X_s^\varepsilon + (\Delta X_s^\varepsilon)^2\right)\\ &= (X_0)^2 + \int_0^t\int_0^1\int_0^\infty\left(2X_{s-}^\varepsilon\left(\tilde{X}_{s-}^\varepsilon(\alpha)\vee\varepsilon\wedge\frac{1}{\varepsilon}\right) + \left(\tilde{X}_{s-}^\varepsilon(\alpha)\vee\varepsilon\wedge\frac{1}{\varepsilon}\right)^2\right)\\ &\qquad \mathbb{1}_{\left\{z\leq\frac{K(X_{s-}^\varepsilon\wedge\frac{1}{\varepsilon},\tilde{X}_{s-}^\varepsilon(\alpha)\wedge\frac{1}{\varepsilon})}{\tilde{X}_{s-}^\varepsilon(\alpha)\vee\varepsilon\wedge\frac{1}{\varepsilon}}\right\}} N(ds,d\alpha,dz).\end{aligned}$$

Hence

$$\begin{aligned}g_2^\varepsilon(t) &= E(X_0^2) + 2\int_0^t EE_\alpha\left[X_s^\varepsilon K\left(X_s^\varepsilon\wedge\frac{1}{\varepsilon},\tilde{X}_s^\varepsilon\wedge\frac{1}{\varepsilon}\right)\right] ds\\ &\quad + \int_0^t EE_\alpha\left[(\tilde{X}_s^\varepsilon\vee\varepsilon)K\left(X_s^\varepsilon\wedge\frac{1}{\varepsilon},\tilde{X}_s^\varepsilon\wedge\frac{1}{\varepsilon}\right)\right] ds \qquad (3.65)\end{aligned}$$

Let us finish the proof for (H_β) with $\beta = 1$, the other case being similar. Using (H_β) with $\beta = 1$, the fact that $\mathcal{L}(X^\varepsilon) = \mathcal{L}_\alpha(\tilde{X}^\varepsilon)$ and (3.52), we obtain the existence of a constant A_T, not depending on ε, such that for all $t \le T$,

$$\begin{aligned} g_2^\varepsilon(t) &\le E(X_0^2) + 3C \int_0^t EE_\alpha \left[\left(\tilde{X}_s^\varepsilon + \varepsilon \right) \left(1 + X_s^\varepsilon + \tilde{X}_s^\varepsilon + X_s^\varepsilon \tilde{X}_s^\varepsilon \right) \right] ds \\ &\le E(X_0^2) + A_T \int_0^t [1 + g_2^\varepsilon(s)]\, ds. \end{aligned} \tag{3.66}$$

The usual Gronwall Lemma allows to conclude. □

The following lemma, associated with Proposition 3.3.3, will conclude the proof of Theorem 3.3.1.

Lemma 3.3.8 *Let Q_0 belong to $\mathcal{P}_1$ and satisfy $\int x^2 Q_0(dx) < \infty$. Assume (H_β) and consider the corresponding T_0. Consider a family $(X_0, X^\varepsilon, \tilde{X}^\varepsilon, N)$ of solutions to $(SDE)_\varepsilon$, and a limiting point Q of the tight family $\mathcal{L}(X^\varepsilon) = \mathcal{L}_\alpha(\tilde{X}^\varepsilon)$. Then Q is a solution to (MP) on $[0,T_0[$, with initial condition $Q_0 = \mathcal{L}(X_0)$.*

Proof We prove the result for $\beta = 1$. The other case is simpler. Let Q be the limit of a sequence of $Q^k = \mathcal{L}(X^{\varepsilon_k})$, ε_k being a sequence of positive real numbers decreasing to 0.
We have to check that for any $\phi \in C_b^1(\mathbb{R}_+)$, any $g_1, \dots, g_l \in C_b(\mathbb{R}_+)$ and any $0 \le s_1 \le \dots \le s_l < s < t < T_0$,

$$\langle Q \otimes Q, F \rangle = 0 \tag{3.67}$$

where F is the map from $\mathbb{D}^\uparrow([0,T_0[, H_{Q_0}) \times \mathbb{D}^\uparrow([0,T_0[, H_{Q_0})$ defined by

$$\begin{aligned} &F(x,y) = g_1(x(s_1)) \times \dots \times g_l(x(s_l)) \times \\ &\left\{ \phi(x(t)) - \phi(x(s)) - \int_s^t [\phi(x(u) + y(u)) - \phi(x(u))] \frac{K(x(u),y(u))}{y(u)} du \right\}. \end{aligned} \tag{3.68}$$

It is clear from the definition of the process X^{ε_k} that for any k,

$$\left\langle Q^k \otimes Q^k, F^k \right\rangle = 0, \tag{3.69}$$

where F^k is defined by :

$$\begin{aligned} &F^k(x,y) = g_1(x(s_1)) \times \dots \times g_l(x(s_l)) \times \left\{ \phi(x(t)) - \phi(x(s)) \right. \\ &\left. - \int_s^t \left[\phi(x(u) + y(u) \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}) - \phi(x(u)) \right] \frac{K(x(u) \wedge \frac{1}{\varepsilon_k}, y(u) \wedge \frac{1}{\varepsilon_k})}{y(u) \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}} du \right\} . \end{aligned} \tag{3.70}$$

It thus suffices to prove that $\langle Q^k \otimes Q^k, F^k \rangle$ tends to $\langle Q \otimes Q, F \rangle$ as k tends to infinity. We split the proof in two steps.

Step 1 : Let us first check that, as k goes to infinity,

$$\langle Q^k \otimes Q^k, |F - F^k| \rangle \longrightarrow 0. \tag{3.71}$$

By definition,

$$\begin{aligned}
\langle Q^k \otimes Q^k, |F - F^k| \rangle &= EE_\alpha\Big[\Big| g_1(X^{\varepsilon_k}(s_1)) \times \cdots \times g_l(X^{\varepsilon_k}(s_l)) \\
\int_s^t \Big\{ \Big[\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})\Big] & \frac{K(X_u^{\varepsilon_k} \wedge \frac{1}{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k} \wedge \frac{1}{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}} \\
- \Big[\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})\Big] & \frac{K(X_u^{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k}} \Big\} du \Big|\Big].
\end{aligned} \tag{3.72}$$

Hence, for some constant A, $\langle Q^k \otimes Q^k, |F - F^k| \rangle$ is smaller than

$$\begin{aligned}
& AEE_\alpha\Big[\int_s^t \left| \frac{\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}} - \frac{\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k}} \right| \\
& \qquad K(X_u^{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k}) du \Big] \\
& + AEE_\alpha\Big[\int_s^t \left| \frac{\varphi(X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k} \vee \varepsilon_k \wedge \frac{1}{\varepsilon_k}} \right| \\
& \qquad \left| K(X_u^{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k}) - K(X_u^{\varepsilon_k} \wedge \frac{1}{\varepsilon_k}, \tilde{X}_u^{\varepsilon_k} \wedge \frac{1}{\varepsilon_k}) \right| du \Big] \\
& = A(I_{\varepsilon_k} + J_{\varepsilon_k}),
\end{aligned} \tag{3.73}$$

with obvious notations for I_{ε_k} and J_{ε_k}. Since φ' is bounded, we obtain, using (H_β) for $\beta = 1$,

$$\begin{aligned}
J_{\varepsilon_k} &\leq 2||\varphi'||_\infty EE_\alpha\Big[\int_s^t \Big(\mathbb{1}_{\{X_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}\}} + \mathbb{1}_{\{\tilde{X}_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}\}} \Big) \\
& \qquad (1 + X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k} + X_u^{\varepsilon_k} \tilde{X}_u^{\varepsilon_k}) du \Big] \\
&\leq A \int_s^t \Big\{ \mathbb{P}(X_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}) + E_\alpha(\tilde{X}_u^{\varepsilon_k}) \mathbb{P}(X_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}) + E(X_u^{\varepsilon_k} \mathbb{1}_{\{X_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}\}}) \\
& \qquad + E(X_u^{\varepsilon_k} \mathbb{1}_{\{X_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}\}}) E_\alpha(\tilde{X}_u^{\varepsilon_k}) \Big\} du.
\end{aligned} \tag{3.74}$$

Since the processes are increasing, and thanks to (3.52), we deduce that

$$J_{\varepsilon_k} \leq A\left[\mathbb{P}(X_t^{\varepsilon_k} > 1/\varepsilon_k) + E\left[X_t^{\varepsilon_k}\mathbb{1}_{\{X_t^{\varepsilon_k} > 1/\varepsilon_k\}}\right]\right]. \tag{3.75}$$

The uniform integrability obtained in Lemma 3.3.7 allows to conclude that J_{ε_k} tends to 0.
Let us now bound I_{ε_k} from above. First,

$$\begin{aligned} I_{\varepsilon_k} \leq\ & AEE_\alpha\Big[\int_s^t \Big\{\mathbb{1}_{\{\tilde{X}_u^{\varepsilon_k} < \varepsilon_k\}} K(X_u^{\varepsilon_k},\tilde{X}_u^{\varepsilon_k}) \\ & \left|\frac{\varphi(X_u^{\varepsilon_k}+\varepsilon_k) - \varphi(X_u^{\varepsilon_k})}{\varepsilon_k} - \frac{\varphi(X_u^{\varepsilon_k}+\tilde{X}_u^{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k}}\right| \\ & + \mathbb{1}_{\{\tilde{X}_u^{\varepsilon_k} > \frac{1}{\varepsilon_k}\}} K(X_u^{\varepsilon_k},\tilde{X}_u^{\varepsilon_k}) \\ & \left|\frac{\varphi(X_u^{\varepsilon_k}+\frac{1}{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\frac{1}{\varepsilon_k}} - \frac{\varphi(X_u^{\varepsilon_k}+\tilde{X}_u^{\varepsilon_k}) - \varphi(X_u^{\varepsilon_k})}{\tilde{X}_u^{\varepsilon_k}}\right| \Big\} du\Big] \\ =\ & I_{\varepsilon_k}^1 + I_{\varepsilon_k}^2, \end{aligned} \tag{3.76}$$

with obvious notations. The second term is similar to J_{ε_k}, and thus goes to 0 as k tends to infinity. Using (H_β) with $\beta = 1$ and (3.52), we see that the first term is smaller than

$$\begin{aligned} I_{\varepsilon_k}^1 & \leq 2A\|\varphi'\|_\infty \int_s^t EE_\alpha\left[\mathbb{1}_{\{\tilde{X}_u^{\varepsilon_k} < \varepsilon_k\}}(1 + X_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k} + \tilde{X}_u^{\varepsilon_k} X_u^{\varepsilon_k})\right] du \\ & \leq A\int_s^t EE_\alpha\left[(1 + X_u^{\varepsilon_k})(1+\varepsilon_k)\mathbb{1}_{\{\tilde{X}_u^{\varepsilon_k} < \varepsilon_k\}}\right] du \\ & \leq A\int_s^t E(1 + X_u^{\varepsilon_k})E_\alpha(\mathbb{1}_{\{\tilde{X}_u^{\varepsilon_k} < \varepsilon_k\}})du \\ & \leq A\int_0^t \mathbb{P}(X_u^{\varepsilon_k} < \varepsilon_k)du \\ & \leq At\,\mathbb{P}(X_0 < \varepsilon_k) \end{aligned}$$

where the last inequality comes from the fact that the process X^{ε_k} is increasing. This goes to 0, because $X_0 > 0$ a.s. Step 1 is finished.

Step 2 : It remains to prove that as k goes to infinity,

$$\langle Q^k \otimes Q^k, F\rangle \longrightarrow \langle Q \otimes Q, F\rangle. \tag{3.78}$$

This convergence would be obvious if F was continuous and bounded on $\mathbb{D}^{\uparrow}([0,T_0[,\, H_{Q_0}) \times \mathbb{D}^{\uparrow}([0,T_0[,\, H_{Q_0})$, thanks to the definition of the convergence in law. The map F is not continuous on $\mathbb{D}^{\uparrow}([0,T_0[,\, H_{Q_0}) \times \mathbb{D}^{\uparrow}([0,T_0[,\, H_{Q_0})$, but only on $\mathcal{C} \times \mathcal{C}$, where

$$\mathcal{C} = \left\{x \in \mathbb{D}^{\uparrow}([0,T_0[,\, H_{Q_0})\ ;\ \Delta x(s_1) = \ldots \Delta x(s_l) = \Delta x(s) = \Delta x(t) = 0\right\}. \quad (3.79)$$

Thanks to Lemma 3.3.6, Q is the law of a quasi-left continuous process, thus $Q(\mathcal{C}) = 1$, and hence F is $Q \otimes Q$-*a.e.* continuous. This implies that for any positive constant A,

$$\left\langle Q^k \otimes Q^k, F \wedge A \vee (-A)\right\rangle \longrightarrow \langle Q \otimes Q, F \wedge A \vee (-A)\rangle \quad (3.80)$$

because $F \wedge A \vee (-A)$ is $Q \otimes Q$-*a.e.* continuous and bounded. Thus (3.78) will hold if we prove that

$$\sup_k \left\langle Q^k \otimes Q^k, |F| 1\!\!1_{|F| \geq A}\right\rangle \longrightarrow 0 \quad (3.81)$$

as A tends to infinity. One can check, after many but easy computations, that

$$\left\langle Q^k \otimes Q^k, |F| 1\!\!1_{|F| \geq A}\right\rangle \leq BE\left[X_t^{\varepsilon} 1\!\!1_{\{X_t^{\varepsilon} > \zeta(A)\}}\right] \quad (3.82)$$

for some constants B and some function $\zeta(A)$ tending to infinity with A. The uniform integrability obtained in Lemma 3.3.7 allows to conclude that (3.81) holds. Hence (3.78) is valid. This concludes the proof of Step 2 and thus the proof of the lemma. □

Let us finally give the proof of the main result of this section.

Proof of Theorem 3.3.1 Thanks to Lemma 3.3.5, there exists a solution $(X_0, X^{\varepsilon}, \tilde{X}^{\varepsilon}, N)$ to $(SDE)_{\varepsilon}$ for each ε. Due to Lemma 3.3.6, the sequence $\{\mathcal{L}(X^{\varepsilon})\}$ is tight, and in particular there exists a sequence ε_k decreasing to 0 such that $\{\mathcal{L}(X^{\varepsilon_k})\}$ tends to some Q. From Lemma 3.3.8, Q satisfies (MP). Finally, Proposition 3.3.3 allows us to build a solution $(X_0, X, \tilde{X}, N)$ to (SDE). □

3.4 Pathwise behaviour of (SDE)

In this short section, we would like to give an idea on the pathwise properties of X_t, for $(X_0, X, \tilde{X}, N)$ a solution to (SDE). We have very few results on this topic, and the study seems to be difficult. However, we hope that new results will arise in a forthcoming paper. Let us begin with a remark concerning the long time behaviour.

Remark 3.4.1 *Let Q_0 belong to $\mathcal{P}_1$, and satisfy $\int x^2 Q_0(dx) < \infty$. Let us assume (H_β) with $\beta = 1/2$, and let $(X_0, X, \tilde{X}, N)$ be a solution to the corresponding (SDE). A natural question is the following. Does the size of every particle in the system tend to infinity when the time grows to infinity? In other words, does X_t tend to infinity a.s. with t? This result, which seems to hold in the cases where $K(x,y) = 1$ and*

$K(x,y) = x+y$ (for which explicit computations are easy), is not obvious. It is clear that a lower bound of K has to be supposed. Indeed, if we assume for example that $K(x,y)$ vanishes for all x,y such that $x \vee y \geq n$ then it is clear that a.s., $\lim_t X_t < \infty$. This comes from the fact that in such a case, if we set

$$\tau = \inf\{t > 0 \; ; \; X_t \geq n\} \tag{3.83}$$

then either $\tau = \infty$ (and hence $\lim_t X_t < \infty$) or τ is finite, and then it is easily deduced from (3.26) that $\lim_t X_t = X_\tau < \infty$.
We are not able yet to express properly the lower bound which has to be assumed on K.

We now present an idea about the frequency of the jumps of X_t. How often does a particle in the system coagulate?
The following result, which says that the number of jumps is finite on every compact interval, is not *a priori* obvious in the continuous case.

Proposition 3.4.2 *Let $Q_0 \in \mathcal{P}_1$ satisfy $\int x^2 Q_0(dx) < \infty$. Assume (H_β) and consider the corresponding T_0. Let $(X_0,X,\tilde{X},N)$ be a solution to the corresponding (SDE). Assume furthermore that*

$$\int_{\mathbb{R}_+} \frac{1}{x} Q_0(dx) < \infty \tag{3.84}$$

which always holds in the discrete case, and which simply means, in the continuous case, that $\int n_0(x)dx < \infty$.
Denote by $J_t = \sum_{s \leq t} \mathbb{1}_{\{\Delta X_s \neq 0\}}$ the number of jumps of X on $[0,t]$. Then for all $t < T_0$, $E[J_t] < \infty$.

Proof Let us again prove the result for $\beta = 1$. Thanks to (3.26), we see that

$$J_t = \int_0^t \int_0^1 \int_0^\infty \mathbb{1}_{\left\{z \leq \frac{K(X_{s-},\tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds,d\alpha,dz) \tag{3.85}$$

and hence

$$\begin{aligned} E[J_t] &= E\left[\int_0^t \int_0^1 \int_0^\infty \mathbb{1}_{\left\{z \leq \frac{K(X_s,\tilde{X}_s(\alpha))}{\tilde{X}_s(\alpha)}\right\}} dz d\alpha ds\right] \\ &= \int_0^t EE_\alpha\left[\frac{K(X_s,\tilde{X}_s)}{\tilde{X}_s}\right] ds. \end{aligned} \tag{3.86}$$

Using (H_β) with $\beta = 1$, we obtain

$$\begin{aligned} E[J_t] &\leq C\int_0^t EE_\alpha\left[1/\tilde{X}_s + X_s/\tilde{X}_s + 1 + X_s\right] ds \\ &\leq C\int_0^t \left[E[1/X_s] + E[X_s]E[1/X_s] + 1 + E[X_s]\right] ds \\ &\leq Ct\left[E[1/X_0] + E[X_t]E[1/X_0] + 1 + E[X_t]\right] \end{aligned} \tag{3.87}$$

where the last inequality comes from the fact that X is *a.s.* increasing. This last upper bound is clearly finite, since $t < T_0$, and since we have assumed that $E(1/X_0) < \infty$. The proof is complete. □

Remark 3.4.3 *If we do not assume (3.84), we do not know what happens. It however seems that in the (non explosive) case where $K(x,y) = 1$ and where $E(1/X_0) = \infty$, then $X_.$ has infinitely many jumps immediately after 0, but that for any $0 < s < t < \infty$, $X_.$ has an a.s. finite number of jumps on $[s,t]$.*

Let us finally talk about the gelification time:

$$T_{gel} = \inf\{t \geq 0\,;\ E(X_t) = \infty\}\,. \tag{3.88}$$

This quantity, which can be seen as a L^1-gelification time, has been much studied by the analysts and physicists. It is easily deduced from Theorem 3.3.1 that under (H_β) with $\beta = 1/2$, $T_{gel} = \infty$ for any initial condition (satisfying $Q_0 \in \mathcal{P}_1$ and $\int x^2 Q_0(dx) < \infty$).
In the case of $\beta = 1$, under the same assumptions on Q_0, Theorem 3.3.1 yields that $T_{gel} \geq T_0 = 1/C(1 + \int xQ_0(dx))$. Of course, we have only proved the existence for (SDE) on $[0,T_0[$, because we have only assumed an upper bound for K. But in any particular case where explicit computations can be done, solutions to (SDE) may be built on $[0,T_{gel}[$. For example, the following proposition holds.

Proposition 3.4.4 *Assume that $Q_0 \in \mathcal{P}_1$ and that $\int x^2 Q_0(dx) < \infty$. Assume that $K(x,y) = A + B(x+y) + Cxy$, for some nonnegative constants A and B, and for some $C > 0$. Then Theorem 3.3.1 still holds replacing T_0 by T_{gel}, where, if $a_0 = \int xQ_0(dx)$,*

1. *If $\Delta = 4(B^2 - AC) = 0$, then $T_{gel} = \dfrac{1}{C(a_0 + B)}$,*
2. *If $\Delta = 4(B^2 - AC) < 0$, then $T_{gel} = \frac{2\pi C}{-\Delta} - \frac{4C}{-\Delta}\arctan\left(4C\dfrac{a_0 + B}{-\Delta}\right)$,*
3. *If $\Delta = 4(B^2 - AC) > 0$, then $T_{gel} = \frac{1}{2\Delta}\ln\left(\frac{a_0 + B/C + \sqrt{\Delta}/C}{a_0 + B/C - \sqrt{\Delta}/C}\right)$.*

Proof This is not hard. It suffices to replace the use of the extended Gronwall Lemma 3.7.3 by solving classical ODEs. The result is easily understood *a posteriori*. If $(X_0,X,\tilde{X},N)$ is a solution to (SDE), one easily checks that

$$E[X_t] = a_0 + \int_0^t EE_\alpha\left[K(X_s,\tilde{X}_s)\right]ds. \tag{3.89}$$

In the present case, by setting $f(t) = E[X_t]$, one easily gets

$$f(t) = a_0 + \int_0^t \left[A + 2Bf(s) + Cf^2(s)\right]ds. \tag{3.90}$$

This equation has an unique solution, which can be explicitly computed, and T_{gel}, which corresponds to its explosion time, can also be computed. We obtain the expressions given in the statement. □

From a probabilistic point of view, the L^1-gelification time is of course important, but we want also to study the stochastic gelification time :

$$\tau_{gel} = \inf\{t \geq 0 \; ; \; X_t = \infty\}. \tag{3.91}$$

Obviously, $\tau_{gel} \geq T_{gel}$ *a.s.* An interesting question is the following. Under which conditions on Q_0 and K do we have

$$P(\tau_{gel} > T_{gel}) \in]0,1[, \;\; P(\tau_{gel} > T_{gel}) = 0 \text{ or } P(\tau_{gel} > T_{gel}) = 1 \text{ ?} \tag{3.92}$$

In other words, are there particles of which the mass is finite (resp. infinite) at the instant T_{gel}? Do all particles have a finite (resp. infinite) mass at the instant T_{gel}? We are not able to give a complete answer for the moment. Let us however state and prove the following result.

Proposition 3.4.5 *Let $Q_0 \in \mathcal{P}_1$ with $\int x^2 Q_0(dx) < \infty$, and let us assume (H_β) with $\beta = 1$. Assume furthermore that $T_{gel} < \infty$, and that there exists a function ζ : $\text{Supp } Q_0 \mapsto \mathbb{R}_+$ such that for all $x \in \text{Supp } Q_0$,*

$$\sup_{y \in H_{Q_0}} \frac{K(x,y)}{y} \leq \zeta(x). \tag{3.93}$$

Consider a solution $(X_0,X,\tilde{X},N)$ to (SDE). Then for any $t \in [0,\infty[$,

$$P(\tau_{gel} > t) > 0. \tag{3.94}$$

This means in particular that there are many particles which have a finite mass at the instant T_{gel}.

Notice that (3.93) is always satisfied in the discrete case, and more generally for any kernel satisfying (H_β) with $\beta = 1$ if $[0,\varepsilon[\cap \text{Supp } Q_0 = \emptyset$ for some $\varepsilon > 0$.
Notice also that (3.93) is satisfied with any initial condition, if $K(x,y) \leq Cxy$ for some constant $C \in \mathbb{R}_+$.

Proof We will prove a much stronger result : for any $t > 0$,

$$P(X_t = X_0) > 0. \tag{3.95}$$

To this end, we study the first jump time

$$T_1 = \inf\{s \geq 0 \; ; \; \Delta X_s \neq 0\}. \tag{3.96}$$

By remarking that thanks to (3.93) and (3.26),

$$X_0 \le X_t \le X_0 + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) 1\!\!1_{\{z \le \zeta(X_{s-})\}} N(ds,d\alpha,dz) \tag{3.97}$$

we deduce that $T_1 \ge S_1$ *a.s.*, where

$$S_1 = \inf\left\{ s \ge 0 \; ; \; \int_0^t \int_0^1 \int_0^\infty 1\!\!1_{\{z \le \zeta(X_0)\}} N(ds,d\alpha,dz) > 0 \right\}. \tag{3.98}$$

Since N is a Poisson measure independent of X_0, the random variable

$$\int_0^t \int_0^1 \int_0^\infty 1\!\!1_{\{z \le \zeta(X_0)\}} N(ds,d\alpha,dz) \tag{3.99}$$

follows, conditionally to X_0, a Poisson distribution of parameter

$$\int_0^t \int_0^1 \int_0^\infty 1\!\!1_{\{z \le \zeta(X_0)\}} ds d\alpha dz = t\zeta(X_0). \tag{3.100}$$

Hence

$$P(S_1 > t) = E\left[P\left(S_1 \ge t \middle| X_0\right)\right] = E\left[e^{-t\zeta(X_0)}\right] > 0. \tag{3.101}$$

Finally, we conclude that

$$P(\tau_{gel} > t) \ge P(X_t = X_0) = P(T_1 > t) \ge P(S_1 > t) > 0 \tag{3.102}$$

which was our aim. □

This concludes the section.

3.5 About the uniqueness for (SDE)

In this section, we deal with the uniqueness in law for (SDE), which is equivalent to the uniqueness for (MP) (see Propositions 3.2.10 and 3.3.3). We are not able to prove such uniqueness results by ourselves (except in the case where $K(x,y) = xy$, see the next section). However, we may prove uniqueness by using the results of the analysts. In other words, we may prove uniqueness in law for (SDE) once we know the uniqueness for the Smoluchowski equation.

We first consider the discrete case.

Proposition 3.5.1 *Let $Q_0 \in \mathcal{P}_1$ satisfy $\int x^2 Q_0(dx) < \infty$. Assume (H_β) and consider the corresponding T_0.*
Assume that $Q_0(\mathbb{N}^*) = 1$, and write Q_0 as $\sum_{k\ge1} \alpha_k \delta_k(dx)$. Set $n_0(k) = \alpha_k/k$. Assume that the uniqueness of a solution to (SC) with the kernel K and the initial condition n_0 holds on $[0,T_0[$. Then the uniqueness of a solution Q to (MP), on $[0,T_0[$ holds. Hence the uniqueness in law holds for (SDE), in the sense that any solution $(X_0,X,\tilde{X},N)$ to (SDE) with $\mathcal{L}(X_0) = Q_0$, satisfies $\mathcal{L}(X) = Q$.

As we will prove below a similar result in the continuous case, we just sketch the proof.

Proof Let $(X_0,X,\tilde{X},N)$ be any solution to (SDE) corresponding to the initial condition Q_0 and to the kernel K. It is clear that for all $t \in [0,T_0[$, $\mathcal{L}(X_t)$ has its support in $\mathbb{N}^*$, and thus can be written as $\sum_{k\geq 1} f(k,t)\delta_k(dx)$. Then, $n(k,t) = f(k,t)/k$ satisfies (SD), thanks to Proposition 3.2.4 and Remark 3.2.7. Thus $\mathcal{L}(X_t)$ is completely determined for each $t \in [0,T_0[$, since the uniqueness holds for (SD). This is of course not sufficient, but one can conclude exactly as in the proof of Proposition 3.5.4 below. □

The following corollary is immediately deduced from Proposition 3.5.1 and from Heilmann [33].

Corollary 3.5.2 *Assume that $Q_0 \in \mathcal{P}_1$ and that $\int x^2 Q_0(dx) < \infty$. Assume also that Q_0 is discrete, i.e. that its support is contained in $\mathbb{N}^*$. Then, if $K(x,y) \leq C(1+x+y)$ for all x, y in $\mathbb{N}^*$, uniqueness holds for (MP), and we have uniqueness in law for (SDE).*

In order to use the results of the analysts in the continuous case, we first have to check that for $(X_0,X,\tilde{X},N)$ a solution to (SDE), $\mathcal{L}(X_t)$ is really a modified solution to (SC): we have to prove that if Q_0 has a density, then for all $t \geq 0$, the law of X_t admits a density.

Proposition 3.5.3 *Assume that $X_0 > 0$ is a random variable of which the law Q_0 belongs to $\mathcal{P}_1$, and such that $E(X_0^2) < \infty$.*
Assume (H_β) and consider the corresponding T_0. Assume also that Q_0 is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}_+$, and that $K(x,y)$ is (not necessarily) increasing (for example in x when y is fixed).
Consider a solution $(X_0,X,\tilde{X},N)$ to (SDE). Then for all $t \in [0,T_0[$, the law of X_t is also absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}_+$. Hence the law of X_t is really a weak solution to (SC), in the sense that if $f(x,t)$ denotes the density of X_t, then $n(x,t) = f(x,t)/x$ is a weak solution to (SC).

Proof Let us denote by $f_0(x)$ the density of the law of X_0. Let $t \in]0,T_0[$ be fixed. Consider a Lebesgue-null set $\mathcal{A}$. Our aim is to check that
$\mathbb{P}(X_t \in \mathcal{A}) = 0$. First notice that

$$\begin{aligned}
\mathbb{P}(X_t \in \mathcal{A}) &= \int_0^\infty \mathbb{P}\big(X_t \in \mathcal{A} \mid X_0 = x\big) f_0(x)dx \\
&= \int_0^\infty \mathbb{P}\big(X_t^x \in \mathcal{A}\big) f_0(x)dx \\
&= E\left(\int_0^\infty \mathbb{1}_{\mathcal{A}}(X_t^x) f_0(x)dx\right) \qquad (3.103)
\end{aligned}$$

where X^x is a solution, on $[0,T_0[$, of the following standard S.D.E. (here $\tilde{X}$ is known, is fixed and behaves as a parameter) :

$$X_t^x = x + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{z \leq \frac{K(X_{s-}^x, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds,d\alpha,dz). \tag{3.104}$$

We will prove that for almost all ω, the map $x \mapsto X_t^x(\omega)$ can be written as $X_t^x(\omega) = x + \phi_{t,\omega}(x)$, for some increasing function $\phi_{t,\omega}$. This will allow to conclude, thanks to Lemma 3.7.2 of the appendix, that for almost all ω,

$$\int_0^\infty \mathbb{1}_{\mathcal{A}}(X_t^x) dx = 0 \tag{3.105}$$

thus that

$$\int_0^\infty \mathbb{1}_{\mathcal{A}}(X_t^x) f_0(x) dx = 0 \tag{3.106}$$

and hence, using (3.103) that $\mathbb{P}(X_t \in \mathcal{A}) = 0$, which was our aim.

It remains to check that for almost all ω, $X_t^x(\omega) = x + \phi_{t,\omega}(x)$, for some increasing function $\phi_{t,\omega}$. It of course suffices to prove that for all $x > y$,
$X_t^x - X_t^y \geq x - y$.
Let thus $x > y$ be fixed. Consider the following stopping time :

$$\tau = \inf \left\{ s \in [0,T_0[\mid X_s^x < X_s^y \right\}. \tag{3.107}$$

Then it is clear that for all $t < \tau$, since K is increasing,

$$\begin{aligned} &\int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{z \leq \frac{K(X_{s-}^x, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds,d\alpha,dz) \\ \geq &\int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{z \leq \frac{K(X_{s-}^y, \tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds,d\alpha,dz) \end{aligned} \tag{3.108}$$

from which we deduce that for all $s < \tau$,

$$X_s^x - X_s^y \geq x - y \tag{3.109}$$

It remains to prove that $\tau = T_0$. Let us assume that for some ω, $\tau(\omega) < T_0$. We deduce from (3.109) that

$$X_{\tau-}^x - X_{\tau-}^y \geq x - y. \tag{3.110}$$

Hence, still using the fact that K is increasing, we obtain that, for some random $\alpha_\tau \in [0,1]$, $z_\tau \in [0,\infty[$,

$$\begin{aligned} \Delta X_\tau^x &= \tilde{X}_{\tau-}(\alpha_\tau) \mathbb{1}_{\left\{z_\tau \leq \frac{K(X_{\tau-}^x, \tilde{X}_{\tau-}(\alpha_\tau))}{\tilde{X}_{\tau-}(\alpha_\tau)}\right\}} \\ &\leq \tilde{X}_{\tau-}(\alpha_\tau) \mathbb{1}_{\left\{z_\tau \leq \frac{K(X_{\tau-}^y, \tilde{X}_{\tau-}(\alpha_\tau))}{\tilde{X}_{\tau-}(\alpha_\tau)}\right\}} \\ &= \Delta X_\tau^y. \end{aligned} \tag{3.111}$$

We deduce that

$$X^x_\tau = X^x_{\tau-} + \Delta X^x_\tau \geq x - y + X^y_{\tau-} + \Delta X^y_\tau \geq x - y + X^y_\tau \tag{3.112}$$

which contradicts the definition of τ. □

Thanks to the previous Proposition, we are able to state the following uniqueness result :

Proposition 3.5.4 *Let $Q_0 \in \mathcal{P}_1$ satisfy $\int x^2 Q_0(dx) < \infty$. Assume (H_β) and consider the corresponding T_0. Assume also that K is increasing and satisfies the regularity condition : there exists a locally bounded function ζ on $[0,\infty[^2$ such that for all $x,x',y \in \mathbb{R}_+$,*

$$|K(x,y) - K(x',y)| \leq |x - x'|\zeta(x,x')(1 + y^2). \tag{3.113}$$

Assume also that Q_0 admits a density $f_0(x)$, and set $n_0(x) = f_0(x)/x$. Assume that the uniqueness of a weak solution to (SC) with initial condition n_0 and kernel K holds. Then there exists a unique solution Q to (MP) with initial condition Q_0. Thus uniqueness in law holds for (SDE), in the sense that any solution $(X_0,X,\tilde{X},N)$ to (SDE) with $\mathcal{L}(X_0) = Q_0$ satisfies $\mathcal{L}(X) = Q$.

Notice that (3.113) always holds when $K(x,y)$ is of the form $A + B(x+y) + Cxy$, for some nonnegative constants A,B,C.

Proof Let Q be a solution to (MP). Thanks to Propositions 3.5.3 and 3.3.3, we know that for all t, $Q_t(dx) = f(t,x)\,dx$, for some function $f \colon [0,T_0[\times\mathbb{R}_+ \to \mathbb{R}_+$. Hence, Proposition 3.2.4 (2) and Remark 3.2.7 show that $f(x,t) = xn(x,t)$, where n is the unique solution of (SC). Since $Q_0 \in \mathcal{P}_1$ and $\int x^2 Q_0(dx) < \infty$, it is easily deduced that for all $T < T_0$,

$$\begin{aligned}&\sup_{t\in[0,T]} \int_0^\infty \left(x + x^2 + x^3\right) n(x,t)dx \\ &= \sup_{t\in[0,T]} \left[1 + E(X_t) + E(X_t^2)\right] < \infty\end{aligned} \tag{3.114}$$

The uniqueness of $\{Q_t\}_{t\in[0,T_0[}$ is proved, but we need more : we want to prove the uniqueness of $Q \in \mathcal{P}_1^\uparrow([0,T_0[,\, H_{Q_0})$.
As Q satisfies (MP) it also satisfies the simple (because linear) martingale problem (MPS) : for all $\phi \in C_b^1(\mathbb{R}_+)$,

$$\phi(Z_t) - \phi(Z_0) - \int_0^t \int_{\mathbb{R}_+} \left(\phi(Z_s + y) - \phi(Z_s)\right) K(Z_s,y) n(y,s)\,dyds \tag{3.115}$$

is a Q-martingale, Z standing for the canonical process of $\mathbb{D}^\uparrow([0,T_0[,\, H_{Q_0})$. We will prove the uniqueness for (MPS). In this way, we will deduce that Q is entirely

determined, since any solution to (MP) satisfies also (MPS). This will conclude the proof.
The uniqueness for (MPS) is equivalent to the uniqueness in law for the following S.D.E. :

$$Y_t = X_0 + \int_0^t \int_{\mathbb{R}_+} \int_0^\infty y \mathbb{1}_{\left\{z \leq \frac{K(Y_{s-},y)}{y}\right\}} \mu(ds,dy,dz) \tag{3.116}$$

$\mu(ds,dy,dz)$ being a Poisson measure on $[0,T_0[\times\mathbb{R}^+ \times [0,\infty[$ with intensity measure $ds\,(yn(y,s)dy)\,dz$. But the strong uniqueness (which implies the uniqueness in law) holds for this equation, thanks to standard arguments : local Lipschitz continuity and at most linear growth. Indeed, for all $u \geq 0$, all $T < T_0$, we obtain, using (H_β) and (3.114),

$$\begin{aligned} \sup_{s\in[0,T]} \int_{\mathbb{R}_+} \int_0^\infty & y \mathbb{1}_{\left\{z \leq \frac{K(u,y)}{y}\right\}} dz\, yn(y,s)\, dy \\ & \leq A(1+u) \sup_{s\in[0,T]} \int_{\mathbb{R}_+} (y+y^3) n(y,s)\, dy \\ & \leq A_T(1+u) \end{aligned} \tag{3.117}$$

the constant A_T depending only on T. We also have, for all u,u' in $[0,\infty[$, all $T < T_0$, by using (3.113) and (3.114),

$$\begin{aligned} \sup_{s\in[0,T]} \int_{\mathbb{R}_+} \int_0^\infty & \left| y \mathbb{1}_{\left\{z \leq \frac{K(u,y)}{y}\right\}} - y \mathbb{1}_{\left\{z \leq \frac{K(u',y)}{y}\right\}} \right| dz\, yn(y,s)\, dy \\ & \leq \sup_{s\in[0,T]} \int_{\mathbb{R}_+} |K(u,y) - K(u',y)|\, yn(y,s)\, dy \\ & \leq \zeta(u,u')|u-u'| \sup_{s\in[0,T]} \int_{\mathbb{R}_+} (y+y^3) n(y,s) dy \\ & \leq A_T \zeta(u,u')|u-u'|. \end{aligned} \tag{3.118}$$

Using these properties, the strong uniqueness is easily checked for equation (3.116). This implies the uniqueness for (MPS) and concludes the proof. □

We finally deduce the following corollary from Aldous [1], Principle 1.

Corollary 3.5.5 *Assume that Q_0 belongs to $\mathcal{P}_1$ and that $\int x^2 Q_0(dx) < \infty$. Assume that $K(x,y) \leq C(1+x+y)$, that K is increasing, and that the regularity condition (3.113) holds.*
In addition, assume that Q_0 admits a density $f_0(x)$ and that $\int \frac{1}{x} Q_0(dx) < \infty$.
Then uniqueness in law holds for (SDE), and so does uniqueness for (MP).

3.6 Study of the exact multiplicative kernel

In this short section, we will make explicit computations for the case $K(x,y) = xy$. In this explicit case, we obtain very satisfying results. In particular, we get rid of the assumption $\int x^2 Q_0(dx) < \infty$. We build directly a solution by using a Picard iteration without cutoff. Uniqueness for (SDE) is proved without using the results of the analysts. Let us begin with the statement.

Theorem 3.6.1 *Assume that $K(x,y) = xy$. Let Q_0 belong to $\mathcal{P}_1$ and $T_0 = 1/\int xQ_0(dx)$. Then the following results hold.*

1. *For any random variable X_0 of law Q_0, any independent Poisson measure $N(dt,d\alpha,dz)$ with intensity measure $dtd\alpha dz$, there exists a solution $(X_0,X,\tilde{X},N)$ to (SDE) on $[0,T_0[$.*
2. *The obtained law $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ is unique, and depends only on Q_0.*
3. *Hence existence and uniqueness for (MP) hold.*

Proof 1. Let X_0 and N be fixed. We consider the following Picard iterations : first, we consider the process $X^0 \equiv X_0$. Then we consider any α-process $\tilde{X}^0$ such that $\mathcal{L}_\alpha(\tilde{X}^0) = \mathcal{L}(X^0)$. Once everything is built up to n, we consider

$$X_t^{n+1} = X_0 + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}^n(\alpha) 1\!\!1_{\{z \leq X_{s-}^n\}} N(ds,d\alpha,dz) \tag{3.119}$$

and we build an α-process $\tilde{X}^{n+1}$ such that

$$\mathcal{L}_\alpha(\tilde{X}^{n+1}|\tilde{X}^0,\dots,\tilde{X}^n) = \mathcal{L}(X^{n+1}|X^0,\dots,X^n). \tag{3.120}$$

One can check that if $f_n(t) = E(X_t^n) = E_\alpha(\tilde{X}_t^n)$, then $f_0(t) = a = E(X_0)$, and for all $n \geq 0$,

$$f_{n+1}(t) = a + \int_0^t f_n^2(s)ds \tag{3.121}$$

which easily implies that for all $t \leq T_0 = 1/a$,

$$\sup_n f_n(t) \leq \frac{a}{1-at}. \tag{3.122}$$

Let now $g_n(t) = E\left[\sup_{s\in[0,t]} |X_s^{n+1} - X_s^n|\right] = E_\alpha\left[\sup_{s\in[0,t]} |\tilde{X}_s^{n+1} - \tilde{X}_s^n|\right]$. A simple computation shows that

$$\begin{aligned} g_n(t) &\leq \int_0^t EE_\alpha\Big(\int_0^\infty |\tilde{X}_s^n 1\!\!1_{\{z\leq X_s^n\}} - \tilde{X}_s^{n-1} 1\!\!1_{\{z\leq X_s^{n-1}\}}|dz\Big)ds \\ &\leq \int_0^t EE_\alpha\Big(X_s^n \left|\tilde{X}_s^n - \tilde{X}_s^{n-1}\right| + \tilde{X}_s^{n-1}\left|X_s^n - X_s^{n-1}\right|\Big)ds \\ &\leq \int_0^t \frac{2a}{1-as} \times g_{n-1}(s)ds. \end{aligned} \tag{3.123}$$

It is now clear that for all $T < T_0$,

$$\sum_{n\geq 1} g_n(T) < \infty. \tag{3.124}$$

Hence there exist a process $X \in L_1^{T_0,\uparrow}(\ H_{Q_0})$ and an α-process $\tilde{X}$ such that for all $T < T_0$,

$$E\left[\sup_{s\in[0,T]} |X_s - X_s^n|\right] = E_\alpha\left[\sup_{s\in[0,T]} |\tilde{X}_s - \tilde{X}_s^n|\right] \qquad (3.125)$$

goes to 0 when n tends to infinity. One easily concludes that $(X_0,X,\tilde{X},N)$ satisfies (SDE).

2. The uniqueness is much more difficult to prove. We can follow the proof of Desvillettes, Graham, Méléard [11] which concerns the Boltzmann equation, and we only give the main steps of the proof.

Step 1 : it is clear that in the existence proof, the obtained law $\mathcal{L}(X) = \mathcal{L}(\tilde{X})$ does not depend on the possible choices for Ω, X_0, N and $\tilde{X}^n(\alpha)$, but only on the law of the initial condition $\mathcal{L}(X_0) = Q_0$.

Step 2 : let thus Ω, X_0 and N be fixed. Consider two solutions $(X_0,X,\tilde{X},N)$ and $(X_0,Y,\tilde{Y},N)$ of (SDE). We have to prove that $\mathcal{L}(X) = \mathcal{L}(Y)$. Let us denote $Q = \mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ and $Q' = \mathcal{L}(Y) = \mathcal{L}_\alpha(\tilde{Y})$. For $T < T_0$, we consider the quantity

$$\rho_T(Q,Q') = \inf_{\tilde{Z},\tilde{Z}'}\left\{E_\alpha\left(\sup_{s\in[0,T]} |\tilde{Z}_s - \tilde{Z}'_s|\right) \ ;\ \mathcal{L}_\alpha(\tilde{Z}) = Q,\, \mathcal{L}_\alpha(\tilde{Z}') = Q'\right\}. \qquad (3.126)$$

For some $\varepsilon > 0$ fixed, we consider α-processes $\tilde{X}^\varepsilon$ and $\tilde{Y}^\varepsilon$ such that $\mathcal{L}_\alpha(\tilde{X}^\varepsilon) = Q$, $\mathcal{L}_\alpha(\tilde{Y}^\varepsilon) = Q'$, and

$$\rho_T(Q,Q') \leq E_\alpha\left(\sup_{s\in[0,T]} |\tilde{X}_s^\varepsilon - \tilde{Y}_s^\varepsilon|\right) + \varepsilon. \qquad (3.127)$$

Then we build X^ε and Y^ε, in such a way that $(X_0,X^\varepsilon,\tilde{X}^\varepsilon,N)$ and $(X_0,Y^\varepsilon,\tilde{Y}^\varepsilon,N)$ be solutions to (SDE). This can be done by solving linear S.D.E.s (because $\tilde{X}^\varepsilon$ and $\tilde{Y}^\varepsilon$ are fixed processes). For all $T < T_0$, one easily obtains the existence of a constant A_T, not depending on ε, such that

$$E\left[\sup_{s\in[0,T]} X_s^\varepsilon\right] + E\left[\sup_{s\in[0,T]} Y_s^\varepsilon\right] \leq A_T. \qquad (3.128)$$

Finally, we obtain, for any $t \leq T < T_0$,

$$\begin{aligned} E\left(\sup_{s\in[0,t]} |X_s^\varepsilon - Y_s^\varepsilon|\right) &\leq \int_0^t EE_\alpha\Big(X_s^\varepsilon \left|\tilde{X}_s^\varepsilon - \tilde{Y}_s^\varepsilon\right| + \tilde{Y}_s^\varepsilon\,|X_s^\varepsilon - Y_s^\varepsilon|\Big)ds \\ &\leq A_T \int_0^t \left[\rho_T(Q,Q') + \varepsilon + E(|X_s^\varepsilon - Y_s^\varepsilon|)\right] ds \qquad (3.129) \end{aligned}$$

which yields that for any $t \leq T$

$$E\left(\sup_{s\in[0,t]} |X_s^\varepsilon - Y_s^\varepsilon|\right) \leq A_T T(\rho_T(Q,Q') + \varepsilon)e^{A_T T}. \tag{3.130}$$

The left hand side member is greater than $\rho_T(Q,Q')$. We thus obtain, making ε go to 0,

$$\rho_T(Q,Q') \leq A_T T \rho_T(Q,Q')e^{A_T T}. \tag{3.131}$$

But X_T is increasing in T. Hence, we can choose τ small enough, such that $A_\tau \tau e^{A_\tau \tau} < 1$ and thus $\rho_\tau(Q,Q') = 0$. The Markov property of the Poisson measure allows to prove that the result remains true on $[0,T_0[$, i.e. that $\rho_T(Q,Q') = 0$ for all $T < T_0$. □

3.7 Appendix

First, we recall the Aldous criterion for tightness (see Jacod, Shiryaev [35]).

Theorem 3.7.1 *Let $\{X_t^n\}_{t\in[0,T_0[}$ be a family of càdlàg adapted processes on $[0,T_0[$, for some $T_0 \leq \infty$. Denote by $Q^n \in \mathcal{P}(\mathbb{D}([0,T_0[,\mathbb{R}))$ the law of X^n. Assume that:*

1. *For all $T < T_0$,*

$$\sup_n E\left[\sup_{t\in[0,T]} |X_t^n|\right] < \infty \tag{3.132}$$

2. *For all $T < T_0$, all $\eta > 0$,*

$$\sup_n \sup_{(S,S')\in ST_T(\delta)} P\left[|X_{S'}^n - X_S^n| \geq \eta\right] \longrightarrow 0 \tag{3.133}$$

when δ goes to 0, where $ST_T(\delta)$ is the set of couples (S,S') of stopping times satisfying a.s. $0 \leq S \leq S' \leq (S+\delta)\wedge T$.

Then the family $\{Q^n\}$ is tight. Furthermore, any limiting point Q of this family is the law of a quasi-left continuous process, i.e. for all $t \in [0,T_0[$ fixed,

$$\int_{\mathbb{D}([0,T_0[,\mathbb{R})} 1\!\!1_{\{\Delta x(t)\neq 0\}} Q(dx) = 0. \tag{3.134}$$

We now prove an easy absolute continuity result.

Lemma 3.7.2 *Let φ be an increasing map from $\mathbb{R}_+$ into itself. Let $\mathcal{A}$ be a Lebesgue-null subset of $\mathbb{R}_+$. Then*

$$\int_0^\infty 1\!\!1_{\mathcal{A}}\big(x+\varphi(x)\big)dx = 0. \tag{3.135}$$

Proof We set $f(x) = x+\varphi(x)$. Since φ is increasing, the Stieljes measure $df^{-1}(x)$ is clearly smaller than the Lebesgue measure dx on $\mathbb{R}_+$. In particular, $df^{-1}(x) << dx$. Hence,

$$\int_0^\infty \mathbb{1}_{\mathcal{A}}\big(x+\varphi(x)\big)dx = \int_{\mathcal{A}} df^{-1}(x) = 0 \tag{3.136}$$

which was our aim. □

We carry on with a generalised Gronwall Lemma (see Beesack [7]).

Lemma 3.7.3 *Let $a, b \geq 0$. Consider a continuous function g on $[0,T]$, satisfying for all $t \in [0,T]$,*

$$g(t) \leq a + b\int_0^t g^2(s)ds. \tag{3.137}$$

Then, for all $t < T_0 = 1/ab$,

$$g(t) \leq \frac{a}{1-abt}. \tag{3.138}$$

Proof Let us denote

$$U(t) = a + b\int_0^t g^2(s)ds.$$

Clearly for all t, $x(t) \leq U(t)$. Let $G(x) = -\frac{1}{x}$. We have :

$$\frac{d}{ds}G(U(s)) = b\frac{x^2(s)}{U^2(s)} \leq b$$

thus :

$$G(U(t)) \leq G(a) + bt$$

which yields

$$U(t) \leq \frac{a}{1-abt}.$$

This inequality ends the proof. □

4

Study of a stochastic particle system associated with the Smoluchowski coagulation equation

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Abstract

The Smoluchowski coagulation equation models the evolution of the density $n(x,t)$ of the particles of size x at the instant $t \geq 0$ for a system in which a coalescence phenomenon occurs. Two versions of this equation exist : the sizes can be discrete ($x \in \mathbb{N}^*$) or continuous ($x \in \mathbb{R}_+$).
In Deaconu, Fournier and Tanré [9], we constructed a nonlinear pure jump Markov process $X = (X_t, t \geq 0)$ which is related to the Smoluchowski equation in the following way : its marginal flow $(\mathcal{L}(X_t))_{t\geq 0}$ satisfies, in a certain sense, the (discrete or continuous) Smoluchowski equation.
In the present paper, we « linearise » the Markov process X by using a simulable interacting particle system. We obtain thus an approximation scheme for the law of X, and also for the solution to the Smoluchowski equation : the empirical probability measure, associated with the particle system converges, as the size of the system grows to infinity, to the solution of the Smoluchowski equation. This convergence result is obtained under satisfying assumptions, and the proof covers both discrete and continuous equations.
Afterwards, for the discrete case and under a quite stringent hypothesis, we prove a central limit theorem associated with our Monte Carlo method.
Numerical results on this approximation scheme are included at the end of the paper.

Key words : Smoluchowski's coagulation equations, interacting stochastic particle systems, Monte Carlo methods.

MSC 2000 : 82C22, 60H30, 60J75, 35M10.

4.1 Introduction

Smoluchowski's coagulation equation writes in the discrete case

$$\begin{cases} \dfrac{d}{dt}n(k,t) = \dfrac{1}{2}\displaystyle\sum_{j=1}^{k-1} K(j,k-j)n(j,t)n(k-j,t) - n(k,t)\sum_{j=1}^{\infty} K(j,k)n(j,t) \\ \\ n(k,0) = n_0(k) \end{cases} \qquad (SD)$$

and describes the time evolution of the average number of particles of mass $k \in \mathbb{N}^*$, in a dynamic particle system where coagulation phenomenons occur. $K(i,j)$ denotes the coagulation rate of clusters of size i and j, we call it coagulation kernel and it is supposed symmetric and positive. In this model we allow only coagulation by pairs. If the clusters' mass take their values in the continuous set $\mathbb{R}_+$, we obtain the continuous version of the Smoluchowski coagulation equation. It writes easily after replacing sums by integrals in (SD):

$$\begin{cases} \dfrac{\partial}{\partial t}n(x,t) = \dfrac{1}{2}\displaystyle\int_0^x K(y,x-y)n(y,t)n(x-y,t)dy - n(x,t)\int_0^{\infty} K(x,y)n(y,t)dy \\ \\ n(x,0) = n_0(x) \end{cases}$$

for all $x \in \mathbb{R}_+$.

Smoluchowski's coagulation equations govern various phenomenons as polymerisation, aggregation of colloidal particles, formation of stars and planets, behaviour of fuel mixtures in engines etc.

The first model of stochastic coalescent was introduced by Marcus [47] and Lushnikov [46] as a model of gelation. Intuitively, the Smoluchowski's coagulation equation is an infinite volume mean field description of coalescence in term of a deterministic equation.
In order to approach this infinite volume mean field description the natural idea is to consider the finite volume mean field description corresponding to. Let us describe it in the discrete situation:
Fix $N \in \mathbb{N}^*$ and consider the state space of the form

$$E = \{\exists m ,\bar{x} = (x_1, \dots ,x_m); x_i \in \mathbb{N}^*, \sum_{i=1}^{m} x_i = N\}.$$

So $\bar{x}$ is a possible configuration of the system with clusters of masses $x_1, \dots ,x_m$.
We can define a continuous time Markov chain by saying that each pair of clusters (x_i,x_j) for $j \neq i$ coalesces into a cluster of size $x_i + x_j$ at rate $\frac{K(x_i,x_j)}{N}$.
Let us express this elementary idea by considering the state space:
$F = \{(n_1,n_2, \dots ,n_N)\}$ where n_x represents the number of clusters of mass x.

Obviously, $\sum_x n_x = N$.
We can regard a coalescence as a transition of the form, for $i < j$:

$$\begin{aligned}(n_1, \dots, n_N) \quad \longrightarrow \quad & (n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \\ & \dots, n_{j-1}, n_j - 1, n_{j+1}, \dots, n_{i+j-1}, n_{i+j} + 1, n_{i+j+1}, \dots, n_N)\end{aligned}$$

at rate $\frac{K(i,j) n_i n_j}{N}$.
This process is called the Marcus-Lushnikov process and we denote it by $ML^{(N)}$. We can regard either $ML^{(N)}(i,t)$ the random number of mass i cluster or $ML_l^{(N)}(t)$ the mass of the l'th largest cluster in the system.
This process is the natural stochastic analog for the discrete Smoluchowski equation, when we are interested in finite-sized clusters.
We can generalise the previous construction, by a normalisation technique. See the survey of Aldous [1] for further details.
In their paper, Lang and Nguyen [42] derived a version of Smoluchowski's coagulation equation from a system driven by a Brownian motion.
Aldous [1] provides a very interesting and up to date survey on what has been done and has to be done from a probabilistic point of view for the Smoluchowski's coagulation equation.
Recently many rigorous results have been obtained for the stochastic approximation of the coagulation equation. Jeon [36] proved that, for a discrete coagulation-fragmentation equation, with coagulation kernel K, and under the hypothesis

$$\lim_{i+j \to \infty} \frac{K(i,j)}{ij} = 0 \tag{4.1}$$

weak limit points of some stochastic particle system exist and they provide a solution to the equation. This result was extended to the continuous case by Norris [52], [53].
Guias [32] introduced a fast method of direct simulation for the stochastic process corresponding to the coagulation-fragmentation dynamics with multiplicative coagulation kernel. Babovsky [3] proposed a Monte Carlo simulation scheme for the Smoluchowski's coagulation equation.

In Deaconu, Fournier and Tanré [9], we have introduced a new stochastic approach to (SD) and (SC). We have built a pure jump Markov process $X = (X_t, t \geq 0)$ whose law is related to the solution of the Smoluchowski equation in the following sense: in the discrete case $P(X_t = k) = kn(k,t)$, while in the continuous case $P(X_t \in dx) = xn(x,t)dx$.
For each ω, $X_t(\omega)$ has to be understood as the evolution of the size of one particle in the system (which has of course an infinite number of particles). At the initial instant $t = 0$ we choose randomly one particle among all the particles. The stochastic dynamic (close to the deterministic usual dynamic) is described as follow: we consider first that, the instant of coagulation of particles are Poissonian, with a jump rate depending on $X_t(\omega)$ and on the coagulation kernel K. At one of these jump instants τ, we choose randomly another particle in the system, and consider

its size $\tilde{X}_\tau$. It is obvious that $\tilde{X}_\tau$ has to be a random variable whose law coincides with the law of X_τ. This approach get ride to a nonlinearity. We control the rate of coagulation between particles of size x and y, by using an acceptance-rejection procedure involving the coagulation kernel K.
The process X is thus naturally the solution of a nonlinear Poisson driven stochastic differential equation.

Our aim in this paper is to present and to study an approximation scheme for the law of X. Due to the nonlinearity of the process X we cannot apply a direct approximation for it. In order to avoid this difficulty, we introduce a particle system, which satisfies a « linearised » version of the S.D.E. verified by X. Furthermore, instead of considering an infinite system of particles, we consider a finite system of particles.
This particle system is not physically « reasonable », because we will see that when a coagulation occurs, the total mass of the system grows. It is however naturally related to the Markov process introduced in [9], which was naturally derived from the expression of the Smoluchowski equation.
The simulation algorithm we obtain is quite simple : the main difference with other particle systems is that the number of particle is constant on any time interval, and also that we are able to treat the continuous situation.

Our approach is closely related to the one introduced recently by Eibeck and Wagner [14], [15]. Eibeck and Wagner introduced a new class of stochastic algorithms for approximation of stochastic dynamics, in which the number of particles is also constant in time. These algorithms are based on the introduction of jumps (expressing the coagulation phenomenon) and an acceptance-rejection technique for distributions depending on particle size. In one of their papers, they studied also numerically the gelation phenomena.

The method used in [9] and also in the present paper are strongly inspired by probabilistic works on the Boltzmann equation, which deals with the distribution of the speeds in a gas. An associated Markov process was initially introduced by Tanaka [62]. Many qualitative results have been derived from this approach. The use of Tanaka's representation in order to introduce particle systems can be found *e.g.* in Graham, Méléard [30], see also Fournier and Méléard [24] for the physical 3D Boltzmann equation without cutoff, and Méléard [50] for a central limit theorem.

The present paper is structured as follows. In Section 4.2, we recall the main notations and results obtained in [9]. We introduce the nonlinear stochastic differential equation, and denote by $(X_t,t \geq 0)$ its solution. $(X_t,t \geq 0)$ has for law the solution of the Smoluchowski equation (in some sense to be defined).
Section 4.3 is devoted to "linearise" the nonlinear SDE. After linearisation we obtain a particle system, easily simulable, whose law converges to the solution of the Smoluchowski's coagulation equation, as the size of the system goes to infinity. We obtain these convergence results under natural and satisfying assumptions. Section

4.4 describes the simulation of the particle system previously introduced. We prove that this procedure is finite in time.
In Section 4.5, we prove a central limit theorem associated to our Monte Carlo method. We consider only the discrete case, because the arguments are very technical, and seem to be much more complicated in the continuous case. We obtain a very precise result, under the quite stringent assumption that the coagulation kernel K is bounded.
Section 4.6 gives numerical results. We remark in these simulations that the central limit theorem seems to apply also in cases of unbounded kernels and continuous ones.
An appendix lies at the end of the paper.

4.2 Notations and previous results

Let us recall in this section the probabilistic interpretation introduced in [9]. We construct a pure jump stochastic process $(X_t, t \geq 0)$ whose law is the solution of the Smoluchowski coagulation equation in the following sense : in the discrete case $P(X_t = k) = kn(k,t)$ for all $t \geq 0$ and $k \in \mathbb{N}^*$, while in the continuous case $P(X_t \in dx) = xn(x,t)dx$ for all $t \geq 0$ and $x \in \mathbb{R}_+$. For each ω, $X_t(\omega)$ can be regarded as the evolution of the size of one « mean » particle in the system.
The jump process satisfies a nonlinear Poisson driven stochastic differential equation. The main property which allows us a probabilistic and unified treatment for (SC) and (SD) is that we have conservation of mass in the system. This means in the discrete case that a solution $(n(k,t), t \geq 0, k \in \mathbb{N}^*)$ of (SD) will satisfy until a time $T_0 \leq \infty$,

$$\text{for all } t \in [0,T_0[, \quad \sum_{k \geq 1} k\, n(k,t) = 1. \tag{4.2}$$

Similarly, in the continuous case, a solution $(n(x,t), t \geq 0, x \in \mathbb{R}_+)$ of (SC) will satisfy until a time $T_0 \leq \infty$,

$$\text{for all } t \in [0,T_0[, \quad \int_0^\infty xn(x,t)dx = 1. \tag{4.3}$$

Thus, either in the discrete or continuous case, the quantity

$$Q_t(dx) = \sum_{k \geq 1} k\, n(k,t) \delta_k(dx) \quad \text{or} \quad Q_t(dx) = x\, n(x,t)\, dx \tag{4.4}$$

(where δ_k denotes the Dirac mass at k) is a probability measure on $\mathbb{R}_+$ for all $t \in [0,T_0[$, and has to be understood as the repartition of particles mass' at some instant t. This leads us to define a modified Smoluchowski equation. Let

$$\mathcal{P}_1 = \left\{ Q \text{ probability measure on } \mathbb{R}_+^*, \text{ and } \int_{\mathbb{R}_+} xQ(dx) < \infty \right\}. \tag{4.5}$$

For $Q_0 \in \mathcal{P}_1$, we denote by

$$H_{Q_0} = \overline{\left\{ \sum_{i=1}^{n} x_i \,;\, x_i \in \text{Supp } Q_0 \,,\, n \in \mathbb{N}^* \right\}}^{\mathbb{R}_+} . \tag{4.6}$$

H_{Q_0} represents the smallest closed subset of $\mathbb{R}_+$ in which the sizes of the particles will always take their values.

Definition 4.2.1 *Let Q_0 belong to $\mathcal{P}_1$ and $T_0 \leq \infty$. We will say that $(Q_t(dx), t \in [0,T_0[)$ is a* **weak solution** *to (MS) on $[0,T_0[$ with initial condition Q_0 if: for all $t \in [0,T_0[$, $\text{Supp } Q_t \subset H_{Q_0}$ and $Q_t \in \mathcal{P}_1$, and for all test function $\varphi \in C_0^1(\mathbb{R}_+)$*

$$\int_0^\infty \varphi(x) Q_t(dx) = \int_0^\infty \varphi(x) Q_0(dx)$$
$$+ \int_0^t \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} [\varphi(x+y) - \varphi(x)] \frac{K(x,y)}{y} Q_s(dy) Q_s(dx) ds. \tag{4.7}$$

This definition allows us to treat together both discrete and continuous cases. To make this assertion clear, let us recall the following result (see [9]) :

Proposition 4.2.2 *Let $(Q_t(dx), t \in [0,T_0[)$ be a weak solution to (MS), with initial condition $Q_0 \in \mathcal{P}_1$, for some $T_0 \leq \infty$.*

1. *If $\text{Supp } Q_0 \subset \mathbb{N}^*$ then $H_{Q_0} \subset \mathbb{N}^*$. Thus for all $t \in [0,T_0[$, $\text{Supp } Q_t \subset \mathbb{N}^*$, and we can write Q_t as :*

$$Q_t(dx) = \sum_{k \geq 1} \alpha_k(t)\, \delta_k(dx) \quad \textit{where} \quad \alpha_k(t) = Q_t(\{k\}). \tag{4.8}$$

   Then, the function $n(k,t) = \alpha_k(t)/k$ is a solution to (SD) on $[0,T_0[$, with initial condition $n_0(k) = \alpha_k(0)/k$.
2. *Assume now that for all $t \in [0,T_0[$, the probability measure Q_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}_+$. We can thus write $Q_0(dx) = f_0(x)\, dx$ and for any $t \in]0,T_0[$, $Q_t(dx) = f(x,t)\, dx$. Then $n(x,t) = f(x,t)/x$ is a weak solution to (SC) on $[0,T_0[$, with initial condition $n_0(x) = f_0(x)/x$.*
3. *Other cases, as mixed cases, are contained in (MS).*

Notice that the assumption $Q_0 \in \mathcal{P}_1$ simply means that the initial condition associated with the Smoluchowski equation admits a second order moment : in the discrete case, $\sum_k k^2 n_0(k) < \infty$, while in the continuous case we write $\int x^2 n_0(x) dx < \infty$. This hypothesis appears to be classical in the study of the Smoluchowski's coagulation equation.

Equation (MS) can be seen as the evolution equation of the time marginals of a pure jump Markov process. In order to exploit this remark, we associate to (MS) a martingale problem.

Notation 4.2.3 *Let $T_0 \leq \infty$ and $Q_0 \in \mathcal{P}_1$ be fixed. We denote by $\mathbb{D}^{\uparrow}([0,T_0[,\ H_{Q_0})$ the set of positive increasing càdlàg functions from $[0,T_0[$ to H_{Q_0}. We denote by $\mathcal{P}_1^{\uparrow}([0,T_0[,\ H_{Q_0})$ the set of probability measures Q on $\mathbb{D}^{\uparrow}([0,T_0[,\ H_{Q_0})$ such that*

$$Q\left(\left\{x \in \mathbb{D}^{\uparrow}([0,T_0[,\ H_{Q_0})\ ;\ x(0) > 0\right\}\right) = 1 \tag{4.9}$$

and such that for all $t < T_0$,

$$\int_{x \in \mathbb{D}^{\uparrow}([0,T_0[,\ H_{Q_0})} x(t) Q(dx) < \infty. \tag{4.10}$$

Definition 4.2.4 *Let $T_0 \leq \infty$, and $Q_0 \in \mathcal{P}_1$ be fixed. Consider $Q \in \mathcal{P}_1^{\uparrow}([0,T_0[,\ H_{Q_0})$. Let Z be the canonical process of $\mathbb{D}^{\uparrow}([0,T_0[,\ H_{Q_0})$. We say that Q is a solution to the martingale problem (MP) on $[0,T_0[$ if for all $\varphi \in C_b^1(\mathbb{R}_+)$ and $t \in [0,T_0[$,*

$$\varphi(Z_t) - \varphi(Z_0) - \int_0^t \int_{\mathbb{R}_+} [\varphi(Z_s + y) - \varphi(Z_s)] \frac{K(Z_s,y)}{y} Q_s(dy) ds \tag{4.11}$$

is a $Q-$martingale, where Q_s denotes the law of Z_s under Q.

Taking expectations in (4.11), we obtain :

Remark 4.2.5 *Let Q be a solution to the martingale problem (MP) on $[0,T_0[$. For $t \in [0,T_0[$, let Q_t denote its time marginal. Then $(Q_t)_{t\in[0,T_0[}$ is a weak solution of (MS) with initial condition Q_0.*

We are now seeking for a pathwise representation of the martingale problem (MP). To this aim, let us introduce some notations. The main ideas of the following notations and definition are taken from Tanaka [62], who was dealing with the Boltzmann equation of Maxwell molecules.

Notation 4.2.6
1. *We consider two probability spaces : $(\Omega,\mathcal{F},\mathbb{P})$ is an abstract space and $([0,1],\mathcal{B}[0,1],d\alpha)$ is an auxiliary space (here $d\alpha$ denotes the Lebesgue measure). In order to avoid confusions, the expectation on $[0,1]$ will be denoted E_α, the laws $\mathcal{L}_\alpha$, the processes will be called α-processes, etc.*
2. *Let $T_0 \leq \infty$ and $Q_0 \in \mathcal{P}_1$ be fixed. An increasing positive càdlàg process $(X_t(\omega), t \in [0,T_0[)$ is said to belong to $L_1^{T_0,\uparrow}(\ H_{Q_0})$ if its law belongs to $\mathcal{P}_1^{\uparrow}([0,T_0[,\ H_{Q_0})$. In the same way, an increasing positive càdlàg α-process $(\tilde{X}_t(\alpha), t \in [0,T_0[)$ is said to belong to $L_1^{T_0,\uparrow}(\ H_{Q_0})$-$\alpha$ if its α-law belongs to $\mathcal{P}_1^{\uparrow}([0,T_0[,\ H_{Q_0})$.*

Definition 4.2.7 *Let $T_0 \leq \infty$ and $Q_0 \in \mathcal{P}_1$ be fixed. We say that $(X_0,X,\tilde{X},N)$ is a solution to the problem (SDE) on $[0,T_0[$ if:*

1. *$X_0 : \Omega \to \mathbb{R}_+$ is a random variable whose law is Q_0.*
2. *$X_t(\omega) : [0,T_0[\times\Omega \to \mathbb{R}_+$ is a $L_1^{T_0,\uparrow}(\ H_{Q_0})$-process*
3. *$\tilde{X}_t(\alpha) : [0,T_0[\times[0,1] \to \mathbb{R}_+$ is a $L_1^{T_0,\uparrow}(\ H_{Q_0})$-$\alpha$-process.*
4. *$N(\omega,dt,d\alpha,dz)$ is a Poisson measure on $[0,T_0[\times[0,1]\times\mathbb{R}_+$ with intensity measure $dt\,d\alpha\,dz$ and is independent of X_0.*
5. *X and $\tilde{X}$ have same law on their respective probability spaces: $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ (this equality holds in $\mathcal{P}_1^{\uparrow}([0,T_0[,\ H_{Q_0})$).*
6. *Finally, the following S.D.E. is satisfied on $[0,T_0[$:*

$$X_t = X_0 + \int_0^t \int_0^1 \int_0^\infty \tilde{X}_{s-}(\alpha) \mathbb{1}_{\left\{z \leq \frac{K(X_{s-},\tilde{X}_{s-}(\alpha))}{\tilde{X}_{s-}(\alpha)}\right\}} N(ds,d\alpha,dz). \qquad (SDE)$$

We recall the following result (see [9]) :

Proposition 4.2.8 *Let $(X_0,X,\tilde{X},N)$ be a solution to (SDE) on $[0,T_0[$. Then the law $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$ satisfies the martingale problem (MP) on $[0,T_0[$ with initial condition $Q_0 = \mathcal{L}(X_0)$. Hence $(\mathcal{L}(X_t))_{t\in[0,T_0[}$ is a solution to the modified Smoluchowski equation (MS) with initial condition Q_0.*

Let us now write the assumption supposed in [9].

Hypothesis (H_β) **:** The initial condition Q_0 belongs to $\mathcal{P}_1$. The symmetric kernel $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz continuous on $(\ H_{Q_0})^2$, and satisfies, for some constant $C < \infty$ and $\beta \in [0,1]$

$$K(x,y) \leq C(1 + x + y + x^\beta y^\beta). \qquad (4.12)$$

Notice that (H_β) is less and less stringent as β grows : if it is satisfied with $\beta_1 > 0$, then it also holds for any $\beta_2 \geq \beta_1$.
The solutions of (SD) or (SC) have different behaviours according to the value of β. In the case $\beta \in [0,1/2]$, the solution is defined on the time interval $[0,\infty[$. This is not the case when $\beta > 1/2$, since it is known that the solution $n(x,t)$ to the Smoluchowski equation may « gel » at some time $T_0 < \infty$. This time T_0 is called the gelification time, and can be written *e.g.* as the first instant where $\sum_{k\geq 1} k^2 n(k,t) = \infty$ (in the discrete case) or $\int x^2 n(x,t)dx = \infty$ (in the continuous case). After the gelification time, the Smoluchowski equation is no more physical, and we are not treating the situation $t > T_0$ here.
In the sequel, we will always assume (H_β) either with $\beta = 1/2$ or $\beta = 1$.

The following existence result is proved in [9].

Theorem 4.2.9 *Let $Q_0 \in \mathcal{P}_1$ satisfy $\int x^2 Q_0(dx) < \infty$. Assume (H_β).*

1. *If $\beta = 1/2$ then there exists a solution $(X_0,X,\tilde{X},N)$ to (SDE) on $[0,\infty[$.*
2. *If $\beta = 1$ then there exists a solution $(X_0,X,\tilde{X},N)$ to (SDE) on $[0,T_0[$, where $T_0 = 1/C(1+\int xQ_0(dx))$.*

Remark that this result also furnishes existence results for (MP) and (MS).

We have also obtained uniqueness results (by making use of those of analysts). Let us state them here. The hypothesis we take is the sub-additive one.

Corollary 4.2.10 *Assume that $Q_0 \in \mathcal{P}_1$, $\operatorname{Supp} Q_0 \subset \mathbb{N}^*$ and that it has a second order moment. Then, if $K(i,j) \le C(1+i+j)$ for all i, j in $\mathbb{N}^*$, uniqueness holds for (MS), (MP) and we have also uniqueness in law for (SDE).*

Corollary 4.2.11 *Assume that Q_0 belongs to $\mathcal{P}_1$ and that $\int x^2 Q_0(dx) < \infty$. Assume that $K(x,y) \le C(1+x+y)$, that K is increasing and that the following regularity condition holds: there exists a locally bounded function ζ on $[0,\infty[^2$ such that for all $x,x',y \in \mathbb{R}_+$,*

$$|K(x,y) - K(x',y)| \le |x-x'|\zeta(x,x')(1+y^2). \tag{4.13}$$

In addition, assume that Q_0 admits a density $f_0(x)$ and that $\int x^{-1}Q_0(dx) < \infty$. Then uniqueness holds for (MS), (MP), and we have uniqueness in law for (SDE).

4.3 An associated particle system

The aim of this section is to solve numerically Smoluchowski's coagulation equation, by building an approximation scheme for $\mathcal{L}(X) = \mathcal{L}_\alpha(\tilde{X})$, where $(X_0,X,\tilde{X},N)$ is a solution to (SDE).
Due to the presence of $\tilde{X}$, the system is nonlinear so we cannot directly simulate X. The natural way to get rid of this nonlinearity is to construct an interacting particle system.
For technical (but rather serious) reasons, we restrict our study, for the moment, to the case of (H_β) with $\beta = 1/2$. We explain at the end of this section how to treat $\beta = 1$.
Let us define a « linearised » version of the nonlinear stochastic differential equation (SDE).

Definition 4.3.1 *Let $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ be a symmetric kernel. Let $Q_0 \in \mathcal{P}_1$ and let $n \in \mathbb{N}^*$ be fixed. Consider a family $\left(X_0^{i,n}\right)_{i\in\{1,\dots,n\}}$ of i.i.d. Q_0-distributed random variables. Consider also a family $(N^i(dt,dj,dz))_{i\in\{1,\dots,n\}}$ of i.i.d. Poisson measures on $[0,\infty[\times\{1,\dots,n\}\times[0,\infty[$ with intensity measures*

$$ds\frac{1}{n}\sum_{k=1}^{n}\delta_k(dj)dz. \tag{4.14}$$

A process $X^n = (X^{1,n},\dots,X^{n,n})$ *with values in* $\left[\mathbb{D}^{\uparrow}([0,\infty[,\ H_{Q_0})\right]^n$ *is said to solve* $(PS)_n$ *if for all* $i \in \{1,\dots,n\}$ *and all* $t \in [0,\infty[$

$$X_t^{i,n} = X_0^{i,n} + \int_0^t \int_j \int_0^\infty X_{s-}^{j,n} \mathbb{1}_{\left\{z \leq \frac{K\left(X_{s-}^{i,n},X_{s-}^{j,n}\right)}{X_{s-}^{j,n}}\right\}} N^i(ds,dj,dz). \qquad (PS)_n$$

For X^n *a solution to* $(PS)_n$*, we will denote by*

$$\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{X^{i,n}} \qquad (4.15)$$

the associated empirical measure.
μ^n *is a random probability measure on* $\mathbb{D}^{\uparrow}([0,\infty[,\ H_{Q_0})$.

Let us first prove that $(PS)_n$ is well-defined.

Proposition 4.3.2 *Let* $Q_0 \in \mathcal{P}_1$*. Assume* (H_β) *with* $\beta = 1/2$ *and let* $n \in \mathbb{N}^*$ *be fixed. Then there exists an unique solution* $X^n = (X^{1,n},\dots,X^{n,n})$ *to* $(PS)_n$*. This solution satisfies, for all* $t \in [0,\infty[$

$$\sup_i E\left[\sup_{s\in[0,t]} X_s^{i,n}\right] = E\left[X_t^{1,n}\right] < \infty \qquad (4.16)$$

Proof The proof is obvious, and thus will just be sketched. For $\varepsilon > 0$, we consider the following system of S.D.E.s : for all $i \in \{1,\dots,n\}$ and all $t \in [0,\infty[$

$$X_t^{i,n,\varepsilon} = X_0^{i,n} + \int_0^t \int_j \int_0^\infty X_{s-}^{j,n,\varepsilon} \mathbb{1}_{\left\{z \leq \frac{K\left(X_{s-}^{i,n,\varepsilon}\wedge\frac{1}{\varepsilon},X_{s-}^{j,n,\varepsilon}\wedge\frac{1}{\varepsilon}\right)}{X_{s-}^{j,n,\varepsilon}}\right\}} N^i(ds,dj,dz). \qquad (4.17)$$

Thanks to (H_β) with $\beta = 1/2$, we deduce that in this equation, the integral on $[0,\infty[$ may be replaced by an integral on $[0,C(1+3/\varepsilon)/\min\{X_0^{1,n},\dots,X_0^{n,n}\}]$. Hence, conditionally to the σ-field generated by the initial system, the Poisson measures N^i can be considered as *i.i.d.* Poisson measures, whose intensities measures are integrable on every compact time interval. S.D.E.s driven by such Poisson measures are easily solved, and the strong existence and uniqueness of (4.17) is deduced. Furthermore, (H_β) with $\beta = 1/2$ allows to prove that

$$\sup_{\varepsilon>0}\ \sup_{i\in\{1,\dots,n\}} E\left[\sup_{s\in[0,t]} |X_s^{i,n,\varepsilon}|\right] = \sup_{\varepsilon>0} E\left[X_t^{1,n,\varepsilon}\right] < \infty. \qquad (4.18)$$

Hence, if we consider the stopping times

$$\tau_\varepsilon = \inf\ \left\{t \geq 0\ ;\ \max\{X_t^{1,n,\varepsilon},\dots,X_t^{n,n,\varepsilon}\} > 1/\varepsilon\right\} \qquad (4.19)$$

it can be easily seen that τ_ε increases *a.s.* to infinity when ε decreases to 0. From the uniqueness for (4.17), we deduce that for all $\varepsilon < \varepsilon'$, for all $i \in \{1,\dots,n\}$ and all $t < \tau_{\varepsilon'}$,

$$X_t^{i,n,\varepsilon} = X_t^{i,n,\varepsilon'}. \tag{4.20}$$

The existence is now easy : for all t and all ω, there exists ε such that $\tau_\varepsilon(\omega) > t$. We set $X_t^{i,n} = X_t^{i,n,\varepsilon}$ for all i. This definition is correct thanks to (4.20).
The uniqueness is also clear, because one can check that any solution Y^n to $(PS)_n$ will satisfy $Y_t^{i,n} = X_t^{i,n,\varepsilon}$ on the set $t < \tau_\varepsilon$.
Finally, (4.16) also follows easily. □

We will see in the next section that this particle system is exactly simulable on $[0,T]$, for any $T < \infty$ fixed.
Our aim is now to prove the following result.

Theorem 4.3.3 *Let $Q_0 \in \mathcal{P}_1$. Assume (H_β) with $\beta = 1/2$, and suppose that $\int x^2 Q_0(dx) < \infty$. Consider, for each n, the solution X^n to $(PS)_n$, and its associated empirical measure μ^n.*

1. *The sequence $(\mathcal{L}(\mu^n))_{n\geq 1}$ is tight in $\mathcal{P}\left(\mathcal{P}\left(\mathbb{D}^\uparrow([0,\infty[,\, H_{Q_0})\right)\right)$ (the set $\mathcal{P}\left(\mathbb{D}^\uparrow([0,\infty[,\, H_{Q_0})\right)$ being endowed with the weak convergence topology associated with the Skorokhod topology on $\mathbb{D}^\uparrow([0,\infty[,\, H_{Q_0})$).*

2. *Any limiting point π of $(\mathcal{L}(\mu^n))_{n\geq 1}$ satisfies*

$$\text{Supp}\ \pi \subset \{\textit{solutions to } (MP)\}\,. \tag{4.21}$$

This implies that there exists a subsequence μ^{n_k} which converges in law, for the weak topology of $\mathcal{P}\left(\mathbb{D}^\uparrow([0,\infty[,\, H_{Q_0})\right)$ (associated with the Skorokhod topology on $\mathbb{D}^\uparrow([0,\infty[,\, H_{Q_0})$), to some random probability measure μ, and that μ is a.s. a solution to (MP).

Let us comment this result.

Remark 4.3.4

1. *If uniqueness holds for (MP), see e.g. Corollaries 4.2.10 and 4.2.11, then we deduce from Theorem 4.3.3 that μ^n goes in law to the unique solution Q of (MP). Since Q is deterministic, we finally conclude that μ^n goes to Q in probability.*
2. *In the discrete case, one are able to obtain approximations of the solution $n(k,t)$, for any k and t, by using*

$$\frac{1}{k}\mu_t^n(k) = \frac{1}{kn}\sum_{i=1}^n \mathbb{1}_{\{X_t^{i,n}=k\}}. \tag{4.22}$$

3. *In the continuous case, we cannot approximate directly the solution $n(x,t)$ of (SC), for t and x fixed, but we are able to obtain approximations of $\int \varphi(x)n(x,t)\,dx$ by using*

$$\langle \mu_t^n, \varphi(x)/x \rangle = \frac{1}{n}\sum_{i=1}^{n} \frac{\varphi(X_t^{i,n})}{X_t^{i,n}}. \tag{4.23}$$

In [14], Eibeck and Wagner have introduced a similar interacting particle system. There are however important differences here : first, Eibeck and Wagner were truncating the kernel K, by replacing $K(x,y)$ by $K(x,y) \wedge b_n$, for some b_n going to infinity as the size of the system tends to infinity. We will see that this cutoff procedure is not useful here, since the particle system we introduce is well defined and directly simulable.

Secondly, they proved their convergence result under the assumptions that $K(x,y) \leq h(x)h(y)$ for some positive function satisfying that $h(x)/x$ is non-increasing, and that

$$\lim_{x+y\to\infty} \frac{K(x,y)}{h(x)h(y)} = 0.$$

It seems that the standard additive kernel $K(x,y) = x + y$ does not satisfy these assumptions.

Finally (this is of course much less important), we do not have to assume, in the continuous case, that $\int x^{-1} Q_0(dx) < \infty$, but only that $Q_0(]0,\infty[) = 1$.

Proof of Theorem 4.3.3. Notice that for evident reasons of symmetry, $\mathcal{L}(X^{i,n})$ is independent of $i \in \{1,\ldots,n\}$, $\mathcal{L}(X^{i,n},X^{j,n})$ is independent of $\{(i,j) \in \{1,\ldots,n\}^2 \; ; \; i \neq j\}$, etc.

We break the proof in several steps. The tightness is proved in Steps 1 and 2, and an uniform integrability result is checked in Step 3. Point 2 of the theorem is proved in Steps 4 to 7.

Step 1 : We first prove that for all $T < \infty$, $\sup_n h_n(T) < \infty$, where

$$h_n(T) = E\left[X_T^{1,n}\right] = \sup_{i\in\{1,\ldots,n\}} E\left[\sup_{s\in[0,T]} X_s^{i,n}\right]. \tag{4.24}$$

Let us compute, using $(PS)_n$, (4.14) and (H_β) with $\beta = 1/2$.

$$\begin{aligned} h_n(t) &= E(X_0^{1,n}) + \int_0^t \frac{1}{n}\sum_{j=1}^{n} E\left(K\left(X_s^{1,n},X_s^{j,n}\right)\right) ds \qquad (4.25) \\ &\leq E(X_0^{1,n}) + C\int_0^t \frac{1}{n}\sum_{j=1}^{n} E\left(1 + X_s^{1,n} + X_s^{j,n} + \sqrt{X_s^{1,n}X_s^{j,n}}\right) ds. \end{aligned}$$

Cauchy-Schwarz inequality yields $E\left(\sqrt{X_s^{1,n}X_s^{j,n}}\right) \le \sqrt{E\left(X_s^{1,n}\right)E\left(X_s^{j,n}\right)} = E\left(X_s^{1,n}\right)$. Hence, setting $a_0 = E(X_0^{1,n})$, we obtain

$$h_n(t) \le a_0 + C\int_0^t \left[1 + 3h_n(s)\right] ds \tag{4.26}$$

which suffices to conclude, by using the usual Gronwall Lemma. We get so

$$\sup_n E\left[X_T^{1,n}\right] < \infty.$$

Step 2 : It is known (see Méléard [49], Lemma 4.5), that the tightness of μ^n is equivalent to that of $X^{1,n}$. We can thus apply the Aldous criterion (see Theorem 4.7.1 of the appendix). Thanks to Step 1, we only have to prove that if S and S' are two stopping times such that *a.s.*, $0 \le S \le S' \le (S+\delta)\wedge T$, for some $T < \infty$ and $\delta > 0$, then

$$E\left[|X_{S'}^{1,n} - X_S^{1,n}|\right] \le C_T\,\delta \tag{4.27}$$

the constant C_T not depending on n, S, S', nor δ. But, using (4.14), $(PS)_n$ and (H_β) with $\beta = 1/2$.

$$\begin{aligned} E\left[|X_{S'}^{1,n} - X_S^{1,n}|\right] &= \frac{1}{n}\sum_{j=1}^n E\left[\int_S^{S'} K\left(X_s^{1,n}, X_s^{j,n}\right) ds\right] \\ &\le \frac{1}{n}\sum_{j=1}^n \delta \sup_{s\in[0,T]} E\left[C\left(1 + X_s^{1,n} + X_s^{j,n} + \sqrt{X_s^{1,n}X_s^{j,n}}\right)\right] \\ &\le \delta\, C\left(1 + 3E(X_T^{1,n})\right) \\ &\le \delta\, C_T \end{aligned} \tag{4.28}$$

where the last inequality is deduced from Step 1. We conclude thus with the tightness of $(\mu^n)_{n\ge1}$.

Step 3 : We will also need a condition of uniform integrability. We deduce it from the following claim : for all $T < \infty$, $\sup_n f_n(T) < \infty$, where

$$f_n(T) = E\left[\left(X_T^{1,n}\right)^2\right] = \sup_{i\in\{1,\dots,n\}} E\left[\sup_{s\in[0,T]}\left(X_s^{i,n}\right)^2\right]. \tag{4.29}$$

Let us prove (4.29). We have

$$\begin{aligned} \left(X_T^{1,n}\right)^2 &= \left(X_0^{1,n}\right)^2 + \sum_{s\le t}\left\{2X_{s-}^{1,n}\Delta X_s^{1,n} + \left(\Delta X_s^{1,n}\right)^2\right\} \\ &= \left(X_0^{1,n}\right)^2 + \int_0^t\int_j\int_0^\infty \left[2X_{s-}^{1,n}X_{s-}^{j,n} + \left(X_{s-}^{j,n}\right)^2\right] \\ &\qquad \mathbb{1}_{\left\{z \le \frac{K\left(X_{s-}^{1,n}, X_{s-}^{j,n}\right)}{X_{s-}^{j,n}}\right\}} N^1(ds,dj,dz). \end{aligned} \tag{4.30}$$

Setting $b_0 = E\left[\left(X_0^{1,n}\right)^2\right]$, using (H_β) with $\beta = 1/2$ and the Hölder inequality, we obtain

$$\begin{aligned} f_n(t) &\leq b_0 + \frac{1}{n}\sum_{j=1}^{n} C \int_0^t E\Big[\left(2X_s^{1,n} + X_s^{j,n}\right) \\ &\qquad \left(1 + X_s^{1,n} + X_s^{j,n} + \sqrt{X_s^{1,n} X_s^{j,n}}\right) \Big] ds \\ &\leq b_0 + \frac{1}{n}\sum_{j=1}^{n} C \int_0^t \left\{3E(X_s^{1,n}) + 9 f_n(s)\right\} ds \\ &\leq b_0 + C\int_0^t \left\{3E(X_s^{1,n}) + 9 f_n(s)\right\} ds \end{aligned} \tag{4.31}$$

which allows to conclude thanks to Step 1 and the usual Gronwall Lemma.

Step 4 : Let us consider a convergent subsequence of μ^n, that we still denote by μ^n, whose limit is μ, a probability measure on $\mathbb{D}^\uparrow([0,\infty[,\, H_{Q_0})$.
We want to prove that μ satisfies *a.s.* the martingale problem (MP). To this aim, we consider $\phi \in C_b^1(\mathbb{R}_+)$, $g_1,\ldots,g_k \in C_b(\mathbb{R}_+)$ and $0 \leq s_1 \leq \cdots \leq s_k < s < t < T_0$. Let F be the map from $\mathbb{D}^\uparrow([0,\infty[,\, H_{Q_0}) \times \mathbb{D}^\uparrow([0,\infty[,\, H_{Q_0})$ into $\mathbb{R}$ defined by

$$\begin{aligned} F(x,y) &= g_1(x(s_1)) \times \cdots \times g_k(x(s_k)) \times \\ &\left\{\phi(x(t)) - \phi(x(s)) - \int_s^t \left[\phi(x(u)+y(u)) - \phi(x(u))\right] \frac{K(x(u),y(u))}{y(u)} du\right\}. \end{aligned} \tag{4.32}$$

We have to prove that *a.s.*,

$$\langle \mu \otimes \mu, F\rangle = 0. \tag{4.33}$$

If this equality is proved for all F satisfying the previous condition, we can deduce easily that *a.s.*, μ solves the martingale problem (MP).

In Step 5, we will prove that

$$E\left[\langle \mu^n \otimes \mu^n, F^2\rangle\right] \underset{n\to\infty}{\longrightarrow} 0. \tag{4.34}$$

Steps 6 and 7 will conclude the proof since we obtain that

$$E\left[|\langle \mu^n \otimes \mu^n, F\rangle|\right] \underset{n\to\infty}{\longrightarrow} E\left[|\langle \mu \otimes \mu, F\rangle|\right]. \tag{4.35}$$

Step 5 : Let us prove (4.34). We set $\alpha_n = \langle \mu^n \otimes \mu^n, F^2\rangle$. Then

$$\begin{aligned} \alpha_n &= \Big[\frac{1}{n^2}\sum_{i,j=1}^{n} g_1(X_{s_1}^{i,n})\ldots g_k(X_{s_k}^{i,n})\Big\{\phi(X_t^{i,n}) - \phi(X_s^{i,n}) \\ &\qquad - \int_s^t \left[\phi(X_u^{i,n} + X_u^{j,n}) - \phi(X_u^{i,n})\right] \frac{K\left(X_u^{i,n}, X_u^{j,n}\right)}{X_u^{j,n}} du\Big\}\Big]^2 \end{aligned} \tag{4.36}$$

which can be rewritten as

$$\alpha_n = \Big[\frac{1}{n}\sum_{i=1}^{n} g_1(X_{s_1}^{i,n})\dots g_k(X_{s_k}^{i,n})\Big\{\phi(X_t^{i,n}) - \phi(X_s^{i,n}) - \frac{1}{n}\sum_{j=1}^{n}\int_s^t \left[\phi(X_u^{i,n}+X_u^{j,n}) - \phi(X_u^{i,n})\right]\frac{K\left(X_u^{i,n},X_u^{j,n}\right)}{X_u^{j,n}}du\Big\}\Big]^2. \tag{4.37}$$

Now we set, for i fixed,

$$M_t^{i,n}(\phi) = \phi(X_t^{i,n}) - \phi(X_0^{i,n}) - \frac{1}{n}\sum_{j=1}^{n}\int_0^t \left[\phi(X_u^{i,n}+X_u^{j,n}) - \phi(X_u^{i,n})\right]\frac{K\left(X_u^{i,n},X_u^{j,n}\right)}{X_u^{j,n}}du. \tag{4.38}$$

Applying the Itô formula to compute $\phi(X_t^{i,n})$, we see that

$$M_t^{i,n}(\phi) = \int_0^t\int_j\int_0^\infty \left[\phi(X_{u-}^{i,n}+X_{u-}^{j,n}) - \phi(X_{u-}^{i,n})\right] \mathbb{1}_{\left\{z\le \frac{K\left(X_{u-}^{i,n},X_{u-}^{j,n}\right)}{X_{u-}^{j,n}}\right\}}\bar{N}^i(ds,dj,dz) \tag{4.39}$$

where $\bar{N}^i(ds,dj,dz) = N^i(ds,dj,dz) - ds\frac{1}{n}\sum_{k=1}^n \delta_k(dj)dz$ is the compensated Poisson measure associated with N^i. It is easily seen, using Step 3, that $M^{i,n}(\phi)$ is an L^2-martingale, for each i. Hence

$$\begin{aligned} E(\alpha_n) &= E\left[\left\{\frac{1}{n}\sum_{i=1}^{n} g_1(X_{s_1}^{i,n})\times\dots\times g_k(X_{s_k}^{i,n})\left[M_t^{i,n}(\phi) - M_s^{i,n}(\phi)\right]\right\}^2\right] \\ &= \frac{1}{n^2}E\left[\sum_{i=1}^{n}\left\{g_1(X_{s_1}^{i,n})\times\dots\times g_k(X_{s_k}^{i,n})\left[M_t^{i,n}(\phi) - M_s^{i,n}(\phi)\right]\right\}^2\right] \\ &\quad +\frac{1}{n^2}E\Big[\sum_{i=1}^{n}\sum_{j\neq i} g_1(X_{s_1}^{i,n})\times\dots\times g_k(X_{s_k}^{i,n})\left[M_t^{i,n}(\phi) - M_s^{i,n}(\phi)\right] \\ &\qquad \times g_1(X_{s_1}^{j,n})\times\dots\times g_k(X_{s_k}^{j,n})\left[M_t^{j,n}(\phi) - M_s^{j,n}(\phi)\right]\Big] \\ &= \alpha_n^1 + \alpha_n^2 \end{aligned} \tag{4.40}$$

with obvious notations for α_n^1 and α_n^2. But, using (4.39) and (H_β) with $\beta = 1/2$, we see that for some constant A,

$$\begin{aligned}
\alpha_n^1 &\leq \frac{1}{n^2} n ||g_1||_\infty \dots ||g_k||_\infty E\left[\left\{M_t^{1,n}(\phi) - M_s^{1,n}(\phi)\right\}^2\right] \\
&\leq \frac{A}{n}\int_s^t \frac{1}{n}\sum_{j=1}^n \int_0^\infty ||\phi'||_\infty^2 E\left\{\left(X_u^{j,n}\right)^2 \mathbb{1}_{\left\{z \leq \frac{K(X_u^{1,n}, X_u^{j,n})}{X_u^{j,n}}\right\}}\right\} dzdu \\
&\leq \frac{A}{n}\int_s^t \frac{1}{n}\sum_{j=1}^n E\left[X_u^{j,n} K\left(X_u^{1,n}, X_u^{j,n}\right)\right] du \\
&\leq \frac{A}{n}\left(1 + \sup_{t\in[0,T]} E\left[\left(X_t^{1,n}\right)^2\right]\right) \\
&\leq \frac{A}{n} \qquad (4.41)
\end{aligned}$$

thanks to Step 2. Thus α_n^1 goes to 0 as n goes to ∞. On the other hand,

$$\begin{aligned}
\alpha_n^2 &= \frac{n(n-1)}{n^2} E\Big(g_1(X_{s_1}^{1,n}) \times \dots \times g_k(X_{s_k}^{1,n}) \times g_1(X_{s_1}^{2,n}) \times \dots \times g_k(X_{s_k}^{2,n}) \\
&\qquad \times \left[M_t^{1,n}(\phi) - M_s^{1,n}(\phi)\right]\left[M_t^{2,n}(\phi) - M_s^{2,n}(\phi)\right]\Big) \\
&= \frac{n(n-1)}{n^2} E\Big(g_1(X_{s_1}^{1,n}) \times \dots \times g_k(X_{s_k}^{1,n}) \times g_1(X_{s_1}^{2,n}) \times \dots \times g_k(X_{s_k}^{2,n}) \\
&\qquad \times E\left\{\left[M_t^{1,n}(\phi) - M_s^{1,n}(\phi)\right]\left[M_t^{2,n}(\phi) - M_s^{2,n}(\phi)\right]\big| \mathcal{F}_s^n\right\}\Big), \qquad (4.42)
\end{aligned}$$

where $\mathcal{F}_s^n$ is the σ-field generated by the initial particles $X_0^{i,n}$, and by the Poisson measures N^i until s.
To conclude, we will prove that $(M_\cdot^{1,n}(\phi) M_\cdot^{2,n}(\phi))_\cdot$ is a martingale for the filtration $\mathcal{F}_s^n$. Applying Itô formula and using that $M^{1,n}(\phi)$ and $M^{2,n}(\phi)$ are pure jump martingales (their continuous part is 0), we obtain

$$\begin{aligned}
M_t^{1,n}(\phi) M_t^{2,n}(\phi) &= \int_0^t M_{s-}^{1,n}(\phi) dM_s^{2,n}(\phi) + \int_0^t M_{s-}^{2,n}(\phi) dM_s^{1,n}(\phi) \\
&\qquad + \sum_{s\leq t} \Delta M_s^{1,n}(\phi) \Delta M_s^{2,n}(\phi). \qquad (4.43)
\end{aligned}$$

Since the Poisson measures N^1 and N^2 are independent, we deduce that $M^{1,n}(\phi)$ and $M^{2,n}(\phi)$ do never jump at same time *a.s.*. Hence, *a.s.*,

$$\sup_{s\in\mathbb{R}_+} |\Delta M_s^{1,n}(\phi) \Delta M_s^{2,n}(\phi)| = 0. \qquad (4.44)$$

We deduce that $M^{1,n}(\phi)M^{2,n}(\phi)$ is a martingale. Thus α_n^2 vanishes identically, which concludes the proof of Step 5.

Step 6 : We now prove (4.35). One can verify that the map from $\mathcal{P}_1^{\uparrow}([0,\infty[,\ H_{Q_0})$ into $\mathbb{R}$, defined by

$$Q \ \mapsto \langle Q \otimes Q, F\rangle \tag{4.45}$$

is continuous for any $Q \in \mathcal{E}$, where $\mathcal{E}$ is defined as the set of all probability measures Q in $\mathcal{P}_1^{\uparrow}([0,\infty[,\ H_{Q_0})$ satisfying

$$Q\Big\{x(0) > 0,\ \Delta x(s_1) = \cdots = \Delta x(s_k) = \Delta x(s) = \Delta x(t) = 0\Big\} = 1. \tag{4.46}$$

It will be shown in the next step that the limiting point μ belongs *a.s.* to $\mathcal{E}$. It is thus clear that for any $C < \infty$, when n tends to infinity,

$$E\left[|\langle \mu^n \otimes \mu^n, F \wedge C \vee (-C)\rangle|\right] \longrightarrow E\left[|\langle \mu \otimes \mu, F \wedge C \vee (-C)\rangle|\right]. \tag{4.47}$$

It only remains to check that

$$\sup_n E\left[\left\langle \mu^n \otimes \mu^n, |F| \mathbb{1}_{|F|\geq C}\right\rangle\right] \underset{C\to\infty}{\longrightarrow} 0. \tag{4.48}$$

Using (H_β) with $\beta = 1/2$, we obtain for some constants A and B, that for all x and y in $\mathbb{D}^{\uparrow}([0,\infty[,\ H_{Q_0})$,

$$\begin{aligned} |F(x,y)| &\leq A\left[2||\phi||_\infty + ||\phi'||_\infty \int_s^t K(x(u),y(u))du\right] \\ &\leq A + B\left(x(t) + y(t) + \sqrt{x(t)y(t)}\right). \end{aligned} \tag{4.49}$$

Thus, by setting

$$Z_C^n = \left\langle \mu^n \otimes \mu^n, |F| \mathbb{1}_{|F|\geq C}\right\rangle \tag{4.50}$$

we obtain

$$\begin{aligned} Z_C^n &\leq \frac{1}{n^2}\sum_{i,j=1}^n \left[A + B\left(X_t^{i,n} + X_t^{j,n} + \sqrt{X_t^{i,n}X_t^{j,n}}\right)\right] \\ &\qquad \times \mathbb{1}_{\left\{A+B\left(X_t^{i,n}+X_t^{j,n}+\sqrt{X_t^{i,n}X_t^{j,n}}\right)\geq C\right\}}. \end{aligned} \tag{4.51}$$

One easily concludes that (4.48) holds, thanks to the uniform integrability obtained in Step 3.

Step 7 : We finally prove that μ belongs *a.s.* to the set $\mathcal{E}$ defined in Step 6. First, it is clear that *a.s.*,

$$\int_{x\in\mathbb{D}^{\uparrow}([0,\infty[,\ H_{Q_0})} \mathbb{1}_{\{x(0)=0\}}\mu(dx) = 0 \tag{4.52}$$

because the initial condition Q_0 of our particle system belongs to $\mathcal{P}_1$. In order to prove that μ belongs to $\mathcal{E}$ *a.s.*, it thus suffices to show that for any $t_0 \in]0,\infty[$ fixed

$$\int_{x\in\mathbb{D}^\uparrow([0,\infty[,\ H_{Q_0})} \mathbb{1}_{\{\Delta x(t_0)\neq 0\}}\mu(dx) = 0 \tag{4.53}$$

a.s., or equivalently that *a.s.*,

$$\int_{x\in\mathbb{D}^\uparrow([0,\infty[,\ H_{Q_0})} \{|\Delta x(t_0)| \wedge 1\}\,\mu(dx) = 0. \tag{4.54}$$

One easily checks that (4.54) holds as soon as

$$E\left[\int_{x\in\mathbb{D}^\uparrow([0,\infty[,\ H_{Q_0})} \sup_{s\in[t_0-r,t_0+r]} \{|\Delta x(s)| \wedge 1\}\,\mu(dx)\right] \underset{r\to 0}{\longrightarrow} 0. \tag{4.55}$$

But, for any $r > 0$ fixed, the mapping from $\mathbb{D}^\uparrow([0,\infty[,\ H_{Q_0})$ into $\mathbb{R}_+$ defined by

$$x \mapsto \sup_{s\in[t_0-r,t_0+r]} \{|\Delta x(s)| \wedge 1\} \tag{4.56}$$

is known to be continuous and bounded, which implies, since μ^n tends to μ in law, that (4.55) holds as soon as

$$\sup_n E\left[\int_{x\in\mathbb{D}^\uparrow([0,\infty[,\ H_{Q_0})} \sup_{s\in[t_0-r,t_0+r]} \{|\Delta x(s)| \wedge 1\}\,\mu^n(dx)\right] \underset{r\to 0}{\longrightarrow} 0. \tag{4.57}$$

But

$$\begin{aligned}
\gamma_n &= E\left[\int_{x\in\mathbb{D}^\uparrow([0,\infty[,\ H_{Q_0})} \sup_{s\in[t_0-r,t_0+r]} \{|\Delta x(s)| \wedge 1\}\,\mu^n(dx)\right] \\
&= \frac{1}{n}\sum_{i=1}^{n} E\left[\sup_{s\in[t_0-r,t_0+r]} \left\{|\Delta X_s^{i,n}| \wedge 1\right\}\right] \\
&= E\left[\sup_{s\in[t_0-r,t_0+r]} \left\{|\Delta X_s^{1,n}| \wedge 1\right\}\right] \\
&\leq E\left[X_{t_0+r}^{1,n} - X_{t_0-r}^{1,n}\right]
\end{aligned} \tag{4.58}$$

the last inequality coming from the fact that $X^{1,n}$ is an increasing process.
Using the definition of $X^{1,n}$, using (H_β) with $\beta = 1/2$ and Step 1, we deduce that for all $r < 1$,

$$\begin{aligned}
\gamma_n &\leq \int_{t_0-r}^{t_0+r} \frac{1}{n}\sum_{j=1}^{n} E\left[K(X_s^{1,n},X_s^{j,n})\right] ds \\
&\leq 2r\,C\left(1 + 3E(X_{t_0+1}^{1,n})\right) \\
&\leq A\,r
\end{aligned} \tag{4.59}$$

the constant A depending only on t_0. Finally, (4.57) is verified. This concludes the proof of Step 7 and ends the proof of Theorem 4.3.3. $\square$

We now prove the following corollary, which is not far obvious, since the projections $x \mapsto x(t)$ are not continuous on $\mathbb{D}^{\uparrow}([0,\infty[, H_{Q_0})$. We cannot a priori conclude, only by Theorem 4.3.3, that for each t fixed, μ_t^n tends to Q_t.

Corollary 4.3.5 *Let $Q_0 \in \mathcal{P}_1$. Assume (H_β) with $\beta = 1/2$, and suppose that $\int x^2 Q_0(dx) < \infty$. Assume also that uniqueness holds for (MP) (see e.g. Corollaries 4.2.10 and 4.2.11). We thus know from Remark 4.3.4 (2) that the empirical measure μ^n goes to the unique solution Q of (MP) in probability. Then the $\mathcal{P}(\mathbb{R}_+)$-valued process $(\mu_t^n)_{t\geq 0} = \left(\frac{1}{n}\sum_{i=1}^n \delta_{X_t^{i,n}}\right)_{t\geq 0}$ converges in probability to $(Q_t)_{t\geq 0}$ in $\mathbb{D}([0,\infty[,\mathcal{P}(\mathbb{R}_+))$ (endowed with the topology of the uniform convergence on every compact associated with the weak topology of $\mathcal{P}(\mathbb{R}_+)$).*

To prove this result, we will use Lemma 4.7.2 of the Appendix, which is due to Méléard, [49].

Proof of Corollary 4.3.5 Thanks to Lemma 4.7.2, we just have to prove that for Q the unique solution to (MP) and any $T < \infty$

$$\sup_{t\in[0,T]} \int_{x\in\mathbb{D}^{\uparrow}([0,T],\ H_{Q_0})} \sup_{s\in[t-r,t+r]} (|\Delta x(s)| \wedge 1)\, Q(dx) \underset{r\to 0}{\longrightarrow} 0. \tag{4.60}$$

To this end, we use the fact that Q is the law of X, $(X_0,X,\tilde{X},N)$ being a solution to (SDE). We thus have to show that

$$\sup_{t\in[0,T]} E\left(\sup_{s\in[t-r,t+r]} (|\Delta X_s| \wedge 1)\right) \underset{r\to 0}{\longrightarrow} 0. \tag{4.61}$$

Since X is *a.s.* increasing, we obtain, by using (H_β) with $\beta = 1/2$,

$$\begin{aligned}
\sup_{t\in[0,T]} E\left(\sup_{s\in[t-r,t+r]} (|\Delta X_s| \wedge 1)\right) &\leq \sup_{t\in[0,T]} E\left(X_{t+r} - X_{t-r}\right) \\
&\leq \sup_{t\in[0,T]} \int_{t-r}^{t+r} EE_\alpha\left[K(X_s,\tilde{X}_s)\right] ds \\
&\leq 2r \times C \sup_{s\in[0,T+r]} E\left(1 + 3X_s\right) \qquad (4.62)
\end{aligned}$$

which clearly goes to 0 as r goes to 0, thanks to the fact that for all $T < \infty$,

$$\sup_{s\in[0,T]} E\left(X_s\right) < \infty. \tag{4.63}$$

The corollary is now proved. $\square$

In order to conclude this section, we present the approximation scheme for the solution of the Smoluchowski equation in the case of (H_β) with $\beta = 1$, by using the previous results.
As a matter of fact, we are not able (and this might be false) to prove that the particle system is well-defined when $\beta > 1/2$, nor that it is simulable. That's why we are led to introduce a double approximation.

Remark 4.3.6 *Assume (H_β) with $\beta = 1$, that $Q_0 \in \mathcal{P}_1$ and $\int x^2 Q_0(dx) < \infty$. Let $T_0 = 1/C(1 + \int x Q_0(dx))$.*
Consider a kernel with cutoff $K^\varepsilon(x,y) = K(x \wedge \frac{1}{\varepsilon}, y \wedge \frac{1}{\varepsilon})$. Then K^ε satisfies (H_β) with $\beta = 1/2$, and we may define the associated particle system : we obtain an empirical measure $\mu^{\varepsilon,n}$. We know from Theorem 4.3.3 that the sequence $\mathcal{L}(\mu^{\varepsilon,n})$ is tight, and that any limiting point μ^ε satisfies a.s. the martingale problem with cutoff $(MP)_\varepsilon$, obtained for (MP) by replacing K with K^ε.
On the other hand, the following result holds, and can be proved by following line by line the proofs of Lemmas 3.6, 3.7 and 3.8 of [9] : the sequence Q^ε is tight in $\mathcal{P}(\mathbb{D}^\uparrow([0,T_0[,\, H_{Q_0}))$, and any limiting point Q is a solution to (MP).
Hence, if ε is small enough, and n large enough, $\mu^{\varepsilon,n}$ « approximates » a solution Q to (MP).

4.4 The simulation algorithm

The aim of this section is to show how to simulate (on $[0,T]$, for $T < \infty$) the particle system described in Definition 4.3.1. We assume (H_β) with $\beta = 1/2$.
Let $X^n = (X^{1,n}, \dots, X^{n,n})$ be the unique solution to $(PS)_n$ associated with the Poisson random measures $\{N^i(ds,dj,dz)\}_{i\in\{1,\dots,n\}}$ and with the initial particles $\left(X_0^{i,n}\right)_{i\in\{1,\dots,n\}}$.
Then X^n belongs *a.s.* to $\left[\mathbb{D}^\uparrow([0,\infty[,\, H_{Q_0})\right]^n$.
The first thing to do is to write a n-dimensional equation satisfied by this system. The fundamental idea is that instead of simulating n Poisson measures and reorder the jump instants, it is more convenient to construct one global Poisson measure. To this aim, we introduce the random measure $\mathcal{N}$ on $[0,\infty[\times\{1,\dots,n\}\times\{1,\dots,n\}\times[0,\infty[$ defined by :

$$\mathcal{N}(ds,di,dj,dz) = \sum_{k=1}^{n} \delta_k(di) N^k(ds,dj,dz). \tag{4.64}$$

Since the Poisson measures N^k are *i.i.d.*, it is not hard to prove that $\mathcal{N}$ is a Poisson measure whose intensity measure is

$$ds \left(\sum_{k=1}^{n} \delta_k(di)\right) \left(\frac{1}{n}\sum_{l=1}^{n} \delta_l(dj)\right) dz. \tag{4.65}$$

The particle system can be seen as the solution of the following n-dimensional S.D.E.

$$\bar{X}_t^n = \bar{X}_0^n + \int_0^t \int_i \int_j \int_0^\infty \left\langle \bar{X}_{s-}^n, e_j \right\rangle . e_i \mathbb{1}_{\left\{ z \leq \frac{K\left(\left\langle \bar{X}_{s-}^n, e_i \right\rangle, \left\langle \bar{X}_{s-}^n, e_j \right\rangle\right)}{\left\langle \bar{X}_{s-}^n, e_j \right\rangle} \right\}} \mathcal{N}(ds,di,dj,dz) \quad (4.66)$$

where $e_i = (0,\ldots,0,1,0,\ldots,0)$, 1 being at the i-th place. $< .,. >$ denotes the scalar product of $\mathbb{R}^n$ and $\bar{X}^n = (X^{1,n},\ldots,X^{n,n})$ is a random vector.

The numerical algorithm for $\bar{X}_t^n$ for $t \in [0,T]$ is the following:

Step 0: At the beginning, we simulate $X_0^{1,n},\ldots,X_0^{n,n}$ *i.i.d.*, according to Q_0.

Step 1: Compute

$$m_1 = \sup_{i,j} \frac{K\left(X_0^{i,n}, X_0^{j,n}\right)}{X_0^{j,n}}. \quad (4.67)$$

Further, simulate a random variable S_1, following an exponential distribution with parameter $n\, m_1$, and set $T_1 = S_1$. T_1 is the first instant when a coagulation may occur. State for all $k \in \{1,\ldots,n\}$ and all $t \in [0,T_1[$

$$X_t^{k,n} = X_0^{k,n}. \quad (4.68)$$

At T_1 choose uniformly a couple (i_1,j_1) in $\{1,\ldots,n\}^2$ (one may have $i_1 = j_1$). Simulate a random variable Z_1, according to the uniform distribution over $[0,m_1]$. Compare Z_1 with $K\left(X_0^{i_1,n}, X_0^{j_1,n}\right)/X_0^{j_1,n}$.
If $Z_1 \leq K\left(X_0^{i_1,n}, X_0^{j_1,n}\right)/X_0^{j_1,n}$, then "accept" the jump and set

$$\begin{aligned} X_{T_1}^{i_1,n} &= X_0^{i_1,n} + X_0^{j_1,n} \\ X_{T_1}^{k,n} &= X_0^{k,n} \text{ for all } k \neq i_1. \end{aligned} \quad (4.69)$$

Otherwise "reject" the jump and set

$$X_{T_1}^{k,n} = X_0^{k,n} \text{ for all } k \in \{1,\ldots,n\}. \quad (4.70)$$

Step p: More generally, suppose that all is constructed up to T_{p-1}. At step p compute

$$m_p = \sup_{i,j} \frac{K\left(X_{T_{p-1}}^{i,n}, X_{T_{p-1}}^{j,n}\right)}{X_{T_{p-1}}^{j,n}}. \quad (4.71)$$

Simulate a random variable S_p, following an exponential distribution with parameter $n\, m_p$ and set $T_p = T_{p-1} + S_p$. Until T_p, nothing happens and for all $k \in \{1,\ldots,n\}$ and all $t \in [T_{p-1},T_p[$ put

$$X_t^{k,n} = X_{T_{p-1}}^{k,n}. \quad (4.72)$$

At T_p, we choose, uniformly, a couple (i_p,j_p) in $\{1,\ldots,n\}^2$ and simulate a random variable Z_p, according to the uniform distribution over $[0,m_p]$.
If $Z_p \leq K\left(X^{i_p,n}_{T_{p-1}},X^{j_p,n}_{T_{p-1}}\right)/X^{j_p,n}_{T_{p-1}}$, then"accept" the jump and set

$$\begin{aligned} X^{i_p,n}_{T_p} &= X^{i_p,n}_{T_{p-1}} + X^{j_p,n}_{T_{p-1}} \\ X^{k,n}_{T_p} &= X^{k,n}_{T_{p-1}} \text{ for all } k \neq i_p. \end{aligned} \tag{4.73}$$

Else "reject" the jump and set

$$X^{k,n}_{T_p} = X^{k,n}_{T_{p-1}} \text{ for all } k \in \{1,\ldots,n\}. \tag{4.74}$$

The algorithm stops when $T_p > T$, in which case we just set, for all $t \in [T_{p-1},T]$,

$$X^{k,n}_{t} = X^{k,n}_{T_{p-1}} \text{ for all } k \in \{1,\ldots,n\}. \tag{4.75}$$

Let us remark that the jumps in the system correspond to the coagulation of particles. Also once we choose a couple (i_p,j_p) the components do not play a symmetrical role. Indeed, if coalescence occurs we replace the i_pth particle with the cluster obtained by coalescing the i_pth particle with the j_pth particle and keeping the j_p particle unchanged. The number of particles is constant in time.

Proposition 4.4.1 *Let $T > 0$ be fixed. Then there exists a.s. p such that $T_p > T$. The particle system is thus simulable.*

Proof Let $n \in \mathbb{N}^*$ and $T \in [0,\infty[$ be fixed. Consider the random variables

$$L^{max} = X^{1,n}_T + \cdots + X^{n,n}_T \tag{4.76}$$

which is finite *a.s.* thanks to Proposition 4.3.2, and

$$L^{min} = \inf_i X^{i,n}_0 \tag{4.77}$$

which is strictly positive *a.s.*, since $Q_0 \in \mathcal{P}_1$. For $k \geq 0$, we denote by μ_k the number of fictive collisions between the $(k-1)$-th and k-th effective collisions. This can be written rigorously by induction. We also denote by $S_1, \ldots, S_k, \ldots$ the instants of effective collisions: $S_1 = T_{\mu_1+1}, \ldots, S_k = T_{(\mu_1+1)+\cdots+(\mu_k+1)}, \ldots$ Then there were ν effective collisions between 0 and T, where $\nu = \sum_{i=1}^{\infty} \mathbb{1}_{\{S_i \leq T\}}$.
The random variable ν is smaller than L^{max}/L^{min}. Indeed, at every real collision, the minimum mass added to the system is L^{min}, and hence the maximum possible number of effective collisions is clearly smaller than the total mass of the system at the instant T divided by its minimum mass L^{min}.
Hence ν is *a.s.* finite.
On the other hand, for each k, the random variable μ_k follows a Poisson distribution (with finite and explicitly computable parameter) conditionally to the σ-field generated by the whole system up to S_{k-1}. It is thus clear that for each k, μ_k is *a.s.* finite. Hence, the total number of (effective or fictive) collisions on $[0,T]$, given by $\nu + \mu_1 + \cdots + \mu_\nu$, is *a.s.* finite. This concludes the proof. □

4.5 A central limit theorem in the discrete case

Our aim in this section is to prove a central limit type theorem, *i.e.* to show that the rate of convergence associated with our simulation algorithm is of order $1/\sqrt{n}$. Let us mention the works of Ferland, Fernique and Giroux [19] and Méléard [50] for similar results concerning the Boltzmann equation and its particles approximating system.
We consider only the discrete case for simplicity, but it seems reasonable that a similar result may hold in the continuous case. However, the technical arguments are clearly much more easy in the discrete case. We will assume the strong hypothesis :

> **Assumption** (A) **:** The initial condition Q_0 has its support in $\mathbb{N}^*$ and admits a second order moment. The positive symmetric coagulation kernel K is bounded.

This assumption is clearly not satisfying : K is in general unbounded. However, we are not able to get rid of this stringent hypothesis.

Under (A), one knows (see Corollary 4.2.10) that uniqueness of a solution Q holds for (MP), and that its marginal flow $(Q_t, t \geq 0)$ is the unique solution to (MS).
We use here the notations of Definition 4.3.1. Consider, for each n, a solution $\bar{X}^n = (X^{1,n}, \dots, X^{n,n})$ to $(PS)_n$, lying *a.s.* in $\left(\mathbb{D}^\uparrow([0,\infty[,\mathbb{N}^*)\right)^n$, associated with the Poisson random measures $\{N^i(ds,dj,dz)\}_{i\in\{1,\dots,n\}}$ and with the initial particles $\left(X_0^{i,n}\right)_{i\in\{1,\dots,n\}}$. Denote by $\mu^n_. = \frac{1}{n}\sum_{i=1}^n \delta_{X^{i,n}_.}$. μ^n takes its values in $\mathcal{P}\left(\mathbb{D}^\uparrow([0,\infty[,\mathbb{N}^*)\right)$.
For each $t \in [0,\infty[$, denote by $\mu^n_t = \frac{1}{n}\sum_{i=1}^n \delta_{X^{i,n}_t}$, the time marginal of $\mu^n_.$. μ^n_t takes its values in $\mathcal{P}(\mathbb{N}^*)$.
We know (see Remark 4.3.4) that the empirical measures μ^n converge in law to Q. $\mathcal{P}(\mathbb{D}^\uparrow([0,\infty[,\mathbb{N}^*))$ is endowed with the weak topology associated with the Skorohod topology on $\mathbb{D}^\uparrow([0,\infty[,\mathbb{N}^*)$. We also know from Corollary 4.3.5 that the $\mathcal{P}(\mathbb{N}^*)$-valued process $(\mu^n_t, t \geq 0)$ goes in probability to $(Q_t, t \geq 0)$ in $\mathbb{D}([0,\infty[,\mathcal{P}(\mathbb{N}^*))$ endowed with the topology of the uniform convergence on every compact subset (of $[0,\infty[$) for the weak topology of $\mathcal{P}(\mathbb{N}^*)$.
Let us now consider the fluctuation process :

$$\eta^n = \sqrt{n}(\mu^n - Q) \tag{4.78}$$

which, for each n, can be seen as a stochastic process with values in the set $\mathcal{M}(\mathbb{N}^*)$ of signed measures on $\mathbb{N}^*$. The aim is to prove that η^n converges weakly to a Gaussian process η, as n goes to infinity. In order to obtain this convergence, we have to introduce a « better » space than $\mathcal{M}(\mathbb{N}^*)$.

Remark 4.5.1 *Let $l^2 = \{(u_k)_{k\geq 0}, u_k \in \mathbb{R}, \sum_k u_k^2 < \infty\}$. l^2 endowed with the norm :*

$$||u||_{l^2} = \sqrt{\sum_{k\geq 1} u_k^2} \tag{4.79}$$

is an Hilbert space. Notice that any bounded (signed) measure $\nu = (\nu(k))_{k\geq 1}$ can be seen as an element of l^2. Remark also that l^2 is not Polish when endowed with the weak topology.
Finally notice that for each n, each t, η_t^n belongs a.s. to l^2, and that, since μ_t^n and Q_t are probability measures,

$$||\eta_t^n||_{l^2} \leq \sqrt{2n}. \tag{4.80}$$

We need to introduce some random objects in order to formulate properly the main result. These objects will serve to define the limit law of η^n.

Definition 4.5.2 *Assume (A) and let $(Q_t, t \geq 0)$ be the unique solution of (MS). Let $\eta_0 = (\eta_0(k))_{k\geq 0}$ be an l^2-valued random variable, and consider an l^2-valued stochastic process, $W = (W_s(k), k \in \mathbb{N}^*, s \geq 0)$. We say that (η_0, W) is of law $(\mathcal{GP})$ if :*

1. *The random infinite vector $(\eta_0(1), \ldots, \eta_0(k), \ldots)$ is a centered Gaussian vector of covariance : for all k, l in $\mathbb{N}^*$,*

$$E(\eta_0(k)\eta_0(l)) = \begin{cases} Q_0(k) - Q_0^2(k) & \text{if} \quad k = l \\ -Q_0(k)Q_0(l) & \text{if} \quad k \neq l. \end{cases} \tag{4.81}$$

2. *The real-valued stochastic « process » $(W_t(k), t \geq 0, k \geq 1)$ is a centered Gaussian process with covariance :*

$$\begin{aligned} E\left(W_t(k)W_s(l)\right) \quad = \quad & \int_0^{s\wedge t} \sum_{i\geq 1}\sum_{j\geq 1} Q_u(i)Q_u(j) \\ & \left(\mathbb{1}_{\{i+j=k\}} - \mathbb{1}_{\{i=k\}}\right)\left(\mathbb{1}_{\{i+j=l\}} - \mathbb{1}_{\{i=l\}}\right)\frac{K(i,j)}{j}du. \end{aligned} \tag{4.82}$$

3. *The random objects η_0 and W are independent.*

The standard theory of Gaussian processes insures that the law $(\mathcal{GP})$ is unique and completely defined.

Let us carry on with the definition of a limit S.D.E.

Proposition 4.5.3 *Assume (A). Let (η_0, W) be a process of law $(\mathcal{GP})$. Then uniqueness of a strongly continuous l^2-valued process η satisfying the following S.D.E.*

$$\eta_t(.) = \eta_0(.) + W_t(.) + \int_0^t \sum_{i\geq 1}\sum_{j\geq 1} L(.)(i,j)\left[\eta_s(i)Q_s(j) + Q_s(i)\eta_s(j)\right] ds \tag{4.83}$$

holds. $L(k)$ being, for each $k \in \mathbb{N}^*$, a map on $\mathbb{N}^* \times \mathbb{N}^*$ defined by

$$L(k)(i,j) = \frac{1\!\!1_{\{i+j=k\}} - 1\!\!1_{\{i=k\}}}{j} K(i,j). \tag{4.84}$$

In particular, the law of the process η is unique.

Let us now state the main result.

Theorem 4.5.4 *Assume (A). Then for each n, the process η^n is a.s. strongly càdlàg from $[0,\infty[$ into l^2.*
Furthermore, η^n converges weakly to the solution η of (4.83) as n goes to infinity.
By « η^n converges weakly to η », we mean the convergence of law of η^n to that of η in the weak topology of $\mathcal{P}(\mathbb{D}([0,\infty[,l^2))$, the space $\mathbb{D}([0,\infty[,l^2)$ being endowed here with the Skorohod topology associated with the weak topology of l^2.

Let us now sketch briefly the main ideas of the proof. First, we will give an useful characterisation of the law $(\mathcal{GP})$. After a technical lemma we will be able to prove the uniqueness result stated in Proposition 4.5.3.
As usual we split the process η^n into satisfying terms, study the tightness and weak convergence of these terms. This technique allows to conclude the proof of Theorem 4.5.4.

We begin with a characterisation of the law $(\mathcal{GP})$.

Proposition 4.5.5 *Assume (A) and let $(Q_t, t \geq 0)$ be the unique solution of (MS). Let $\eta_0 = (\eta_0(k))_{k\geq 0}$ be an l^2-valued random variable, and let $W = (W_s(k), k \geq 1, s \geq 0)$ be an l^2-valued stochastic process. Then (η_0,W) has the law $(\mathcal{GP})$ as soon as the following condition are satisfied.*

1. *For each $k_0 \geq 1$, the random vector $(\eta_0(1), \ldots ,\eta_0(k_0))$ is a centered Gaussian vector of covariance: for all k, l in $\mathbb{N}^*$,*

$$E(\eta_0(k)\eta_0(l)) = \begin{cases} Q_0(k) - Q_0^2(k) & \text{if} \quad k = l \\ -Q_0(k)Q_0(l) & \text{if} \quad k \neq l. \end{cases} \tag{4.85}$$

2. *The process W is strongly continuous from $[0,\infty[$ into l^2. For all $k_0 \in \mathbb{N}^*$, the real-valued process $W_.(k_0)$ is an $(\mathcal{F}_t, t \geq 0)$-martingale starting from 0, where for each $t \geq 0$,*

$$\mathcal{F}_t = \sigma\left\{\eta_0(k)\ ;\ k \in \mathbb{N}^*\right\} \vee \sigma\left\{W_s(k)\ ;\ 0 \leq s \leq t,\, k \in \mathbb{N}^*\right\}. \tag{4.86}$$

For all k_1, k_2 in $\mathbb{N}^*$, $W(k_1)$ and $W(k_2)$ have the following (deterministic)

Doob-Meyer bracket

$$\langle W(k_1),W(k_2)\rangle_t = \int_0^t \sum_{i\geq 1}\sum_{j\geq 1} Q_s(i)Q_s(j) \left(\mathbb{1}_{\{i+j=k_1\}} - \mathbb{1}_{\{i=k_1\}}\right)\left(\mathbb{1}_{\{i+j=k_2\}} - \mathbb{1}_{\{i=k_2\}}\right)\frac{K(i,j)}{j}ds. \tag{4.87}$$

Proof We consider a couple (η_0,W) satisfying the conditions in the proposition, and prove that its law is $(\mathcal{GP})$.
We first prove that for each $k_1, \ldots, k_l$ in $\mathbb{N}^*$, and each $\alpha_1, \ldots, \alpha_l$ in $\mathbb{R}$, the $\mathbb{R}^l$-valued process $Z = (\alpha_1 W(k_1), \ldots, \alpha_l W(k_l))$ has $(\mathcal{F}_t, t \geq 0)$-independent increments and is Gaussian. This is immediate, by using the usual theory of martingales : Z is a continuous $(\mathcal{F}_t, t \geq 0)$-martingale starting from 0 and of which all the Doob-Meyer brackets are deterministic, hence it is a centered Gaussian process with $(\mathcal{F}_t, t \geq 0)$-independent increments. Let us now verify these results.
Let us first check that the real valued « process » $(W_t(k), t \geq 0, k \geq 1)$ is Gaussian. Let $k_1, \ldots, k_l$ in $\mathbb{N}^*$, $\alpha_1, \ldots, \alpha_l$ in $\mathbb{R}$, and $0 < t_1 < t_2 < \ldots < t_l$ be fixed. Then the random variable $V = \alpha_1 W_{t_1}(k_1) + \cdots + \alpha_l W_{t_l}(k_l)$ is Gaussian, since it can be written as the sum of l independent Gaussian random variables :

$$\begin{aligned} V &= \alpha_1 W_{t_1}(k_1) + \cdots + \alpha_l W_{t_1}(k_l) \\ &\quad + \alpha_2\left(W_{t_2}(k_2) - W_{t_1}(k_2)\right) + \cdots + \alpha_l\left(W_{t_2}(k_l) - W_{t_1}(k_l)\right) \\ &\quad + \ldots \\ &\quad + \alpha_l\left(W_{t_l}(k_l) - W_{t_{l-1}}(k_l)\right). \end{aligned} \tag{4.88}$$

It remains to prove (4.82). Let $0 \leq s \leq t$ and k, l in $\mathbb{N}^*$ be fixed. Since the $\mathbb{R}^2$-valued process $(W(k),W(l))$ has independent increments and is centered, we deduce that

$$E\left[W_t(k)W_s(l)\right] = E\left[W_s(k)W_s(l)\right] = E\left[\langle W(k),W(l)\rangle_s\right]. \tag{4.89}$$

(4.87) allows to conclude.
We finally notice that W and η_0 are obviously independent, since W has $(\mathcal{F}_t, t \geq 0)$-independent increments, starts from 0, and since for each t, $\mathcal{F}_t$ contains the σ-field generated by η_0. □

We now state a technical lemma. The hypothesis K bounded appears to be useful in this statement only.

Lemma 4.5.6 *Assume (A). There exists a constant A such that for every couple of probability measures q, μ in $\mathcal{P}(\mathbb{N}^*)$ and all $\alpha \in l^2$, we have*

$$\left|\left|\sum_{i\geq 1}\sum_{j\geq 1} L(.)(i,j)\left[\alpha(i)q(j) + \mu(i)\alpha(j)\right]\right|\right|_{l^2} \leq A\,||\alpha||_{l^2} \tag{4.90}$$

and

$$\left|\left|\sum_{i\geq 1}\sum_{j\geq 1} L(.)(i,j)\alpha(i)\left[q(j)-\mu(j)\right]\right|\right|_{l^2} \leq A\,||\alpha||_{l^2}\,||q-\mu||_{l^2}\,. \tag{4.91}$$

Proof Let us first check (4.90). First notice that

$$\left|\left|\sum_{i\geq 1}\sum_{j\geq 1} L(.)(i,j)\left[\alpha(i)q(j)+\mu(i)\alpha(j)\right]\right|\right|^2_{l^2} \leq 2I_1+2I_2 \tag{4.92}$$

where

$$I_1=\sum_{k\geq 1}\left\{\sum_{i\geq 1}\sum_{j\geq 1}\frac{\mathbb{1}_{\{i+j=k\}}-\mathbb{1}_{\{i=k\}}}{j}K(i,j)\alpha(i)q(j)\right\}^2 \tag{4.93}$$

and

$$I_2=\sum_{k\geq 1}\left\{\sum_{i\geq 1}\sum_{j\geq 1}\frac{\mathbb{1}_{\{i+j=k\}}-\mathbb{1}_{\{i=k\}}}{j}K(i,j)\mu(i)\alpha(j)\right\}^2. \tag{4.94}$$

Thanks to the Cauchy-Schwarz inequality applied to the probability measure q,

$$\begin{aligned}
I_1 &\leq \sum_{k\geq 1}\sum_{j\geq 1}q(j)\left\{\sum_{i\geq 1}\frac{\mathbb{1}_{\{i+j=k\}}-\mathbb{1}_{\{i=k\}}}{j}K(i,j)\alpha(i)\right\}^2\\
&= \sum_{k\geq 1}\sum_{j\geq 1}\frac{q(j)}{j^2}\left\{K(k-j,j)\alpha(k-j)\mathbb{1}_{\{k>j\}}-K(k,j)\alpha(k)\right\}^2\\
&\leq A\sum_{j\geq 1}q(j)\sum_{k\geq 1}\left[\alpha^2(k-j)\mathbb{1}_{\{k>j\}}+\alpha^2(k)\right]\\
&\leq A\,||\alpha||^2_{l^2}
\end{aligned} \tag{4.95}$$

since K is bounded and since q is a probability measure. On the other hand, using twice the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned}
I_2 &\leq \sum_{k\geq 1}\sum_{i\geq 1}\mu(i)\left\{\sum_{j\geq 1}\frac{\mathbb{1}_{\{i+j=k\}}-\mathbb{1}_{\{i=k\}}}{j}K(i,j)\alpha(j)\right\}^2\\
&\leq A\sum_{k\geq 1}\sum_{i\geq 1}\mu(i)\left(\sum_{j\geq 1}\alpha^2(j)\right)\sum_{j\geq 1}\frac{\mathbb{1}_{\{i+j=k\}}+\mathbb{1}_{\{i=k\}}}{j^2}K(i,j)\\
&\leq A\,||\alpha||^2_{l^2}\sum_{i\geq 1}\mu(i)\sum_{j\geq 1}\frac{1}{j^2}\sum_{k\geq 1}\left[\mathbb{1}_{\{i+j=k\}}+\mathbb{1}_{\{i=k\}}\right]\\
&\leq A\,||\alpha||^2_{l^2}\,.
\end{aligned} \tag{4.96}$$

Let us now check (4.91). Since K is bounded, we obtain, using the Cauchy-Schwarz inequality, that for some constant A, the square of the left hand side member of

(4.91) is smaller than

$$
\begin{aligned}
& A\sum_{k\geq 1}\left(\sum_{i\geq 1}\sum_{j\geq 1}\frac{\mathbb{1}_{\{i+j=k\}}+\mathbb{1}_{\{i=k\}}}{j}|\alpha(i)||q(j)-\mu(j)|\right)^2 \\
& \leq A\sum_{k\geq 1}\left(\sum_{j\geq 1}\frac{1}{j^2}\right)\sum_{j\geq 1}\left(\sum_{i\geq 1}\left[\mathbb{1}_{\{i+j=k\}}+\mathbb{1}_{\{i=k\}}\right]|\alpha(i)||q(j)-\mu(j)|\right)^2 \\
& \leq A\sum_{k\geq 1}\sum_{j\geq 1}\left[q(j)-\mu(j)\right]^2\left[\alpha^2(k-j)\mathbb{1}_{\{k>j\}}+\alpha^2(k)\right] \\
& \leq A\,||q-\mu||_{l^2}^2\,||\alpha||_{l^2}^2\,, \qquad (4.97)
\end{aligned}
$$

the last inequality being immediate when exchanging the order of the sums. The lemma is proved. □

The uniqueness for (4.83) is now straightforward.

Proof of Proposition 4.5.3 Consider two solutions η and η' of the equation (4.83), and let $T>0$ be fixed. Then one gets immediately the existence of a constant A_T such that

$$
\begin{aligned}
& \sup_{[0,t]}||\eta_s-\eta'_s||_{l^2}^2 \\
& \leq A_T\int_0^t\left|\left|\sum_{i\geq 1}\sum_{j\geq 1}L(.)(i,j)\left[(\eta_s(i)-\eta'_s(i))Q_s(j)+Q_s(i)(\eta_s(j)-\eta'_s(j)\right]\right|\right|_{l^2}^2 ds \\
& \leq A_T\int_0^t||\eta_s-\eta'_s||_{l^2}^2\,ds \qquad (4.98)
\end{aligned}
$$

thanks to Lemma 4.5.6. Gronwall's Lemma allows to conclude. □

In order to prove Theorem 4.5.4, we have to split η^n into satisfying terms.

Notation 4.5.7 *Thanks to Definition 4.3.1 and the fact that $(Q_t, t\geq 0)$ satisfies (MS), we can write for any $k\geq 1$*

$$\eta_t^n(k)=\eta_0^n(k)+M_t^n(k)+F_t^n(k) \qquad (4.99)$$

where

$$\eta_0^n(k)=\frac{1}{\sqrt{n}}\sum_{i=1}^n\left(\mathbb{1}_{\{X_0^{i,n}=k\}}-Q_0(k)\right), \qquad (4.100)$$

where

$$M_t^n(k) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_0^t \int_j \int_0^\infty \left(\mathbb{1}_{\{X_{s-}^{i,n}+X_{s-}^{j,n}=k\}} - \mathbb{1}_{\{X_{s-}^{i,n}=k\}} \right)$$

$$\mathbb{1}_{\left\{z \le \frac{K(X_{s-}^{i,n},X_{s-}^{j,n})}{X_{s-}^{j,n}}\right\}} \bar{N}^i(ds,dj,dz). \tag{4.101}$$

$\bar{N}^i$ denotes the Poisson compensated measure associated with N^i and where

$$F_t^n(k) = \int_0^t \sum_{i\ge1} \sum_{j\ge1} L(k)(i,j)\, [\eta_s^n(i)\mu_s^n(j) + Q_s(i)\eta_s^n(j),]\, ds, \tag{4.102}$$

$L(k)$ being defined in (4.84).

Let us study first the asymptotic behaviour of the initial condition.

Lemma 4.5.8 *Assume (A). For all n,*

$$E\left[||\eta_0^n||_{l^2}^2\right] \le 1. \tag{4.103}$$

Furthermore, η_0^n converges weakly (for the weak topology on l^2) to a random variable η_0 with the law defined in Definition 4.5.2 1.

Proof The first estimate is easily proved, using an independence argument. Indeed,

$$\begin{aligned} E\left[||\eta_0^n||_{l^2}^2\right] &\le \frac{1}{n} E\left[\sum_{k\ge1} \left\{ \sum_{i=1}^{n} \left(\mathbb{1}_{\{X_0^{i,n}=k\}} - Q_0(k) \right) \right\}^2 \right] \\ &= \sum_{k\ge1} E\left[\left(\mathbb{1}_{\{X_0^{1,n}=k\}} - E\left[\mathbb{1}_{\{X_0^{1,n}=k\}}\right] \right)^2 \right] \\ &\le \sum_{k\ge1} P\left(X_0^{1,n}=k\right) = 1. \end{aligned} \tag{4.104}$$

To prove the convergence result, it suffices to check that the sequence η_0^n is tight in l^2 (endowed with the weak topology), and that for any k_0 fixed, the random vector $(\eta_0^n(1),\ldots,\eta_0^n(k_0))$ converges weakly to $(\eta_0(1),\ldots,\eta_0(k_0))$. The first point is obvious thanks to (4.103), since the balls are weakly compact in l^2. This is an essential property of l^2 that we use here. The second point is a straightforward consequence of the standard central limit theorem. Indeed, for each n, the vector $(\eta_0^n(1),\ldots,\eta_0^n(k_0))$ can be written as

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \begin{pmatrix} \mathbb{1}_{\{X_0^{i,n}=1\}} \\ \cdots \\ \mathbb{1}_{\{X_0^{i,n}=k_0\}} \end{pmatrix} - E\left[\begin{pmatrix} \mathbb{1}_{\{X_0^{i,n}=1\}} \\ \cdots \\ \mathbb{1}_{\{X_0^{i,n}=k_0\}} \end{pmatrix} \right] \right\} \tag{4.105}$$

and thus converges weakly to a centered Gaussian distribution on $\mathbb{R}^{k_0}$ with covariance matrix

$$\left[\text{Cov}\ \left(1\!\!1_{\{X_0^{i,n}=k\}}, 1\!\!1_{\{X_0^{i,n}=l\}}\right)\right]_{k,l\in\{1,\ldots,k_0\}} \tag{4.106}$$

which is the same as (4.81) □

We now state some easy moment calculus and trajectorial estimates for M^n and η^n.

Lemma 4.5.9 *Assume (A).*

1. *For all $T > 0$, there exists a constant A_T, such that for any n*

$$E\left[\sup_{[0,T]} ||M_t^n||_{l^2}^2\right] \le E\left[\sum_{k\ge 1} \sup_{[0,T]} (M_t^n(k))^2\right] \le A_T. \tag{4.107}$$

2. *For all $T > 0$, there exists a constant A_T, such that for any n*

$$E\left[\sup_{[0,T]} ||\eta_t^n||_{l^2}^2\right] \le A_T. \tag{4.108}$$

3. *For all n, M^n and η^n are a.s. strongly càdlàg from $[0,\infty[$ into l^2.*
4. *For each $T > 0$, there exists a constant A_T, such that for any n and $k_0 \in \mathbb{N}^*$*

$$E\left[\sup_{[0,T]} (M_t^n(k_0))^4\right] \le A_T. \tag{4.109}$$

Proof **1.** A simple computation using Doob's inequality and the expression of M^n shows that

$$\begin{aligned}
&\sum_{k\ge 1} E\left[\sup_{[0,T]} (M_t^n(k))^2\right] \\
&\le A \sum_{k\ge 1} \frac{1}{n} E\left[\sum_{i=1}^n \frac{1}{n} \sum_{j=1}^n \int_0^T \left(1\!\!1_{\{X_s^{i,n}+X_s^{j,n}=k\}} - 1\!\!1_{\{X_s^{i,n}=k\}}\right)^2 \frac{K(X_s^{i,n}, X_s^{j,n})}{X_s^{j,n}} ds\right] \\
&\le A \frac{1}{n^2} \sum_{i,j=1}^n \int_0^T \sum_{k\ge 1} \left(1\!\!1_{\{X_s^{i,n}+X_s^{j,n}=k\}} + 1\!\!1_{\{X_s^{i,n}=k\}}\right) ds \\
&\le AT
\end{aligned} \tag{4.110}$$

since K is bounded, and since $X^{i,n}$ takes its values in $\mathbb{N}^*$ and thus is greater than 1.

2. We deduce from 1, equation (4.99), Lemmas 4.5.8 and 4.5.6 that for any n and $t \leq T$,

$$E\left[\sup_{[0,t]} ||\eta_s^n||_{l^2}^2\right] \leq A_T + A_T \int_0^t E\left[||\eta_s^n||_{l^2}^2\right] ds. \tag{4.111}$$

Gronwall's Lemma allows to conclude.

3. Recall that $\eta_t^n = \eta_0^n + M_t^n + F_t^n$. Let us first notice that thanks to Lemma 4.5.6, we obtain, for all $s < t$

$$||F_t^n - F_s^n||_{l^2}^2 \leq t \int_0^t ||\eta_u^n||_{l^2}^2 \, du \tag{4.112}$$

and it is clear from 1 that F^n is strongly continuous.
We thus just have to check that M^n is càdlàg. Let us for example show that it is càd. Let t_m be a sequence decreasing to t and $\varepsilon > 0$ be fixed. Then for all $q \in \mathbb{N}$,

$$\left|\left|M_{t_m}^n - M_t^n\right|\right|_{l^2}^2 \leq \sum_{k=1}^{q} \left(M_{t_m}^n(k) - M_t^n(k)\right)^2 + 2 \sum_{k=q+1}^{\infty} \sup_{[0,t+1]} \left(M_u^n(k)\right)^2 \tag{4.113}$$

(at least if m is large enough, which ensures that $t_m \leq t+1$). Now, choosing q large enough will imply that the second term of the right hand side member is smaller than $\varepsilon/2$, thanks to 1. It is also clear that for each k fixed $M^n(k)$ is càdlàg, since it is a finite sum of integrals against Poisson measures. We can now conclude: it is sufficient to choose m large enough, in order to obtain that for all k in $\{1,\ldots,q\}$, $(M_{t_m}^n(k) - M_t^n(k))^2 \leq \varepsilon/(2q)$.

4. First notice that the quadratic variation of $M^n(k_0)$ is given, since the Poisson measures N^i are independent (and thus never jump at the same moment *a.s.*), by

$$[M^n(k_0)]_t = \frac{1}{n}\sum_{i=1}^{n} \int_0^t \int_j \int_0^\infty \left(\mathbb{1}_{\{X_{s-}^{i,n}+X_{s-}^{j,n}=k\}} - \mathbb{1}_{\{X_{s-}^{i,n}=k\}}\right)^2$$

$$\mathbb{1}_{\left\{z \leq \frac{K(X_{s-}^{i,n},X_{s-}^{j,n})}{X_{s-}^{j,n}}\right\}} N^i(ds,dj,dz). \tag{4.114}$$

In particular, since K is bounded and $X^{j,n}$ takes its values in $\mathbb{N}^*$ (and is thus in particular greater than 1),

$$[M^n(k_0)]_t \leq \frac{1}{n}\sum_{i=1}^{n} \int_0^t \int_j \int_0^\infty \mathbb{1}_{\{z \leq ||K||_\infty\}} N^i(ds,dj,dz). \tag{4.115}$$

But the random variables $\displaystyle\int_0^t \int_j \int_0^\infty \mathbb{1}_{\{z \leq ||K||_\infty\}} N^i(ds,dj,dz)$ are independent and have a Poisson law of parameter $t\,||K||_\infty$. Hence

$$\sum_{i=1}^{n} \int_0^t \int_j \int_0^\infty \mathbb{1}_{\{z \leq ||K||_\infty\}} N^i(ds,dj,dz) \tag{4.116}$$

has a Poisson law of parameter $nt\,||K||_\infty$, from which we deduce that

$$E\left[[M^n(k_0)]_t^2\right] \leq \frac{1}{n^2}\left(nt\,||K||_\infty + [nt\,||K||_\infty]^2\right) \leq A_T. \tag{4.117}$$

Finally, using the Burkholder-Davis-Gundy inequality, we obtain

$$E\left[\sup_{[0,T]}(M_t^n(k_0))^4\right] \leq AE\left([M^n(k_0)]_T^2\right) \leq A_T. \tag{4.118}$$

The Lemma 4.5.9 is proved. □

It remains to prove tightness results. Notice that l^2, endowed with the weak topology, is not Polish. We thus have to use a specific Lemma which is due to Fernique [20] (see lemma 4.7.3 in the appendix). This lemma allows us to state and prove the following result.

Lemma 4.5.10 *Assume* (A).

1. *The sequences of processes* η^n *and* M^n *are tight in* $\mathbb{D}([0,\infty[,l^2)$, *(endowed with the Skorohod topology associated with the weak topology of* l^2*).*
2. *Any limit point* η *(resp.* M*) of the sequence* η^n *(resp.* M^n*) is a.s. strongly continuous from* $[0,\infty[$ *into* l^2.

Proof **1.** We prove first 1. We apply Lemma 4.7.3 to the sequences η^n and M^n. Condition (i) is satisfied thanks to Lemma 4.5.9, 1 and 2.
To prove the second condition, we fix k, and apply the Aldous criterion. We just have to show, for example, that there exists a constant A_T such that for all n, all $\delta > 0$, all couple of stopping times $0 < S < S' < (S+\delta)\wedge T$,

$$E\left[(M_{S'}^n(k) - M_S^n(k))^2\right] + E\left[(\eta_{S'}^n(k) - \eta_S^n(k))^2\right] \leq A_T\delta. \tag{4.119}$$

First, using (4.101),

$$\begin{aligned}
&E\left[(M_{S'}^n(k) - M_S^n(k))^2\right] \\
&\leq \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n E\left[\int_S^{S'}\left\{1\!\!1_{\{X_s^{i,n}+X_s^{j,n}=k\}} - 1\!\!1_{\{X_s^{i,n}=k\}}\right\}^2\frac{K(X_s^{i,n},X_s^{j,n})}{X_s^{j,n}}ds\right] \\
&\leq A_T E\left[S'-S\right] \leq A_T\delta
\end{aligned} \tag{4.120}$$

thanks to (A). Using (4.99), Lemmas 4.5.6 and 4.5.9, we obtain

$$\begin{aligned}
E\left[(\eta_{S'}^n(k) - \eta_S^n(k))^2\right] &\leq 2E\left[(M_{S'}^n(k) - M_S^n(k))^2\right] \\
&\quad +2E\left[\int_S^{S'} ds \times \sup_{[0,T]}||\eta_s^n||_{l^2}^2\right] \leq A_T\delta
\end{aligned} \tag{4.121}$$

which was our aim.

2. To prove the continuity results, it suffices to check that

$$E\left[\sup_{[0,\infty[}||\Delta M^n_t||_{l^2}\right]\underset{n\to 0}{\longrightarrow} 0 \tag{4.122}$$

and that the same limit result holds for η^n. First notice that for any ω, n, t, $\Delta M^n_t = \Delta\eta^n_t$, so that it suffices to prove (4.122). Remark also that as the Poisson measures N^i are independent, it is obvious using (4.101) that any jump of M^n is of the form

$$\Delta M^n_t(.) = \frac{1}{\sqrt{n}}\left[1\!\!1_{\{X^{i,n}_{t-}+X^{j,n}_{t-}=.\}} - 1\!\!1_{\{X^{i,n}_{t-}=.\}}\right] \tag{4.123}$$

from which we deduce that

$$||\Delta M^n_t||^2_{l^2} \leq \frac{1}{n}\sum_{k\geq 1}\left[1\!\!1_{\{X^{i,n}_{t-}+X^{j,n}_{t-}=k\}} - 1\!\!1_{\{X^{i,n}_{t-}=k\}}\right]^2 \leq \frac{1}{n}, \tag{4.124}$$

which implies (4.122). The Lemma is now proved. □

Let us study now the law of the limiting points.

Lemma 4.5.11 *Assume (A). Consider a subsequence η^{n_k} of η^n, weakly convergent to a strongly continuous l^2-valued process η , in $\mathbb{D}([0,\infty[,l^2)$ (endowed with the Skorohod topology associated with the weak topology on l^2). Consider the processes*

$$M^n = \eta^n - \eta^n_0 - F^n \tag{4.125}$$

and

$$W_.(\centerdot) = \eta_.(\centerdot) - \eta_0(\centerdot) - \int_0^. \sum_{i\geq 1}\sum_{j\geq 1} L(\centerdot)(i,j)\left\{\eta_s(i)Q_s(j) + Q_s(i)\eta_s(j)\right\}ds. \tag{4.126}$$

Finally set, for $t\geq 0$,

$$\mathcal{F}_t = \sigma\left\{\eta_0(k)\ ;\ k\in\mathbb{N}^*\right\}\vee\sigma\left\{W_s(k)\ ;\ 0\leq s\leq t,\, k\in\mathbb{N}^*\right\}. \tag{4.127}$$

We have :

1. *$(\eta^{n_k}_0,M^{n_k})$ converges weakly to (η_0,W) in $l^2\times\mathbb{D}([0,\infty[,l^2)$ (endowed with the product topology, l^2 being endowed with the weak topology and $\mathbb{D}([0,\infty[,l^2)$ with the associated Skorohod topology).*
2. *The process W is a.s. strongly continuous from $[0,\infty[$ into l^2. For each $k_0\in\mathbb{N}^*$, the real-valued process $W(k_0)$ is an $(\mathcal{F}_t, t\geq 0)$-martingale.*

3. For each q_1, q_2 in $\mathbb{N}^*$, the Doob-Meyer bracket of $W(q_1)$ and $W(q_2)$ is given by

$$\begin{aligned}\langle W(q_1),W(q_2)\rangle_t &= \int_0^t \sum_{i\geq 1}\sum_{j\geq 1} Q_s(i)Q_s(j) \\ &\quad \left(\mathbb{1}_{\{i+j=q_1\}} - \mathbb{1}_{\{i=q_1\}}\right)\left(\mathbb{1}_{\{i+j=q_2\}} - \mathbb{1}_{\{i=q_2\}}\right)\frac{K(i,j)}{j}ds.\end{aligned} \tag{4.128}$$

Proof 1. We divide the proof of 1 in two steps. Consider the process

$$G^n_\cdot(\centerdot) = \eta^n_\cdot(\centerdot) - \eta^n_0(\centerdot) - \int_0^\cdot \sum_{i\geq 1}\sum_{j\geq 1} L(\centerdot)(i,j)\left\{\eta^n_s(i)Q_s(j) + Q_s(i)\eta^n_s(j)\right\}ds. \tag{4.129}$$

(i) We will first prove that for any $T>0$, the process $M^n - G^n$ goes to 0, as n tends to infinity, in L^1 (and thus in probability), for the uniform norm on $[0,T]$ and the strong norm of l^2.
(ii) Then we will check that $(\eta_0^{n_k},G^{n_k})$ goes in law, as k tends to infinity, to (η_0,W) in $l^2\times\mathbb{D}([0,\infty[,l^2)$, the space l^2 being endowed (twice) with the weak topology. Once the two points are proved, 1 is straightforward.
Let us check the first point (i). We deduce from the expression of F^n (see (4.102)) that

$$\begin{aligned}&E\left[\sup_{[0,T]}||M^n_t - G^n_t||_{l^2}\right] \\ &\leq E\left[\int_0^T\left|\left|\sum_{i\geq 1}\sum_{j\geq 1}L(.)(i,j)\eta^n_s(i)\left\{\mu^n_s(j) - Q_s(j)\right\}\right|\right|_{l^2} ds\right].\end{aligned} \tag{4.130}$$

Using Lemma 4.5.6, we get the existence of a constant A_T such that

$$\begin{aligned}E\left[\sup_{[0,T]}||M^n_t - G^n_t||_{l^2}\right] &\leq A_T E\left[\int_0^T ||\eta^n_s||_{l^2}\,||\mu^n_s - Q_s||_{l^2}\,ds\right] \\ &\leq \frac{1}{n^{1/2}}A_T E\left[\sup_{[0,T]}||\eta^n_s||^2_{l^2}\right]\end{aligned} \tag{4.131}$$

which goes to 0 as n tends to infinity thanks to Lemma 4.5.9.
To prove that $(\eta_0^{n_k},M^{n_k})$ goes in law to (η_0,W), it suffices to prove that the map $\mathcal{G}$ from $\mathbb{D}([0,\infty[,l^2)$ into $l^2\times\mathbb{D}([0,\infty[,l^2)$, defined by

$$\begin{aligned}\mathcal{G}(\alpha) &= (\mathcal{G}^{(1)},\mathcal{G}^{(2)}) \\ &= \left(\alpha_0,\alpha - \alpha_0 - \int_0^\cdot \sum_{i\geq 1}\sum_{j\geq 1}L(\centerdot)(i,j)\left\{\alpha_s(i)Q_s(j) + Q_s(i)\alpha_s(j)\right\}ds\right)\end{aligned} \tag{4.132}$$

is continuous at any point α that is strongly continuous. Indeed, we know that η^{n_k} goes to η and that η is strongly continuous.
We thus consider a sequence α^n of $\mathbb{D}([0,\infty[,l^2)$, converging (for the Skorohod topology associated with the weak topology of l^2) to a strongly continuous l^2-valued function α. It is well-known that since the limit α is continuous, the convergence holds also for the topology of the uniform convergence on compacts, *i.e.* that for any $T > 0$, any γ in l^2,

$$\sup_{[0,T]} \left| \sum_{k\geq 1} \gamma(k)(\alpha_t^n(k) - \alpha_t(k)) \right| \underset{n\to 0}{\longrightarrow} 0. \tag{4.133}$$

We first deduce that $\mathcal{G}^{(1)}(\alpha_n)$ tends to $\mathcal{G}^{(1)}(\alpha)$ for the weak topology of l^2. It remains to prove that for any β in l^2 and any $T > 0$

$$\delta_n = \sup_{[0,T]} \left| \sum_{k\geq 1} \beta(k) \left(\mathcal{G}_t^{(2)}(\alpha_n) - \mathcal{G}_t^{(2)}(\alpha) \right) \right| \underset{n\to\infty}{\longrightarrow} 0. \tag{4.134}$$

First, it is clear that

$$\begin{aligned}
\delta_n &\leq 2 \sup_{[0,T]} \left| \sum_{k\geq 1} \beta(k)(\alpha_t^n(k) - \alpha_t(k)) \right| \\
&\quad + \int_0^T \left| \sum_{k\geq 1} \beta(k) \left(\sum_{i\geq 1} \sum_{j\geq 1} L(k)(i,j)(\alpha_s^n(i) - \alpha_s(i)) Q_s(j) \right) \right| ds \\
&\quad + \int_0^T \left| \sum_{k\geq 1} \beta(k) \left(\sum_{i\geq 1} \sum_{j\geq 1} L(k)(i,j) Q_s(i)(\alpha_s^n(j) - \alpha_s(j)) \right) \right| ds \\
&= 2\delta_n^{(1)} + \delta_n^{(2)} + \delta_n^{(3)}
\end{aligned} \tag{4.135}$$

with obvious notations in the last equality. First, $\delta_n^{(1)}$ tends to 0 thanks to (4.133) applied with $\gamma = \beta$. On the other hand,

$$\begin{aligned}
\delta_n^{(2)} &= \int_0^T \left| \sum_{k\geq 1} \beta(k) \sum_{i\geq 1} \sum_{j\geq 1} \left(\mathbb{1}_{\{i+j=k\}} - \mathbb{1}_{\{i=k\}} \right) \frac{K(i,j)}{j} \left[\alpha_s^n(i) - \alpha_s(i) \right] Q_s(j) \right| ds \\
&= \int_0^T \Big| \sum_{i\geq 1} [\alpha_s^n(i) - \alpha_s(i)] \\
&\qquad \times \left\{ \sum_{j\geq 1} \beta(i+j) \frac{K(i,j)}{j} Q_s(j) - \beta(i) \sum_{j\geq 1} \frac{K(i,j)}{j} Q_s(j) \right\} \Big| ds.
\end{aligned} \tag{4.136}$$

For each s, the integrand tends to 0, thanks to (4.133) applied with

$$\gamma(i) = \sum_{j\geq 1} \beta(i+j) \frac{K(i,j)}{j} Q_s(j) - \beta(i) \sum_{j\geq 1} \frac{K(i,j)}{j} Q_s(j), \tag{4.137}$$

which belongs to l^2. On the other hand, the Cauchy-Schwarz inequality and Lemma 4.5.6 allow to obtain that the integrand in (4.136) is smaller than $A \times ||\beta||_{l^2} \, ||\alpha_s^n - \alpha_s||_{l^2}$ which is clearly bounded (uniformly in n) on $[0,T]$. By Lebesgue theorem we get that $\delta_n^{(2)}$ tends to 0. One proves in the same way that $\delta_n^{(3)}$ tends to 0. This ends the proof of 1.

2. From Lemma 4.5.10 the process W is strongly continuous since it is a limit point of M^n. To check that $W(k_0)$ is a $(\mathcal{F}_t, t \geq 0)$-martingale, let us consider $0 \leq s_1 \leq \cdots \leq s_l \leq t$, $k_1, \ldots, k_l, m_1, \ldots, m_p$ in $\mathbb{N}^*$, and a family $\phi_1, \ldots, \phi_l$, $\psi_1, \ldots, \psi_p$ of continuous bounded functions from $\mathbb{R}$ into itself. We have to check that

$$E\Big[\{W_t(k_0) - W_{s_l}(k_0)\} \, \psi_1(\eta_0(m_1)) \ldots \psi_p(\eta_0(m_p)) \\ \phi_1(M_{s_1}(k_1)) \ldots \phi_l(M_{s_l}(k_l))\Big] = 0. \tag{4.138}$$

By notation 4.5.7, we see that for each n,

$$E\Big[\{M_t^n(k_0) - M_{s_l}^n(k_0)\} \, \psi_1(\eta_0^n(m_1)) \ldots \psi_p(\eta_0^n(m_p)) \\ \phi_1(M_{s_1}^n(k_1)) \ldots \phi_l(M_{s_l}^n(k_l))\Big] = 0, \tag{4.139}$$

so that we « just » have to take the limit. We know from 1 that $(\eta_0^{n_k}, M^{n_k})$ converges weakly to (η_0, W), and that W is strongly continuous. Furthermore, the map $F : l^2 \times \mathbb{D}([0,\infty[, l^2) \to \mathbb{R}$, defined by

$$F(\zeta,\alpha) = \{\alpha_t(k_0) - \alpha_{s_l}(k_0)\} \, \psi_1(\zeta(m_1)) \ldots \psi_p(\zeta(m_p))\phi_1(\alpha_{s_1}(k_1)) \ldots \phi_l(\alpha_{s_l}(k_l)) \tag{4.140}$$

is continuous at any point (ζ,α) such that α is strongly continuous. We deduce that for each $0 \leq A < \infty$,

$$E\left[F(\eta_0^{n_k}, M^{n_k}) \wedge A \vee (-A)\right] \underset{k\to\infty}{\longrightarrow} E\left[F(\eta_0, W) \wedge A \vee (-A)\right]. \tag{4.141}$$

Finally, using the fact that for some constant B, $|F(\zeta,\alpha)| \leq B \sup_{[0,t]} |\alpha_s(k_0)|$ and the uniform integrability (concerning M^n) obtained in Lemma 4.5.9, we can make A go to infinity, and obtain that

$$E\left[F(\eta_0^{n_k}, M^{n_k})\right] \underset{k\to\infty}{\longrightarrow} E\left[F(\eta_0, W)\right]. \tag{4.142}$$

By using (4.139), we get $E\left[F(\eta_0, W)\right] = 0$, which was our aim.

3. To prove (4.128) it suffices to check that, for Y the process denoting the right hand side member of (4.128), $W(q_1)W(q_2) - Y$ is a $(\mathcal{F}_t, t \geq 0)$-martingale. To this aim, we proceed exactly as in 2. We have to prove, with the same notations as in 2, that

$$E\Big[\{W_t(q_1)W_t(q_2) - Y_t - W_{s_l}(q_1)W_{s_l}(q_2) + Y_{s_l}\}$$

$$\psi_1(\eta_0(m_1))\dots\psi_p(\eta_0(m_p))\phi_1(M_{s_1}(k_1))\dots\phi_l(M_{s_l}(k_l))\Big] = 0 \tag{4.143}$$

This equality holds when replacing everywhere W by M^{n_k}, Y by $\langle M^{n_k}(q_1), M^{n_k}(q_2)\rangle$, and η_0 by $\eta_0^{n_k}$. We just have to make k tend to infinity. Using (4.101) we obtain for each n

$$\begin{aligned}\langle M^n(q_1), M^n(q_2)\rangle_t &= \int_0^t \sum_{i\geq 1}\sum_{j\geq 1} \mu_s^n(i)\mu_s^n(j) \\ &\quad \left(\mathbb{1}_{\{i+j=q_1\}} - \mathbb{1}_{\{i=q_1\}}\right)\left(\mathbb{1}_{\{i+j=q_2\}} - \mathbb{1}_{\{i=q_2\}}\right)\frac{K(i,j)}{j} ds.\end{aligned} \tag{4.144}$$

A simple computation shows that for any $T>0$, for some constant A,

$$\begin{aligned}
&E\left[\sup_{[0,T]} |\langle M^n(q_1), M^n(q_2)\rangle_t - Y_t|\right] \\
&\leq E\Big[\int_0^T \sum_{i\geq 1}\sum_{j\geq 1} \left|\mathbb{1}_{\{i+j=q_1\}} - \mathbb{1}_{\{i=q_1\}}\right| \left|\mathbb{1}_{\{i+j=q_2\}} - \mathbb{1}_{\{i=q_2\}}\right| \frac{K(i,j)}{j} \\
&\qquad \{|Q_s(i)||Q_s(j)-\mu_s^n(j)| + |Q_s(j)||Q_s(i)-\mu_s^n(i)|\}\, ds\Big] \\
&\leq AE\Big[\int_0^T \sum_{j\geq 1}\frac{1}{j}\sum_{i\geq 1}\left\{\mathbb{1}_{\{i+j=q_1\}} + \mathbb{1}_{\{i=q_1\}} + \mathbb{1}_{\{i+j=q_2\}} + \mathbb{1}_{\{i=q_2\}}\right\} \\
&\qquad \{|Q_s(i)||Q_s(j)-\mu_s^n(j)| + |Q_s(j)||Q_s(i)-\mu_s^n(i)|\}\, ds\Big] \\
&\leq AE\Big[\int_0^T \sum_{j\geq 1}\frac{1}{j} 4\sup_{i\geq 1}\Big\{|Q_s(i)||Q_s(j)-\mu_s^n(j)| + |Q_s(j)||Q_s(i)-\mu_s^n(i)|\Big\} ds\Big] \\
&\leq AE\Big[\int_0^T \sum_{j\geq 1}\frac{1}{j}\{|Q_s(j)-\mu_s^n(j)| + Q_s(j)\,||\mu_s^n - Q_s||_{l^2}\}\, ds\Big] \\
&\leq AE\left[\int_0^T ||\mu_s^n - Q_s||_{l^2}\, ds\right] \\
&\leq A\,T\frac{1}{\sqrt{n}} E\left[\sup_{[0,T]} ||\eta_s^n||_{l^2}\right]
\end{aligned}$$

which tends to 0 thanks to Lemma 4.5.9. Finally notice that for any $T>0$,

$$\begin{aligned}
&\sup_n E\left[\sup_{[0,T]} |M_t^n(q_1)M_t^n(q_2) - \langle M^{n_k}(q_1), M^{n_k}(q_2)\rangle_t|^2\right] \\
&\leq 2\sup_n E\left[\sup_{[0,T]} (M_t^n(q_1))^4\right] E\left[\sup_{[0,T]} (M_t^n(q_2))^4\right] \\
&+2\sup_n E\left[|M_t^n(q_1)M_t^n(q_2)|^2\right] < \infty
\end{aligned} \tag{4.145}$$

thanks to Lemma 4.5.9 4. and since one obviously deduces from (4.144) (recall that μ^n is a probability measure) that for all $t \in [0,T]$,

$$| < M^n(q_1),M^n(q_2) >_t | \leq T\, ||K||_\infty .$$

Using the convergence in L^1 of $< M^n(q_1),M^n(q_2) >$ to Y, the convergence in law of $(\eta_0^{n_k},M^{n_k})$ to (η_0,W), the uniform integrability obtained in (4.145) and the equality

$$\begin{aligned} E\Big[\Big\{ & M_t^{n_k}(q_1)M_t^{n_k}(q_2) - \langle M^{n_k}(q_1),M^{n_k}(q_2)\rangle_t \\ & -M_{s_l}^{n_k}(q_1)M_{s_l}^{n_k}(q_2) + \langle M^{n_k}(q_1),M^{n_k}(q_2)\rangle_{s_l} \Big\} \\ & \times\psi_1(\eta_0^{n_k}(m_1))\dots\psi_p(\eta_0^{n_k}(m_p))\phi_1(M_{s_1}^{n_k}(k_1))\dots\phi_l(M_{s_l}^{n_k}(k_l))\Big] = 0, \end{aligned} \tag{4.146}$$

we obtain (4.143) by letting k go to infinity. This concludes the proof of the lemma. □

We finally are able to provide the

Proof of Theorem 4.5.4 Recall the Notation 4.5.7, for each n, $\eta^n = \eta_0^n + M^n + F^n$. We know from Lemma 4.5.10 that the sequence η^n is tight, and that any limit point is strongly continuous. Let us consider a converging subsequence η^{n_k}, going to some process η. From Lemma 4.5.11 we know that $\eta_t^{n_k} - \eta_0^{n_k} - F_t^{n_k}$ goes to the process

$$W_\cdot(\centerdot) = \eta_\cdot(\centerdot) - \eta_0(\centerdot) - \int_0^\cdot \sum_{i\geq 1}\sum_{j\geq 1} L(\centerdot)(i,j)\left\{\eta_s(i)Q_s(j) + Q_s(i)\eta_s(j)\right\} ds. \tag{4.147}$$

It is clear from Lemmas 4.5.11, 4.5.8, and Definition 4.5.2 that (η_0,W) has the law $(\mathcal{GP})$. Hence, η can be written as a solution to equation (4.83). Since the uniqueness in law for this S.D.E. holds thanks to Proposition 4.5.3, we deduce that the whole sequence η^n goes, in law, to the solution η of equation (4.83), which concludes the proof of Theorem 4.5.4. □

4.6 Numerical results

In this section we test numerically our algorithm. The main questions we treat here are :

(i) test the convergence of the particle system for classical kernels for which the solution is known,
(ii) we answer numerically the question : is the particle system still simulable when the coagulation kernel K does not satisfy $(H_{1/2})$ but only (H_1)?

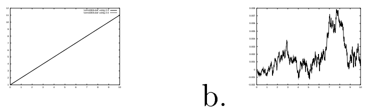
(iii) if the answer to *(ii)* is yes, does the convergence result of Theorem 4.3.3 remain valid under (H_1)?
(iv) does the central limit type result obtained in Theorem 4.5.4 hold when the coagulation kernel does not satisfy any more assumption (A), but only $(H_{1/2})$ or even (H_1)?
(v) can we hope that similar results hold in the continuous case?

Let us first recall that $\mu_t^n = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_t^{i,n}}$ is defined in Definition 4.3.1, and approximates the solution Q_t to the equation (MS). In the discrete case $Q_t = \sum_{k=1}^{\infty} kn(k,t)\delta_k$, where $(n(k,t), k \in \mathbb{N}^*, t \geq 0)$ satisfies (SD), while in the continuous case $Q_t(dx) = xn(x,t)dx$, where $(n(x,t), x \in \mathbb{R}_+, t \geq 0)$ satisfies (SC).

We first consider the discrete case, more precisely the case where $Q_0 = \delta_1$, *i.e.* where the initial system contains only particles of mass 1. We compare first $m(t) = \sum_{k=1}^{\infty} k^2 n(k,t)$ to $m_n(\omega,t) = \int x\mu_t^n(\omega,dx) = \frac{1}{n}\sum_{i=1}^{n} X_t^{i,n}$. This will give a « global idea » of the rate of convergence. The function $m(t)$ can be explicitly computed in any case where $K(x,y)$ is of the form $A + B(x+y) + Cxy$.

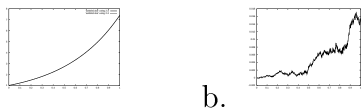
We first consider the case $K(x,y) = 1$. Figure 1 (a) represents $m(t)$ and $m_n(t)$ (obtained with one simulation) for $n = 10^6$ particles, as functions of $t \in [0,10]$. On 1 (b), we draw the error $m_n(t,\omega) - m(t)$: it really looks like one path of a continuous « Brownian type » process, which illustrates Theorem 4.5.4.
The same quantities are studied in Figure 2, for the case $K(x,y) = x + y$, $n = 10^6$ and $t \in [0,1]$.
In Figure 3, we study the multiplicative kernel $K(x,y) = xy$. In this case, one has an explicit expression of the solution $(n(k,t), 0 \leq t < 1, k \in \mathbb{N}^*)$ to (SC). The first part 3 (a) represents $n(2,t)$ and its approximation $\frac{1}{2}\sum_{i=1}^{n} \mathbb{1}_{\{X_t^{i,n}=2\}}$ as functions of $t \in [0,0.98]$, for $n = 10^5$ particles. The second part 3 (b) represents the corresponding error.

We study now the rate of convergence of our scheme, as the number of particles increases. On Figure 4 (a) each cross is obtained for one simulation, and represents $m_n(\omega,t) - m(t)$, for $t = 1$ and $K(x,y) = 1$, as a function of the number $n \in \{2,\ldots,10^6\}$ of particles. We remark that the obtained « cloud » is almost surely contained in $[-C\sqrt{n}, C\sqrt{n}]$, for some constant C, which illustrates again Theorem 4.5.4.
Figure 4 (b) (resp. 4 (c)) treats the case $K(x,y) = x+y$, $t = 1$ and $n \in \{2,\ldots,5.10^5\}$ (resp. $K(x,y) = xy$, $t = 0.9$ and $n \in \{2,\ldots,7.10^5\}$).



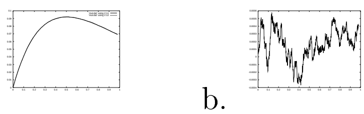
a. b.

FIG. 4.1 – *K(x,y)=1.*



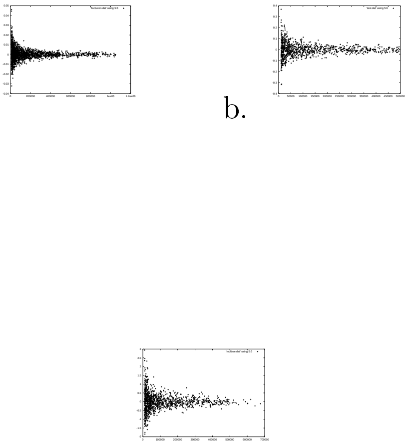
a. b.

FIG. 4.2 – *K(x,y)=x+y.*



a. b.

FIG. 4.3 – *K(x,y)=xy.*



a. b. c.

FIG. 4.4 – $a: K(x,y) = 1,\ b: K(x,y) = x + y,\ c: K(x,y) = xy$

Figures 1 to 4 answer the questions from *(i)* to *(iv)*.

Let us finally treat the continuous case and answer so the question *(v)*. We consider the uniform distribution $Q_0(dx) = \mathbb{1}_{[0,1]}(x)dx$, and the coagulation kernel $K(x,y) = 1+x+y+xy$. Notice that $\int x^{-1}Q_0(dx) = \infty$. The corresponding solution $(n(x,t), x, t)$ to the Smoluchowski equation (SC) is then known to have a gelification time which equals 2/3.
We study again $m(t) = \int x^2 n(x,t)dx$, that we compare with

$$m_n(\omega,t) = \int x\mu_t^n(\omega,dx) = \frac{1}{n}\sum_{i=1}^{n} X_t^{i,n}.$$

Figure 5 (a) represents $m(t)$ and $m_n(\omega,t)$ (obtained by one simulation) as functions of $t \in [0,0.65]$ for $n = 25000$. Figure 5 (b) shows the corresponding error. Finally, each cross of Figure 6 is obtained by one simulation, and represents $m_n(\omega,t) - m(t)$ for $t = 0.5$, as a function of the number $n \in \{2,\ldots,10^4\}$ of particles.

a. b.

FIG. 4.5 – *K(x,y)=1+x+y+xy.*

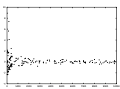


FIG. 4.6 – *K(x,y)=1+x+y+xy.*

The main conclusion of this numerical study is that our assumptions in Theorems 4.3.3 and 4.5.4 are too stringent, and that these results seem to hold true in a more general context. Indeed, the particle system really seems to be simulable under (H_1) (we have not used any cutoff procedure to obtain Figures 3, 4 (c), 5 and 6). The convergence result of Theorem 4.3.3 seems to hold also under (H_1).
Finally, Figures 1 (b), 2 (b), 3 (b) and 4 show that the result of Theorem 4.5.4 may hold, in the discrete case, under (H_1), while Figures 5 (b) and 6 show that a similar result might remain true in the continuous case.

4.7 Appendix

First, we recall the Aldous criterion for tightness (see Jacod, Shiryaev [35]).

Theorem 4.7.1 *Let $(X_t^n, t \in [0,T_0[)$ be a family of càdlàg adapted processes on $[0,T_0[$, for some $T_0 \le \infty$. Denote by $Q^n \in \mathcal{P}(\mathbb{D}([0,T_0[,\mathbb{R}))$ the law of X^n. Assume that:*

1. *For all $T < T_0$,*
$$\sup_n E\left[\sup_{t\in[0,T]} |X_t^n|\right] < \infty. \tag{4.148}$$
2. *For all $T < T_0$, all $\eta > 0$,*
$$\sup_n \sup_{(S,S')\in ST_T(\delta)} P\left[|X_{S'}^n - X_S^n| \ge \eta\right] \underset{\delta\to 0}{\longrightarrow} 0 \tag{4.149}$$
where $ST_T(\delta)$ is the set of couples (S,S') of stopping times satisfying a.s. $0 \le S \le S' \le (S+\delta)\wedge T$.

Then the family $(Q^n)_n$ is tight. Furthermore, any limiting point Q of this family is the law of a quasi-left continuous process, i.e. for all $t \in [0,T_0[$ fixed,
$$\int_{\mathbb{D}([0,T_0[,\mathbb{R})} \mathbb{1}_{\{\Delta x(t)\neq 0\}} Q(dx) = 0. \tag{4.150}$$

The following Lemma can be found in Méléard [49].

Lemma 4.7.2 *Let $T > 0$ be fixed and let ν^n be a sequence of random probability measures on $\mathbb{D}([0,T],\mathbb{R})$, which converges in law to a deterministic probability measure $R \in \mathcal{P}(\mathbb{D}([0,T],\mathbb{R}))$. Assume moreover that*
$$\sup_{t\in[0,T]} \int_{x\in\mathbb{D}([0,T],\mathbb{R})} \sup_{s\in[t-r,t+r]} (|\Delta x(s)| \wedge 1)\, R(dx) \underset{r\to 0}{\longrightarrow} 0. \tag{4.151}$$

Then $(\nu_t^n, t \in [0,T])$ converges in probability to $(R_t, t \in [0,T])$ in $\mathbb{D}([0,T],\mathcal{P}(\mathbb{R}))$ endowed with the topology of the uniform convergence.

We state here a Lemma due to Fernique [20], which we write in the particular situation of l^2-valued processes.

Lemma 4.7.3 *Let α^n be a sequence of strongly càdlàg processes with values in l^2. Then the sequence α^n is tight in $\mathbb{D}([0,\infty[,l^2)$ (endowed with the Skorohod topology associated with the weak topology of l^2) if the following conditions are satisfied :*
(i) For any $T < \infty$, there exists a sequence of weakly compact subsets K_m of l^2 such that for any n, any m,

$$P\,(\forall\, t \in [0,T],\, \alpha_t^n \in K_m) \geq 1 - 2^{-m}. \tag{4.152}$$

In particular this condition is always satisfied if for all T,

$$\sup_n E\left[\sup_{[0,T]} ||\alpha_t^n||_{l^2}^2\right] < \infty. \tag{4.153}$$

(ii) For each $k \geq 1$, the sequence of real-valued processes $\alpha^n(k)$ is tight (for the usual Skorohod topology on $\mathbb{D}([0,\infty[,\mathbb{R})$.

Deuxième partie

Approximation de l'espérance de fonctionnelles de la trajectoire d'une diffusion par le schéma d'Euler

5

Introduction

Cette étude a fait l'objet d'une collaboration avec Jean-Sébastien Giet.

Pour certaines fonctionnelles $\Phi \ : \ \mathcal{C}([0,\infty[\ ; \ \mathbb{R}^d) \to \mathbb{R}$, l'espérance de la variable aléatoire
$\Phi\left(X(s)_{s\geq 0}\right)$, où $\left(X(s)\right)_{s\geq 0}$ est la solution de l'E.D.S. (6.1), représente une quantité intervenant dans de nombreux problèmes d'E.D.P. et de mathématiques financières.

$$X(t) = X(0) + \int_0^t \sigma(X(s))\ dB(s) + \int_0^t b(X(s))\ ds \qquad (t \geq 0). \qquad (6.1)$$

La section suivante rappelle différentes équations dont les solutions peuvent s'exprimer comme l'espérance de fonctionnelles de trajectoires stochastiques, ainsi que certaines grandeurs liées à la finance.

5.1 Quelques exemples de fonctionnelles intéressantes

5.1.1 L'équation de la chaleur

L'exemple le plus simple est sans doute celui d'un mouvement brownien $(B(s)_{s\geq 0})$ issu de zéro.
La solution de l'équation de la chaleur,

$$\left\{ \begin{array}{rcll} \dfrac{\partial u}{\partial t}(t,x) & = & \frac{1}{2}\Delta u(t,x) & (t > 0\ ,\ x \in \mathbb{R}^d) \\ u(0,x) & = & f(x) & (x \in \mathbb{R}^d), \end{array} \right. \qquad (5.1)$$

où f est une fonction continue, s'exprime sous la forme d'une espérance de ce mouvement brownien :

$$u(t,x) = \mathbb{E}\left\{ f(x + B(t)) \right\} \qquad (t \geq 0\ ,\ x \in \mathbb{R}^d). \qquad (5.2)$$

En considérant la diffusion $(X(t))_{t\geq 0}$ à la place du mouvement brownien, on obtient la solution de l'équation de la chaleur généralisée, au sens où le Laplacien Δ est remplacé par l'opérateur de différentiation $\mathcal{A}$ associé à l'E.D.S. (6.1) :

$$\mathcal{A}g(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2 g}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} b_i(x)\frac{\partial g}{\partial x_i}(x) \qquad (g \in C^2) \quad (x \in \mathbb{R}^d), \quad (5.3)$$

où

$$a_{i,j}(x) = \sum_{k=1}^{l} \left(\sigma_{i,k}\sigma_{j,k}\right)(x) \qquad (x \in \mathbb{R}^d). \quad (5.4)$$

Si on note $(X^x(t))_{t\geq 0}$ la solution de l'E.D.S. (6.1) pour laquelle $X(0) = x$, la solution de l'équation,

$$\left\{ \begin{array}{rcll} \dfrac{\partial u}{\partial t}(t,x) & = & \mathcal{A}u(t,x) & (t > 0 \ , \ x \in \mathbb{R}^d) \\ u(0,x) & = & f(x) & (x \in \mathbb{R}^d), \end{array} \right. \quad (5.5)$$

où f est une fonction continue, s'écrit d'une manière analogue à la solution (5.2) de l'équation (5.1) :

$$u(t,x) = \mathbb{E}\left\{f(X^x(t))\right\} \qquad (t \geq 0 \ , \ x \in \mathbb{R}). \quad (5.6)$$

Il est également possible d'interpréter le membre de droite de (5.6) en terme de prix d'options financières européennes en particularisant la fonction f : $f(y) = (K - y)_+$.

Tous ces exemples présentent le même type de fonctionnelles qui ne dépendent que de la seule valeur à un instant t de la trajectoire. Ces fonctionnelles élémentaires ne sont toutefois pas les seules à fournir des solutions d'E.D.P. ou des grandeurs économiques.

5.1.2 Le problème de Cauchy avec potentiel et Lagrangien

Deux exemples, où la totalité de la trajectoire du processus stochastique intervient, peuvent être considérés en parallèle : ils utilisent le même type de fonctionnelles. Il s'agit des solutions du problème de Cauchy présentant un potentiel non nul et du prix des options asiatiques.

Le problème de Cauchy avec potentiel r et Lagrangien g, est une extension de l'équation de la chaleur (5.5) :

$$\left\{ \begin{array}{rcll} \dfrac{\partial u}{\partial t}(t,x) & = & \mathcal{A}u(t,x) - r(x)u(t,x) + g(x) & (t > 0 \ , \ x \in \mathbb{R}^d) \\ u(0,x) & = & u_0(x) & (x \in \mathbb{R}^d), \end{array} \right. \quad (5.7)$$

où les fonctions r et g sont continues ; r est de plus supposée minorée.
La formule de Feynman-Kac [38] permet d'exprimer la solution du problème de Cauchy sous la forme suivante :

$$\begin{aligned} u(t,x) \;=\; & \mathbb{E}\left\{ u_0(X^x(t)) \exp\left(- \int_0^t r(X^x(s))\, ds \right) \right. \\ & \left. + \int_0^t g(X^x(\theta)) \exp\left(- \int_0^\theta r(X^x(s))\, ds \right)\, d\theta \right\} \end{aligned} \qquad (t \geq 0\ ,\ x \in \mathbb{R}^d), \tag{5.8}$$

Quant aux prix des options asiatiques, l'allure de leur expression ressemble à celle des solutions ci-dessus. Plus précisément, soit $(r(t)_{0 \leq t \leq T})$ un processus stochastique représentant le taux d'intérêt. On suppose, pour ne pas avoir à changer de probabilité, qu'il compense exactement le taux de croissance de l'ensemble des actions. On considère une option ("contingent claim") de valeur terminale de paiement u_T et de taux de paiement $(g(t)_{0 \leq t \leq T})$; il est alors démontré [38] que le prix initial qu'il faut investir dans un portefeuille d'actions pour couvrir le paiement de l'option est donné par :

$$\mathbb{E}\left\{ \exp\left(- \int_0^T r(u)\, du \right)\, u_T + \int_0^T \exp\left(- \int_0^s r(u)\, du \right)\, g(s)\, ds \right\}. \tag{5.9}$$

5.1.3 Le problème de Dirichlet

D'autres types de fonctionnelles apparaissent à la fois dans l'analyse des E.D.P. et dans les mathématiques financières.
Si D est un domaine régulier de l'espace d'état d'un processus X^x issu de x, le temps de sortie τ_D^x de l'ensemble D est une information contenue dans l'ensemble de la trajectoire du processus : $\tau_D^x = \inf\ \{s > 0\ ;\ X_s^x \notin D\}$.
La solution du problème de Dirichlet (5.10) :

$$\left\{ \begin{array}{rcll} \mathcal{A}u & = & 0 & (x \in D) \\ u(x) & = & f(x) & (x \in \partial D), \end{array} \right. \tag{5.10}$$

où f est une fonction continue sur ∂D, s'exprime sous la forme suivante :

$$u(x) = \mathbb{E}\left\{ f(X^x_{\tau_D^x}) \right\}. \tag{5.11}$$

Contenant lui aussi la v.a. τ_D^x du temps de sortie du domaine D pour la diffusion issue de x, le prix d'une option barrière admet comme expression :

$$\mathbb{E}\left\{ \mathbb{1}_{T < \tau_D^x} f(X_T^x) \right\}. \tag{5.12}$$

Ces différents exemples illustrent l'intérêt qu'il peut y avoir à estimer, à défaut de calculer, des espérances de fonctionnelles de la trajectoire de diffusions puisque ces quantités représentent des solutions d'E.D.P. ou des prix de couverture d'options financières.

5.2 Calcul de l'espérance des fonctionnelles

5.2.1 Principe

Le point de vue adopté ici ne consiste pas à résoudre numériquement les éventuelles E.D.P. dont ces espérances sont solutions. D'ailleurs, l'existence et l'unicité des solutions des E.D.P. décrites ci-dessus ne sont assurées que si la fonction f est suffisamment régulière. L'optique est d'approcher ces quantités par des méthodes probabilistes, moins sensibles à la régularité des fonctions mises en jeu.

La méthode de Monte-Carlo constitue un moyen classique pour calculer des espérances de v.a., et il est naturel d'y avoir recours pour estimer ces espérances de fonctionnelles de la trajectoire de processus. Toutefois, cette approche présente un problème : il est en effet impossible de simuler des trajectoires du processus initial $(X(s)_{s\geq 0})$. On est alors amené à remplacer ce dernier par un processus proche de X et que l'on peut simuler. Une méthode est privilégiée par sa simplicité : le schéma d'Euler.

L'objectif est donc ici d'apprécier les vitesses de convergence de quantités calculables, en faisant intervenir le schéma d'Euler et des approximations de fonctionnelles, vers les espérances contenant le processus initial de diffusion.

5.2.2 Études réalisées dans la littérature

Plusieurs travaux ont déjà été conduits dans cette voie. En particulier, les études des cas où la fonctionnelle est réduite à la valeur en un seul point de la trajectoire ont permis d'obtenir des développements en δ de l'erreur commise en remplaçant le processus initial par celui obtenu grâce au schéma d'Euler de pas δ [61], [6]. Ce dernier article se distingue du premier par un affaiblissement des hypothèses portant sur la fonction f qui apparaît dans (5.6). En effet, aucune régularité n'est demandée à cette fonction si ce n'est la mesurabilité ; une hypothèse de bornitude peut être relaxée pour ne conserver qu'une croissance polynomiale en $-\infty$ et $+\infty$.
Plus récemment, des travaux portant sur l'erreur lors de l'évaluation du prix des options barrière par une méthode impliquant le schéma d'Euler conduisent également à des développements de l'erreur [Seumen, (1997)], [Gobet, (1998)].

5.2.3 Fonctionnelles étudiées

L'étude présentée ici s'articule autour de plusieurs sortes de fonctionnelles. Dans un premier temps, une étude de fonctionnelles faisant intervenir plusieurs points de la trajectoire est réalisée. Cette extension, sans applications directes, permet dans un second temps d'aboutir à des résultats concernant des fonctionnelles plus intéressantes, qui reprennent l'information de l'intégrale de la trajectoire de la diffusion. Ceci permet alors de connaître les comportements de l'erreur, lors du calcul du prix d'options tel que le laisse apparaître (5.9), ou lors du calcul de solutions de l'équation (5.7) qui s'expriment par (5.8).

5.2.4 Résultats

Plus précisément, on démontre (théorème 7.1.1) que pour un nombre fixé n de points de la trajectoire, l'écart existant entre les espérances $\mathbb{E}\left\{\Phi(X(t_1),\ldots,X(t_n))\right\}$ et $\mathbb{E}\left\{\Phi(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_n))\right\}$, où Φ désigne une fonction réelle définie sur $\left(\mathbb{R}^d\right)^n$ et $(t_i)_{(0\leq i\leq n)}$ une subdivision, se comporte en δ. Cette démonstration a déjà été présentée lorsque ces espérances se ramènent au membre de droite de (5.6), avec f très régulière [61] et avec f seulement supposée mesurable et bornée [6]. C'est cette dernière hypothèse qui est imposée dans le cadre de cette étude.

Ce premier résultat permet d'aboutir à des fonctionnelles ayant la forme $x(.) \longmapsto \left(\int_0^1 f(x(s))\ ds\right)^l$ et $x(.) \longmapsto \exp\left(\int_0^1 f(x(s))\ ds\right)$ au travers de l'étude des convergences des fonctionnelles du type précédent $x(.) \longmapsto \left(\frac{1}{n}\sum_{i=1}^n f\left(x(\frac{i}{n})\right)\right)^l$ et $x(.) \longmapsto \exp\left(\frac{1}{n}\sum_{i=1}^n f\left(x(\frac{i}{n})\right)\right)$.

Il est ensuite démontré dans les théorèmes 8.2.1, 8.4.1 et 8.8.1 que chaque erreur est majorée par une quantité comportant un terme en δ et un autre en $\frac{1}{n}$. Toutefois, si l'on sait majorer uniformément en n le coefficient devant δ, il n'a pas été possible de contrôler uniformément le reste $\varepsilon_{\delta,n}$. Par conséquent, la vitesse de convergence des quantités contenant l'approximation d'Euler du processus n'est pas connue.

Enfin, en imposant l'égalité $n = \frac{1}{\delta}$, on obtient une majoration de l'erreur, énoncée dans le théorème 9.1, concernant l'intégrale de la trajectoire du processus.

Théorème 5.2.1 *Soit f une fonction mesurable et bornée.*
Soit $X(.)$ la solution de (6.1) ; soit $\bar{X}^{\frac{1}{n}}(.)$ l'approximation de cette solution donnée par le schéma d'Euler ; $\bar{X}^{\frac{1}{n}}(.)$ est la solution de (6.2).
Alors il existe C tel que :

$$\left|\mathbb{E}\left\{\int_0^1 f(X(s))\ ds\right\} - \mathbb{E}\left\{\frac{1}{n}\sum_{i=1}^n f(\bar{X}^{\frac{1}{n}}(\frac{i}{n}))\right\}\right| \leq C\ \|f\|_\infty \frac{1}{n}. \tag{5.13}$$

Ce théorème fournit des majorations des vitesses de convergence pour les algorithmes permettant de calculer des prix d'options ou des solutions du problème de Cauchy.

5.2.5 Plan de l'étude

• Quelques notations et hypothèses intervenant dans l'ensemble de l'étude sont précisées dans la partie 6.

• À partir de là, le théorème 7.1.1, indiquant la vitesse de convergence pour des fonctionnelles faisant intervenir plusieurs points de la trajectoire, est énoncé dans la partie 7.1. Sa démonstration s'articule autour de trois parties : 7.2, 7.3 et 7.4.

• Avant l'étude des fonctionnelles de la trajectoire, qui sont cette fois-ci des fonctions de l'intégrale de la trajectoire, la partie 8.1 est consacrée au rappel d'un résultat concernant la dérivation des noyaux de transition pour des diffusions suivant les hypothèses décrites en 6.1. On y démontre la majoration de noyaux dérivés que l'on peut trouver dans les travaux de Kusuoka-Stroock [41] ou Sanchez-Calle [55].
Ensuite, un développement de l'erreur par rapport à δ est établi pour les moments d'ordre l de l'intégrale de la trajectoire (partie 8.2) ainsi que pour les moments exponentiels (partie 8.8). La démonstration de ce développement pour les moments (théorème 8.4.1) est réalisée dans le cas des ordres 1, 2, puis l dans les paragraphes 8.5, 8.6 et 8.7 respectivement. La partie 8.3 est réservée à la majoration de la vitesse de convergence des sommes de Riemann $\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left\{f\left(X(\frac{i}{n})\right)\right\}$ vers $\int_0^1 \mathbb{E}\left\{f(X(s))\right\}\, ds$.

• La partie 9 est réservée à l'étude de majorations uniformes en $\delta = \frac{1}{n}$.

6

Notations et hypothèses

6.1 Le processus de diffusion

Soit $\{B(t)\}_{t\geq 0}$ un mouvement brownien de dimension 1, issu de zéro et défini sur un espace de probabilité $(\Omega,\mathcal{F},\mathbb{P})$.

Définition :
On désigne par $X(.)$ la solution de l'E.D.S. :

$$X(t) = X(0) + \int_0^t \sigma(X(s))\, dB(s) + \int_0^t b(X(s))\, ds \qquad (t \geq 0), \qquad (6.1)$$

où $X(0)$ est une v.a. possédant des moments de tout ordre, et où σ, b et $a = \sigma\sigma^t$ (5.3) sont des fonctions vérifiant les hypothèses suivantes :

(H 1)
$b\ :\ \mathbb{R}^d \to \mathbb{R}^d$ et $\sigma\ :\ \mathbb{R}^d \to \mathbb{R}^d$ sont de classe C^∞
les dérivées b', b'', σ', σ'' sont bornées sur $\mathbb{R}^d$
il existe η tel que $\displaystyle\sum_{i,j} \xi_i a_{i,j}(x)\xi_j \geq \eta > 0 \quad (x \in \mathbb{R}^d) \quad (\xi \in \mathbb{R}^d\ ;\ \|\xi\| = 1)$.

Notation :
On note, pour $r \in [0,1]$ et $x \in \mathbb{R}^d$, $X^{r,x}(.)$ la diffusion correspondant à $X(.)$ partant à l'instant r du point x ; $X^{r,x}(.)$ est la solution de l'E.D.S. :

$$X^{r,x}(t) = x + \int_r^t \sigma(X^{r,x}(s))\, dB(s) + \int_r^t b(X^{r,x}(s))\, ds \qquad (t \geq r) \quad (x \in \mathbb{R}^d). \quad (6.2)$$

Définitions :
i) $\{\varphi_\delta\}_{\delta>0}$ désigne une famille de fonctions réelles, définies sur [0,1] vérifiant les hypothèses :

(H 2) Pour tout $\delta > 0$, φ_δ est une fonction en escalier croissante telle que pour tout $t \in [0,1]$, $\varphi_\delta(t) \leq t$ et $\frac{t-\varphi_\delta(t)}{\delta} \leq 1$, $\lim\limits_{\delta \to 0} \varphi_\delta(t) = t$

ii) On définit, pour toute fonction φ_δ, le processus $\bar{X}^\delta(.)$ comme la solution de l'E.D.S. :

$$\bar{X}^\delta(t) = X(0) + \int_0^t \sigma(\bar{X}^\delta \circ \varphi_\delta(s))\ dB(s) + \int_0^t b(\bar{X}^\delta \circ \varphi_\delta(s))\ ds \qquad (t \geq 0). \quad (6.3)$$

$\bar{X}^\delta(.)$ correspond à la solution approchée par le schéma d'Euler de l'équation (6.1) si $\varphi_\delta(t) = \left\lfloor \frac{t}{\delta} \right\rfloor \delta$, où $\lfloor v \rfloor$ représente la partie entière de $v \in [0, +\infty[$.

Notation :

On note $t^{(\delta)}$ la quantité $\left\lfloor \frac{t}{\delta} \right\rfloor \delta$.

Remarque :

On ne s'interdit pas, sauf mention contraire, de choisir des familles de fonctions vérifiant seulement les hypothèses (H 2).

Un résultat classique (voir par exemple [27] ou [38]) consiste en une majoration des moments de $Y(t)$ où (Y,W) est une solution faible de l'E.D.S. générale :

$$Y(t) = Y(0) + \int_0^t f(s,Y)\ dW(s) + \int_0^t g(s,Y)\ ds, \qquad (6.4)$$

les fonctionnelles f et g étant définies sur $[0, +\infty[\times \mathcal{C}([0, +\infty[, \mathbb{R}^d)$.

Proposition 6.1.1 *On suppose que les fonctionnelles f et g vérifient l'hypothèse de majoration suivante :*
il existe $K \in \mathbb{R}$ tel que :

$$\|f(t,y)\|^2 + \|g(t,y)\|^2 \leq K \left(1 + \sup_{0<s<t} \|y(s)\|^2\right). \qquad (6.5)$$

Alors pour toute solution faible (Y,W) de (6.4) :

$$\mathbb{E}\left\{\sup_{0 \leq s \leq t} \|Y(s)\|^{2m}\right\} \leq C(1 + \mathbb{E}\left\{\|Y(0)\|^{2m}\right\}) \exp(Ct) \qquad (0 \leq t \leq T), \qquad (6.6)$$

le nombre C dépendant uniquement de m, T et K.

Les E.D.S. (6.3) entrent dans le cadre des E.D.S. (6.4), puisqu'il suffit de choisir $f(s,y) = \sigma(y(\varphi_\delta(s)))$ et $g(s,y) = b(y(\varphi_\delta(s)))$ pour obtenir l'équation suivie par $\bar{X}^\delta$.

Les conclusions de la proposition 6.5 permettent d'établir un résultat concernant la famille de processus $\left(\bar{X}^\delta(.)\right)_{\delta>0}$:

Corollaire 6.1.2 *Sous les hypothèses (H 1) vérifiées par b et σ, il existe un nombre C tel que pour tout $\delta > 0$:*

$$\mathbb{E}\left\{\sup_{0\le s\le t}\|\bar{X}^\delta(s)\|^{2m}\right\} \le C(1+\mathbb{E}\left\{\|X(0)\|^{2m}\right\})\exp(Ct) \qquad (0\le t\le T). \tag{6.7}$$

Démonstration du corollaire 6.1.2.
L'hypothèse (6.5) de la proposition 6.1.1 est réalisée pour la famille $\left(\bar{X}^\delta(.)\right)_{\delta>0}$:

$$\begin{array}{l}\|\sigma(y(\varphi_\delta(t)))\|^2+\|b(y(\varphi_\delta(t)))\|^2\\ \le K\left(1+\sup\limits_{0<s<t}\|y(s)\|^2)\right)\end{array} \qquad \left(y\in\mathcal{C}([0,+\infty[,\mathbb{R}^d)\right) \quad (t\ge 0), \tag{6.8}$$

où K est un réel indépendant de δ.
Donc la majoration (6.8) est uniforme en $\delta > 0$, et le nombre C de la majoration (6.7) ne dépend pas de δ. □

Définition :
Pour toute famille $\{\varphi_\delta\}_{\delta>0}$ et tout $r \in\ ]0,1[$, on définit le processus $\bar{X}^\delta_r(.)$ comme étant la solution $\bar{X}^\delta(.)$ de l'équation (6.3) jusqu'au temps r, puis la solution $X^{r,\bar{X}^\delta(r)}$ de l'équation (6.2) au-delà de l'instant r.
$\bar{X}^\delta_r(.)$ est ainsi la solution de l'E.D.S. suivante :

$$\begin{array}{rcl}\bar{X}^\delta_r(t) &=& X(0)+\displaystyle\int_0^t\left(\sigma(\bar{X}^\delta_r\circ\varphi_\delta(s))\mathbb{1}_{]0,r[}(s)+\sigma(\bar{X}^\delta_r(s))\mathbb{1}_{]r,1[}(s)\right)\,dB(s)\\ && +\displaystyle\int_0^t\left(b(\bar{X}^\delta_r\circ\varphi_\delta(s))\mathbb{1}_{]0,r[}(s)+b(\bar{X}^\delta_r(s))\mathbb{1}_{]r,1[}(s)\right)\,ds\end{array} \qquad (t\ge 0). \tag{6.9}$$

Cette famille de processus a été introduite par Kurtz et Protter. Elle est au cœur du principe de la démonstration du théorème 7.1.1.

6.2 La fonctionnelle de la trajectoire

Les résultats des parties 7, 8 et 9 correspondent à un horizon de temps fini. Pour simplifier les notations, ils sont énoncés et démontrés sur l'intervalle [0,1].

Nous nous intéressons, dans un premier temps (partie 7), à des fonctionnelles qui ne font intervenir qu'un nombre fini de points de la trajectoire.

Définition :

Une fonctionnelle $\tilde{\Phi} \ : \ \mathcal{C}([0,1],\mathbb{R}^d) \to \mathbb{R}$, ne dépendant que d'un nombre fini n de points de la trajectoire, est entièrement déterminée par la donnée :
- d'une subdivision $\{t_i \ ; \ 1 \leq i \leq n\}$ de l'intervalle $[0,1]$ avec $0 \leq t_1 < \ldots < t_n \leq 1$,
- d'une fonction $\Phi \ : \ \left(\mathbb{R}^d\right)^n \to \mathbb{R}$.

Ainsi,

$$\tilde{\Phi}(x(.)) = \Phi(x(t_1), \ldots, x(t_n)) \qquad \left(x(.) \in \mathcal{C}([0,1],\mathbb{R}^d)\right). \tag{6.10}$$

La fonction Φ vérifie l'hypothèse (H 3) :

(H 3) Φ est une fonction mesurable,
il existe $C \in \mathbb{R}$ et $p > 0$ tels que :

$$|\Phi(x_1, \ldots, x_n)| \leq C \left(1 + \sup_{1 \leq j \leq n} \|x_j\|^p\right). \tag{M_p}$$

7

Développement en δ de l'erreur pour des fonctionnelles à n points

7.1 Résultat principal

Le théorème 7.1.1 propose un développement de l'erreur induite par le remplacement de X par $\bar{X}^\delta$ dans l'expression $\mathbb{E}\left\{\Phi(X(t_1),\ldots,X(t_n))\right\}$. La dimension de la diffusion est restreinte à $d = 1$. Cependant, lors de la démonstration du théorème 7.1.1, le théorème 7.2.1 dont le résultat sera également utilisé dans la partie 9 est énoncé dans le cas général $d \in \mathbb{N}^*$.

Théorème 7.1.1 *On considère une fonctionnelle $\tilde{\Phi}$ définie à partir d'une fonction*
$\Phi \; : \; \mathbb{R}^n \longrightarrow \mathbb{R}$ *et d'une subdivision* $\{t_i \; ; \; 1 \leq i \leq n\}$ *de* $[0,1]$.
On suppose que Φ vérifie l'hypothèse (H 3).
Soit $X(.)$ la solution de (6.1) *avec $d = 1$; soit $\bar{X}^\delta(.)$ l'approximation de cette solution donnée par le schéma d'Euler.*
Alors,

$$\mathbb{E}\left\{\Phi(X(t_1),\ldots,X(t_n))\right\} - \mathbb{E}\left\{\Phi(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_n))\right\} = R(\tilde{\Phi})\delta + \varepsilon_{\delta,n}, \tag{7.1}$$

la quantité $\varepsilon_{\delta,n}$ vérifiant pour tout entier n : $\lim_{\delta\to 0} \dfrac{\varepsilon_{\delta,n}}{\delta} = 0$, et la quantité $R(\tilde{\Phi})$ pouvant s'exprimer comme la somme de trois intégrales faisant intervenir la fonction $H_{r,n}$ définie plus avant par (7.3) :

$$\begin{aligned} R(\tilde{\Phi}) \;=\; & \int_0^1 \mathbb{E}\left\{\frac{\partial H_{r,n}}{\partial z}(X(t_1),\ldots,X(t_{n_r}),X(r)) \left(\frac{ab''}{2} + bb'\right)(X(r))\right\} dr \\ & + \int_0^1 \mathbb{E}\Bigg\{\frac{\partial^2 H_{r,n}}{\partial z^2}(X(t_1),\ldots,X(t_{n_r}),X(r)) \\ & \left(ab' + \frac{aa''}{4} + \frac{ba'}{2}\right)(X(r))\Bigg\} dr \\ & + \int_0^1 \mathbb{E}\left\{\frac{\partial^3 H_{r,n}}{\partial z^3}(X(t_1),\ldots,X(t_{n_r}),X(r)) \; \frac{aa'}{2}(X(r))\right\} dr. \end{aligned} \tag{7.2}$$

Lorsque k est égal à 1, ce théorème est une version assez proche du théorème 3.1 de Bally-Talay [6]. Il faut noter que nos hypothèses concernant les coefficients de la diffusion sont plus restrictives, et que celles portant sur la fonction f sont identiques ; l'hypothèse (H 3), dans le cas $k = 1$, correspond à la mesurabilité de f et à sa croissance polynomiale.

Par contre, la démonstration du théorème 7.1.1 ne reprend pas celle de Bally-Talay [6] mais s'inspire d'une idée de Kurtz et Protter. En particulier, l'introduction de la famille de processus paramétrés, qui sont les solutions de l'E.D.S. (6.9), leur revient. De plus, le théorème 7.2.1 et le lemme 7.3.2 ont été adaptés ici à des valeurs de n supérieures à 1.

Plan de la démonstration du théorème 7.1.1.

La démonstration qui suit se compose de trois parties.
Dans une première partie, l'écart entre $\mathbb{E}\left\{\Phi(X(t_1,\ldots,t_n))\right\}$ et $\mathbb{E}\left\{\Phi(\bar{X}^\delta(t_1,\ldots,t_n))\right\}$ est représenté comme l'intégrale d'une dérivée ; cette dérivée est calculée, dans le théorème 7.2.1, en fonction du seul processus $\bar{X}^\delta$.
Une seconde partie est consacrée au calcul des limites des quotients des termes obtenus au théorème 7.2.1 pour cette dérivée par δ, quand δ tend vers zéro.
Une dernière partie regroupe les résultats précédents pour établir (7.1) et l'expression (7.2) de $R(\tilde{\Phi})$.

7.2 Dérivation de l'espérance d'une fonction du processus $\bar{X}_r^\delta(.)$

Les processus $\bar{X}_r^\delta(.)$, lorsque r décrit [0,1], constituent une paramétrisation dans l'espace des solutions des équations (6.3) où φ_δ vérifie (H 2), allant de la solution $X(.)$ de (6.1) pour $r = 0$ à la solution $\bar{X}^\delta(.)$ de (6.3) pour $r = 1$. La différence $\mathbb{E}\left\{\Phi(X(t_1),\ldots,X(t_n))\right\} - \mathbb{E}\left\{\Phi(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_n))\right\}$ peut ainsi s'écrire comme l'intégrale entre $r = 0$ et $r = 1$ de la dérivée $\dfrac{\partial}{\partial r}\mathbb{E}\left\{\Phi(\bar{X}_r^\delta(t_1),\ldots,\bar{X}_r^\delta(t_n))\right\}$.
Le théorème suivant exprime cette dérivée en fonction des coefficients σ et b.

Théorème 7.2.1 *On considère une fonctionnelle $\tilde{\Phi}$ définie à partir d'une fonction $\Phi : \left(\mathbb{R}^d\right)^n \longrightarrow \mathbb{R}$ et d'une subdivision $\{t_i\ ;\ 1 \le i \le n\}$ de* [0,1].
On suppose que Φ vérifie l'hypothèse (H 3).
Soit $X(.)$ la solution de (6.1) *avec $d \in \mathbb{N}^*$; soit $\bar{X}^\delta(.)$ l'approximation de cette solution donnée par le schéma d'Euler. On définit la famille de fonctions à valeurs réelles indexées par r et n :*

$$\begin{aligned} H_{r,n} \quad : \quad & \left(\mathbb{R}^d\right)^{n_r} \times \mathbb{R}^d \longrightarrow \mathbb{R} \\ & (x_1,\ldots,x_{n_r},z) \longmapsto \mathbb{E}\left\{\Phi(x_1,\ldots,x_{n_r},X^{r,z}(t_{n_r+1}),\ldots,X^{r,z}(t_n))\right\}, \end{aligned} \tag{7.3}$$

où $n_r = \sup\{i\ ;\ t_i \leq r\}$.
Alors pour tout $r \in]0,1[\setminus \{t_i\ ;\ 1 \leq i \leq n\}$,

$$\begin{aligned}
&\frac{\partial}{\partial r}\Big(\mathbb{E}\left\{\Phi(\bar{X}^{\delta}_r(t_1),\ldots,\bar{X}^{\delta}_r(t_n))\right\}\Big) \\
&= \mathbb{E}\left\{\sum_{i=1}^{d}\left[b_i(\bar{X}^{\delta}\circ\varphi_\delta(r)) - b_i(\bar{X}^{\delta}(r))\right]\frac{\partial H_{r,n}}{\partial z_i}(\bar{X}^{\delta}(t_1),\ldots,\bar{X}^{\delta}(t_{n_r}),\bar{X}^{\delta}(r))\right\} \\
&\quad + \frac{1}{2}\ \mathbb{E}\left\{\sum_{i,j=1}^{d}\left[a_{ij}(\bar{X}^{\delta}\circ\varphi_\delta(r)) - a_{ij}(\bar{X}^{\delta}(r))\right]\frac{\partial^2 H_{r,n}}{\partial z_i \partial z_j}(\bar{X}^{\delta}(t_1),\ldots,\bar{X}^{\delta}(t_{n_r}),\bar{X}^{\delta}(r))\right\}.
\end{aligned} \tag{7.4}$$

Notations :

Si g désigne une fonction réelle C^∞ définie pour $x = (y,z) \in \left(\mathbb{R}^d\right)^i \times \mathbb{R}^d$ dans $\mathbb{R}$, on écrira dans la suite :

$b(x)\dfrac{\partial g}{\partial z}(x)$ pour $\displaystyle\sum_{i=1}^{d} b_i(x)\frac{\partial g}{\partial z_i}(x)$,

$a(x)\dfrac{\partial^2 g}{\partial z^2}(x)$ pour $\displaystyle\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2 g}{\partial z_i \partial z_j}(x)$.

La conclusion du théorème 7.2.1 s'écrit alors :

$$\begin{aligned}
&\frac{\partial}{\partial r}\Big(\mathbb{E}\left\{\Phi(\bar{X}^{\delta}_r(t_1),\ldots,\bar{X}^{\delta}_r(t_n))\right\}\Big) \\
&= \mathbb{E}\left\{\left[b(\bar{X}^{\delta}\circ\varphi_\delta(r)) - b(\bar{X}^{\delta}(r))\right]\frac{\partial H_{r,n}}{\partial z}(\bar{X}^{\delta}(t_1),\ldots,\bar{X}^{\delta}(t_{n_r}),\bar{X}^{\delta}(r))\right\} \\
&\quad + \frac{1}{2}\ \mathbb{E}\left\{\left[a(\bar{X}^{\delta}\circ\varphi_\delta(r)) - a(\bar{X}^{\delta}(r))\right]\frac{\partial^2 H_{r,n}}{\partial z^2}(\bar{X}^{\delta}(t_1),\ldots,\bar{X}^{\delta}(t_{n_r}),\bar{X}^{\delta}(r))\right\}.
\end{aligned} \tag{7.5}$$

L'énoncé du théorème appelle quelques commentaires avant d'aborder sa démonstration.

• Premièrement, la fonction $H_{r,n}$ est correctement définie en tant qu'espérance d'une v.a. intégrable. En effet, en appliquant successivement (M_p) et (6.6), on obtient :

$$\begin{aligned}
&\mathbb{E}\left\{|\Phi(x_1,\ldots,x_{n_r},X^{r,z}(t_{n_r+1}),\ldots,X^{r,z}(t_n))|\right\} \\
&\leq C\left(1 + \sup_{1\leq j\leq n_r}\|x_j\|^p + \mathbb{E}\left\{\sup_{n_r+1\leq j\leq n}\|X^{r,z}(t_j)\|^p\right\}\right) \\
&\leq \tilde{C}\left(1 + \sup_{1\leq j\leq n_r}\|x_j\|^p + \|z\|^p\right),
\end{aligned}$$

où C et $\tilde{C}$ sont des constantes.

• Deuxièmement, la fonction $H_{r,n}$ est dérivable par rapport à la dernière variable. Il

suffit d'utiliser le théorème de dérivation sous le signe d'intégration pour le démontrer.
Plus précisément, soit $a > 0$, les fonctions $\Phi(x_1,\ldots,x_{n_r},X^{r,z}(t_{n_r+1}),\ldots,X^{r,z}(t_n))\}$ sont $\mathbb{P}$-intégrables pour tout $z \in]-a,a[$.
En introduisant le noyau de transition du processus $X^{r,z}(.)$ entre les temps r et t_{n_r+1}, on isole le paramètre z par rapport auquel on veut dériver.

Notation :
La densité de la probabilité de transition du processus $X(.)$ entre les instants s_1 et s_2 est notée q_{s_1,s_2}, parfois $q_{s_2-s_1}$:

$$\mathbb{P}\left(\{X^{s_1,x}(s_2) \in dy\}\right) = q_{s_1,s_2}(x,y)\ dy. \tag{7.6}$$

La fonction $H_{r,n}$ s'écrit alors :

$$\begin{aligned} H_{r,n}(x_1,\ldots,x_{n_r},z) &= \mathbb{E}\Bigg\{ \int_{\mathbb{R}} \Phi(x_1,\ldots,x_{n_r+1},X^{t_{n_r+1},x_{n_r+1}}(t_{n_r+2}),\ldots,X^{t_{n_r+1},x_{n_r+1}}(t_n)) \\ &\qquad q_{r,t_{n_r+1}}(z,x_{n_r+1})\ dx_{n_r+1} \Bigg\}. \end{aligned}$$

Pour tout entier strictement positif γ, l'application de l'inégalité (8.2) permet de majorer la famille des dérivées

$$\left\{ \frac{\partial^\gamma}{\partial z^\gamma} \left(\Phi(x_1,\ldots,x_{n_r+1},X^{t_{n_r+1},x_{n_r+1}}(t_{n_r+2}),\ldots,X^{t_{n_r+1},x_{n_r+1}}(t_n)) q_{r,t_{n_r+1}}(z,x_{n_r+1})\right) \right\}_{z\in]-a,a[}$$

par :

$$\begin{aligned} &C\left|\Phi(x_1,\ldots,x_{n_r+1},X^{t_{n_r+1},x_{n_r+1}}(t_{n_r+2}),\ldots,X^{t_{n_r+1},x_{n_r+1}}(t_n))\right| \\ &\times \frac{1}{(t_{n_r+1}-r)^{\frac{\gamma+1}{2}}} \frac{\exp\left(-\dfrac{x_{n_r+1}^2 - 2|a|x_{n_r+1}}{2M(t_{n_r+1}-r)}\right)}{\sqrt{2\pi M}}\ ; \end{aligned}$$

cette fonction est intégrable par rapport à $\mathbb{P} \otimes dx_{n_r+1}$.
Par conséquent, pour $\gamma \in \mathbb{N}^*$,

$$\begin{aligned} \frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(x_1,\ldots,x_{n_r},z) &= \mathbb{E}\Bigg\{ \int_{\mathbb{R}} \Phi(x_1,\ldots,x_{n_r+1},X^{t_{n_r+1},x_{n_r+1}}(t_{n_r+2}), \\ &\qquad \ldots,X^{t_{n_r+1},x_{n_r+1}}(t_n)) \frac{\partial^l q_{r,t_{n_r+1}}}{\partial z^l}(z,x_{n_r+1})dx_{n_r+1} \Bigg\}. \end{aligned} \tag{7.7}$$

• Enfin, le lemme suivant, qui interviendra lors de la démonstration du lemme 7.3.2, démontre en outre que les espérances apparaissant dans le membre de droite de (7.4), impliquant les dérivées $\dfrac{\partial^\gamma H_{r,n}}{\partial z^\gamma}$, sont finies.

Lemme 7.2.1 *Les hypothèses du théorème 7.2.1 sont restreintes à $d = 1$; q désigne un entier naturel non nul.*
Alors, pour $\gamma \in \mathbb{N}^*$, il existe un réel C et un entier N tels que :

$$\begin{aligned}&\mathbb{E}\left\{\sup_{0\leq\theta\leq 1}\left|\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),(1-\theta)\bar{X}^\delta\circ\varphi_\delta(r)+\theta\bar{X}^\delta(r))\right|^q\right\}\\ &\leq C\ \mathbb{E}\left\{1+\sup_{0\leq s\leq r}\left|\bar{X}^\delta(s)\right|^N\right\}.\end{aligned}\tag{7.8}$$

Démonstration du lemme 7.2.1.
On note :

$$I = \left|\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(x_1,\ldots,x_{n_r},z)\right|.$$

Par (7.7), I s'exprime sous la forme suivante :

$$\begin{aligned}I \ = \ &\left|\mathbb{E}\left\{\int_{\mathbb{R}}\Phi(x_1,\ldots,x_{n_r+1},X^{t_{n_r+1},x_{n_r+1}}(t_{n_r+2}),\ldots,X^{t_{n_r+1},x_{n_r+1}}(t_n))\right.\right.\\ &\left.\left.\frac{\partial^\gamma q_{r,t_{n_r+1}}}{\partial z^\gamma}(z,x_{n_r+1})dx_{n_r+1}\right\}\right|.\end{aligned}$$

On majore la fonction Φ et la dérivée du noyau $q_{r,t_{n_r+1}}$ en utilisant respectivement (M_p) et (8.2) :

$$\begin{aligned}I \ \leq \ & C_1\ \mathbb{E}\left\{\int_{\mathbb{R}}\left(1+\sup_{1\leq j\leq n_r}|x_j|^p+\sup_{n_r+2\leq j\leq n}\left|X^{t_{n_r+1},x_{n_r+1}}(t_j)\right|^p\right)\right.\\ &\left.\frac{\exp\left(-\dfrac{(x_{n_r+1}-z)^2}{2M(t_{n_r+1}-r)}\right)}{\sqrt{2\pi M}(t_{n_r+1}-r)^{\frac{\gamma+1}{2}}}\ dx_{n_r+1}\right\}.\end{aligned}$$

On échange espérance et intégrale, puis on utilise la majoration (6.6) avant qu'un changement de variable dans l'intégrale ne conduise à :

$$I \leq C_2\left(1+\sup_{1\leq j\leq n_r}|x_j|^p+\int_{\mathbb{R}}|x|^p\frac{\exp\left(-\dfrac{(x-z)^2}{2}\right)}{\sqrt{2\pi}}\ dx\right),$$

où le réel C_2 dépend de M et du temps $t_{n_r+1}-r$.
L'intégrale en x est clairement dominée par une fonction polynomiale en $|z|$ de degré p.
On obtient alors successivement :

$$I \leq C_3\left(1+\sup_{1\leq j\leq n_r}|x_j|^p+|z|^p\right)$$

et

$$\mathbb{E}\left\{\sup_{0\leq\theta\leq 1}\left|\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),(1-\theta)\bar{X}^\delta\circ\varphi_\delta(r)+\theta\bar{X}^\delta(r))\right|^q\right\}$$
$$\leq C\,\mathbb{E}\left\{1+\sup_{0\leq s\leq r}\left|\bar{X}^\delta(s)\right|^{qp}\right\}.$$

□

Démonstration du théorème 7.2.1.
On considère la différence suivante dans laquelle le réel strictement positif h est choisi pour que $r+h<t_{n_r+1}$, de sorte que $n_{r+h}=n_r$:

$$\mathbb{E}\left\{\Phi(\bar{X}^\delta_{r+h}(t_1),\ldots,\bar{X}^\delta_{r+h}(t_n))\right\}-\mathbb{E}\left\{\Phi(\bar{X}^\delta_r(t_1),\ldots,\bar{X}^\delta_r(t_n))\right\}. \tag{7.9}$$

Chacun des deux termes peut s'exprimer au moyen de la fonction $H_{r+h,n}$.
En effet, la variable aléatoire $\bar{X}^\delta_{r+h}(s)$ étant égale à $\bar{X}^\delta(s)$ ou à $X^{r+h,\bar{X}^\delta(r+h)}(s)$, suivant que le temps s est respectivement inférieur ou supérieur à $r+h$, le premier terme s'écrit :

$$\mathbb{E}\left\{\Phi(\bar{X}^\delta_{r+h}(t_1),\ldots,\bar{X}^\delta_{r+h}(t_n))\right\}$$
$$=\mathbb{E}\left\{\Phi(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),X^{r+h,\bar{X}^\delta_{r+h}(r+h)}(t_{n_r+1}),\ldots,X^{r+h,\bar{X}^\delta_{r+h}(r+h)}(t_n))\right\}.$$

Un conditionnement par rapport à la tribu $\mathcal{F}_{r+h}$ laisse apparaître la fonction $H_{r+h,n}$:

$$\mathbb{E}\left\{\Phi(\bar{X}^\delta_{r+h}(t_1),\ldots,\bar{X}^\delta_{r+h}(t_n))\right\}=\mathbb{E}\left\{H_{r+h,n}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta_{r+h}(r+h))\right\}. \tag{7.10}$$

En procédant à l'identique, on peut écrire le second terme de (7.9) comme suit :

$$\mathbb{E}\left\{\Phi(\bar{X}^\delta_r(t_1),\ldots,\bar{X}^\delta_r(t_n))\right\}=\mathbb{E}\left\{H_{r+h,n}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta_r(r+h))\right\}.$$

On applique alors à la fonction $H_{r+h,n}$ la formule d'Itô [38] avec les processus $\bar{X}^\delta_r(.)$ et $\bar{X}^\delta_{r+h}(.)$, entre les instants r et $r+h$:

$$\begin{aligned}
&H_{r+h,n}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta_{r+h}(r+h))=H_{r+h,n}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta_{r+h}(r))\\
&+\int_r^{r+h}\frac{\partial H_{r+h,n}}{\partial z}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta_{r+h}(s))\;\sigma(\bar{X}^\delta_{r+h}(s^{(\delta)}))\;dB(s)\\
&+\int_r^{r+h}\frac{\partial H_{r+h,n}}{\partial z}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta_{r+h}(s))\;b(\bar{X}^\delta_{r+h}(s^{(\delta)}))\;ds\\
&+\frac{1}{2}\int_r^{r+h}\frac{\partial^2 H_{r+h,n}}{\partial z^2}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta_{r+h}(s))\;a(\bar{X}^\delta_{r+h}(s^{(\delta)}))\;ds,
\end{aligned} \tag{7.11}$$

$$
\begin{aligned}
&H_{r+h,n}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}_r^\delta(r+h)) = H_{r+h,n}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(r))\\
&+\int_r^{r+h} \frac{\partial H_{r+h,n}}{\partial z}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}_r^\delta(s))\ \sigma(\bar{X}_r^\delta(s))\ dB(s)\\
&+\int_r^{r+h} \frac{\partial H_{r+h,n}}{\partial z}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}_r^\delta(s))\ b(\bar{X}_r^\delta(s))\ ds\\
&+\frac{1}{2}\int_r^{r+h} \frac{\partial^2 H_{r+h,n}}{\partial z^2}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}_r^\delta(s))\ a(\bar{X}_r^\delta(s))\ ds.
\end{aligned}
\tag{7.12}
$$

On choisit le réel positif h suffisamment petit pour que $s^{(\delta)} \leq r$ lorsque $r < s < r+h$; ainsi, on peut remplacer $\bar{X}_{r+h}^\delta(s^{(\delta)})$ par $\bar{X}^\delta(s^{(\delta)})$ dans les intégrales de (7.11).
Par ailleurs, comme $s < r + h$ dans les intégrales de (7.11), on peut remplacer $\bar{X}_{r+h}^\delta(s)$ par $\bar{X}^\delta(s)$.
Soustrayant (7.12) à (7.11), on obtient :

$$
\begin{aligned}
&H_{r+h,n}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}_{r+h}^\delta(r+h)) - H_{r+h,n}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}_r^\delta(r+h))\\
&=\int_r^{r+h} \frac{\partial H_{r+h,n}}{\partial z}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(s))\left[\sigma(\bar{X}^\delta(s^{(\delta)}))\ dB(s) + b(\bar{X}^\delta(s^{(\delta)}))\ ds\right]\\
&-\int_r^{r+h} \frac{\partial H_{r+h,n}}{\partial z}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}_r^\delta(s))\left[\sigma(\bar{X}_r^\delta(s))\ dB(s) + b(\bar{X}_r^\delta(s))\ ds\right]\\
&+\frac{1}{2}\int_r^{r+h} \frac{\partial^2 H_{r+h,n}}{\partial z^2}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(s))\ a(\bar{X}^\delta(s^{(\delta)}))\ ds\\
&-\frac{1}{2}\int_r^{r+h} \frac{\partial^2 H_{r+h,n}}{\partial z^2}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}_r^\delta(s))\ a(\bar{X}_r^\delta(s))\ ds.
\end{aligned}
\tag{7.13}
$$

Il resulte du lemme 7.2.1 et du corollaire 6.1.2 que les intégrales en $dB(s)$ sont des martingales de carré intégrable. Par conséquent, en prenant l'espérance dans l'égalité (7.13), on fait disparaître ces intégrales en $dB(s)$, et par (7.10) on obtient :

$$
\begin{aligned}
&\mathbb{E}\left\{\Phi(\bar{X}_{r+h}^\delta(t_1),\ldots,\bar{X}_{r+h}^\delta(t_n))\right\} - \mathbb{E}\left\{\Phi(\bar{X}_r^\delta(t_1),\ldots,\bar{X}_r^\delta(t_n))\right\}\\
&=\mathbb{E}\Bigg\{\int_r^{r+h}\Big(\frac{\partial H_{r+h,n}}{\partial z}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(s))b(\bar{X}^\delta(s^{(\delta)}))\\
&\quad-\frac{\partial H_{r+h,n}}{\partial z}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}_r^\delta(s))b(\bar{X}_r^\delta(s))\Big)\Bigg\}ds\\
&\quad+\frac{1}{2}\,\mathbb{E}\Bigg\{\int_r^{r+h}\Big(\frac{\partial^2 H_{r+h,n}}{\partial z^2}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(s))\ a(\bar{X}^\delta(s^{(\delta)}))\\
&\quad-\frac{\partial^2 H_{r+h,n}}{\partial z^2}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}_r^\delta(s))\ a(\bar{X}_r^\delta(s))\Big)\ ds\Bigg\}.
\end{aligned}
\tag{7.14}
$$

La conclusion du théorème s'obtient en divisant cette dernière égalité par h et en faisant tendre h vers zéro. □

Les résultats de cette première partie sont utilisés dans la dernière partie de la démonstration. Par ailleurs, l'égalité (7.4) oriente les calculs menés dans la deuxième partie.

7.3 Limite en δ

L'égalité (7.4) fournit une expression de la dérivée par rapport au paramètre r, de la fonctionnelle $\tilde{\Phi}$ appliquée au processus $\bar{X}^\delta_r$. L'écriture de cette dérivée ne fait apparaître que le seul processus $\bar{X}^\delta$.
Le but de cette section est de préciser les comportements des espérances du membre de droite de l'égalité (7.4), lorsque le paramètre δ converge vers zéro. Les résultats de cette section sont restreints au cas où la diffusion est de dimension un.

Avant d'étudier ces comportements, nous comparons les vitesses de convergence vers zéro, de la norme L^p de $\bar{X}^\delta(r) - \bar{X}^\delta \circ \varphi_\delta(r)$ et de $r - \varphi_\delta(r)$.

Lemme 7.3.1 *Soit un réel $p \geq 1$. Il existe un nombre C tel que pour tout $\delta > 0$:*

$$\mathbb{E}\left\{\left|\bar{X}^\delta(r) - \bar{X}^\delta \circ \varphi_\delta(r)\right|^{2p}\right\} \leq C\,(r - \varphi_\delta(r))^p\,. \tag{7.15}$$

Démonstration du lemme 7.3.1.

La fonction φ_δ reste constante entre $\varphi_\delta(r)$ et r ; par conséquent, l'équation (6.3) se simplifie :

$$\bar{X}^\delta(r) - \bar{X}^\delta \circ \varphi_\delta(r) = \sigma(\bar{X}^\delta \circ \varphi_\delta(r))(B(r) - B \circ \varphi_\delta(r)) + b(\bar{X}^\delta \circ \varphi_\delta(r))(r - \varphi_\delta(r)). \tag{7.16}$$

$$||\bar{X}^\delta(r) - \bar{X}^\delta \circ \varphi_\delta(r)||_p \leq ||\sigma(\bar{X}^\delta \circ \varphi_\delta(r))(B(r) - B \circ \varphi_\delta(r))||_p + ||b(\bar{X}^\delta \circ \varphi_\delta(r))(r - \varphi_\delta(r))||_p \tag{7.17}$$

Le corollaire 6.1.2 et le contrôle des moments d'ordre p d'une gaussienne nous permettent de conclure. □

Remarque :

L'application de l'inégalité de Cauchy-Schwarz et de la majoration (7.15) lorsque $p = 1$ implique l'existence d'un nombre C tel que pour tout $\delta > 0$:

$$\mathbb{E}\left\{\left|\bar{X}^\delta(r) - \bar{X}^\delta \circ \varphi_\delta(r)\right|\right\} \leq C\sqrt{r - \varphi_\delta(r)}. \tag{7.18}$$

Lemme 7.3.2 *Soit g une fonction définie de $\mathbb{R}$ dans $\mathbb{R}$, de classe C^2 telle qu'il existe une constante C :*

$$|g(x) - g(y)| \leq C|x - y|(1 + |x| + |y|) \qquad \left((x,y) \in \mathbb{R}^2\right). \tag{7.19}$$

Soient $n \in \mathbb{N}^*$, $\gamma \in \{1,2\}$ *et* $r \in]0,1[\setminus\{t_i\ ;\ 1 \le i \le n\}$.
Alors :

$$\lim_{\delta\to 0} \frac{1}{r-\varphi_\delta(r)}\mathbb{E}\left\{\left[g(\bar{X}_r^\delta(r))-g(\bar{X}_r^\delta\circ\varphi_\delta(r))\right]\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}_r^\delta(t_1),\ldots,\bar{X}_r^\delta(t_{n_r}),\bar{X}_r^\delta(r))\right\}$$
$$= \mathbb{E}\left\{\left(\frac{ag''}{2}+bg'\right)(X(r))\ \frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(X(t_1),\ldots,X(t_{n_r}),X(r))\right\}$$
$$+\ \mathbb{E}\left\{a(X(r))g'(X(r))\frac{\partial^{\gamma+1}H_{r,n}}{\partial z^{\gamma+1}}(X(t_1),\ldots,X(t_{n_r}),X(r))\right\}. \tag{7.20}$$

Démonstration du lemme 7.3.2.

On développe l'expression apparaissant dans la limite à calculer, en utilisant la dérivée par rapport à la dernière variable de la fonction $H_{r,n}$.

$$\begin{aligned}
&\left[g(\bar{X}^\delta(r)) - g(\bar{X}^\delta\circ\varphi_\delta(r))\right]\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(r))\\
&= \left[g(\bar{X}^\delta(r)) - g(\bar{X}^\delta\circ\varphi_\delta(r))\right]\left\{\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(r))\right.\\
&\quad -\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\\
&\quad \left.-\frac{\partial^{\gamma+1}H_{r,n}}{\partial z^{\gamma+1}}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))(\bar{X}^\delta(r)-\bar{X}^\delta\circ\varphi_\delta(r))\right\}\\
&\quad +\left[g(\bar{X}^\delta(r)) - g(\bar{X}^\delta\circ\varphi_\delta(r))\right]\left\{\frac{\partial^{\gamma+1}H_{r,n}}{\partial z^{\gamma+1}}(\bar{X}^\delta(t_1),\ldots,\right.\\
&\quad \left.\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))(\bar{X}^\delta(r)-\bar{X}^\delta\circ\varphi_\delta(r))\right\}\\
&\quad +\left[g(\bar{X}^\delta(r)) - g(\bar{X}^\delta\circ\varphi_\delta(r))\right]\left\{\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\right\}.
\end{aligned} \tag{7.21}$$

Chacun des termes du membre de droite de (7.21) va être étudié.

• On va démontrer que l'espérance du premier terme du membre de droite de (7.21) divisée par $r-\varphi_\delta(r)$ converge vers zéro lorsque δ tend vers zéro.
L'espérance de ce premier terme s'écrit :

$$\begin{aligned}
&\mathbb{E}\left\{(g(\bar{X}^\delta(r)) - g(\bar{X}^\delta\circ\varphi_\delta(r)))\right.\\
&\left[\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(r)) - \frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\right.\\
&\left.\left.-\left(\bar{X}^\delta(r)-\bar{X}^\delta\circ\varphi_\delta(r)\right)\frac{\partial^{\gamma+1}H_{r,n}}{\partial z^{\gamma+1}}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\right]\right\}.
\end{aligned} \tag{7.22}$$

En appliquant à plusieurs reprises l'inégalité de Cauchy-Schwarz et en utilisant l'hypothèse (7.19) suivie par g, on obtient une majoration de (7.22) par :

$$
\begin{aligned}
&C\left(\mathbb{E}\left\{(\bar{X}^\delta(r)-\bar{X}^\delta\circ\varphi_\delta(r))^4\right\}\right)^{\frac14}\left(\mathbb{E}\left\{\left(1+\left|\bar{X}^\delta(r)\right|+\left|\bar{X}^\delta\circ\varphi_\delta(r)\right|\right)^4\right\}\right)^{\frac14}\\
&\Bigg(\mathbb{E}\Big\{\Big(\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\dots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(r))\\
&-\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\dots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\\
&-\left(\bar{X}^\delta(r)-\bar{X}^\delta\circ\varphi_\delta(r)\right)\frac{\partial^{\gamma+1}H_{r,n}}{\partial z^{\gamma+1}}(\bar{X}^\delta(t_1),\dots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\Big)^2\Big\}\Bigg)^{\frac12}.
\end{aligned}
\tag{7.23}
$$

Le premier facteur de (7.23) se comporte comme $\sqrt{r-\varphi_\delta(r)}$ lorsque δ tend vers zéro. Le second facteur est borné en vertu de la proposition 6.1.1. Il reste à démontrer que le dernier facteur converge vers zéro avec δ.
Le développement de Taylor à l'ordre 2 permet d'écrire :

$$
\begin{aligned}
&\Big(\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\dots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(r))-\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\dots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\\
&-\left(\bar{X}^\delta(r)-\bar{X}^\delta\circ\varphi_\delta(r)\right)\frac{\partial^{\gamma+1}H_{r,n}}{\partial z^{\gamma+1}}(\bar{X}^\delta(t_1),\dots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\Big)^2\\
&=\Bigg(\left(\bar{X}^\delta(r)-\bar{X}^\delta\circ\varphi_\delta(r)\right)^2\int_0^1(1-\theta)\frac{\partial^{\gamma+2}H_{r,n}}{\partial z^{\gamma+2}}(\bar{X}^\delta(t_1),\dots,\bar{X}^\delta(t_{n_r}),\\
&\qquad(1-\theta)\bar{X}^\delta\circ\varphi_\delta(r)+\theta\bar{X}^\delta(r))\,d\theta\Bigg)^2\\
&\le\left(\bar{X}^\delta(r)-\bar{X}^\delta\circ\varphi_\delta(r)\right)^4\\
&\qquad\sup_{0\le\theta\le1}\left(\frac{\partial^{\gamma+2}H_{r,n}}{\partial z^{\gamma+2}}(\bar{X}^\delta(t_1),\dots,\bar{X}^\delta(t_{n_r}),(1-\theta)\bar{X}^\delta\circ\varphi_\delta(r)+\theta\bar{X}^\delta(r))\right)^2.
\end{aligned}
\tag{7.24}
$$

En prenant l'espérance et en appliquant l'inégalité de Cauchy-Schwarz, puis la majoration (7.8), on obtient :

$$
\begin{aligned}
&\mathbb{E}\Big\{\Big(\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\dots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(r))-\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\dots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\\
&-\left(\bar{X}^\delta(r)-\bar{X}^\delta\circ\varphi_\delta(r)\right)\frac{\partial^{\gamma+1}H_{r,n}}{\partial z^{\gamma+1}}(\bar{X}^\delta(t_1),\dots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\Big)^2\Big\}\\
&\le C\left(\mathbb{E}\left\{\left(\bar{X}^\delta(r)-\bar{X}^\delta\circ\varphi_\delta(r)\right)^8\right\}\right)^{\frac14}\left(1+\mathbb{E}\left\{\sup_{0\le s\le r}\left|\bar{X}^\delta(s)\right|^N\right\}\right).
\end{aligned}
$$

Le corollaire 6.1.2 et le lemme 7.15 impliquent l'existence d'une constante C permettant la majoration de l'espérance ci-dessus par $C\,\sqrt{r-\varphi_\delta(r)}$.

• L'espérance du second terme du membre de droite de (7.21) est développée grâce à la formule d'Itô.

$$\begin{aligned}
&\mathbb{E}\Bigg\{\frac{\partial^{\gamma+1}H_{r,n}}{\partial z^{\gamma+1}}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\\
&\big[g(\bar{X}^\delta(r)-g(\bar{X}^\delta\circ\varphi_\delta(r))\big]\,(\bar{X}^\delta(r)-\bar{X}^\delta\circ\varphi_\delta(r))\Bigg\}\\
&=\mathbb{E}\Bigg\{\frac{\partial^{\gamma+1}H_{r,n}}{\partial z^{\gamma+1}}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\int_{\varphi_\delta(r)}^r(\bar{X}^\delta(s)-\bar{X}^\delta\circ\varphi_\delta(r))\\
&\Big[g'(\bar{X}^\delta(s))b(\bar{X}^\delta\circ\varphi_\delta(s))+\frac{1}{2}g''(\bar{X}^\delta(s))a(\bar{X}^\delta\circ\varphi_\delta(s))\Big]ds\Bigg\}\\
&+\ \mathbb{E}\Bigg\{\frac{\partial^{\gamma+1}H_{r,n}}{\partial z^{\gamma+1}}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\\
&\int_{\varphi_\delta(r)}^r(g(\bar{X}^\delta(s))-g(\bar{X}^\delta\circ\varphi_\delta(r)))b(\bar{X}^\delta\circ\varphi_\delta(s))\ ds\Bigg\}\\
&+\ \mathbb{E}\Bigg\{\frac{\partial^{\gamma+1}H_{r,n}}{\partial z^{\gamma+1}}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\\
&\int_{\varphi_\delta(r)}^r g'(\bar{X}^\delta(s))a(\bar{X}^\delta\circ\varphi_\delta(s))\ ds\Bigg\}.
\end{aligned}\tag{7.26}$$

Dans le membre de droite de (7.26), l'application du lemme 7.2.1 et du corollaire 6.1.2 permet de vérifier les hypothèses des théorèmes de Fubini et de Lebesgue ; il est alors possible d'echanger les signes d'intégration et d'espérance, puis d'obtenir la limite lorsque δ converge vers zéro des espérances. Les deux premiers termes divisés par $r-\varphi_\delta(r)$ convergent vers zéro avec δ. Seul le troisième terme divisé par $r-\varphi_\delta(r)$ possède une limite non nulle :

$$\mathbb{E}\left\{a(X(r))g'(X(r))\frac{\partial^{\gamma+1}H_{r,n}}{\partial z^{\gamma+1}}(X(t_1),\ldots,X(t_{n_r}),X(r))\right\}.$$

• L'espérance du troisième terme du membre de droite de (7.21) s'écrit grâce à la formule d'Itô :

$$\begin{aligned}
&\mathbb{E}\left\{\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))(g(\bar{X}^\delta(r))-g(\bar{X}^\delta\circ\varphi_\delta(r)))\right\}\\
&=\mathbb{E}\Bigg\{\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta\circ\varphi_\delta(r))\\
&\int_{\varphi_\delta(r)}^r\Big[g'(\bar{X}^\delta(s))b(\bar{X}^\delta\circ\varphi_\delta(r))+\frac{1}{2}g''(\bar{X}^\delta(s))a(\bar{X}^\delta\circ\varphi_\delta(r))\Big]\ ds\Bigg\}.
\end{aligned}\tag{7.27}$$

L'égalité (7.27) et le troisième terme de (7.26) donnent les termes de (7.20) ; ce qui achève la démonstration du lemme 7.3.2 et clôt la deuxième partie de la démonstration du théorème 7.1.1. □

7.4 Expression du coefficient devant δ

Il reste à établir l'expression (7.2) de la quantité $R(\tilde{\Phi})$ pour achever la démonstration du théorème 7.1.1.
Le lemme 7.3.2 appliqué à $g = b$ puis $g = a$ et le théorème 7.2.1 permettent de majorer $R(\tilde{\Phi})$, pour tout $n \in \mathbb{N}^*$:

$$\begin{aligned}
\left|R(\tilde{\Phi})\right| &= \lim_{\delta\to 0}\frac{1}{\delta}\left|\mathbb{E}\left\{\Phi(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_n))\right\} - \mathbb{E}\left\{\Phi(X(t_1),\ldots,X(t_n))\right\}\right| \\
&= \lim_{\delta\to 0}\frac{1}{\delta}\left|\int_0^1 \frac{\partial}{\partial r}\mathbb{E}\left\{\Phi(\bar{X}_r^\delta(t_1),\ldots,\bar{X}_r^\delta(t_n))\right\}\,dr\right| \\
&= \lim_{\delta\to 0}\left|\int_0^1 \frac{r-\varphi_\delta(r)}{\delta}\frac{1}{r-\varphi_\delta(r)}\right. \\
&\quad \mathbb{E}\left\{(b(\bar{X}^\delta\circ\varphi_\delta(r)) - b(\bar{X}^\delta(r)))\frac{\partial H_{r,n}}{\partial z}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(r))\right. \\
&\quad \left.\left.+\frac{1}{2}(a(\bar{X}^\delta\circ\varphi_\delta(r)) - a(\bar{X}^\delta(r)))\frac{\partial^2 H_{r,n}}{\partial z^2}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(r))\right\}dr\right| \\
&\leq \left|\int_0^1 E\left\{\frac{\partial H_{r,n}}{\partial z}(X(t_1),\ldots,X(t_{n_r}),X(r))\left(\frac{ab''}{2}+bb'\right)(X(r))\right\}dr\right. \\
&\quad + \int_0^1 E\left\{\frac{\partial^2 H_{r,n}}{\partial z^2}(X(t_1),\ldots,X(t_{n_r}),X(r))\right. \\
&\quad \left.\left(ab' + \frac{aa''}{4} + \frac{ba'}{2}\right)(X(r))\right\}dr \\
&\quad \left.+ \int_0^1 E\left\{\frac{\partial^3 H_{r,n}}{\partial z^3}(X(t_1),\ldots,X(t_{n_r}),X(r))\left(\frac{aa'}{2}\right)(X(r))\right\}dr\right|.
\end{aligned} \tag{7.28}$$

Si l'on choisit la fonction φ_δ correspondant au schéma d'Euler, c'est-à-dire si $\varphi_\delta(r) = \left\lfloor\frac{r}{\delta}\right\rfloor\delta = r^{(\delta)}$, on peut établir un lemme qui permettra d'obtenir une égalité dans (7.28) :

Lemme 7.4.1 *Soit $\{g_\delta\ ;\ \delta > 0\}$ une famille uniformément bornée de fonctions réelles continues sur $[0,1]$ convergeant simplement vers une fonction continue g ; c'est-à-dire que pour tout $r \in [0,1]$, $\lim_{\delta\to 0} g_\delta(r) = g(r)$.*

Alors

$$\lim_{\delta\to 0}\int_0^1 \frac{r-r^{(\delta)}}{\delta}\; g_\delta(r)\; dr = \frac{1}{2}\int_0^1 g(r)\; dr. \tag{7.29}$$

Démonstration du lemme 7.4.1.

La famille de fonctions $\{g_\delta\ ;\ \delta > 0\}$ étant bornée, la convergence simple de g_δ vers g entraîne la convergence en norme L^1.

Ce qui, après que la quantité $\dfrac{r-r^{(\delta)}}{\delta}$ est majorée par 1, permet d'établir :

$$\lim_{\delta\to 0}\left[\int_0^1 \frac{r-r^{(\delta)}}{\delta}\; g_\delta(r)\; dr - \int_0^1 \frac{r-r^{(\delta)}}{\delta}\; g(r)\; dr\right] = 0.$$

On calcule maintenant la limite du second terme :

$$\begin{aligned}\lim_{\delta\to 0}\int_0^1 \frac{r-r^{(\delta)}}{\delta}\; g(r)\; dr &= \lim_{\substack{\delta\to 0\\ \{\frac{1}{\delta}\in\mathbb{N}\}}}\sum_{i=1}^{\frac{1}{\delta}}\int_{(i-1)\delta}^{i\delta}\Big(\frac{r}{\delta}-(i-1)\Big)\; g(r)\; dr\\ &= \lim_{N\to\infty}\sum_{i=1}^{N}\int_{\frac{i-1}{N}}^{\frac{i}{N}}(rN-(i-1))\, g(r)\, dr\end{aligned}$$

On peut remplacer dans cette expression $g(r)$ par $g\left(\frac{i-1}{N}\right)$. En effet, en majorant $rN-(i-1)$ par 1 et la différence impliquant la fonction g, on obtient :

$$\sum_{i=1}^{N}\int_{\frac{i-1}{N}}^{\frac{i}{N}}(rN-(i-1))\;\left|g(r)-g\left(\frac{i-1}{N}\right)\right|\; dr \le \sup_{|s-t|\le\frac{1}{N}}|g(s)-g(t)|\,,$$

et comme la fonction g est uniformément continue puisque continue sur un compact, le module de continuité converge vers zéro lorsque N tend vers l'infini.

Il reste à remplacer l'intégrale $\displaystyle\int_{\frac{i-1}{N}}^{\frac{i}{N}}(rN-(i-1))\; dr$ par sa valeur $\dfrac{1}{2N}$ pour obtenir :

$$\lim_{\delta\to 0}\int_0^1 \frac{r-r^{(\delta)}}{\delta}\; g(r)\; dr = \frac{1}{2}\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} g\left(\frac{i-1}{N}\right) = \frac{1}{2}\int_0^1 g(r)\; dr.$$

□

On peut reprendre le calcul de la limite (7.28) sans avoir recours à des majorations :

$$
\begin{aligned}
R(\tilde{\Phi}) &= \lim_{\delta\to 0} \frac{1}{\delta}\mathbb{E}\left\{\Phi(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_n))\right\} - \mathbb{E}\left\{\Phi(X(t_1),\ldots,X(t_n))\right\} \\
&= \lim_{\delta\to 0} \frac{1}{\delta}\int_0^1 \frac{\partial}{\partial r}\mathbb{E}\left\{\Phi(\bar{X}_r^\delta(t_1),\ldots,\bar{X}_r^\delta(t_n))\right\}\, dr \\
&= \lim_{\delta\to 0} \int_0^1 \frac{r-r^{(\delta)}}{\delta}\frac{1}{r-r^{(\delta)}}\mathbb{E}\Bigg\{(b(\bar{X}^\delta(r^{(\delta)}))-b(\bar{X}^\delta(r))) \\
&\quad \frac{\partial H_{r,n}}{\partial z}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(r)) \\
&\quad +\frac{1}{2}\,(a(\bar{X}^\delta(r^{(\delta)}))-a(\bar{X}^\delta(r)))\frac{\partial^2 H_{r,n}}{\partial z^2}(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_{n_r}),\bar{X}^\delta(r))\Bigg\}dr \\
&= \int_0^1 \mathbb{E}\Bigg\{\frac{\partial H_{r,n}}{\partial z}(X(t_1),\ldots,X(t_{n_r}),X(r))\left(\frac{ab''}{2}+bb'\right)(X(r))\Bigg\}\, dr \\
&\quad +\int_0^1 \mathbb{E}\Bigg\{\frac{\partial^2 H_{r,n}}{\partial z^2}(X(t_1),\ldots,X(t_{n_r}),X(r)) \\
&\quad \left(ab'+\frac{aa''}{4}+\frac{b(\sigma^2)'}{2}\right)(X(r))\Bigg\}\, dr \\
&\quad +\int_0^1 \mathbb{E}\Bigg\{\frac{\partial^3 H_{r,n}}{\partial z^3}(X(t_1),\ldots,X(t_{n_r}),X(r))\left(\frac{aa'}{2}\right)(X(r))\Bigg\}\, dr.
\end{aligned}
\tag{7.30}
$$

Pour tout entier naturel n fixé, les trois intégrales composant la quantité $R(\tilde{\Phi})$ et apparaissant dans (7.30) sont finies.
Le théorème 7.1.1 est démontré. □

Le paramètre n reste fixé dans toute la partie précédente. Notre objectif est d'obtenir des résultats de convergence pour des fonctionnelles qui dépendent de la trajectoire complète du processus stochastique et non plus d'un nombre fini de points de la trajectoire. Pour cela, on va étudier l'influence de n dans le coefficient $R(\tilde{\Phi})$.

8

Majoration en δ et n de l'erreur pour l'intégrale de la trajectoire

Ce qui suit est consacré à plusieurs exemples de fonctionnelles dépendant de la totalité de la trajectoire. Imaginons que les fonctionnelles $\tilde{\Phi}$ soient des approximations d'une fonctionnelle Ψ. Il est alors possible de faire tendre n vers l'infini dans le membre de gauche de (7.1). Le membre de droite de (7.1) possède donc une limite, mais on ignore s'il en est de même pour chacun de ses deux termes $R(\tilde{\Phi})\delta$ et $\varepsilon_{\delta,n}$.

La suite de l'étude se restreint à l'utilisation de l'approximation usuelle $\bar{X}^\delta$ donnée par le schéma d'Euler. C'est-à-dire que la famille de fonctions $(\varphi_\delta)_{\delta>0}$ est définie par :

$$\varphi_\delta(r) = \left\lfloor \frac{r}{\delta} \right\rfloor \delta \qquad (r \in [0,1])$$

Les deux sections 8.2 et 8.3 ont pour objet de démontrer que la quantité $R(\tilde{\Phi})$ reste bornée uniformément en n, pour plusieurs types de fonctionnelles spécifiques. En effet, toutes les v.a. $\tilde{\Phi}(X(.))$ étudiées ne dépendent que de l'intégrale de la trajectoire $s \longmapsto f(X(s))$.

Mais auparavant, on va énoncer le résultat fondamental permettant de contrôler le coefficient $R(\tilde{\Phi})$ par rapport au paramètre n.

Le développement (7.1) de l'erreur $\mathbb{E}\left\{\Phi(X(t_1),\ldots,X(t_n))\right\} - \mathbb{E}\left\{\Phi(\bar{X}^\delta(t_1),\ldots,\bar{X}^\delta(t_n))\right\}$ présente une expression du coefficient $R(\tilde{\Phi})$ devant le paramètre δ. Au vu de (7.2), le comportement de $R(\tilde{\Phi})$ lorsque n tend vers l'infini est lié à ceux des trois premières dérivées de la fonction $H_{r,n}$. La fonction $H_{r,n}$, définie par (7.3), est égale à l'espérance de la fonctionnelle $\tilde{\Phi}$ sur une trajectoire, aléatoire seulement au-delà du temps r. Ce morceau de trajectoire aléatoire est $(X^{r,z}(s))_{r\leq s\leq 1}$. Ainsi, les dérivées de la fonction $H_{r,n}$ mesure l'impact d'une variation du point de départ de la diffusion à l'instant r. Ces dérivations pourront être contrôlées au travers des majorations des dérivées du noyau de transition du processus X.

8.1 Majorations des dérivées du noyau de la diffusion

Une des conséquences de la définition (7.3) de $H_{r,n}$ est que l'expression (7.2) du coefficient $R(\tilde{\Phi})$ fait intervenir des dérivées de l'espérance de la trajectoire de la diffusion $X^{r,z}$ par rapport à son point de départ z. Et si l'on utilise le noyau de transition $q_t(x,y)$ de la diffusion $X(.)$ pour écrire la fonction $H_{r,n}$,

$$\begin{aligned} H_{r,n}(x_1,\ldots,x_{n_r},z) &= \int_{\mathbb{R}^{n-n_r}} \Phi(x_1,\ldots,x_n) q_{r,t_{n_r+1}}(z,x_{n_r+1})\ldots \\ &\quad q_{t_{n-1},t_n}(x_n - x_{n-1})\, dx_{n_r+1}\ldots dx_n, \end{aligned} \tag{8.1}$$

cette dérivation se concentre exclusivement sur un des noyaux de transition. C'est clairement le cas pour l'expression (8.31) correspondant à l'un des trois termes de $R(\tilde{\Phi})$.

Par conséquent, il est possible de contrôler $R(\tilde{\Phi})$ si l'on peut majorer les dérivées en espace du noyau de transition. Les hypothèses imposées à l'opérateur différentiel $\mathcal{A} = \frac{1}{2}a\frac{\partial^2}{\partial x^2} + b\frac{\partial}{\partial x}$ au travers des fonctions σ et b (H 1) permettent d'obtenir de telles majorations. L'opérateur $\mathcal{A}$ est en effet uniformément elliptique. Il s'ensuit que l'on peut utiliser des résultats classiques sur les noyaux [41] ou [55]. Le théorème II de ce dernier article démontre la proposition suivante dans un cadre plus général.

Proposition 8.1.1 *Sous les hypothèses (H 1) de régularité des coefficients de l'opérateur associé à l'E.D.S. (6.1) et de son uniforme ellipticité, pour tout entier naturel l, il existe deux constantes C et M telles que :*

$$\left\| \frac{\partial^l q_t}{\partial z^l}(z,x) \right\| \leq \frac{C}{t^{\frac{l}{2}}} \frac{1}{\sqrt{2\pi M t}} \exp\left(-\frac{(z-x)^2}{2Mt}\right), \tag{8.2}$$

$$\left\| \frac{\partial^l}{\partial t^l}\big(q_t(z,x)\big) \right\| \leq \frac{C}{t^l} \frac{1}{\sqrt{2\pi M t}} \exp\left(-\frac{(z-x)^2}{2Mt}\right). \tag{8.3}$$

Le fait remarquable tient à ce que l'ordre de la dérivation en espace apparaît, au dénominateur du majorant, comme exposant de la racine carrée du temps. C'est cette racine qui évite l'explosion de $R(\tilde{\Phi})$, lorsque n tend vers l'infini et que les dérivations portent sur des noyaux dont le paramètre du temps converge vers zéro. La majoration (8.3) de la dérivée par rapport au temps intervient lors de l'étude sur la convergence des fonctionnelles $\tilde{\Phi}_{n,l}$, définies à partir de la fonction (8.4) et de la subdivision $\left\{\frac{i}{n}\ ;\ 1 \leq i \leq n\right\}$, vers l'intégrale de la trajectoire.

8.2 Les moments d'ordre l

On se propose d'étudier le coefficient $R(\tilde{\Phi})$ au travers des dérivées de la fonction $H_{r,n}$ dans le cas où la fonctionnelle $\tilde{\Phi}$ représente la moyenne arithmétique considérée aux points $\left\{\frac{i}{n}\ ;\ 1 \le i \le n\right\}$ de la trajectoire de $(f(X(s)))_{(0\le s\le 1)}$. Lorsque n tend vers l'infini, cette moyenne converge vers l'intégrale de $(f(X(s)))_{(0\le s\le 1)}$. Les hypothèses à imposer et la vitesse de convergence sont discutées dans la section (8.3).

Définition :

Soit f une fonction borélienne, définie sur $\mathbb{R}^d$, vérifiant la condition de croissance (M_p) :

(M_p) Il existe une constante $C > 0$ et un entier naturel p tels que :
$|f(x)| \le C\,(1 + \|x\|^p)$ pour tout $x \in \mathbb{R}^d$.

Pour $l \in \mathbb{N}^*$ et $n \in \mathbb{N}^*$, on considère la fonction réelle $\Phi_{n,l}$ définie sur $\left(\mathbb{R}^d\right)^n$ par :

$$\Phi_{n,l}(x_1, \dots, x_n) = \left(\frac{f(x_1) + \dots + f(x_n)}{n}\right)^l. \tag{8.4}$$

$\tilde{\Phi}_{n,l}$ désigne la fonctionnelle associée à la fonction $\Phi_{n,l}$ et à la subdivision $\left\{\frac{i}{n}\ ;\ 1 \le i \le n\right\}$.

Ainsi définie, la fonction $\Phi_{n,l}$ vérifie la condition (M_{lp}). Il suffit en effet d'appliquer la majoration (M_p) à chaque fonction f rencontrée dans l'expression de $\Phi_{n,l}$:

$$\begin{aligned}
|\Phi_{n,l}(x_1, \dots, x_n)| &\le \left(\frac{|f(x_1)| + \dots + |f(x_n)|}{n}\right)^l \\
&\le C^l \left(1 + \frac{\|x_1\|^p + \dots + \|x_n\|^p}{n}\right)^l \\
&\le C^l \left(1 + \sup_{1\le j\le n} \|x_j\|^p\right)^l \\
&\le \tilde{C}\left(1 + \sup_{1\le j\le n} \|x_j\|^{lp}\right).
\end{aligned}$$

L'application du théorème 7.1.1 permet d'obtenir (7.1) qui s'écrit pour la fonctionnelle $\tilde{\Phi}_{n,l}$ associée à la fonction $\Phi_{n,l}$ et à la subdivision $\left\{t_i = \frac{i}{n}\ ;\ 1 \le i \le n\right\}$:

$$\mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^n f(X(\frac{i}{n}))\right)^l\right\} - \mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^n f(\bar{X}^\delta(\frac{i}{n}))\right)^l\right\} = R(\tilde{\Phi}_{n,l})\ \delta + \varepsilon_{\delta,n,l}, \tag{8.5}$$

avec

$$\sup_{n,l\in\mathbb{N}^*} \lim_{\delta\to 0} \frac{|\varepsilon_{\delta,n,l}|}{\delta} = 0. \tag{8.6}$$

Le théorème suivant explique dans quelle mesure on peut faire tendre n vers $+\infty$ dans le premier terme du membre de gauche de (8.5) pour obtenir l'intégrale de la trajectoire $s \longmapsto f(X(s))$.

Théorème 8.2.1 *Soit f une fonction mesurable et bornée.*
Soit $X(.)$ la solution de (6.1) avec $d = 1$; soit $\bar{X}^\delta(.)$ l'approximation de cette solution donnée par le schéma d'Euler .
Alors il existe C, $(\tilde{\varepsilon}_{\delta,n,l})_{\substack{n,l\in\mathbb{N}^*\\ 0<\delta<1}}$ et $(\tilde{C}_{n,l})_{n,l\in\mathbb{N}^*}$ tels que :

$$\left|\mathbb{E}\left\{\left(\int_0^1 f(X(s))\ ds\right)^l\right\} - \mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^n f(\bar{X}^\delta(\frac{i}{n}))\right)^l\right\}\right| \leq \tilde{C}_{n,l}\ \delta + \tilde{\varepsilon}_{\delta,n,l} \\ + C\ l^2\ \|f\|_\infty^l \frac{1}{n} \qquad (0<\delta<1)\ \ (n,l\in\mathbb{N}^*)\ \ (1\leq l\leq \frac{n}{6}). \tag{8.7}$$

Le nombre C est indépendant de l, n et δ ; les quantités $\tilde{C}_{n,l}$ et $\tilde{\varepsilon}_{\delta,n,l}$ vérifient :

$$\sup_{n,l} \frac{\tilde{C}_{n,l}}{l^2} < \infty\ , \qquad \sup_{\delta,n,l} \frac{\tilde{\varepsilon}_{\delta,n,l}}{l^2(\|f\|_\infty^l \vee 1)} < \infty, \tag{8.8}$$

$$\sup_{n,l}\ \lim_{\delta\to 0} \frac{\tilde{\varepsilon}_{\delta,n,l}}{\delta} = 0. \tag{8.9}$$

Démonstration du théorème 8.2.1.

L'inégalité triangulaire appliquée au membre de gauche de (8.7) définit le schéma de la démonstration.

$$\begin{aligned} &\left|\mathbb{E}\left\{\left(\int_0^1 f(X(s))\ ds\right)^l\right\} - \mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^n f(\bar{X}^\delta(\frac{i}{n}))\right)^l\right\}\right| \\ &\leq \left|\mathbb{E}\left\{\left(\int_0^1 f(X(s))\ ds\right)^l\right\} - \mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^n f(X(\frac{i}{n}))\right)^l\right\}\right| \\ &\quad + \left|\mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^n f(X(\frac{i}{n}))\right)^l\right\} - \mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^n f(\bar{X}^\delta(\frac{i}{n}))\right)^l\right\}\right|. \end{aligned} \tag{8.10}$$

Cette inégalité décompose l'erreur en une partie indépendante du schéma d'Euler (et donc du pas δ), et en une autre sans l'intégrale sur [0,1].

Le second terme du membre de droite de (8.10), sans la valeur absolue, est égal à $R(\tilde{\Phi}_{n,l})\delta + \varepsilon_{\delta,n,l}$ d'après (8.5) ; cette expression est transformée dans le théorème 8.4.1 pour appréhender le comportement en l et n. Il est ainsi démontré que

$$\mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^n f(X(\frac{i}{n}))\right)^l\right\} - \mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^n f(\bar{X}^\delta(\frac{i}{n}))\right)^l\right\} = C_{n,l}\ \delta + \varepsilon_{\delta,n,l}\ ; \tag{8.11}$$

et par conséquent que sont réalisées l'inégalité (8.12) et les conditions (8.8) et (8.9).

$$\left|\mathbb{E}\left\{\left(\int_0^1 f(X(s))\ ds\right)^l\right\} - \mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^n f(\bar{X}^\delta(\frac{i}{n}))\right)^l\right\}\right| \leq \tilde{C}_{n,l}\ \delta + \tilde{\varepsilon}_{\delta,n,l}. \quad (8.12)$$

La majoration du premier terme est liée à l'évaluation de la vitesse de convergence des sommes de Riemann vers l'intégrale, pour la fonction C^∞ sur $]0,1]$, $s \longmapsto \mathbb{E}\{f(X(s))\}$.
Cette majoration est réalisée dans le lemme (8.3.1) et ainsi (8.7) est une conséquence de (8.10), (8.11) et (8.13). □

8.3 Convergence des sommes de Riemann

La partie de la majoration de l'erreur comportant le facteur $\frac{1}{n}$ provient de l'approximation de l'intégrale $\displaystyle\int_0^1 f(X(s))\ ds$ par les sommes de Riemann $\displaystyle\frac{1}{n}\sum_{i=1}^n f\left(X(\frac{i}{n})\right)$, ou plus exactement de $\displaystyle\int_0^1 \mathbb{E}\{f(X(s))\}\ ds$ par $\displaystyle\frac{1}{n}\sum_{i=1}^n \mathbb{E}\left\{f\left(X(\frac{i}{n})\right)\right\}$.

Lemme 8.3.1 *Soit f une fonction mesurable et bornée.*
Soit $X(.)$ la solution de (6.1) avec $d \in \mathbb{N}^*$.
Alors il existe un réel C tel que :

$$\left|\mathbb{E}\left\{\int_0^1 f(X(s))\ ds\right\} - \mathbb{E}\left\{\frac{1}{n}\sum_{i=1}^n f(X(\frac{i}{n}))\right\}\right| \leq C\ \ \|f\|_\infty \frac{1}{n} \quad (8.13)$$

Démonstration du lemme 8.3.1.
Notons $\varphi(t) = E(f(X(t)))$. La régularité de X assure que :

- φ est bornée,
- $|\varphi'(s)| \leq \dfrac{C}{s}$
- $|\varphi''(s)| \leq \dfrac{C}{s^2}$.

Notons à présent I_1 la quantité suivante :

$$I_1 = \int_0^1 f(X(s))\, ds - \frac{1}{n}\sum_{i=1}^n f(X(\frac{i}{n})). \quad (8.14)$$

Nous avons :

$$\begin{aligned}\mathbb{E}(I_1) &= \int_0^t \varphi(s)\,ds - \frac{1}{n}\sum_{i=1}^{n}\varphi\left(\frac{i}{n}\right) \\ &= \sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} \varphi(s) - \varphi\left(\frac{i}{n}\right)\,ds\end{aligned} \tag{8.15}$$

Écrivons ici le développement de Taylor de φ :

$$\varphi(s) - \varphi(\frac{i}{n}) = \varphi'(\frac{i}{n})\left(s - \frac{i}{n}\right) + \varphi''(y)\left(s - \frac{i}{n}\right)^2 \tag{8.16}$$

où $y \in [s,\frac{i}{n}]$. Bien entendu, y dépend de s. On déduit que :

$$\begin{aligned}|E(I_1)| &\leq \left|\int_0^{\frac{1}{n}} \varphi(s) - \varphi(\frac{1}{n})\,ds\right| + \left|\sum_{i=2}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} \varphi'(\frac{i}{n})\left(s - \frac{i}{n}\right)\,ds\right| \\ &\quad + \sum_{i=2}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} |\varphi''(y)|\left(s - \frac{i}{n}\right)^2\,ds \\ &\leq \frac{2||f||_\infty}{n} + \frac{1}{2n^2}\left|\sum_{i=2}^{n}\varphi'(\frac{i}{n})\right| \\ &\quad + \sum_{i=2}^{n}\frac{n^2}{(i-1)^2}\frac{1}{3n^3} \\ &\leq \frac{C}{n} + \frac{1}{2n^2}\left|\sum_{i=2}^{n}\varphi'(\frac{i}{n})\right|.\end{aligned} \tag{8.17}$$

On montre maintenant que $\frac{1}{n}\sum_{i=2}^{n}\varphi'(\frac{i}{n})$ est borné.

$$\begin{aligned}\frac{1}{n}\sum_{i=2}^{n}\varphi'(\frac{i}{n}) - \int_{\frac{1}{n}}^{1}\varphi'(s)\,ds &= \sum_{i=2}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} \varphi'(\frac{i}{n}) - \varphi'(s)\,ds \\ &= \sum_{i=2}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} \varphi''(\tilde{y})\left(\frac{i}{n} - s\right)\,ds \\ \left|\frac{1}{n}\sum_{i=2}^{n}\varphi'(\frac{i}{n}) - \int_{\frac{1}{n}}^{1}\varphi'(s)\,ds\right| &\leq \sum_{i=2}^{n}\frac{n^2}{(i-1)^2}\frac{1}{2n^2} \\ &\leq \frac{1}{2}\sum_{i>0}\frac{1}{i^2}.\end{aligned} \tag{8.18}$$

On a de plus :

$$\left|\int_{\frac{T}{n}}^{T}\varphi'(s)\,ds\right| = \left|\varphi(T) - \varphi(\frac{T}{n})\right| \leq 2||f||_\infty. \tag{8.19}$$

Ce dernier résultat achève la preuve.

8.4 Convergence du schéma d'Euler pour le moment d'ordre l

Comme annoncé dans la démonstration du théorème 8.2.1, le théorème 8.4.1 permet d'obtenir l'erreur commise en remplaçant X par $\bar{X}^\delta$ dans l'expression du moment d'ordre l de la somme de Riemann d'ordre n.

Théorème 8.4.1 *Soit f une fonction mesurable et bornée.*
Soit $X(.)$ la solution de (6.1) avec $d = 1$; soit $\bar{X}^\delta(.)$ l'approximation de $X(.)$ donnée par le schéma d'Euler.
Alors l'erreur commise en remplaçant X par $\bar{X}^\delta$, pour calculer le moment d'ordre l de la v.a. $\frac{1}{n}\sum_{i=1}^{n} f(X(\frac{i}{n}))$, s'exprime sous la forme suivante :

$$\mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^{n} f(X(\frac{i}{n}))\right)^l\right\} - \mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^{n} f(\bar{X}^\delta(\frac{i}{n}))\right)^l\right\} = C_{n,l}\ \delta + \varepsilon_{\delta,n,l}, \tag{8.20}$$

les quantités $C_{n,l}$ et $\varepsilon_{\delta,n,l}$ vérifiant les conditions (8.8) et (8.9) du théorème 8.2.1 :

$$\sup_{n,l} \frac{|C_{n,l}|}{l^2} < \infty\ , \qquad \sup_{\delta,n,l} \frac{|\varepsilon_{\delta,n,l}|}{l^2(\|f\|_\infty^l \vee 1)} < \infty, \tag{8.8}$$

$$\sup_{n,l}\ \lim_{\delta\to 0} \frac{|\varepsilon_{\delta,n,l}|}{\delta} = 0. \tag{8.9}$$

8.5 Démonstration dans le cas du moment d'ordre un

Lorsque $l = 1$, les résultats (8.20), (8.8) et (8.9) du théorème 8.4.1 s'écrivent :

$$\mathbb{E}\left\{\frac{1}{n}\sum_{i=1}^{n} f(X(\frac{i}{n}))\right\} - \mathbb{E}\left\{\frac{1}{n}\sum_{i=1}^{n} f(\bar{X}^\delta(\frac{i}{n}))\right\} = C_n\ \delta + \varepsilon_{\delta,n}, \tag{8.21}$$

$$\sup_{n\in\mathbb{N}^*} |C_n| < \infty\ , \qquad \sup_{\delta\in]0,1],n\in\mathbb{N}^*} |\varepsilon_{\delta,n}| < \infty, \tag{8.22}$$

$$\sup_{n\in\mathbb{N}^*}\ \lim_{\delta\to 0} \frac{|\varepsilon_{\delta,n}|}{\delta} = 0. \tag{8.23}$$

Compte-tenu de (8.5) et (8.6), il suffit de montrer que la quantité $R(\tilde{\Phi}_{n,1})$, qui correspond à C_n dans (8.21), est bornée uniformément en n.
En effet, on peut alors transformer (8.21) pour obtenir :

$$\varepsilon_{\delta,n} \;=\; \mathbb{E}\left\{\frac{1}{n}\sum_{i=1}^{n} f(X(\frac{i}{n}))\right\} - \mathbb{E}\left\{\frac{1}{n}\sum_{i=1}^{n} f(\bar{X}^\delta(\frac{i}{n}))\right\} - C_n\ \delta,$$

puis

$$|\varepsilon_{\delta,n}| \quad \leq \quad 2\|f\|_\infty + \delta \sup_{n\in\mathbb{N}^*} |C_n|,$$

et ainsi

$$\sup_{\delta\in]0,1],n\in\mathbb{N}^*} |\varepsilon_{\delta,n}| < \infty. \tag{8.24}$$

Démontrons donc que $R(\tilde{\Phi}_{n,1})$ est bornée en n.
L'expression de $R(\tilde{\Phi}_{n,1})$ apparaissant en (7.2) conduit à examiner la fonction $H_{r,n}$ définie en (7.3) et associée à la fonction $\Phi_{n,1}$.
Pour tout $r \in]0,1[$ et tout entier strictement positif n, la fonction $H_{r,n}$ s'écrit pour $\Phi_{n,1}$:

$$\begin{aligned} H_{r,n}(x_1,\ldots,x_{n_r},z) \quad = \quad & \frac{1}{n}\Big[f(x(t_1)) + \cdots + f(x(t_{n_r})) \\ & +\mathbb{E}\left\{f(X^{r,z}(t_{n_r+1})) + \cdots + f(X^{r,z}(t_n))\right\}\Big]. \end{aligned} \tag{8.25}$$

La linéarité de l'expression de $H_{r,n}$ élimine les n_r premiers termes des dérivées par rapport à z ; pour $\gamma = 1$, 2, ou 3 :

$$\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(x_1,\ldots,x_{n_r},z) \quad = \quad \frac{1}{n}\sum_{j=n_r+1}^{n} \frac{\partial^\gamma}{\partial z^\gamma}\mathbb{E}\left\{f(X^{r,z}(t_j))\right\}. \tag{8.26}$$

Le lemme suivant permet de contrôler les trois termes apparaissant dans le membre de droite de l'égalité (7.2).

Lemme 8.5.1 *Soit g une fonction réelle définie sur $\mathbb{R}$, de classe C^3 à croissance au plus polynomiale ainsi que ses dérivées ; c'est-à-dire qu'il existe $\beta \in \mathbb{N}$ et $C \in \mathbb{R}$ tels que :*

$$\left|g^{(j)}(x)\right| \leq C\left(1+|x|^\beta\right) \qquad (x\in\mathbb{R}) \quad (j\in\{0,1,2,3\}). \tag{8.27}$$

Sous les hypothèses du théorème 8.4.1 et si la fonction $\dfrac{\partial^\gamma H_{r,n}}{\partial z^\gamma}$ est définie par (8.26). Alors pour $\gamma = 1$, 2 ou 3, la famille d'intégrales $\displaystyle\int_0^1 \mathbb{E}\left\{\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(X(t_1),\ldots,X(t_{n_r}),X(r))g(X(r))\right\} dr$ paramétrée par n est uniformément bornée.

Démonstration du lemme 8.5.1.

Si $]a,b[$ désigne un intervalle de $]0,1[$, on note :

$$\mathcal{I}_\gamma(]a,b[) = \int_a^b \mathbb{E}\left\{\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(X(t_1),\ldots,X(t_{n_r}),X(r))g(X(r))\right\} dr. \tag{8.28}$$

En tenant compte de (8.26) et en remarquant que $n_r = i$ lorsque $t_i < r < t_{i+1}$, on obtient :

$$\mathcal{I}_\gamma(]t_i,t_{i+1}[) = \frac{1}{n} \sum_{j=i+1}^{n} \int_{t_i}^{t_{i+1}} \mathbb{E}\left\{\mathcal{I}_{\gamma,j}(X(r))\right\} dr,$$

où

$$\mathcal{I}_{\gamma,j}(z) = g(z) \frac{\partial^\gamma}{\partial z^\gamma}\Big(\mathbb{E}\left\{f(X^{r,z}(t_j))\right\}\Big).$$

On écrit $\mathcal{I}_\gamma(]t_i,t_{i+1}[)$ en utilisant les noyaux de transition :

$$\begin{aligned}\mathcal{I}_\gamma(]t_i,t_{i+1}[) \quad = \quad & \frac{1}{n} \sum_{j=i+1}^{n} \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^i \times \mathbb{R}} q_{t_1}(x_0,x_1) \dots q_{t_i - t_{i-1}}(x_{i-1},x_i) \;\; g(z) \; q_{r-t_i}(x_i,z) \\ & \frac{\partial^\gamma}{\partial z^\gamma} \left(\int_{\mathbb{R}^{n-i}} f(x_j) q_{t_{i+1}-r}(z,x_{i+1}) q_{t_{i+2}-t_{i+1}}(x_{i+1},x_{i+2}) \dots \right. \\ & \left. q_{t_n - t_{n-1}}(x_{n-1},x_n) \; dx_{i+1} \dots dx_n \right) \; dx_1 \dots dx_n \; dz \; dr.\end{aligned}$$

L'application successive de l'égalité de Chapman-Kolmogorov permet d'établir :

$$\mathcal{I}_\gamma(]t_i,t_{i+1}[) = \frac{1}{n} \sum_{j=i+1}^{n} \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} q_r(x_0,z) g(z) \frac{\partial^\gamma}{\partial z^\gamma} \left(\int_{\mathbb{R}} q_{t_j - r}(z,x_j) f(x_j) \; dx_j \right) dz \; dr.$$

En sommant les contributions de chaque intervalle $]t_i,t_{i+1}[$, et en inversant les sommes en i et j, on obtient :

$$\mathcal{I}_\gamma(]0,1[) = \frac{1}{n} \sum_{j=1}^{n} \int_0^{t_j} \int_{\mathbb{R}} \frac{\partial^\gamma}{\partial z^\gamma} \left(\int_{\mathbb{R}} q_{t_j - r}(z,x_j) f(x_j) \; dx_j \right) q_r(x_0,z) g(z) \; dz \; dr.$$

L'intégrale en temps est scindée en deux parties et l'une d'elles est transformée par une intégration par parties. Les termes de bord sont nuls puisqu'ils comportent le facteur $q_r(x_0,z)$ qui tendent vers zéro à une vitesse exponentielle lorsque z tend vers l'infini.

$$\begin{aligned}\mathcal{I}_\gamma(]0,1[) \quad = \quad & \frac{1}{n} \sum_{j=1}^{n} \left(\int_0^{\frac{t_j}{2}} \int_{\mathbb{R}} \frac{\partial^\gamma}{\partial z^\gamma} \left(\int_{\mathbb{R}} q_{t_j - r}(z,x_j) f(x_j) \; dx_j \right) q_r(x_0,z) g(z) \; dz \; dr \right. \\ & \left. + \, (-1)^\gamma \int_{\frac{t_j}{2}}^{t_j} \int_{\mathbb{R}} \int_{\mathbb{R}} q_{t_j - r}(z,x_j) f(x_j) \; dx_j \; \frac{\partial^\gamma}{\partial z^\gamma} \left(q_r(x_0,z) g(z) \right) \; dz \; dr \right).\end{aligned}$$

On utilise la majoration (8.2) de la valeur absolue de la dérivée en la variable d'espace du noyau $q_t(x,y)$. Il existe par conséquent un réel K indépendant de r, j et n tel que :

$$\mathcal{I}_\gamma(]0,1[) \;\leq\; \frac{K}{n}\sum_{j=1}^{n}\left(\int_0^{\frac{t_j}{2}} \frac{1}{(t_j-r)^{\frac{\gamma}{2}}}\,dr + \sum_{i=0}^{\gamma}\int_{\frac{t_j}{2}}^{t_j} \frac{1}{r^{\frac{i}{2}}}\,dr\right).$$

Par deux changements de variables, on obtient :

$$\mathcal{I}_\gamma(]0,1[) \;\leq\; \frac{K}{n}\sum_{j=1}^{n}\left(t_j^{1-\frac{\gamma}{2}}\int_{\frac{1}{2}}^{1} \frac{1}{u^{\frac{\gamma}{2}}}\,du + \sum_{i=0}^{\gamma} t_j^{1-\frac{i}{2}}\int_{\frac{1}{2}}^{1} \frac{1}{u^{\frac{i}{2}}}\,du\right).$$

On majore alors les intégrales par 1 et les sommes par des intégrales.

$$\mathcal{I}_\gamma(]0,1[) \;\leq\; K\left(\int_0^1 x^{1-\frac{\gamma}{2}}\,dx + \sum_{i=0}^{\gamma}\int_0^1 x^{1-\frac{i}{2}}\,dx\right),$$

$$\mathcal{I}_\gamma(]0,1[) \;\leq\; K\left(\frac{2}{4-\gamma} + \sum_{i=0}^{\gamma}\frac{2}{4-i}\right).$$

□

8.6 Démonstration dans le cas du moment d'ordre deux

La démonstration se différencie du cas $l=1$ par l'expression de la fonction $\dfrac{\partial^\gamma H_{r,n}}{\partial z^\gamma}$. En particulier, $\dfrac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(x_1,\ldots,x_{n_r},z)$ ne dépend plus seulement de z comme il apparaissait dans (8.26).

Nous allons démontrer un lemme, analogue au lemme 8.5.1, mais où la fonction $H_{r,n}$ est définie par :

$$\begin{aligned} H_{r,n}(x_1,\ldots,x_{n_r},z) \;=\; & \frac{1}{n^2}\Big[f(x(t_1))+\cdots+f(x(t_{n_r})) \\ & +\mathbb{E}\left\{f(X^{r,z}(t_{n_r+1}))+\cdots+f(X^{r,z}(t_n))\right\}\Big]^2. \end{aligned} \tag{8.29}$$

Lemme 8.6.1 *Soit g une fonction réelle définie sur $\mathbb{R}$, de classe C^3 à croissance polynomiale ainsi que ses dérivées ; c'est-à-dire qu'il existe $\beta \in \mathbb{N}$ et $C \in \mathbb{R}$ tels que :*

$$\left|g^{(j)}(x)\right| \leq C\left(1+|x|^\beta\right) \qquad (x\in\mathbb{R}) \quad (j\in\{0,1,2,3\}). \tag{8.27}$$

Sous les hypothèses du théorème 8.4.1 et si la fonction $H_{r,n}$ est définie par (8.29). Alors pour $\gamma = 1$, 2 ou 3 :

$$\sup_{n\in\mathbb{N}^*}\left|\int_0^1 \mathbb{E}\left\{\frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(X(t_1),\ldots,X(t_{n_r}),X(r))g(X(r))\right\}dr\right| < \infty. \tag{8.30}$$

Démonstration du lemme 8.6.1.

En utilisant les noyaux de transition, on peut écrire la quantité $\mathcal{I}_\gamma(]0,1[)$ comme suit :

$$\begin{aligned}\mathcal{I}_\gamma(]0,1[) &= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}^i\times\mathbb{R}}\frac{\partial^\gamma}{\partial z^\gamma}\left(\int_{\mathbb{R}^{n-i}}\left(\frac{f(x_1)+\cdots+f(x_n)}{n}\right)^2 q_{t_{i+1}-r}(z,x_{i+1})\right.\\ &\quad\left. q_{t_{i+2}-t_{i+1}}(x_{i+1},x_{i+2})\ldots q_{t_n-t_{n-1}}(x_{n-1},x_n)\ dx_{i+1}\ldots dx_n\right)g(z) \qquad (8.31)\\ &\quad q_{t_1}(x_0,x_1)\ldots q_{t_i-t_{i-1}}(x_{i-1},x_i)q_{r-t_i}(x_i,z)\ dx_1\ldots dx_i\ dz\ dr.\end{aligned}$$

Le carré de la somme peut se développer de la façon suivante :

$$(f(x_1)+\cdots+f(x_n))^2 = \sum_{1\le j_1\le j_2\le n} C_{j_1,j_2} f(x_{j_1})f(x_{j_2}), \qquad (8.32)$$

où les coefficients C_{j_1,j_2} valent 1 ou 2 suivant que $j_1 = j_2$ ou non.
L'interversion des sommes et des intégrales entraîne :

$$\begin{aligned}\mathcal{I}_\gamma(]0,1[) &= \frac{1}{n^2}\sum_{1\le j_1\le j_2\le n} C_{j_1,j_2}\sum_{i=0}^{n}\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}^i\times\mathbb{R}}\frac{\partial^\gamma}{\partial z^\gamma}\left(\int_{\mathbb{R}^{n-i}} f(x_{j_1})f(x_{j_2})q_{t_{i+1}-r}(z,x_{i+1})\right.\\ &\quad\left. q_{t_{i+2}-t_{i+1}}(x_{i+1},x_{i+2})\ldots q_{t_n-t_{n-1}}(x_{n-1},x_n)\ dx_{i+1}\ldots dx_n\right)g(z)\\ &\quad q_{t_1}(x_0,x_1)\ldots q_{t_i-t_{i-1}}(x_{i-1},x_i)q_{r-t_i}(x_i,z)\ dx_1\ldots dx_i\ dz\ dr.\end{aligned}$$

En distinguant les sommes sur les indices vérifiant $0 \le i \le j_1 - 1$, $j_1 \le i \le j_2 - 1$ et $j_2 \le i \le n$, on applique plusieurs fois l'identité de Chapman-Kolmogorov pour simplifier l'écriture des deux premières sommes et montrer que la dernière est nulle :

$$\begin{aligned}\mathcal{I}_\gamma(]0,1[) &= \frac{1}{n^2}\sum_{1\le j_1\le j_2\le n} C_{j_1,j_2}\left[\sum_{i=0}^{j_1-1}\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}}\frac{\partial^\gamma}{\partial z^\gamma}\left(\int_{\mathbb{R}^2} f(x_{j_1})f(x_{j_2})q_{t_{j_1}-r}(z,x_{j_1})\right.\right.\\ &\quad\left. q_{t_{j_2}-t_{j_1}}(x_{j_1},x_{j_2})\ dx_{j_1}\ dx_{j_2}\right)g(z)q_r(x_0,z)\ dz\ dr\\ &\quad+\sum_{i=j_1}^{j_2-1}\int_{t_i}^{t_{i+1}}\int_{\mathbb{R}^2}\frac{\partial^\gamma}{\partial z^\gamma}\left(\int_{\mathbb{R}} f(x_{j_1})f(x_{j_2})q_{t_{j_2}-r}(z,x_{j_2})dx_{j_2}\right)\\ &\quad\left. g(z)q_{t_{j_1}}(x_0,x_{j_1})q_{r-t_{j_1}}(x_{j_1},z)\ dx_{j_1}\ dz\ dr\right].\end{aligned} \qquad (8.33)$$

On remarque que l'indice de sommation i n'apparaît plus que dans les bornes des intégrales en temps. On peut, par conséquent, faire totalement disparaître l'indice i

de l'expression ci-dessus.
Par ailleurs, on découpe chacune des intégrales de temps en deux. En prenant soin d'intégrer par parties en la variable z deux des quatre termes obtenus, on échange ensuite les signes d'intégration et de dérivation.

$$\begin{aligned}
\mathcal{I}_\gamma(]0,1[) \;=\; & \frac{1}{n^2}\sum_{1\leq j_1\leq j_2\leq n} C_{j_1,j_2}\Bigg[\int_0^{\frac{t_{j_1}}{2}}\int_{\mathbb{R}}\int_{\mathbb{R}^2} f(x_{j_1})f(x_{j_2})\frac{\partial^\gamma}{\partial z^\gamma} \\
& \big(q_{t_{j_1}-r}(z,x_{j_1})\big)\, q_{t_{j_2}-t_{j_1}}(x_{j_1},x_{j_2})\ dx_{j_1}\ dx_{j_2} g(z) q_r(x_0,z)\ dz\ dr \\
& +(-1)^\gamma \int_{\frac{t_{j_1}}{2}}^{t_{j_1}}\int_{\mathbb{R}}\int_{\mathbb{R}^2} f(x_{j_1})f(x_{j_2}) \\
& \times q_{t_{j_1}-r}(z,x_{j_1}) q_{t_{j_2}-t_{j_1}}(x_{j_1},x_{j_2})\ dx_{j_1}\ dx_{j_2}\frac{\partial^\gamma}{\partial z^\gamma}\left(g(z)q_r(x_0,z)\right)\ dz\ dr \\
& +\int_{t_{j_1}}^{\frac{t_{j_1}+t_{j_2}}{2}}\int_{\mathbb{R}^2}\int_{\mathbb{R}} f(x_{j_1})f(x_{j_2})\frac{\partial^\gamma}{\partial z^\gamma}\big(q_{t_{j_2}-r}(z,x_{j_2})\big)\ dx_{j_2} g(z) \\
& \times q_{t_{j_1}}(x_0,x_{j_1}) q_{r-t_{j_1}}(x_{j_1},z)\ dx_{j_1}\ dz\ dr \\
& +(-1)^\gamma \int_{\frac{t_{j_1}+t_{j_2}}{2}}^{t_{j_2}}\int_{\mathbb{R}^2}\int_{\mathbb{R}} f(x_{j_1})f(x_{j_2}) q_{t_{j_2}-r}(z,x_{j_2})\ dx_{j_2} q_{t_{j_1}}(x_0,x_{j_1}) \\
& \frac{\partial^\gamma}{\partial z^\gamma}\big(g(z)q_{r-t_{j_1}}(x_{j_1},z)\big)\ dx_{j_1}\ dz\ dr\Bigg].
\end{aligned}$$

On utilise la majoration (8.2) de la valeur absolue de la dérivée en la variable d'espace du noyau $q_t(x,y)$. On est alors en mesure de majorer les intégrales en les variables d'espace x_{j_1}, x_{j_2} et z ; par exemple pour la première des quatre intégrales, on peut écrire :

$$\begin{aligned}
& \left|\int_{\mathbb{R}^3} f(x_{j_1})f(x_{j_2})g(z)q_r(x_0,z)\frac{\partial^\gamma q_{t_{j_1}-r}}{\partial z^\gamma}(z,x_{j_1})q_{t_{j_2}-t_{j_1}}(x_{j_1},x_{j_2})\ dz\ dx_{j_1}\ dx_{j_2}\right| \\
& \leq \frac{C}{(t_{j_1}-r)^{\frac{l}{2}}}\int_{\mathbb{R}^3}\left|f(x_{j_1})f(x_{j_2})g(z)\right| q_r(x_0,z) \\
& \frac{\exp\left(-\frac{(z-x_{j_1})^2}{2M(t_{j_1}-r)}\right)}{\sqrt{2\pi M(t_{j_1}-r)}} q_{t_{j_2}-t_{j_1}}(x_{j_1},x_{j_2})\ dz\ dx_{j_1}\ dx_{j_2},
\end{aligned}$$

et cette dernière intégrale peut s'interpréter en terme de l'espérance d'une diffusion $\tilde{X}(.)$ solution de l'équation :

$$\begin{aligned}
\tilde{X}(t) \;=\; & \tilde{X}(0)+\int_0^t\left(\sigma(\tilde{X}(s))\mathbb{1}_{[0,r[\cup]t_{j_1},1]}(s)+\sqrt{M}\mathbb{1}_{[r,t_{j_1}]}(s)\right)dB(s) \\
& +\int_0^t\left(b(\tilde{X}(s))\mathbb{1}_{[0,r[\cup]t_{j_1},1]}(s)\right)ds
\end{aligned} \qquad (t\geq 0),$$

ainsi,

$$\int_{\mathbb{R}^3} |f(x_{j_1})f(x_{j_2})g(z)|\, q_r(x_0,z)\frac{\exp\left(-\frac{(z-x_{j_1})^2}{2M(t_{j_1}-r)}\right)}{\sqrt{2\pi M(t_{j_1}-r)}} q_{t_{j_2}-t_{j_1}}(x_{j_1},x_{j_2})\ dz\ dx_{j_1}\ dx_{j_2}$$
$$= \mathbb{E}\left\{\left|f(\tilde{X}(t_{j_1}))f(\tilde{X}(t_{j_2}))g(\tilde{X}(r)\right|\right\}.$$

Comme la fonction f est bornée et la fonction g à croissance polynomiale, la majoration (6.6) implique que cette espérance est contrôlée par une quantité indépendante de j_1, j_2, n ou r. Il s'ensuit l'existence d'un nombre K ne dépendant que de f, b et σ permettant d'écrire :

$$|\mathcal{I}_\gamma(]0,1[)| \leq \frac{K}{n^2} \sum_{1\leq j_1\leq j_2\leq n} C_{j_1,j_2}\left[\int_0^{\frac{t_{j_1}}{2}} \frac{1}{(t_{j_1}-r)^{\frac{\gamma}{2}}}\ dr + \int_{\frac{t_{j_1}}{2}}^{t_{j_1}} \sum_{i=0}^{\gamma} \frac{1}{r^{\frac{i}{2}}}\ dr\right.$$
$$\left. + \int_{t_{j_1}}^{\frac{t_{j_1}+t_{j_2}}{2}} \frac{1}{(t_{j_2}-r)^{\frac{\gamma}{2}}}\ dr + \int_{\frac{t_{j_1}+t_{j_2}}{2}}^{t_{j_2}} \sum_{i=0}^{\gamma} \frac{1}{(r-t_{j_1})^{\frac{i}{2}}}\ dr\right]. \tag{8.34}$$

Ce qui s'écrit, après quatre changements de variables :

$$|\mathcal{I}_\gamma(]0,1[)| \ \leq\ \frac{K}{n^2} \sum_{1\leq j_1\leq j_2\leq n} C_{j_1,j_2}\left[t_{j_1}^{1-\frac{\gamma}{2}} \int_{\frac{1}{2}}^1 \frac{1}{u^{\frac{\gamma}{2}}}\ du + \sum_{i=0}^{\gamma} t_{j_1}^{1-\frac{i}{2}} \int_{\frac{1}{2}}^1 \frac{1}{u^{\frac{i}{2}}}\ du\right]$$
$$+\frac{K}{n^2} \sum_{1\leq j_1< j_2\leq n} C_{j_1,j_2}\left[(t_{j_2}-t_{j_1})^{1-\frac{\gamma}{2}} \int_{\frac{1}{2}}^1 \frac{1}{u^{\frac{\gamma}{2}}}\ du + \sum_{i=0}^{\gamma} (t_{j_2}-t_{j_1})^{1-\frac{i}{2}} \int_{\frac{1}{2}}^1 \frac{1}{u^{\frac{i}{2}}}\ du\right].$$

Toutes les intégrales ci-dessus sont majorées par 1.
Les sommes en j_1, j_2 divisées par n^2 possèdent des limites simples à calculer lorsque n tend vers l'infini.

$$\frac{1}{n^2} \sum_{1\leq j_1\leq j_2\leq n} \left(\frac{j_1}{n}\right)^{1-\frac{\gamma}{2}} \ =\ \frac{1}{n} \sum_{j_1=1}^{n} \left(\frac{j_1}{n}\right)^{1-\frac{\gamma}{2}} \left(1-\frac{j_1}{n}\right)$$
$$\simeq\ \int_0^1 x^{1-\frac{\gamma}{2}}(1-x)\ dx = \frac{4}{(4-\gamma)(6-\gamma)},$$

et

$$\frac{1}{n^2} \sum_{1\leq j_1< j_2\leq n} \left(\frac{j_2-j_1}{n}\right)^{1-\frac{\gamma}{2}} \ \simeq\ \int_0^1 \int_y^1 (x-y)^{1-\frac{\gamma}{2}}\ dx\ dy = \frac{4}{(4-\gamma)(6-\gamma)}.$$

Par conséquent, $|\mathcal{I}_\gamma(]0,1[)|$ est majorée par un nombre indépendant de n et le lemme 8.6.1 est démontré. □

La réunion de (7.2) et (8.30) entraîne que $R(\tilde{\Phi}_{n,2}) = C_{n,2}$ est uniformément bornée en n.

8.7 Démonstration dans le cas général

Démonstration du théorème 8.4.1.

Il suffit de démontrer que la valeur absolue de la quantité $R(\tilde{\Phi}_{n,l}) = C_{n,l}$ est majorée par une quantité indépendante de n et inférieure à un polynôme de degré deux en l :

$$\sup_{n,l} \frac{|C_{n,l}|}{l^2} < \infty. \tag{8.35}$$

En effet, si l'on exprime $\varepsilon_{\delta,n,l}$ grâce à (8.20) et que l'on majore :

$$\begin{aligned}
\varepsilon_{\delta,n,l} &= \mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^{n} f(X(\frac{i}{n}))\right)^l\right\} - \mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^{n} f(\bar{X}^\delta(\frac{i}{n}))\right)^l\right\} - C_{n,l}\ \delta, \\
|\varepsilon_{\delta,n,l}| &\leq 2\left(\|f\|_\infty^l \vee 1\right) + l^2\ \delta\ \frac{|C_{n,l}|}{l^2}, \\
\frac{|\varepsilon_{\delta,n,l}|}{l^2\left(\|f\|_\infty^l \vee 1\right)} &\leq \frac{2}{l^2} + \delta\ \sup_{n,l} \frac{|C_{n,l}|}{l^2},
\end{aligned}$$

et ainsi, (8.8) est démontrée.

Nous allons maintenant démontrer (8.35) en reprenant la démonstration du lemme 8.6.1.

On exprime $\mathcal{I}_\gamma(]0,1[)$ à l'aide des noyaux de transition, comme il a été fait en (8.31) dans le cas $l = 2$:

$$\begin{aligned}
\mathcal{I}_\gamma(]0,1[) = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^i\times\mathbb{R}} \frac{\partial^\gamma}{\partial z^\gamma}\Bigg(\int_{\mathbb{R}^{n-i}} \left(\frac{f(x_1)+\cdots+f(x_n)}{n}\right)^l q_{t_{i+1}-r}(z,x_{i+1}) \\
q_{t_{i+2}-t_{i+1}}(x_{i+1},x_{i+2})\dots q_{t_n-t_{n-1}}(x_{n-1},x_n)\ dx_{i+1}\dots dx_n\Bigg) g(z)\dots \\
q_{t_1}(x_0,x_1) q_{t_i-t_{i-1}}(x_{i-1},x_i) q_{r-t_i}(x_i,z)\ dx_1\dots dx_i\ dz\ dr.
\end{aligned} \tag{8.36}$$

La puissance l se développe comme suit :

$$\left(f(x_1)+\cdots+f(x_n)\right)^l = \sum_{1\leq j_1\leq\cdots\leq j_l\leq n} C_{j_1,\dots,j_l} f(x_{j_1})\dots f(x_{j_l}), \tag{8.37}$$

où le coefficient $C_{j_1,\dots,j_l}$ vérifie

$$C_{j_1,\dots,j_l} \leq l!, \qquad \sum_{1\leq j_1\leq\cdots\leq j_l\leq n} C_{j_1,\dots,j_l} = n^l. \tag{8.38}$$

Comme dans le cas $l = 2$, on échange les sommations en i, les sommations en $j_1,\dots,j_l$ et les intégrales. Puis, on prend soin de distinguer les sommes sur i telles que $0 \leq i \leq j_1 - 1$, $j_1 \leq i \leq j_2 - 1$, ..., $j_{p-1} \leq i \leq j_p$ et $j_p \leq i \leq n$.
On obtient alors l'analogue de l'expression (8.33) de $\mathcal{I}_\gamma(]0,1[)$:

$$\begin{aligned}
\mathcal{I}_\gamma(]0,1[) = \frac{1}{n^l} \sum_{1\leq j_1\leq\cdots\leq j_p\leq n} C_{j_1,\dots,j_l} \Bigg[& \sum_{i=0}^{j_1-1} \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}} \frac{\partial^\gamma}{\partial z^\gamma} \Bigg(\int_{\mathbb{R}^l} f(x_{j_1}) \dots f(x_{j_l}) \\
& q_{t_{j_1}-r}(z,x_{j_1}) q_{t_{j_2}-t_{j_1}}(x_{j_1},x_{j_2}) \dots q_{t_{j_l}-t_{j_{l-1}}}(x_{j_{l-1}},x_{j_l}) \; dx_{j_1} \dots \; dx_{j_l} \Bigg) \\
& g(z) q_r(x_0,z) \; dz \; dr \\
& + \sum_{i=j_1}^{j_2-1} \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^2} \frac{\partial^\gamma}{\partial z^\gamma} \Bigg(\int_{\mathbb{R}^{l-1}} f(x_{j_1}) \dots f(x_{j_l}) q_{t_{j_2}-r}(z,x_{j_2}) \\
& q_{t_{j_3}-t_{j_2}}(x_{j_2},x_{j_3}) \dots q_{t_{j_l}-t_{j_{l-1}}}(x_{j_{l-1}},x_{j_l}) dx_{j_2} \dots dx_{j_l} \Bigg) g(z) q_{t_{j_1}}(x_0,x_{j_1}) \\
& q_{r-t_{j_1}}(x_{j_1},z) dx_{j_1} \; dz \; dr \\
& + \; \dots \\
& + \sum_{i=j_{l-1}}^{j_l-1} \int_{t_i}^{t_{i+1}} \int_{\mathbb{R}^l} \frac{\partial^\gamma}{\partial z^\gamma} \Bigg(\int_{\mathbb{R}} f(x_{j_1}) \dots f(x_{j_l}) q_{t_{j_l}-r}(z,x_{j_l}) dx_{j_l} \Bigg) g(z) \\
& q_{t_{j_1}}(x_0,x_{j_1}) \dots q_{r-t_{j_{l-1}}}(x_{j_{l-1}},z) dx_{j_1} \dots dx_{j_{l-1}} \; dz \; dr \Bigg].
\end{aligned} \tag{8.39}$$

Comme dans la démonstration correspondant au cas $l = 2$, on peut regrouper les sommes en i avec les intégrales en temps pour n'obtenir que des intégrales sur $]0,t_{j_1}[$, $]t_{j_1},t_{j_2}[$, ..., $]t_{j_{l-1}},t_{j_l}[$.

De même, on intègre par parties une intégrale sur deux. Puis on utilise pour chacun des termes la majoration (8.2), de sorte que l'on peut écrire une majoration analogue à (8.34) :

$$\begin{aligned}
|\mathcal{I}_\gamma(]0,1[)| \leq\ & \frac{K}{n^l} \sum_{1\leq j_1 \leq \cdots \leq j_l \leq n} C_{j_1,\ldots,j_l} \Bigg[\int_0^{\frac{t_{j_1}}{2}} \frac{1}{(t_{j_1}-r)^{\frac{\gamma}{2}}}\, dr + \int_{\frac{t_{j_1}}{2}}^{t_{j_1}} \sum_{i=0}^{\gamma} \frac{1}{r^{\frac{i}{2}}}\, dr \\
& + \int_{t_{j_1}}^{\frac{t_{j_1}+t_{j_2}}{2}} \frac{1}{(t_{j_2}-r)^{\frac{\gamma}{2}}}\, dr + \int_{\frac{t_{j_1}+t_{j_2}}{2}}^{t_{j_2}} \sum_{i=0}^{\gamma} \frac{1}{(r-t_{j_1})^{\frac{i}{2}}}\, dr \\
& + \ \ldots \\
& + \int_{t_{j_{l-1}}}^{\frac{t_{j_{l-1}}+t_{j_l}}{2}} \frac{1}{(t_{j_l}-r)^{\frac{\gamma}{2}}}\, dr + \int_{\frac{t_{j_{l-1}}+t_{j_l}}{2}}^{t_{j_l}} \sum_{i=0}^{\gamma} \frac{1}{\left(r-t_{j_{l-1}}\right)^{\frac{i}{2}}}\, dr \Bigg].
\end{aligned} \tag{8.40}$$

Un changement de variable dans chaque intégrale permet de faire sortir les indices de sommations.
Les intégrales obtenues sont du type $\displaystyle\int_{\frac{1}{2}}^{1} \frac{1}{u^{\frac{\alpha}{2}}}\, du$ où α est un entier compris entre zéro et γ. Elles sont ainsi toutes majorées par 1 ; ce qui permet d'écrire en utilisant (8.38) :

$$\begin{aligned}
|\mathcal{I}_\gamma(]0,1[)| \leq\ & \frac{K}{n^l}\, l! \sum_{1\leq j_1 \leq \cdots \leq j_l \leq n} \Bigg[t_{j_1}^{1-\frac{\gamma}{2}} + \sum_{i=0}^{\gamma} t_{j_1}^{1-\frac{i}{2}} \\
& + (t_{j_2}-t_{j_1})^{1-\frac{\gamma}{2}} + \sum_{i=0}^{\gamma} (t_{j_2}-t_{j_1})^{1-\frac{i}{2}} \\
& + \ \ldots \\
& + (t_{j_l}-t_{j_{l-1}})^{1-\frac{\gamma}{2}} + \sum_{i=0}^{\gamma} (t_{j_l}-t_{j_{l-1}})^{1-\frac{i}{2}} \Bigg].
\end{aligned} \tag{8.41}$$

Les sommes sur les indices croissants $j_1,\ldots,j_l$ vont compenser en partie le facteur $l!$. Nous allons examiner ces sommes lorsque le paramètre n devient grand.

L'exposant $1-\dfrac{\gamma}{2}$ ou $1-\dfrac{i}{2}$ peut prendre les valeurs 1, $\dfrac{1}{2}$, 0 ou $-\dfrac{1}{2}$.
Dans les trois premiers cas, on peut majorer chaque facteur de ces sommes par 1, et ainsi chaque somme se comporte comme $\dfrac{n^l}{l!}$. Par conséquent, on a :

$$\sup_{n,l} \frac{|\mathcal{I}_\gamma(]0,1[)|}{l} < \infty \qquad (\gamma \in \{1,2\}). \tag{8.42}$$

Dans le cas $\gamma = 3$, les sommes apparaissant dans le membre de droite de (8.41) sont des sommes de Riemann qui approximent des intégrales sur le domaine $\{(x_1,\ldots,x_l) \in$

$\mathbb{R}^l\ ; 0 < x_1 < \ldots < x_l < 1\}$:

$$\frac{1}{n^l} \sum_{1 \le j_1 \le \cdots \le j_l \le n} \left(\frac{j_1}{n}\right)^{-\frac{1}{2}} \simeq \int_{0<x_1<\ldots<x_l<1} x_1^{-\frac{1}{2}} dx_1 \ldots dx_l \le \frac{2}{(l-1)!}, \tag{8.43}$$

et pour tout entier q compris entre 2 et l :

$$\frac{1}{n^l} \sum_{1 \le j_1 \le \cdots \le j_l \le n} \left(\frac{j_q - j_{q-1}}{n}\right)^{-\frac{1}{2}} \simeq \int_{0<x_1<\ldots<x_l<1} (x_q - x_{q-1})^{-\frac{1}{2}}\, dx_1 \ldots dx_l \le \frac{2}{(l-1)!}\ ; \tag{8.44}$$

cette dernière inégalité est obtenue en intégrant par parties l'intégrale en x_{q-1}.
Par conséquent, (8.41), (8.43) et (8.44) donnent :

$$\sup_{n,l} \frac{|\mathcal{I}_3(]0,1[)|}{l(l+1)} < \infty. \tag{8.45}$$

La conclusion du théorème vient alors de la réunion de (7.2), (8.28), (8.42) et (8.45).
□

8.8 Les moments exponentiels

Cette partie est consacrée à l'estimation de l'erreur commise en approchant $\mathbb{E}\left\{\exp\left(\int_0^1 f(X(s))\, ds\right)\right\}$ par $\mathbb{E}\left\{\exp\left(\frac{1}{n}\sum_{i=1}^{n} f(\bar{X}^\delta(\frac{i}{n}))\right)\right\}$ en δ et n.

Théorème 8.8.1 *Soit f une fonction mesurable et bornée.*
Soit $X(.)$ la solution de (6.1) avec $d = 1$; soit $\bar{X}^\delta(.)$ l'approximation, définie pour tout réel $\delta > 0$, de $X(.)$ par le schéma d'Euler ; $\bar{X}^\delta(.)$ est solution de (6.2).
Alors il existe C_0, C_1 et $(\varepsilon_{\delta,n,l})_{\substack{n,l\in\mathbb{N}^* \\ 0<\delta<1}}$ tels que :

$$\begin{aligned} &\left|\mathbb{E}\left\{\exp\left(\int_0^1 f(X(s))\, ds\right)\right\} - \mathbb{E}\left\{\exp\left(\frac{1}{n}\sum_{i=1}^{n} f(\bar{X}^\delta(\frac{i}{n}))\right)\right\}\right| \\ &\le C_0\, \delta + \left(\sum_{l\ge 1} \frac{1}{(l-1)!}\, \frac{\varepsilon_{\delta,n,l}}{\delta}\right) \delta + \frac{C_1}{n}, \end{aligned} \tag{8.46}$$

C_0 et C_1 sont deux nombres indépendants de l, n et δ et la quantité $(\varepsilon_{\delta,n,l})_{\substack{n,l\in\mathbb{N}^* \\ 0<\delta<1}}$ vérifie les conditions (8.8) et (8.9) du théorème 8.2.1 :

$$\sup_{\delta,n,l} \frac{|\varepsilon_{\delta,n,l}|}{l^2(\|f\|_\infty^l \vee 1)} < \infty, \tag{8.8}$$

$$\sup_{n,l} \lim_{\delta\to 0} \frac{|\varepsilon_{\delta,n,l}|}{\delta} = 0. \tag{8.9}$$

Démonstration du théorème 8.8.1.

La démonstration du résultat est basée sur un développement en série de l'exponentielle et sur une exploitation des résultats obtenus dans la section précédente.

Soit p un entier positif,

$$\begin{aligned}
&\left|\mathbb{E}\left\{\sum_{l=1}^{p}\frac{1}{l!}\left(\int_0^1 f(X(s))\,ds\right)^l\right\}-\mathbb{E}\left\{\sum_{l=1}^{p}\frac{1}{l!}\left(\frac{1}{n}\sum_{i=1}^{n}f(\bar{X}^{\delta}(\frac{i}{n}))\right)^l\right\}\right|\\
&\leq\left|\sum_{l=1}^{p}\frac{1}{l!}\left[\mathbb{E}\left\{\left(\int_0^1 f(X(s))\,ds\right)^l\right\}-\mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^{n}f(X(\frac{i}{n}))\right)^l\right\}\right]\right|\\
&\quad+\left|\sum_{l=1}^{p}\frac{1}{l!}\left[\mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^{n}f(\bar{X}^{\delta}(\frac{i}{n}))\right)^l\right\}-\mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^{n}f(X(\frac{i}{n}))\right)^l\right\}\right]\right|.
\end{aligned}\tag{8.47}$$

Le second terme du membre de droite de (8.47) est contrôlé par (8.20).

$$\begin{aligned}
&\sum_{l=1}^{p}\frac{1}{l!}\left[\mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^{n}f(\bar{X}^{\delta}(\frac{i}{n}))\right)^l\right\}-\mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^{n}f(X(\frac{i}{n}))\right)^l\right\}\right]\\
&=\sum_{l=1}^{p}\frac{1}{l!}[C_{n,l}\;\delta+\varepsilon_{\delta,n,l}],
\end{aligned}\tag{8.48}$$

et par suite,

$$\begin{aligned}
&\left|\sum_{l=1}^{p}\frac{1}{l!}\left[\mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^{n}f(\bar{X}^{\delta}(\frac{i}{n}))\right)^l\right\}-\mathbb{E}\left\{\left(\frac{1}{n}\sum_{i=1}^{n}f(X(\frac{i}{n}))\right)^l\right\}\right]\right|\\
&\leq 2\exp(1)\left(\sup_{n,l}\frac{|C_{n,l}|}{l^2}\right)\;\delta+\left(\sum_{l\geq 1}\frac{1}{(l-1)!}\;\frac{\varepsilon_{\delta,n,l}}{\delta}\right)\;\delta.
\end{aligned}\tag{8.49}$$

La convergence du premier terme est quant à elle démontrée dans le lemme suivant :

Lemme 8.8.1 *Soit f une fonction mesurable et bornée.*
Soit $X(.)$ la solution de (6.1).
Alors pour tout entier $p>0$ et tout entier $l>0$, il existe C_1 tel que :

$$\left|\mathbb{E}\left\{\sum_{l=1}^{p}\frac{1}{l!}\left(\int_0^1 f(X(s))\,ds\right)^l\right\}-\mathbb{E}\left\{\sum_{l=1}^{p}\frac{1}{l!}\left(\frac{1}{n}\sum_{i=1}^{n}f\left(X(\frac{i}{n})\right)\right)^l\right\}\right|\leq\frac{C_1}{n}.\tag{8.50}$$

Démonstration du lemme 8.8.1.

Si p_n désigne une suite d'entiers positifs, on peut écrire :

$$\begin{aligned}
&\mathbb{E}\left\{\sum_{l=1}^{p}\frac{1}{l!}\left(\int_0^1 f(X(s))\,ds\right)^l\right\}-\mathbb{E}\left\{\sum_{l=1}^{p}\frac{1}{l!}\left(\frac{1}{n}\sum_{i=1}^{n} f\left(X(\frac{i}{n})\right)\right)^l\right\}\\
&=\sum_{l=1}^{p}\frac{1}{l!}\int_0^1\dots\int_0^1 \mathbb{E}\left\{f(X(s_1))\dots f(X(s_l))\right\}\,ds_1\dots ds_l\\
&\quad-\frac{1}{n^l}\sum_{i_l=1}^{n}\cdots\sum_{i_1=1}^{n}\mathbb{E}\left\{f\left(X(\frac{i_1}{n})\right)\dots f\left(X(\frac{i_l}{n})\right)\right\}\\
&=\sum_{l=1}^{p\wedge(p_n-1)}\frac{1}{l!}D_{n,l}+\sum_{p_n\le l\le p}\frac{1}{l!}D_{n,l}.
\end{aligned}\tag{8.51}$$

On utilise (8.13) pour majorer la somme dont l'indice l va de zéro à $p\wedge(p_n-1)$.

$$\sum_{l=1}^{p_n-1}\frac{1}{l!}|D_{n,l}|\le\|f\|_\infty\exp(\|f\|_\infty)\left(\frac{C_1}{n}+(C_2+C_3\|f\|_\infty)\frac{1}{n}\right)\qquad\left(p_n\le\frac{n}{6}\right).\tag{8.52}$$

Pour la somme allant de p_n à p, on choisit p_n tel que l'on puisse se contenter de la majoration triviale $|D_{n,l}|\le 2\|f\|_\infty^l$; ainsi :

$$\sum_{l=p_n}^{p}\frac{1}{l!}|D_{n,l}|\le 2\sum_{l=p_n}^{p}\frac{\|f\|_\infty^l}{l!}\le 2\exp(\|f\|_\infty)\frac{\|f\|_\infty^{p_n}}{p_n!}.\tag{8.53}$$

On remarque que ces deux majorations (8.52) et (8.53) sont indépendantes de p. En les réunissant dans (8.51), on obtient :

$$\begin{aligned}
&\left|\mathbb{E}\left\{\sum_{l=1}^{p}\frac{1}{l!}\left(\int_0^1 f(X_s)\,ds\right)^l\right\}-\mathbb{E}\left\{\sum_{l=1}^{p}\frac{1}{l!}\left(\frac{1}{n}\sum_{i=1}^{n} f\left(X(\frac{i}{n})\right)\right)^l\right\}\right|\\
&\le 2\exp(\|f\|_\infty)\frac{\|f\|_\infty^{p_n}}{p_n!}+\ \|f\|_\infty\exp(\|f\|_\infty)\left(\frac{C_1}{n}+(C_2+C_3\|f\|_\infty)\frac{1}{n}\right).
\end{aligned}\tag{8.54}$$

Il reste à choisir la suite $(p_n)_{n>0}$ pour préciser la vitesse de convergence en n. La seule restriction concernant cette suite est la majoration $p_n\le\frac{n}{6}$. En choisissant cette borne pour p_n, on obtient

$$\begin{aligned}
&\left|\mathbb{E}\left\{\sum_{l=1}^{p}\frac{1}{l!}\left(\int_0^1 f(X(s))\,ds\right)^l\right\}-\mathbb{E}\left\{\sum_{l=1}^{p}\frac{1}{l!}\left(\frac{1}{n}\sum_{i=1}^{n} f(\bar{X}^\delta(\frac{i}{n}))\right)^l\right\}\right|\\
&\le 2\exp(\|f\|_\infty)\frac{\|f\|_\infty^{\frac{n}{6}}}{(\frac{n}{6})!}+\ \|f\|_\infty\exp(\|f\|_\infty)\left(\frac{C_1}{n}+(C_2+C_3\|f\|_\infty)\frac{1}{n}\right)
\end{aligned}\tag{8.55}$$

et la conclusion (8.50) du lemme. □

Remarque :

Le terme $\dfrac{\|f\|_\infty^{\frac{n}{6}}}{(\frac{n}{6})!}$ est négligeable devant les autres termes du membre de droite de (8.55) lorsque n tend vers l'infini : par exemple, dès que $n \geq 12\ \|f\|_\infty\ e$, on a la majoration suivante :

$$\frac{\|f\|_\infty^{\frac{n}{6}}}{(\frac{n}{6})!} \leq \frac{1}{2^{\frac{n}{6}}}. \tag{8.56}$$

En injectant (8.48) et (8.55) dans (8.47), on obtient l'existence de trois nombres C_1, C_2 et C_3 tels que :

$$\begin{aligned}
&\left|\mathbb{E}\left\{\sum_{l=1}^{p}\frac{1}{l!}\left(\int_0^1 f(X(s))\,ds\right)^l\right\} - \mathbb{E}\left\{\sum_{l=1}^{p}\frac{1}{l!}\left(\frac{1}{n}\sum_{i=1}^{n} f(\bar{X}^\delta(\frac{i}{n}))\right)^l\right\}\right| \\
&\leq 2\exp(1)\left(\sup_{n,l}\frac{|C_{n,l}|}{l^2}\right)\ \delta + \left(\sum_{l\geq 1}\frac{1}{(l-1)!}\ \frac{\varepsilon_{\delta,n,l}}{\delta}\right)\ \delta \\
&\quad + 2\exp(\|f\|_\infty)\frac{\|f\|_\infty^{\frac{n}{6}}}{(\frac{n}{6})!} + \ \|f\|_\infty \exp(\|f\|_\infty)\left(\frac{C_1}{n} + (C_2 + C_3\|f\|_\infty)\frac{1}{n}\right).
\end{aligned} \tag{8.57}$$

Le membre de droite est indépendant de p. En faisant tendre p vers $+\infty$ dans (8.57) et en ne conservant que les termes dominants, on obtient la conclusion (8.46). □

9

Majoration en $\delta = \frac{1}{n}$ de l'erreur pour l'intégrale de la trajectoire

Dans la section précédente, nous avons présenté un développement en δ de l'erreur. Il a été montré que le coefficient $R(\tilde{\Phi}(n,l))$ devant δ est borné par rapport à n. Mais ceci ne suffit pas pour obtenir la vitesse de convergence de l'erreur puisque l'on ne contrôle pas le reste $\frac{\varepsilon_{\delta,n,l}}{\delta}$ uniformément en n.
Nous allons reprendre l'étude de l'erreur en faisant tendre, en même temps et à la même vitesse ($\delta = \frac{1}{n}$), les paramètres δ vers zéro et n vers l'infini.

Théorème 9.1.2 *Soit f une fonction mesurable et bornée.*
Soit $X(.)$ la solution de (6.1) avec $d \in \mathbb{N}^*$; soit $\bar{X}^{\frac{1}{n}}(.)$ l'approximation de cette solution donnée par le schéma d'Euler .
Alors il existe C tel que :

$$\begin{aligned}&\left|\mathbb{E}\left\{\int_0^1 f(X(s))\,ds\right\} - \mathbb{E}\left\{\frac{1}{n}\sum_{i=1}^{n} f(\bar{X}^{\frac{1}{n}}(\frac{i}{n}))\right\}\right| \\ &\leq C\ \|f\|_\infty \frac{1}{n}.\end{aligned} \tag{9.1}$$

Démonstration du théorème 9.1.2.

On applique l'inégalité triangulaire (8.10) au membre de gauche de (9.1), puis la conclusion (8.13) du lemme 8.3.1 et enfin celle (9.3) du théorème 9.1.3 ci-dessous. Ainsi il existe C_1 et C_2 tels que :

$$\begin{aligned}&\left|\mathbb{E}\left\{\int_0^1 f(X(s))\,ds\right\} - \mathbb{E}\left\{\frac{1}{n}\sum_{i=1}^{n} f(\bar{X}^{\frac{1}{n}}(\frac{i}{n}))\right\}\right| \\ &\leq \|f\|_\infty \left(C_1\ \frac{1}{n} + \ C_2\ \frac{1}{n}\right).\end{aligned} \tag{9.2}$$

□

Théorème 9.1.3 *Soit f une fonction mesurable et bornée.*
Soit $X(.)$ la solution de (6.1) ; soit $\bar{X}^{\frac{1}{n}}(.)$ l'approximation de cette solution donnée par le schéma d'Euler .
Alors il existe un réel C tel que pour tous les entiers strictement positifs n :

$$\left| \mathbb{E}\left\{ \frac{1}{n} \sum_{i=1}^{n} f(X(\frac{i}{n})) \right\} - \mathbb{E}\left\{ \frac{1}{n} \sum_{i=1}^{n} f(\bar{X}^{\frac{1}{n}}(\frac{i}{n})) \right\} \right| \leq C \, \|f\|_\infty \, \frac{1}{n}. \tag{9.3}$$

Démonstration du théorème 9.1.3.

Le théorème 7.2.1 donne une expression de la dérivée de l'erreur comprenant deux termes où interviennent les dérivées première et seconde de la fonction $H_{r,n}$. Ces deux termes sont de la forme suivante, où tantôt $g = b$ et $\gamma = 1$, tantôt $g = \frac{a}{2}$ et $\gamma = 2$:

$$I = \int_0^1 \mathbb{E}\left\{ \left[g(\bar{X}^\delta(r)) - g(\bar{X}^\delta(r^{(\delta)})) \right] \frac{\partial^\gamma H_{r,n}}{\partial z^\gamma}(\bar{X}^\delta(t_1), \ldots, \bar{X}^\delta(t_{n_r}), \bar{X}^\delta(r)) \right\} dr. \tag{9.4}$$

Notation 9.1.2 *Pour $\gamma = 1$ ou 2, on pose :*

$$Q_\gamma^{(n)}(i,j,r) = \mathbb{E}\left\{ \int_{\mathbb{R}} f(x_j) \left[g_\gamma(\bar{X}(r)) - g_\gamma(\bar{X}(\frac{i}{n})) \right] \frac{\partial^\gamma}{\partial z^\gamma} \left(q_{r,\frac{j}{n}}(z,x_j) \right) \Big|_{z=\bar{X}(r)} dx_j \right\} \tag{9.5}$$

où les entiers i, j, n et le réel r vérifient les conditions :
$n \geq 1$ et $0 \leq i/n < r < (i+1)/n \leq j/n \leq 1$;
g_1 et g_2 désignent respectivement les fonctions b et $\dfrac{a}{2}$.

Avec ces notations, l'erreur commise en utilisant le schéma d'Euler de pas $1/n$ est la somme des intégrales $\dfrac{1}{n} \displaystyle\int_{i/n}^{(i+1)/n} Q_1^{(n)}(i,j,r)\, dr$ et $\dfrac{1}{n} \displaystyle\int_{i/n}^{(i+1)/n} Q_2^{(n)}(i,j,r)\, dr$ où (i,j) décrivent $\Delta(n) = \{(i,j) \; ; \; 0 \leq i \leq j-1, \, 1 \leq j \leq n\}$.
Ce domaine peut être découpé en deux parties :

$$\begin{aligned} \Delta_1(n) &= \bigcup_{p=2}^{k+1} \left\{ (i,j) \; ; \; 0 \leq i \leq 2^{p-1} - 2, \, 2^{p-1} \leq j \leq 2^p - 1 \right\}, \\ \Delta_2(n) &= \bigcup_{p=2}^{k+1} \left\{ (i,j) \; ; \; 2^{p-1} - 1 \leq i \leq j - 1, \, 2^{p-1} \leq j \leq 2^p - 1 \right\}. \end{aligned}$$

où $k = \left\lfloor \dfrac{\log n}{\log 2} \right\rfloor$. ($\lfloor x \rfloor$ désigne la partie entière de x)
Nous avons : $\Delta(n) \subset (\Delta_1(n) \cup \Delta_2(n))$.

Les deux lemmes suivants donnent des majorations de $Q_\gamma^{(n)}(i,j,r)$. Le Lemme 9.1.3 sera utile pour la somme sur $\Delta_1(n)$, le Lemme 9.1.4 pour celle sur $\Delta_2(n)$.

Lemme 9.1.3 *Il existe un entier n_0 et une constante K_1 indépendante de n, i, j et r tels que :*

$$|Q_\gamma^{(n)}(i,j,r)| \leq K_1 \, ||f||_\infty \frac{\left(r - \frac{i}{n}\right)}{\left(\frac{j}{n} - r\right)^{\frac{\gamma+1}{2}}} \qquad (\gamma \in \{1,2\}) \quad (n > n_0) \tag{9.6}$$

Lemme 9.1.4 *Il existe un entier n_0 et une constante K_2 indépendante de n, i, j et r tels que :*

$$|Q_\gamma^{(n)}(i,j,r)| \leq K_2 \, ||f||_\infty \frac{\left(r - \frac{i}{n}\right)}{\sqrt{\frac{j}{n} - r}} \left(\frac{n}{i}\right)^{\frac{\gamma}{2}} \qquad (\gamma \in \{1,2\}) \quad (n > n_0) \tag{9.7}$$

Montrons ici que ces deux majorations permettent de conclure.

Somme sur $\Delta_1(n)$

Comme $r - i/n$ et $j/n - r$ sont plus petit que 1, le terme dominant dans la majoration obtenue dans le Lemme 9.1.3 provient du cas $\gamma = 2$ et son ordre est $(r - i/n)(j/n - r)^{-3/2}$. Ainsi il existe une constante $\tilde{K}_1$ indépendante de n, i, j et r telle que pour tout $n > n_0$:

$$\begin{aligned}
\tilde{K}_1^{-1} \frac{1}{n} \sum_{(i,j)\in\Delta_1(n)} \left| \int_{i/n}^{(i+1)/n} Q_\gamma^{(n)}(i,j,r)\, dr \right| &\leq \frac{1}{n} \sum_{(i,j)\in\Delta_1(n)} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \frac{r - \frac{i}{n}}{\left(\frac{j}{n} - r\right)^{\frac{3}{2}}}\, dr \\
&\leq \frac{1}{n^2} \sum_{p=2}^{k+1} \sum_{j=2^{p-1}}^{2^p-1} \sum_{i=0}^{2^{p-1}-2} \left[\frac{2}{\sqrt{\frac{j}{n} - r}} \right]_{\frac{i}{n}}^{\frac{i+1}{n}} \\
&\leq \frac{2}{n^{\frac{3}{2}}} \sum_{p=2}^{k+1} \sum_{j=2^{p-1}}^{2^p-1} \frac{1}{\sqrt{j - 2^{p-1} + 1}} - \frac{1}{\sqrt{j}} \\
&\leq \frac{2}{n^{\frac{3}{2}}} \sum_{p=2}^{k+1} \left(\int_0^{2^{p-1}} \frac{1}{\sqrt{x}}\, dx - \int_{2^{p-1}}^{2^p} \frac{1}{\sqrt{x}}\, dx \right) \\
&\leq \frac{8}{n^{\frac{3}{2}}} \sqrt{2}^k
\end{aligned}$$

Le choix de k nous permet de conclure pour la somme sur $\Delta_1(n)$:

$$\frac{1}{n} \sum_{(i,j)\in\Delta_1(n)} \left| \int_{i/n}^{(i+1)/n} Q_\gamma^{(n)}(i,j,r)\, dr \right| \leq \frac{8\, \tilde{K}_1}{n} \qquad (\gamma \in \{1,2\}) \quad (n > n_0) \tag{9.8}$$

Somme sur $\Delta_2(n)$

La majoration obtenue dans le Lemme 9.1.4 va nous permettre de contrôler la somme sur $\Delta_2(n)$ de la manière suivante : le terme prédominant est à nouveau obtenu pour $\gamma = 2$ et l'on obtient l'existence d'une constante $\tilde{K}_2$ indépendante de n, i, j et r telle que pour tout $n > n_0$:

$$\begin{aligned}
\tilde{K}_2^{-1}\frac{1}{n}\sum_{(i,j)\in\Delta_2(n)}\left|\int_{i/n}^{(i+1)/n} Q_\gamma^{(n)}(i,j,r)\,dr\right| &\leq \frac{1}{n}\sum_{(i,j)\in\Delta_2(n)}\int_{\frac{i}{n}}^{\frac{i+1}{n}}\frac{n}{i}\frac{r-\frac{i}{n}}{\sqrt{\frac{j}{n}-r}}\,dr \\
&\leq \frac{1}{n}\sum_{p=2}^{k+1}\sum_{j=2^{p-1}}^{2^p-1}\sum_{i=2^{p-1}-1}^{j-1}\frac{1}{i}\left[-2\sqrt{\frac{j}{n}-r}\right]_{\frac{i}{n}}^{\frac{i+1}{n}} \\
&\leq \frac{2}{n^{\frac{3}{2}}}\sum_{p=2}^{k+1}\sum_{i=2^{p-1}-1}^{2^p-1}\frac{\sqrt{2^p-i-1}}{i} \\
&\leq \frac{2}{n^{\frac{3}{2}}}\left(2\sqrt{2}+\sum_{p=3}^{k+1}\sqrt{2}^{p-1}\int_{2^{p-1}-2}^{2^p-1}\frac{1}{x}\,dx\right) \\
&\leq \frac{2}{n^{\frac{3}{2}}}\left(2\sqrt{2}+\sum_{p=3}^{k+1}\sqrt{2}^{p-1}2\log 2\right) \\
&\leq \frac{4}{n^{\frac{3}{2}}}\left(\sqrt{2}+2\log 2\frac{\sqrt{2}^{k-2}-1}{\sqrt{2}-1}\right)
\end{aligned}$$

Le choix de k nous donne la majoration suivante :

$$\frac{1}{n}\sum_{(i,j)\in\Delta_2(n)}\left|\int_{i/n}^{(i+1)/n} Q_\gamma^{(n)}(i,j,r)\,dr\right| \leq \frac{4\tilde{K}_2\log 2}{\sqrt{2}-1}\,\frac{1}{n}+\frac{4\tilde{K}_2\sqrt{2}}{n^{\frac{3}{2}}} \qquad (\gamma\in\{1,2\}) \quad (n>n_0) \tag{9.9}$$

Les majorations (9.8) et (9.9) donnent un majorant de l'erreur en $\frac{1}{n}$.

Avant de prouver les Lemmes 9.1.3 et 9.1.4, montrons un lemme préliminaire

Notation 9.1.5 *Soit G une variable aléatoire gaussienne centrée réduite. On pose pour tout $x \in \mathbb{R}$:*

$$H(x) = \sigma(x)\sqrt{r-i/n}\,G + b(x)\,(r-i/n)\,, \tag{9.10}$$

$$\Psi(x) = (g_\gamma(x+H(x)) - g_\gamma(x))\,q_{i/n}^{(n)}(x_0,x). \tag{9.11}$$

Lemme 9.1.6 *Sous les hypothèses (H 1) sur les fonctions σ et b, les fonctions H et Ψ vérifient :*
pour tout α, β, β_1, $\beta_2 \in \{0,1,2\}$, q, q_1, $q_2 \in \mathbb{N}$, la suite suivante est bornée :

$$\left(\sup_{(i,r)\in J_n} \int_{\mathbb{R}} \mathbb{E}\left\{ \left(\frac{i}{n}\right)^{\alpha/2} \left| \frac{\Psi^{(\alpha)}(x)}{\sqrt{r-i/n}} \right| \left| \frac{H^{(\beta)}(x)}{\sqrt{r-i/n}} \right|^{q} \left|a^{(\beta_1)}(x)\right|^{q_1} \left|b^{(\beta_2)}(x)\right|^{q_2} \mathbb{1}_{\left\{|G|\le \frac{(r-i/n)^{-1/2}}{2||\sigma'||_\infty}\right\}} \right\} dx \right)_{n\ge 1}$$

où $J_n = \{(i,r) \in \mathbb{N}\times[0,1]\ ;\ i/n < r < (i+1)/n\}$

Preuve du Lemme 9.1.6

D'après la définition (9.11) de Ψ, il est immédiat que :

$$|\Psi(x)| \le |H(x)|\, ||g'_\gamma||_\infty\, q^{(n)}_{i/n}(x_0,x) \tag{9.12}$$

De la même manière, on calcule les deux premières dérivées de Ψ :

$$\begin{aligned} \Psi'(x) &= \big((1+H'(x))g'_\gamma(x+H(x)) - g'_\gamma(x)\big)\, q^{(n)}_{i/n}(x_0,x) \\ &\quad + (g_\gamma(x+H(x)) - g_\gamma(x))\, \frac{\partial q^{(n)}_{i/n}}{\partial x}(x_0,x) \\ \Psi'(x) &= \big(g'_\gamma(x+H(x)) - g'_\gamma(x)\big)\, q^{(n)}_{i/n}(x_0,x) \\ &\quad + H'(x) g'_\gamma(x+H(x)) q^{(n)}_{i/n}(x_0,x) \\ &\quad + (g_\gamma(x+H(x)) - g_\gamma(x))\, \frac{\partial q^{(n)}_{i/n}}{\partial x}(x_0,x). \end{aligned}$$

Ceci nous donne la majoration suivante :

$$|\Psi'(x)| \le |H(x)|\, ||g'_\gamma||_\infty \left| \frac{\partial q^{(n)}_{i/n}}{\partial x}(x_0,x) \right| + \big(|H(x)|\, ||g''_\gamma||_\infty + |H'(x)|\, ||g'_\gamma||_\infty\big)\, q^{(n)}_{i/n}(x_0,x), \tag{9.13}$$

Calculons à présent la dérivée seconde de Ψ :

$$\begin{aligned}
\Psi''(x) \;=\;& \big((1+H'(x))g''_\gamma(x+H(x)) - g''_\gamma(x)\big)\, q^{(n)}_{i/n}(x_0,x) \\
&+ \big(g'_\gamma(x+H(x)) - g'_\gamma(x)\big)\, \frac{\partial q^{(n)}_{i/n}}{\partial x}(x_0,x) \\
&+ H''(x) g'_\gamma(x+H(x)) q^{(n)}_{i/n}(x_0,x) \\
&+ H'(x)(1+H'(x)) g''_\gamma(x+H(x)) q^{(n)}_{i/n}(x_0,x) \\
&+ H'(x) g'_\gamma(x+H(x)) \frac{\partial q^{(n)}_{i/n}}{\partial x}(x_0,x) \\
&+ \big((1+H'(x))g'_\gamma(x+H(x)) - g'_\gamma(x)\big)\, \frac{\partial q^{(n)}_{i/n}}{\partial x}(x_0,x) \\
&+ \left(g_\gamma(x+H(x)) - g_\gamma(x)\right) \frac{\partial^2 q^{(n)}_{i/n}}{\partial x^2}(x_0,x).
\end{aligned}$$

Après factorisation, on obtient :

$$\begin{aligned}
\Psi''(x) \;=\;& \Big(\big(g''_\gamma(x+H(x)) - g''_\gamma(x)\big) + 2H'(x) g''_\gamma(x+H(x)) + \\
& H''(x) g'_\gamma(x+H(x)) + H'(x)^2 g''_\gamma(x+H(x))\Big) q^{(n)}_{i/n}(x_0,x) \\
&+ 2\Big(\big(g'_\gamma(x+H(x)) - g'_\gamma(x)\big) + H'(x) g'_\gamma(x+H(x))\Big) \frac{\partial q^{(n)}_{i/n}}{\partial x}(x_0,x) \\
&+ \Big(g_\gamma(x+H(x)) - g_\gamma(x)\Big) \frac{\partial^2 q^{(n)}_{i/n}}{\partial x^2}(x_0,x).
\end{aligned}$$

Nous obtenons ainsi :

$$\begin{aligned}
|\Psi''(x)| \;\le\;& \Big(|H(x)|\,||g^{(3)}_\gamma||_\infty + |(H'(x)+2)H'(x)|\,||g''_\gamma||_\infty + |H''(x)|\,||g'_\gamma||_\infty\Big)\, q^{(n)}_{i/n}(x_0,x) \\
&+2\left(|H(x)|\,||g''_\gamma||_\infty + |H'(x)|\,||g'_\gamma||_\infty\right) \left| \tfrac{\partial q^{(n)}_{i/n}}{\partial x}(x_0,x)\right| + |H(x)|\,||g'_\gamma||_\infty \left| \tfrac{\partial^2 q^{(n)}_{i/n}}{\partial x^2}(x_0,x)\right|
\end{aligned} \tag{9.14}$$

Dans un premier temps, on peut remplacer dans (9.13) et (9.14) les dérivées d'ordre κ du noyau $q^{(n)}_{i/n}(x_0,x)$ par les majorations données dans (8.2) : $C\left(\frac{n}{i}\right)^{\kappa/2} p_{Mi/n}(x-x_0)$. ($p_{\sigma^2}(x)$ désigne la densité d'une gaussienne centrée de variance σ^2).

Par ailleurs, sur l'événement $\left\{|G| \le \Big(2||\sigma'||_\infty \sqrt{r - i/n}\Big)^{-1}\right\}$, on obtient les majorations des dérivées de H :

$$H'(x) = \sigma'(x)(r-i/n)^{1/2} G + b'(x)(r-i/n)$$

$$|H'(x)| \le \frac{1}{2} + \frac{||b'||_\infty}{n} \tag{9.15}$$

On obtient également
$$|H''(x)| \leq \frac{||\sigma''||_\infty}{2||\sigma'||_\infty} + \frac{||b''||_\infty}{n} \tag{9.16}$$
On sait donc que H' et H'' sont bornées sur l'événement considéré.

On en déduit l'existence d'une constante C indépendante de i,n,r tel que pour tout $\alpha \in \{0,1,2\}$ et pour n assez grand, on ait :
$$\left(\frac{i}{n}\right)^{\alpha/2} \left|\Psi^{(\alpha)}(x)\right| \leq C\,|H(x)|\,||g'_\gamma||_\infty p_{Mi/n}(x-x_0), \tag{9.17}$$
ainsi, la famille du Lemme 9.1.6 est majorée par la famille suivante :
$$\Bigg(C\,||g'_\gamma||_\infty \sup_{(i,r)\in J_n} \int_{\mathbb{R}} \mathbb{E}\left\{ \left|\frac{H(x)}{\sqrt{r-i/n}}\right| \left|\frac{H^{(\beta)}(x)}{\sqrt{r-i/n}}\right|^q \left|a^{(\beta_1)}(x)\right|^{q_1} \left|b^{(\beta_2)}(x)\right|^{q_2} \right\} p_{Mi/n}(x-x_0)\,dx \Bigg)_{n\geq 1}$$
Grâce à un changement de variable, l'intégrale précédente s'écrit :
$$\int_{\mathbb{R}^2} \left|\sigma(x_0+x\sqrt{\frac{Mi}{n}})y + b(x_0+x\sqrt{\frac{Mi}{n}})\sqrt{r-\frac{i}{n}}\right| \left|a^{(\beta_1)}(x_0+x\sqrt{\frac{Mi}{n}})\right|^{q_1}$$
$$\times \left|\sigma^{(\beta)}(x_0+x\sqrt{\frac{Mi}{n}})y + b^{(\beta)}(x_0+x\sqrt{\frac{Mi}{n}})\sqrt{r-\frac{i}{n}}\right|^q$$
$$\times \left|b^{(\beta_2)}(x_0+x\sqrt{\tfrac{Mi}{n}})\right|^{q_2} p_1(x)p_1(y)\,dx\,dy$$
où p_1 désigne la densité de la gaussienne centrée réduite. Le noyau $p_1(x)p_1(y)$ assure que l'intégrale est uniformément bornée en r, i et n.

Remarque 9.1.7 *Une petite variante dans la preuve précédente nous permet de conclure que sous les hypothèses du Lemme 9.1.6, la famille suivante est également bornée :*
$$\Bigg(\sup_{(i,r)\in J_n} \int_{\mathbb{R}} \mathbb{E}\Bigg\{ \left(\frac{i}{n}\right)^{\alpha/2} \left|\frac{\partial^\alpha p^{(n)}_{i/n}}{\partial x^\alpha}(x_0,x)\right| \left|\frac{H^{(\beta)}(x)}{\sqrt{r-i/n}}\right|^q \left|a^{(\beta_1)}(x)\right|^{q_1} \left|b^{(\beta_2)}(x)\right|^{q_2} \mathbb{1}_{\left\{|G|\leq \frac{(r-i/n)^{-1/2}}{2||\sigma'||_\infty}\right\}} \Bigg\} dx \Bigg)_{n\geq 1}$$
Montrons à présent les Lemmes 9.1.3 et 9.1.4.

Preuve du Lemme 9.1.3

Nous allons écrire l'expression (9.5) de $Q_\gamma^{(n)}(i,j,r)$ en utilisant une variable aléatoire gaussienne : en effet, conditionnellement à $\bar{X}(\frac{i}{n}) = x$, la *v.a.* $\bar{X}(r)$ s'écrit :

$$x + \underbrace{\sigma(x)\sqrt{r-\frac{i}{n}}\,G + b(x)\left(r-\frac{i}{n}\right)}_{H(x)}$$

où G désigne une gaussienne centrée réduite.
L'expression (9.5) de $Q_\gamma^{(n)}(i,j,r)$ devient :

$$\begin{aligned} Q_\gamma^{(n)}(i,j,r) &= \int_{\mathbb{R}} \mathbb{E}\Bigg\{ \int_{\mathbb{R}} f(x_j)\,[g_\gamma(x+H(x)) - g_\gamma(x)] \\ &\quad \frac{\partial^\gamma}{\partial z^\gamma}\left(q_{r,\frac{j}{n}}(z,x_j)\right)\Bigg|_{z=x+H(x)} dx_j \Bigg\} q_{\frac{i}{n}}^{(n)}(x_0,x)\,dx \end{aligned} \tag{9.18}$$

Écrivons le développement de Taylor avec reste intégrale des fonctions g_γ et $z \longmapsto \frac{\partial^\gamma}{\partial z^\gamma}\left(q_{r,\frac{j}{n}}(z,x_j)\right)$, nous obtenons alors :

$$\begin{aligned} Q_\gamma^{(n)}(i,j,r) &= \int_{\mathbb{R}} \mathbb{E}\Bigg\{ \int_{\mathbb{R}} f(x_j)\left[g_\gamma'(x)\,H(x) + \int_x^{x+H(x)} g_\gamma''(u)\,(x+H(x)-u)\,du\right] \\ &\left[\frac{\partial^\gamma}{\partial x^\gamma}\left(q_{r,\frac{j}{n}}(x,x_j)\right) + \int_x^{x+H(x)} \frac{\partial^{\gamma+1}}{\partial u^{\gamma+1}}\left(q_{r,\frac{j}{n}}(u,x_j)\right)du\right] dx_j\Bigg\} q_{\frac{i}{n}}^{(n)}(x_0,x)\,dx \end{aligned} \tag{9.19}$$

Le développement du produit des deux crochets donne quatre termes T_1, T_2, T_3 et T_4.
Dans T_1, une simplification apparaît grâce à la symétrie de G.

$$\begin{aligned} T_1 &= \int_{\mathbb{R}} \mathbb{E} \int_{\mathbb{R}} f(x_j) g_\gamma'(x) H(x) \frac{\partial^\gamma}{\partial x^\gamma}\left(q_{r,\frac{j}{n}}(x,x_j)\right) dx_j\, q_{\frac{i}{n}}^{(n)}(x_0,x)\,dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(x_j) g_\gamma'(x) b(x)\left(r-\frac{i}{n}\right) \frac{\partial^\gamma}{\partial x^\gamma}\left(q_{r,\frac{j}{n}}(x,x_j)\right) dx_j\, q_{\frac{i}{n}}^{(n)}(x_0,x)\,dx \end{aligned} \tag{9.20}$$

On obtient alors une majoration de T_1 :

$$|T_1| \leq K||f||_\infty \left(r-\frac{i}{n}\right) \frac{1}{\left(\frac{j}{n}-r\right)^{\frac{\gamma}{2}}} \tag{9.21}$$

Intéressons nous maintenant à T_2 :

$$\begin{aligned}
T_2 &= \int_{\mathbb{R}} \mathbb{E} \int_{\mathbb{R}} f(x_j) g'_\gamma(x) H(x) \int_x^{x+H(x)} \frac{\partial^{\gamma+1}}{\partial u^{\gamma+1}} \left(q_{r,\frac{j}{n}}(u,x_j) \right) du\, dx_j\, q^{(n)}_{\frac{i}{n}}(x_0,x)\, dx \\
|T_2| &\leq \frac{||f||_\infty ||g'_\gamma||_\infty}{\left(\frac{j}{n}-r\right)^{\frac{\gamma+1}{2}}} \int_{\mathbb{R}} \mathbb{E}\left(H^2(x)\right) q^{(n)}_{\frac{i}{n}}(x_0,x)\, dx \\
&\leq K||f||_\infty \left(r-\frac{i}{n}\right) \frac{1}{\left(\frac{j}{n}-r\right)^{\frac{\gamma+1}{2}}} \qquad (9.22)
\end{aligned}$$

Nous obtenons pour T_3 et T_4 :

$$\begin{aligned}
T_3 &= \int_{\mathbb{R}} \mathbb{E} \int_{\mathbb{R}} f(x_j) \frac{\partial^\gamma}{\partial x^\gamma} \left(q_{r,\frac{j}{n}}(x,x_j) \right) \\
&\quad \int_x^{x+H(x)} g''_\gamma(u)\, (x+H(x)-u)\; du\, dx_j\, q^{(n)}_{\frac{i}{n}}(x_0,x)\, dx \\
|T_3| &\leq K||f||_\infty \left(r-\frac{i}{n}\right) \frac{1}{\left(\frac{j}{n}-r\right)^{\frac{\gamma}{2}}} \qquad (9.23)
\end{aligned}$$

$$\begin{aligned}
T_4 &= \int_{\mathbb{R}} \mathbb{E} \int_{\mathbb{R}} f(x_j) \int_x^{x+H(x)} \frac{\partial^{\gamma+1}}{\partial u^{\gamma+1}} \left(q_{r,\frac{j}{n}}(u,x_j) \right) du \\
&\quad \int_x^{x+H(x)} g''_\gamma(u)\, (x+H(x)-u)\, du\, dx_j\, q^{(n)}_{\frac{i}{n}}(x_0,x)\, dx \\
|T_4| &\leq K||f||_\infty \left(r-\frac{i}{n}\right)^{\frac{3}{2}} \frac{1}{\left(\frac{j}{n}-r\right)^{\frac{\gamma+1}{2}}} \qquad (9.24)
\end{aligned}$$

Grâce aux inégalités (9.21),(9.22),(9.23) et (9.24), nous avons terminé la preuve du Lemme 9.1.3.

Preuve du Lemme 9.1.4

La majoration obtenue dans le Lemme 9.1.3 n'est pas bonne lorsque i et j sont trop proches ; on ne peut pas intégrer en r à cause du terme $\frac{1}{(j/n-r)^{3/2}}$. Le but de ce Lemme 9.1.4 est de trouver une majoration de $Q^{(n)}_\gamma(i,j,r)$ qui soit 'bonne' pour les indices i et j dans $\Delta_2(n)$.
Dans ce cas, nous allons grâce à une intégration par partie faire porter l'irrégularité due à la dérivée sur la transition entre l'instant initial et l'instant $\frac{i}{n}$.
Nous allons écrire la preuve complète dans le cas $\gamma = 2$. (le cas $\gamma = 1$ étant similaire et plus simple)

On découpe $Q_2^{(n)}(i,j,r)$ en une somme de deux termes suivant que l'événement $\left\{|G| \leq \left(2||\sigma'||_\infty\sqrt{r-i/n}\right)^{-1}\right\}$ est réalisé ou non.

$$Q_2^{(n)}(i,j,r) = Q_{2,1} + Q_{2,2} \tag{9.25}$$

En suivant l'écriture de (9.18), on obtient :

$$Q_{2,1} = \mathbb{E}\left\{\int_{\mathbb{R}^2} f(x_j)\frac{\partial^2}{\partial z^2}\left(q_{r,\frac{j}{n}}(z,x_j)\right)\Big|_{z=x+H(x)} \Psi(x)\mathbb{1}_{\left\{|G|\leq \frac{||\sigma'||_\infty^{-1}}{2\sqrt{r-\frac{i}{n}}}\right\}} dx_j\right\} dx \tag{9.26}$$

$$\begin{aligned} Q_{2,2} \;=\; & \int_{\mathbb{R}^3} f(x_j)\,[g_2(z)-g_2(x)]\,\frac{\partial^2}{\partial z^2}\left(q_{r,\frac{j}{n}}(z,x_j)\right) \\ & \mathbb{1}_{B(i,n,r,x)}(z)p_{\sigma^2(x)(r-i/n)}(z-x-b(x)(r-i/n))\,dz\,dx_j\,q^{(n)}_{\frac{i}{n}}(x_0,x)\,dx \end{aligned} \tag{9.27}$$

avec $B(i,n,r,x) = \left\{z \in \mathbb{R}\ ;\ \left|\frac{z-x-b(x)\left(r-\frac{i}{n}\right)}{\sigma(x)}\right| > \frac{1}{2||\sigma'||_\infty}\right\}$.

Majoration de $|Q_{2,1}|$

D'abord on scinde l'expression (9.26) de $Q_{2,1}$ en une somme de trois termes.

$$\begin{aligned} Q_{2,1} \;=\; & \mathbb{E}\left\{\int_{\mathbb{R}^2}\frac{\partial^2 q_{r,\frac{j}{n}}}{\partial x^2}(x,x_j)g_2'(x)H(x)\mathbb{1}_{\left\{|G|\leq \frac{||\sigma'||_\infty^{-1}}{2\sqrt{r-\frac{i}{n}}}\right\}} q^{(n)}_{\frac{i}{n}}(x_0,x)\,dx\,f(x_j)\,dx_j\right\} \\ & +\mathbb{E}\left\{\int_{\mathbb{R}^2}\frac{\partial^2 q_{r,\frac{j}{n}}}{\partial x^2}(x,x_j)\int_x^{x+H(x)} g_2''(u)(x+H(x)-u)\,du\,\mathbb{1}_{\left\{|G|\leq \frac{||\sigma'||_\infty^{-1}}{2\sqrt{r-\frac{i}{n}}}\right\}} q^{(n)}_{\frac{i}{n}}(x_0,x)\,dx\,f(x_j)\,dx_j\right\} \\ & +\mathbb{E}\left\{\int_{\mathbb{R}^2}\left(\frac{\partial^2}{\partial z^2}\left(q_{r,\frac{j}{n}}(z,x_j)\right)\Big|_{z=x+H(x)} - \frac{\partial^2 q_{r,\frac{j}{n}}}{\partial x^2}(x,x_j)\right)\mathbb{1}_{\left\{|G|\leq \frac{||\sigma'||_\infty^{-1}}{2\sqrt{r-\frac{i}{n}}}\right\}}\Psi(x)\,dx\,f(x_j)\,dx_j\right\} \end{aligned} \tag{9.28}$$

Avec des notations évidentes, $Q_{2,1} = I_1 + I_2 + I_3$.
Dans I_1, on utilise l'expression (9.10) de H.
On remarque que la *v.a.* $G\,\mathbb{1}_{\left\{|G|\leq\left(2||\sigma'||_\infty\sqrt{r-\frac{i}{n}}\right)^{-1}\right\}}$ est centrée et il vient :

$$I_1 = \mathbb{E}\left\{\int_{\mathbb{R}^2}\frac{\partial^2 q_{r,\frac{j}{n}}}{\partial x^2}(x,x_j)g_2'(x)(r-\frac{i}{n})b(x)\mathbb{1}_{\left\{|G|\leq \frac{||\sigma'||_\infty^{-1}}{2\sqrt{r-\frac{i}{n}}}\right\}} q^{(n)}_{\frac{i}{n}}(x_0,x)\,dx\,f(x_j)\,dx_j\right\}$$

Intégrons par parties en x :

$$\begin{aligned} I_1 &= -\mathbb{E}\left\{ \int_{\mathbb{R}^2} \frac{\partial q_{r,\frac{j}{n}}}{\partial x}(x,x_j)(r-\frac{i}{n})g_2'(x)b(x)\mathbb{1}_{\left\{|G|\leq \frac{||\sigma'||_\infty^{-1}}{2\sqrt{r-\frac{i}{n}}}\right\}} \frac{\partial q^{(n)}_{\frac{i}{n}}}{\partial x}(x_0,x)\,dx\,f(x_j)\,dx_j \right\} \\ & -\mathbb{E}\left\{ \int_{\mathbb{R}^2} \frac{\partial q_{r,\frac{j}{n}}}{\partial x}(x,x_j)(r-\frac{i}{n})(g_2''(x)b(x)+g_2'(x)b'(x)) \right. \\ & \left. \qquad \times \mathbb{1}_{\left\{|G|\leq \frac{||\sigma'||_\infty^{-1}}{2\sqrt{r-\frac{i}{n}}}\right\}} q^{(n)}_{\frac{i}{n}}(x_0,x)\,dx\,f(x_j)\,dx_j \right\} \end{aligned}$$

La remarque 9.1.7 et les majorations sur les dérivées des noyaux permettent d'obtenir le contrôle suivant sur I_1 : il existe une constante K indépendant de i, j, n et r telle que :

$$|I_1| \leq \frac{K\,||f||_\infty}{\sqrt{\frac{j}{n}-r}}\left(r-\frac{i}{n}\right)\left(1+\sqrt{\frac{n}{i}}\right) \tag{9.29}$$

Pour estimer un majorant de I_2, nous allons à nouveau procéder à une intégration par parties en x et nous allons avoir besoin de la dérivée de $\varphi(x) = \int_x^{x+H(x)} g_2''(u)(x+H(x)-u)\,du$.

$$\begin{aligned} \varphi(x) &= (x+H(x))\int_x^{x+H(x)} g_2''(u)\,du - \int_x^{x+H(x)} ug_2''(u)\,du \\ \varphi'(x) &= (1+H'(x))\int_x^{x+H(x)} g_2''(u)\,du \\ & +(x+H(x))((1+H'(x))g_2''(x+H(x))-g_2''(x)) \\ & -(1+H'(x))(x+H(x))g_2''(x+H(x))+xg_2''(x) \\ \varphi'(x) &= (1+H'(x))\int_x^{x+H(x)} g_2''(u)\,du - H(x)g_2''(x) \end{aligned}$$

Nous obtenons alors

$$|I_2| \leq \frac{K\,||f||_\infty}{\sqrt{\frac{j}{n}-r}}\left(r-\frac{i}{n}\right)\left(1+\sqrt{\frac{n}{i}}\right) \tag{9.30}$$

L'expression (9.15) nous assure que sur l'événement $\left\{|G| \leq \left(2||\sigma'||_\infty\sqrt{r-i/n}\right)^{-1}\right\}$, la quantité $1+H'(x)$ est suffisamment éloignée de 0 pour n assez grand. On peut ainsi procéder à une intégration par partie (double) dans I_3.

$$
\begin{aligned}
I_3 &= \mathbb{E}\Bigg\{ \int_{\mathbb{R}^2}\Bigg(\frac{\partial^2}{\partial z^2}\left(q_{r,\frac{j}{n}}(z,x_j)\right)\Bigg|_{z=x+H(x)} - \frac{\partial^2 q_{r,\frac{j}{n}}}{\partial x^2}(x,x_j)\Bigg)\,\Psi(x)\mathbb{1}_{\left\{|G|\leq \frac{||\sigma'||_\infty^{-1}}{2\sqrt{r-\frac{i}{n}}}\right\}}\,dx\, f(x_j)\,dx_j\Bigg\} \\
&= -\mathbb{E}\Bigg\{ \int_{\mathbb{R}^2}\frac{\partial}{\partial z}\left(q_{r,\frac{j}{n}}(z,x_j)\right)\Bigg|_{z=x+H(x)} \left(\frac{\Psi}{1+H'}\right)'(x)\mathbb{1}_{\left\{|G|\leq \frac{||\sigma'||_\infty^{-1}}{2\sqrt{r-\frac{i}{n}}}\right\}}\,dx\, f(x_j)\,dx_j\Bigg\} \\
&\quad +\mathbb{E}\Bigg\{ \int_{\mathbb{R}^2}\frac{\partial}{\partial z}\left(q_{r,\frac{j}{n}}(z,x_j)\right)\Psi'(x)\mathbb{1}_{\left\{|G|\leq \frac{||\sigma'||_\infty^{-1}}{2\sqrt{r-\frac{i}{n}}}\right\}}\,dx\, f(x_j)dx_j\Bigg\} \\
&= \mathbb{E}\Bigg\{ \int_{\mathbb{R}^2} q_{r,\frac{j}{n}}(x+H(x),x_j)\left(\frac{1}{1+H'}\left(\frac{\Psi}{1+H'}\right)'\right)'(x)\mathbb{1}_{\left\{|G|\leq \frac{||\sigma'||_\infty^{-1}}{2\sqrt{r-\frac{i}{n}}}\right\}}\,dx\, f(x_j)\,dx_j\Bigg\} \\
&\quad -\mathbb{E}\Bigg\{ \int_{\mathbb{R}^2} q_{r,\frac{j}{n}}(x,x_j)\Psi''(x)\mathbb{1}_{\left\{|G|\leq \frac{||\sigma'||_\infty^{-1}}{2\sqrt{r-\frac{i}{n}}}\right\}}\,dx\, f(x_j)\,dx_j\Bigg\}
\end{aligned}
\tag{9.31}
$$

Dans (9.31), on remplace $q_{r,\frac{j}{n}}(x+H(x),x_j)$ par l'expression :

$$
q_{r,\frac{j}{n}}(x,x_j) + \int_x^{x+H(x)} \frac{\partial q_{r,\frac{j}{n}}}{\partial u}(u,x_j)\,du
$$

Nous obtenons ainsi :

$$
\begin{aligned}
I_3 &= \mathbb{E}\Bigg\{ \int_{\mathbb{R}}\Bigg[\int_{\mathbb{R}} q_{r,\frac{j}{n}}(x,x_j)\left(\left(\frac{1}{1+H'}\left(\frac{\Psi}{1+H'}\right)'\right)' - \Psi''\right)(x)\,dx \qquad (9.32)\\
&\quad + \int_{\mathbb{R}}\int_x^{x+H(x)} \frac{\partial q_{r,\frac{j}{n}}}{\partial u}(u,x_j)\,du \left(\left(\frac{1}{1+H'}\left(\frac{\Psi}{1+H'}\right)'\right)'\right)(x)\,dx\Bigg] \\
&\quad \mathbb{1}_{\left\{|G|\leq \frac{||\sigma'||_\infty^{-1}}{2\sqrt{r-\frac{i}{n}}}\right\}} f(x_j)\,dx_j\Bigg\}
\end{aligned}
$$

Un calcul des dérivées dans (9.32) et l'application des majorations du Lemme 9.1.6 nous donne le contrôle suivant sur I_3 :

$$
|I_3| \leq \frac{K||f||_\infty}{\sqrt{\frac{j}{n}-r}}\left(r-\frac{i}{n}\right)\left(\frac{n}{i}\right) \tag{9.33}
$$

En rassemblant les résultats obtenus par (9.29), (9.30) et (9.33) nous avons finalement

$$|Q_{2,1}| \ \leq \ K\left(r-\frac{i}{n}\right)\binom{n}{i}\frac{1}{\sqrt{\frac{j}{n}-r}} \tag{9.34}$$

Majoration de $|Q_{\gamma,2}|$

Nous allons à nouveau prouver le résultat pour $\gamma = 2$, le cas $\gamma = 1$ se faisant d'une manière analogue et plus simple. $Q_{2,2}$ est donnée par l'expression (9.27) :

$$\begin{aligned} Q_{2,2} \ &= \ \int_{\mathbb{R}^3} f(x_j)\,[g_2(z)-g_2(x)]\,\frac{\partial^2}{\partial z^2}\left(q_{r,\frac{j}{n}}(z,x_j)\right) \\ &\quad \mathbb{1}_{B(i,n,r,x)}(z) p_{\sigma^2(x)(r-i/n)}(z-x-b(x)(r-i/n))\,dz\,dx_j\,q^{(n)}_{\frac{i}{n}}(x_0,x)\,dx \end{aligned}$$

On découpe l'intégrale en z suivant le signe de $z-x-b(x)(r-i/n)$.
On définit :

$$\begin{aligned} Q^+_{2,2} \ &= \ \int_{\mathbb{R}^2} f(x_j)\,q^{(n)}_{\frac{i}{n}}(x_0,x)\int^{\infty}_{\frac{|\sigma(x)|}{2||\sigma'||_\infty}+x+b(x)(r-\frac{i}{n})}[g_2(z)-g_2(x)]\,\frac{\partial^2}{\partial z^2}\left(q_{r,\frac{j}{n}}(z,x_j)\right) \\ &\quad p_{\sigma^2(x)(r-i/n)}(z-x-b(x)(r-i/n))\,dz\,dx_j\ dx \end{aligned} \tag{9.35}$$

et $Q^-_{2,2}$ d'une manière analogue. Ainsi,

$$Q_{2,2} = Q^-_{2,2} + Q^+_{2,2} \tag{9.36}$$

Pour $Q^+_{2,2}$, une intégration par parties en z donne :

$$\begin{aligned} Q^+_{2,2} \ &= \ \int_{\mathbb{R}^2} dx_j\,dx\,f(x_j)\,q^{(n)}_{\frac{i}{n}}(x_0,x) \\ &\quad \Bigg\{\Bigg[(g_2(z)-g_2(x))p_{\sigma^2(x)(r-i/n)}(z-x-b(x)(r-i/n)) \\ &\qquad\qquad \frac{\partial}{\partial z}q_{r,\frac{j}{n}}(z,x_j)\Bigg]^{\infty}_{\frac{|\sigma(x)|}{2||\sigma'||_\infty}+x+b(x)(r-\frac{i}{n})} \\ &\quad -\int^{\infty}_{\frac{|\sigma(x)|}{2||\sigma'||_\infty}+x+b(x)(r-\frac{i}{n})}\frac{\partial}{\partial z}q_{r,\frac{j}{n}}(z,x_j) \\ &\qquad \Big((g_2(z)-g_2(x))p'_{\sigma^2(x)(r-i/n)}(z-x-b(x)(r-i/n)) \\ &\quad +g'_2(z)\,p_{\sigma^2(x)(r-i/n)}(z-x-b(x)(r-i/n))\Big)\Bigg\} \end{aligned}$$

Après un changement de variable, nous obtenons :

$$
\begin{aligned}
Q_{2,2}^{+} = & \int_{\mathbb{R}^2} dx_j\, dx\, f(x_j)\, q^{(n)}_{\frac{i}{n}}(x_0,x) \\
& \Bigg\{ \Bigg[(g_2(x) - g_2(\frac{|\sigma(x)|}{2||\sigma'||_\infty} + x + b(x)(r - \frac{i}{n}))) \exp(-\frac{1}{8||\sigma'||_\infty (r - i/n)}) \Bigg] \\
& \frac{\partial}{\partial z} q_{r,\frac{j}{n}}(\frac{|\sigma(x)|}{2||\sigma'||_\infty} + x + b(x)(r - \frac{i}{n}), x_j) \\
& - \int_{\frac{1}{2||\sigma'||_\infty}\sqrt{r-\frac{i}{n}}}^{\infty} \frac{\partial^1}{\partial z^1} q_{r,\frac{j}{n}}(y|\sigma(x)|\sqrt{r - \frac{i}{n}} + x + b(x)(r - \frac{i}{n}), x_j) \\
& \Big[(g_2(x) - g_2(y|\sigma(x)|\sqrt{r - \frac{i}{n}} + x + b(x)(r - \frac{i}{n}))) \frac{y}{|\sigma(x)|\sqrt{r - i/n}} \\
& + g_2'(y|\sigma(x)|\sqrt{r - \frac{i}{n}} + x + b(x)(r - \frac{i}{n})) \Big] \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy
\end{aligned}
$$

On utilise la majoration de la dérivée du noyau q donnée par (8.2) et l'on intègre en x_j, ce qui nous donne :

$$
\begin{aligned}
\left|Q_{2,2}^{+}\right| \leq & \frac{C||f||_\infty ||g_2'||_\infty}{\left(\frac{j}{n} - r\right)^{\frac{1}{2}}} \int_{\mathbb{R}} dx\, q^{(n)}_{\frac{i}{n}}(x_0,x) \Bigg\{ \left| \frac{|\sigma(x)|}{2||\sigma'||_\infty} + x + b(x)\left(r - \frac{i}{n}\right) \right| \\
& \qquad \exp(-\tfrac{1}{8||\sigma'||_\infty (r-i/n)}) \\
& + \int_{\frac{1}{2||\sigma'||_\infty}\sqrt{r-\frac{i}{n}}}^{\infty} \left(\left| y^2 + y\frac{b(x)}{|\sigma(x)|}\sqrt{r - \frac{i}{n}} \right| + 1 \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy \Bigg\}
\end{aligned}
$$

On peut donc trouver des constantes C_1 et C_2 telles que :

$$
\left|Q_{2,2}^{+}\right| \leq \frac{||f||_\infty}{\left(\frac{j}{n} - r\right)^{\frac{1}{2}}} \exp(-\frac{1}{8||\sigma'||_\infty (r - i/n)}) \left[C_1 + C_2 \left(\sqrt{r - \frac{i}{n}} + \frac{1}{\sqrt{r - \frac{i}{n}}} \right) \right] \tag{9.37}
$$

On procède de la même manière pour majorer $\left|Q_{2,2}^{-}\right|$

$$
\left|Q_{2,2}^{-}\right| \leq \frac{||f||_\infty}{\left(\frac{j}{n} - r\right)^{\frac{1}{2}}} \exp(-\frac{1}{8||\sigma'||_\infty (r - i/n)}) \left[C_1 + C_2 \left(\sqrt{r - \frac{i}{n}} + \frac{1}{\sqrt{r - \frac{i}{n}}} \right) \right] \tag{9.38}
$$

La conclusion (9.7) du Lemme 9.1.4 découle de (9.25), (9.34), (9.36), (9.37) et (9.38).

9.2 Résultats numériques

Nous allons illustrer dans cette partie le résultat que nous venons de montrer. Nous approchons la quantité $\mathbb{E}(\int_0^1 f(X(s))ds)$ où la diffusion $X(.)$ est donnée par l'E.D.S. suivante :

$$X(t) = \frac{1}{2} + \int_0^t \frac{X(s)}{1+X^2(s)}ds + \int_0^t X(s)\,dB_s.$$

La fonction f que l'on considère est la fonction indicatrice de l'évênement « *le premier chiffre après la virgule est un 5* » :

$$f(x) = \sum_{m\in\mathbb{Z}} \mathbb{1}_{[0.5,0.6[}(x-m).$$

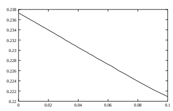


FIG. 9.1 – *Approximation par le schéma d'Euler*

Nous représentons sur cette figure l'approximation obtenue par notre méthode en fonction de l'inverse du pas de discrétisation.
Cette simulation amène plusieurs remarques :

- l'erreur semble être une fonction linéaire de l'inverse du pas de discrétisation ; nous avons seulement montré qu'elle est majorée par une fonction linéaire.
- la condition d'uniforme ellipticité imposée pour la diffusion semble pouvoir être relaxée puisque la diffusion X que nous considérons ici ne l'est pas.

Troisième partie

Amplitude du mouvement brownien avec dérive

10

Amplitude du mouvement brownien avec dérive

10.1 Introduction

Le but est d'étudier le processus de l'amplitude d'un mouvement brownien réel avec dérive δ. Le cas $\delta = 0$ a été étudié par Imhof [34], Vallois [64].
Nous donnerons les résultats avec $\delta > 0$, le cas $\delta < 0$ s'en déduisant immédiatement par symétrie.

Puisque $\lim_{t\to\infty} B_\delta(t) = +\infty$, nous allons décrire les amplitudes successives de B_δ en « remontant le temps » à partir de l'instant où B_δ a atteint son minimum absolu. Pour cela, on va considérer :

$$\begin{aligned} S_1 &= -\inf_{t\geq 0}\{B_\delta(t)\} \\ \rho_1 &= \sup\{t \geq 0, B_\delta(t) = -S_1\} \end{aligned} \tag{10.1}$$

$$\begin{aligned} S_2 &= \sup_{0\leq t\leq \rho_1}\{B_\delta(t)\} \\ \rho_2 &= \sup\{0 \leq t \leq \rho_1, B_\delta(t) = S_2\} \end{aligned} \tag{10.2}$$

et plus généralement pour tout $k \in \mathbb{N}^*$:

$$\begin{aligned} S_k &= \sup_{t\in[0,\rho_{k-1}]} (-1)^k B_\delta(t) \\ \rho_k &= \sup\{t \in [0,\rho_{k-1}], B_\delta(t) = (-1)^k\, S_k\} \end{aligned} \tag{10.3}$$

(avec la convention $\rho_0 = +\infty$).

Remarque 10.1.1 – *S_1 est le minimum absolu de $B_\delta(t)$ pour $t \geq 0$,*
– *ρ_1 est le dernier instant où ce minimum est atteint,*
– *S_2 est le maximum absolu de $B_\delta(t)$ pour $t \leq \rho_1$,*
– *ρ_2 est le dernier instant dans $[0,\rho_1]$ où ce maximum est atteint.*

Afin d'énoncer le résultat principal, introduisons la notation :

Notation 10.1.2 *$Z^{(\delta)}(t)$ désigne la solution positive de l'équation différentielle stochastique suivante :*

$$Z^{(\delta)}(t) = B(t) + \delta \int_0^t \coth\left(\delta Z^{(\delta)}(s)\right)\, ds. \tag{10.4}$$

Théorème 10.1.1 *Soit B_δ un mouvement brownien avec dérive positive δ.*

1. *S_1 suit une loi exponentielle de paramètre 2δ (i.e. de densité $2\delta e^{-2\delta x} 1\!\!1_{\{x\geq 0\}}$).*
2. *Conditionnellement à $\{S_1 = a_1\}$, S_2 a pour densité*

$$\delta e^{\delta a_1} \operatorname{sh}(\delta a_1) \frac{1}{\operatorname{sh}^2(\delta(x+a_1))} 1\!\!1_{\{x\geq 0\}} \tag{10.5}$$

3. *Pour tout $k \geq 3$, conditionnellement à $\{S_1 = a_1, \ldots, S_{k-1} = a_{k-1}\}$:*
   S_k a pour densité :

$$\delta \frac{\operatorname{sh} \delta a_{k-1} \operatorname{sh} \delta(a_{k-1} + a_{k-2})}{\operatorname{sh} \delta a_{k-2}} \frac{1}{\operatorname{sh}^2 \delta(a_{k-1} + x)} 1\!\!1_{\{0\leq x\leq a_{k-2}\}}$$

4. *Pour tout $k \geq 1$, conditionnellement à $\{S_k = a_k, \ldots, S_1 = a_1\}$,*
   - *$(B_\delta(t))_{t\leq\rho_k}$ et $(B_\delta(t+\rho_k) - B_\delta(\rho_k))_{0\leq t\leq \rho_{k-1}-\rho_k}$ sont indépendants*
   - *$(B_\delta(t))_{t\leq\rho_k}$ a même loi qu'un mouvement brownien avec dérive $(-1)^k\delta$ arrêté au premier temps d'atteinte du niveau $(-1)^k a_k$ et conditionné à ne pas atteindre $(-1)^{k+1} a_{k-1}$*
   - *$\left((-1)^{k+1}(B_\delta(t+\rho_k) - B_\delta(\rho_k))\right)_{0\leq t\leq\rho_{k-1}-\rho_k}$ est un processus (positif) qui a même loi que $Z^{(\delta)}$ défini par (10.4) arrêté au premier instant où il atteint le niveau $a_k + a_{k-1}$.*

Remarque 10.1.3 *On a pris comme convention : $a_0 = \rho_0 = +\infty$.*

La suite du travail s'articulera ainsi :
dans la partie 10.2, nous donnerons les résultats connus que nous utiliserons, puis nous donnerons les deux propositions centrales dans la partie suivante, enfin nous prouverons le théorème 10.1.1.

10.2 Résultats préliminaires

Donnons ici les notations utilisées par la suite.

Notation 10.2.1 • *$B(t)$ est un mouvement brownien de dimension 1 partant de 0*

- *$\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ désigne la filtration engendrée par $(B(t))_{t\geq 0}$*

- $B_\alpha(t)$ *désigne un mouvement brownien avec dérive* α *issu de* 0
  $B_\alpha(t) = B(t) + \alpha\, t$
- T_a^α *est le temps d'atteinte d'un niveau a par* B_α *:*
  $T_a^\alpha = \inf\{t > 0\ ,\ B_\alpha(t) = a\}$
- A^δ *désigne le processus de l'amplitude de* B_δ *:*
  $A^\delta(t) = \sup_{s\in[0,t]} B_\delta(s) - \inf_{s\in[0,t]} B_\delta(s).$
- θ^δ *est l'inverse continu à droite de* A^δ *:*
  $\theta^\delta(a) = \inf\{u \geq 0; A^\delta(u) > a\}$

Donnons ici un premier résultat classique ([54] page 301) relatif aux temps d'atteinte d'un niveau donné par un processus de Markov en fonction de sa fonction d'échelle :

Lemme 10.2.2 *Soit* (X_t) *un processus de Markov réel continu, s une fonction d'échelle associée à* X.
On note $T_a = \inf\{t > 0, X_t = a\}$.
On a :

$$\mathbb{P}_x(T_b < T_a) = \frac{s(x) - s(a)}{s(b) - s(a)}\ ,\ \textit{pour } x \textit{ compris entre } a \textit{ et } b.$$

En appliquant le lemme 10.2.2 au mouvement brownien avec dérive non nulle, on a :

Corollaire 10.2.3 *Soit* $\alpha \neq 0$. *On a :*

$$\mathbb{P}_x\left(T_b^\alpha < T_a^\alpha\right) = \frac{e^{-2\alpha x} - e^{-2\alpha a}}{e^{-2\alpha b} - e^{-2\alpha a}}\ ,\ \textit{pour } x \textit{ compris entre } a \textit{ et } b. \tag{10.6}$$

Démonstration Une fonction d'échelle associée au mouvement brownien avec dérive $\alpha \neq 0$ est :

$$s(y) = e^{-2\alpha y}.$$

□

Donnons une conséquence de ce corollaire 10.2.3 :

Corollaire 10.2.4 *Soit* $\delta > 0$. *Alors*

$$\mathbb{P}_x\left(T_b^\delta < \infty\right) = e^{-2\delta(x-b)}\ ,\ \textit{pour } b < x \tag{10.7}$$

Ajoutons ici un résultat d'égalité en loi qui nous sera utile par la suite :

Proposition 10.2.5 *Soient* δ, x *et* y *des réels positifs.*
Les 2 processus suivants ont même loi :

- $(B_\delta(t), 0 \leq t \leq T_x^\delta)$ *conditionné par* $\left\{T_x^\delta < T_{-y}^\delta\right\}$
- $(B_{-\delta}(t), 0 \leq t \leq T_x^{-\delta})$ *conditionné par* $\left\{T_x^{-\delta} < T_{-y}^{-\delta}\right\}$

Démonstration Nous allons montrer ce résultat en appliquant le théorème de Girsanov.
Soit F une fonction test.

$$\begin{aligned} A &= \mathbb{E}\left[F\left(B(t)+\delta t, 0 \leq t \leq T_x^\delta\right) | T_x^\delta < T_{-y}^\delta\right] \\ &= \mathbb{E}\left[F\left(B(t)+\delta t, 0 \leq t \leq T_x^\delta\right) \mathbb{1}_{\{T_x^\delta < T_{-y}^\delta\}}\right] \frac{1}{\mathbb{P}(T_x^\delta < T_{-y}^\delta)}. \end{aligned}$$

On utilise le corollaire 10.2.3 et on applique le théorème de Girsanov en remarquant que $F\left(B(t)+\delta t, 0 \leq t \leq T_x^\delta\right) \mathbb{1}_{\{T_x^\delta < T_{-y}^\delta\}}$ est $\mathcal{F}_{T_x^\delta}$ mesurable et $T_x^\delta < \infty$ presque sûrement ([54] proposition VIII 1.3).

$$\begin{aligned} A &= \mathbb{E}\left[F\left(B(t), 0 \leq t \leq T_x^0\right) \mathbb{1}_{\{T_x^0 < T_{-y}^0\}} \exp\left(\delta B(T_x^0) - \frac{\delta^2}{2} T_x^0\right)\right] \frac{e^{-2\delta x} - e^{2\delta y}}{1 - e^{2\delta y}} \\ &= \mathbb{E}\left[F\left(B(t), 0 \leq t \leq T_x^0\right) \mathbb{1}_{\{T_x^0 < T_{-y}^0\}} \exp\left(-\frac{\delta^2}{2} T_x^0\right)\right] \frac{e^{-\delta x} - e^{2\delta y + \delta x}}{1 - e^{2\delta y}} \end{aligned}$$

Effectuons le même calcul pour $B(t) - \delta t$.

$$\begin{aligned} \tilde{A} &= \mathbb{E}\left[F\left(B(t)-\delta t, 0 \leq t \leq T_x^{-\delta}\right) | T_x^{-\delta} < T_{-y}^{-\delta}\right] \\ &= \mathbb{E}\left[F\left(B(t)-\delta t, 0 \leq t \leq T_x^{-\delta}\right) \mathbb{1}_{\{T_x^{-\delta} < T_{-y}^{-\delta}\}}\right] \frac{1}{\mathbb{P}(T_x^{-\delta} < T_{-y}^{-\delta})} \\ &= \mathbb{E}\left[F\left(B(t), 0 \leq t \leq T_x^0\right) \mathbb{1}_{\{T_x^0 < T_{-y}^0\}} \exp\left(-\delta B(T_x^0) - \frac{\delta^2}{2} T_x^0\right)\right] \frac{e^{2\delta x} - e^{-2\delta y}}{1 - e^{-2\delta y}} \\ &= \mathbb{E}\left[F\left(B(t), 0 \leq t \leq T_x^0\right) \mathbb{1}_{\{T_x^0 < T_{-y}^0\}} \exp\left(-\frac{\delta^2}{2} T_x^0\right)\right] \frac{e^{\delta x} - e^{-2\delta y - \delta x}}{1 - e^{-2\delta y}} \\ &= A \end{aligned}$$

□

On déduit de la proposition 10.2.5 le

Corollaire 10.2.6 *Soit $\delta > 0$ et $x > 0$.*
Les deux processus suivants ont même loi :

- $(B_\delta(t), 0 \leq t \leq T_x^\delta)$
- $(B_{-\delta}(t), 0 \leq t \leq T_x^{-\delta})$ *conditionné par* $\{T_x^{-\delta} < \infty\}$

Donnons ici une conséquence immédiate de la proposition 10.2.5 (et du corollaire 10.2.6) :

Corollaire 10.2.7 *Soient δ, x et y des réels positifs.*
Les 2 processus suivants ont même loi :

- $(B_\delta(t), 0 \leq t \leq T_{-x}^\delta)$ *conditionné par* $\left\{T_{-x}^\delta < T_y^\delta\right\}$
- $(B_{-\delta}(t), 0 \leq t \leq T_{-x}^{-\delta})$ *conditionné par* $\left\{T_{-x}^{-\delta} < T_y^{-\delta}\right\}$.

D'autre part, les deux processus suivants ont même loi :

- $(B_\delta(t), 0 \le t \le T^{\delta}_{-x})$ *conditionné par* $\{T^{-\delta}_{-x} < \infty\}$
- $(B_{-\delta}(t), 0 \le t \le T^{-\delta}_{-x})$

Démonstration Le résultat est immédiat en utilisant la proposition 10.2.5, le corollaire 10.2.6 et la symétrie du mouvement brownien. □

Nous rappelons brièvement quelques résultats concernant la projection duale prévisible et le grossissement de filtration.

Rappel : Soit $A^{(0)}_t$ un processus croissant, nul en 0, continu à droite.
La projection duale prévisible de $A^{(0)}_t$ est le processus prévisible croissant continu à droite A_t caractérisé par :

$$\mathbb{E}\left[\int_0^\infty U_s \, dA^{(0)}_s\right] = \mathbb{E}\left[\int_0^\infty U_s \, dA_s\right]. \tag{10.8}$$

pour tout processus U prévisible positif.
Puisque nous travaillons avec la filtration brownienne, un processus est prévisible si et seulement s'il est adapté.
Soit T un temps d'arrêt borné, le processus $U = \mathbb{1}_{[0,T]}$ est un processus prévisible, (10.8) s'écrit :

$$\mathbb{E}\left[A^{(0)}_T\right] = \mathbb{E}\left[A_T\right]. \tag{10.9}$$

En raisonnant par classe monotone, il est facile de montrer que si un processus $(A_t)_{t\ge 0}$ croissant et adapté, nul en 0, vérifie (10.9) pour tout temps d'arrêt borné T, alors A est la projection duale prévisible de $A^{(0)}$.

Pour finir, nous rappelons un résultat de grossissement progressif que l'on peut trouver dans Jeulin ([37] théorème 5.10 page 80). Pour cela, nous avons besoin de la définition suivante :

Definition 10.2.8 *Une variable aléatoire ρ est dite honnête si pour tout $t > 0$, ρ est égale à une variable $\mathcal{F}_t$-mesurable sur $\{\rho < t\}$.*

Théorème 10.2.1 [Jeulin] *Soit ρ une variable aléatoire honnête. On note $(\mathcal{F}^\rho_t)_{t\ge 0}$ la plus petite filtration continue à droite contenant $(\mathcal{F}_t)_{t\ge 0}$ pour laquelle ρ est un temps d'arrêt.*
Soit $(Y^\rho_t)_{t\ge 0}$ le processus

$$Y^\rho_t = \mathbb{P}\left(\rho \le t | \mathcal{F}_t\right); t \ge 0. \tag{10.10}$$

$(Y^\rho_t)_{t\ge 0}$ est une sous-martingale de partie martingale $(M^\rho_t)_{t\ge 0}$

$$\chi(t) = -\int_0^{t\wedge\rho} \frac{1}{1-Y^\rho_s} d\left\langle B, M^\rho\right\rangle_s + \int_0^t \mathbb{1}_{\{\rho<s\}} \frac{1}{Y^\rho_s} d\left\langle B, M^\rho\right\rangle_s \tag{10.11}$$

$B - \chi = \overline{B}$ est un $\mathcal{F}^\rho$ mouvement brownien.

10.3 Preuve du théorème 10.1.1

10.3.1 Une première décomposition

Dans un premier temps, nous allons prouver les résultats du théorème 10.1.1 pour $k = 1$; ils font l'objet de la proposition 10.3.1.
Rappelons les notations :

$$\begin{aligned} S_1 &= -\inf_{t\geq 0}\{B_\delta(t)\} \\ \rho_1 &= \sup\{t \geq 0, B_\delta(t) = -S_1\} \end{aligned} \tag{10.1}$$

Comme $\delta > 0$, S_1 est p.s. fini.
On cherche : la loi de $(S_1,(B_\delta(t),0 \leq t \leq \rho_1),(B_\delta(t),\rho_1 \leq t))$ en conditionnant par les valeurs de S_1.
Le résultat qui suit n'est pas nouveau, il figure par exemple dans Williams [66], toutefois, notre approche est différente, elle repose sur le grossissement progressif ; elle présente l'avantage de pouvoir s'appliquer à l'étude du processus B_δ arrêté aux temps ρ_k.

Proposition 10.3.1 *Soit S_1 et ρ_1 les deux variables aléatoires définies par* (10.1). *Alors*

- *S_1 suit une loi exponentielle de paramètre 2δ (i.e. de densité $2\delta e^{-2\delta x}\mathbb{1}_{\{x\geq 0\}}$).*
- *Conditionnellement à $\{S_1 = a_1\}$,*
  1. *$(B_\delta(t))_{t\leq\rho_1}$ et $(B_\delta(t+\rho_1) - B_\delta(\rho_1))_{t>0}$ sont deux processus indépendants*
  2. *$(B_\delta(t))_{t\leq\rho_1}$ a même loi qu'un mouvement brownien avec dérive $-\delta$ arrêté au premier temps d'atteinte du niveau $-a_1$*
  3. *$(B_\delta(t+\rho_1) - B_\delta(\rho_1))_{t\geq 0}$ est un processus (positif) qui a même loi que le processus $(Z^{(\delta)})$ défini par* (10.4).

Notre approche repose sur la théorie du grossissement ; plus précisément nous réalisons un grossissement progressif avec ρ_1. D'après le rappel, nous sommes conduits dans un premier temps à calculer la projection duale prévisible de $\mathbb{1}_{\{t\geq\rho_1\}}$.

Lemme 10.3.2 *La projection duale prévisible de $A_t^{(0)} = \mathbb{1}_{\{t\geq\rho_1\}}$*

est : $\quad A_t = -2\delta \underline{B_\delta}(t) \quad$ *où :* $\quad \underline{B_\delta}(t) = \inf_{s\leq t} B_\delta(s)$

Démonstration D'après le rappel, on cherche A_t processus croissant tel que pour tout temps d'arrêt borné T,

$$\mathbb{E}[A_T] = \mathbb{P}(\rho_1 \leq T).$$

$\rho_1 \leq T$ signifie qu'après l'instant T, B_δ ne va plus visiter $\underline{B_\delta}(T)$.
On utilise le corollaire 10.2.4 :

$$\mathbb{P}(\rho_1 \leq T|\mathcal{F}_T) = \frac{e^{-2\delta\underline{B_\delta}(T)} - e^{-2\delta B_\delta(T)}}{e^{-2\delta\underline{B_\delta}(T)}}.$$

On pose :

$$
\begin{aligned}
Y_t^{\rho_1} &= \frac{e^{-2\delta \underline{B_\delta}(t)} - e^{-2\delta B_\delta(t)}}{e^{-2\delta \underline{B_\delta}(t)}} \\
&= 1 - e^{-2\delta(B_\delta(t) - \underline{B_\delta}(t))}. \qquad (10.12)
\end{aligned}
$$

$(Y_t^{\rho_1})$ est une semi-martingale continue ; on cherche sa décomposition canonique en remarquant que $\exp(-2\delta B_\delta(t)) = \exp(-2\delta B(t) - 2\delta^2 t)$ est une martingale.

$$
dY_t^{\rho_1} = 2\delta e^{-2\delta(B_\delta(t)-\underline{B_\delta}(t))}\, dB(t) - 2\delta e^{-2\delta(B_\delta(t)-\underline{B_\delta}(t))}\, d\underline{B_\delta}(t) \qquad (10.13)
$$

Remarquons que $(B_\delta(t) - \underline{B_\delta}(t)) \geq 0$ donc $\exp(-2\delta(B_\delta(t) - \underline{B_\delta}(t))) \leq 1$ et on peut appliquer le théorème d'arrêt. On obtient :

$$
\begin{aligned}
\mathbb{E}\left[Y_T^{\rho_1}\right] &= -2\delta \mathbb{E}\left[\int_0^T \exp(-2\delta(B_\delta(t) - \underline{B_\delta}(t)))\, d\underline{B_\delta}(t)\right] \\
&= -2\delta \mathbb{E}\left[\int_0^T d\underline{B_\delta}(t)\right]. \qquad (10.14)
\end{aligned}
$$

Ce résultat provient du fait que la mesure aléatoire $d\underline{B_\delta}$ ne charge que les instants t tels que $B_\delta(t) = \underline{B_\delta}(t)$.

$$
\mathbb{E}\left[Y_T^{\rho_1}\right] = -2\delta \mathbb{E}\left[\underline{B_\delta}(T)\right]
$$

□

Introduisons la notation suivante :

$$
M_t^{\rho_1} = \int_0^t 2\delta \exp(-2\delta(B_\delta(s) - \underline{B_\delta}(s)))\, dB(s). \qquad (10.15)
$$

M^{ρ_1} est la partie martingale de Y^{ρ_1}.

Démonstration de la proposition 10.3.1

1) Nous allons dans un premier temps montrer les deux résultats suivants :

- S_1 suit une loi exponentielle de paramètre 2δ .
- Conditionnellement à $\{S_1 = a_1\}$ $(B_\delta(t))_{t\leq\rho_1}$ a même loi qu'un mouvement brownien avec dérive δ arrêté au premier temps d'atteinte du niveau $-a_1$ et conditionné par $\{T_{-a_1}^\delta < \infty\}$

Soit f et F des fonctions test. Posons :

$$
\theta = \mathbb{E}\left[f(S_1)F\left(B_\delta(s), 0 \leq s \leq \rho_1\right)\right] = \mathbb{E}\left[f(-\underline{B_\delta}(\rho_1))F\left(B_\delta(s), 0 \leq s \leq \rho_1\right)\right]
$$

On pose : $U_t = f(-\underline{B_\delta}(t))F\left(B_\delta(s), 0 \leq s \leq t\right)$; U_t est un processus prévisible donc on applique le lemme 10.3.2 et (10.8) :

$$
\begin{aligned}
\theta &= \mathbb{E}\left[U_{\rho_1}\right] = \mathbb{E}\left[\int_0^\infty U_t\, dA_t^{(0)}\right] = \mathbb{E}\left[\int_0^\infty U_t\, dA_t\right] \\
\theta &= -2\delta \mathbb{E}\left[\int_0^\infty f(-\underline{B_\delta}(t))F\left(B_\delta(s), 0 \leq s \leq t\right)\, d\underline{B_\delta}(t)\right] \qquad (10.16)
\end{aligned}
$$

Particularisons avec $F = 1$ pour obtenir la loi de S_1.

$$\mathbb{E}\left[f(S_1)\right] = -2\delta\mathbb{E}\left[\int_0^{\infty} f(-\underline{B_\delta}(t))\, d\underline{B_\delta}(t)\right]$$

Puisque $-\underline{B_\delta}$ est une fonction croissante continue, le 'changement de variable' $u = -\underline{B_\delta}(t)$ donne :

$$\begin{aligned}\mathbb{E}\left[f(S_1)\right] &= 2\delta\mathbb{E}\left[\int_0^{S_1} f(u)\, du\right] \\ &= 2\delta\int_0^{\infty} f(u)\,\mathbb{P}(S_1 \geq u)\, du.\end{aligned}$$

On déduit de ce calcul :

$$\mathbb{P}(S_1 \in du) = 2\delta\mathbb{P}(S_1 \geq u)\, du.$$

Donc S_1 admet pour densité $2\delta e^{-2\delta u}1\!\!1_{\{u\geq 0\}}$. Revenons à (10.16) ; $(T^\delta_{-u}, u \geq 0)$ est l'inverse continu à droite de $-\underline{B_\delta}$ donc le 'changement de variable' $u = -\underline{B_\delta}(t)$ nous permet d'obtenir :

$$\begin{aligned}\theta &= 2\delta\mathbb{E}\left[\int_0^{+\infty} f(u)\, F\left(B_\delta(s), 0 \leq s \leq T^\delta_{-u}\right) 1\!\!1_{\{T^\delta_{-u}<\infty\}}\, du\right] \\ &= 2\delta\int_0^{\infty} f(u)\,\mathbb{E}\left[F\left(B_\delta(s), 0 \leq s \leq T^\delta_{-u}\right) 1\!\!1_{\{T^\delta_{-u}<\infty\}}\right] du \\ &= 2\delta\int_0^{\infty} f(u)\,\mathbb{E}\left[F\left(B_\delta(s), 0 \leq s \leq T^\delta_{-u}\right) | T^\delta_{-u} < \infty\right]\mathbb{P}\left(T^\delta_{-u} < \infty\right) du \\ &= \int_0^{\infty} f(u)\,\mathbb{E}\left[F\left(B_\delta(s), 0 \leq s \leq T^\delta_{-u}\right) | T^\delta_{-u} < \infty\right] 2\delta e^{-2\delta u}\, du \qquad (10.17)\end{aligned}$$

Cette dernière égalité termine la preuve de ces premiers résultats. Notons qu'une application immédiate du corollaire 10.2.7 donne le deuxième résultat de la proposition 10.3.1.

2) Nous nous intéressons à présent à l'étude de $(B_\delta(t+\rho_1) - B_\delta(t))_{t\geq 0}$, nous allons appliquer le théorème 10.2.1 avec ρ_1. Remarquons que la variable aléatoire ρ_1 est un dernier temps de passage donc est une variable aléatoire honnête : sur $\{\rho_1 \leq t\}$, $\rho_1 = \sup\left\{u \in [0,t], B_\delta(u) = \underline{B_\delta}(t)\right\}$.

Nous obtenons :

$$\begin{aligned}\chi(t) &= -\int_0^{t\wedge\rho_1} \frac{1}{e^{-2\delta(B_\delta(t)-\underline{B_\delta}(t))}} 2\delta e^{-2\delta(B_\delta(s)-\underline{B_\delta}(s))}\, ds \\ &\quad + \int_0^t 1\!\!1_{\{\rho_1<s\}} \frac{e^{-2\delta\underline{B_\delta}(s)}}{e^{-2\delta\underline{B_\delta}(s)} - e^{-2\delta B_\delta(s)}} 2\delta e^{-2\delta(B_\delta(s)-\underline{B_\delta}(s))}\, ds \\ \chi(t) &= -2\delta(t\wedge\rho_1) + 2\delta\int_0^t 1\!\!1_{\{\rho_1<s\}} \frac{e^{-2\delta B_\delta(s)}}{e^{-2\delta\underline{B_\delta}(s)} - e^{-2\delta B_\delta(s)}} ds \qquad (10.18)\end{aligned}$$

Utilisons (10.18) pour étudier $(B_\delta(t))_{t\geq\rho_1}$

$$\chi(t+\rho_1)-\chi(\rho_1)=2\delta\int_{\rho_1}^{\rho_1+t}\frac{e^{-2\delta B_\delta(s)}}{e^{-2\delta \underline{B_\delta}(s)}-e^{-2\delta B_\delta(s)}}\,ds \tag{10.19}$$

On obtient ainsi $(B(t)-\chi(t)=B^1(t))$ est un $\mathcal{F}^{\rho_1}$ mouvement brownien.

$$B_\delta(t)=B(t)+\delta t=B^1(t)+\delta t+\chi(t)$$

$$B_\delta(t+\rho_1)-B_\delta(\rho_1)=B^1(t+\rho_1)-B^1(\rho_1)+\delta t+\chi(t+\rho_1)-\chi(\rho_1)$$

Posons :

$$Z_1(t)=B_\delta(t+\rho_1)-B_\delta(\rho_1)=B_\delta(t+\rho_1)+S_1 \tag{10.20}$$

$$\begin{aligned} Z_1(t) &= B^1(t+\rho_1)-B^1(\rho_1)\\ &+ \delta\int_{\rho_1}^{\rho_1+t}1+2\frac{\exp\left(-2\delta B_\delta(s)\right)}{\exp\left(-2\delta\underline{B_\delta}(s)\right)-\exp\left(-2\delta B_\delta(s)\right)}\,ds\\ &= B^1(t+\rho_1)-B^1(\rho_1)\\ &+ \delta\int_{\rho_1}^{\rho_1+t}\frac{\exp\left(-2\delta\underline{B_\delta}(s)\right)+\exp\left(-2\delta B_\delta(s)\right)}{\exp\left(-2\delta\underline{B_\delta}(s)\right)-\exp\left(-2\delta B_\delta(s)\right)}\,ds\\ &= B^1(t+\rho_1)-B^1(\rho_1)\\ &+ \delta\int_{\rho_1}^{\rho_1+t}\frac{\exp\left(2\delta B_\delta(s)-2\delta\underline{B_\delta}(s)\right)+1}{\exp\left(2\delta B_\delta(s)-2\delta\underline{B_\delta}(s)\right)-1}\,ds \end{aligned} \tag{10.21}$$

Puisque ρ_1 est un $\mathcal{F}^{\rho_1}$ temps d'arrêt, le processus

$$\tilde{B}^1(t)=B^1(t+\rho_1)-B^1(\rho_1) \tag{10.22}$$

est un $\mathcal{F}^{\rho_1}$ mouvement brownien indépendant de $\mathcal{F}^{\rho_1}_{\rho_1}$.
Revenons à (10.21) en nous souvenant que pour $t\geq\rho_1$, $\underline{B_\delta}(t)=-S_1$

$$Z_1(t) = \tilde{B}^1(t)+\delta\int_0^t\frac{\exp\left(2\delta Z_1(s)\right)+1}{\exp\left(2\delta Z_1(s)\right)-1}\,ds \tag{10.23}$$

$$Z_1(t) = \tilde{B}^1(t)+\delta\int_0^t\coth\left(\delta Z_1(s)\right)ds \tag{10.24}$$

Z_1 est la solution forte de (10.24) à valeurs dans $\mathbb{R}_+$ et est donc indépendant de $\mathcal{F}^{\rho_1}_{\rho_1}$ donc de $(B_\delta(t))_{t\leq\rho_1}$. □

D'après la proposition 10.3.1, l'étude de $(B_{-\delta}(t))_{t\leq T^{-\delta}_{-a_1}}$ nous donnera le comportement de B_δ sur $[0,\rho_1]$ où

$$T^{-\delta}_{-a_1}=\inf\{t>0,B_{-\delta}(t)=-a_1\}$$

10.3.2 Une deuxième décomposition : étude de $(B_\delta(t))_{t\le T_a^\delta}$

Pour des raisons de symétrie, nous allons étudier le processus $(B_\delta(t))_{t\le T_a^\delta}$ pour obtenir $(B_{-\delta}(t))_{t\le T_{-a_1}^{-\delta}}$ comme indiqué a la fin de la partie précédente.
Les techniques seront très proches de celles utilisées pour la proposition 10.3.1.
On notera :

$$S = -\inf_{t\in[0,T_a^\delta]} B_\delta(t) \tag{10.25}$$

On souhaite étudier le processus $(B_\delta(t),0\le t\le T_a^\delta)$ conditionnellement à S. L'outil sera à nouveau le grossissement progressif de filtration. A cet effet, on introduit :

$$\rho = \sup\{t\in[0,T_a^\delta], B_\delta(t) = -S\}. \tag{10.26}$$

Donnons ici le résultat principal :

Proposition 10.3.3 *Soient S et ρ les variables aléatoires définies par* (10.25) *et* (10.26). *Alors :*

- *S a pour densité*

$$\varphi(x) = \delta e^{\delta a}\,\mathrm{sh}\,\delta a \frac{1}{\mathrm{sh}^2\,\delta(x+a)} \mathbb{1}_{\{x\ge 0\}} \tag{10.5}$$

- *Conditionnellement à $\{S=b\}$,*
  1. *$(B_\delta(t))_{t\le\rho}$ et $(B_\delta(t+\rho)-B_\delta(\rho))_{0\le t\le T_{a-\rho}^\delta}$ sont indépendants,*
  2. *$(B_\delta(t))_{t\le\rho}$ a même loi qu'un mouvement brownien avec dérive $-\delta$ arrêté au premier temps d'atteinte du niveau $-b$ et conditionné à ne pas atteindre a*
  3. *$(B_\delta(t+\rho)-B_\delta(\rho)=Z(t))$ est un processus (positif) qui a même loi que le processus $(Z^{(\delta)})$ défini par* (10.4) *arrêté au premier instant où il atteint le niveau $a+b$*

ρ est un temps honnête, il s'agit de calculer la projection duale prévisible de $\mathbb{1}_{[\rho,\infty[}$. :

Lemme 10.3.4 *La projection duale prévisible de $A_t^{(0)} = \mathbb{1}_{\{t\ge\rho\}}$*

est : $$A_t = \ln\left(\frac{e^{-2\delta \underline{B_\delta}(t\wedge T_a^\delta)} - e^{-2\delta a}}{1-e^{-2\delta a}}\right)$$

où : $$\underline{B_\delta}(t) = -\inf_{s\le t} B_\delta(s)$$

Démonstration Soit T un temps d'arrêt. $U_s = \mathbb{1}_{[0,T]}$ est un processus prévisible élémentaire.

$$\mathbb{E}\left[\int_0^\infty U_s\,dA_s^{(0)}\right] = \mathbb{E}\left[U_\rho\right] = \mathbb{P}(\rho\le T)$$

On a :

$$\mathbb{1}_{\{\rho\le T\}} = \mathbb{1}_{\{\rho\le T\le T_a^\delta\}} + \mathbb{1}_{\{T_a^\delta < T\}} \tag{10.27}$$

Sur $\{T \leq T_a^\delta\}$, $\rho \leq T$ signifie qu'après l'instant T, B_δ va visiter a avant $\min_{u \leq T} B_\delta(u)$. En utilisant le corollaire 10.2.3, nous obtenons :

$$\mathbb{P}(\rho \leq T \leq T_a^\delta | \mathcal{F}_T) = \frac{e^{-2\delta \underline{B_\delta}(T)} - e^{-2\delta B_\delta(T)}}{e^{-2\delta \underline{B_\delta}(T)} - e^{-2\delta a}} \mathbb{1}_{\{T \leq T_a^\delta\}} \tag{10.28}$$

On pose :

$$Y_t^\rho = \frac{e^{-2\delta \underline{B_\delta}(t)} - e^{-2\delta B_\delta(t)}}{e^{-2\delta \underline{B_\delta}(t)} - e^{-2\delta a}} \text{ pour } t \geq 0 \tag{10.29}$$

Mais $Y_{T_a^\delta}^\rho = 1$ donc

$$\mathbb{P}(\rho \leq T | \mathcal{F}) = Y_{T \wedge T_a^\delta}^\rho \tag{10.30}$$

On sait que $e^{-2\delta B_\delta(t)} = e^{-2\delta B(t) - 2\delta^2 t}$ est une martingale, on trouve ainsi la décomposition canonique de la semi-martingale (Y_t^ρ) :

$$\begin{aligned} dY_t^\rho &= \frac{1}{e^{-2\delta \underline{B_\delta}(t)} - e^{-2\delta a}} e^{-2\delta B_\delta(t)} 2\delta dB(t) \\ &- \frac{e^{-2\delta B_\delta(t)} - e^{-2\delta a}}{\left(e^{-2\delta \underline{B_\delta}(t)} - e^{-2\delta a}\right)^2} 2\delta e^{-2\delta \underline{B_\delta}(t)} d\underline{B_\delta}(t) \end{aligned} \tag{10.31}$$

On utilise maintenant le fait que la mesure aléatoire $d\underline{B_\delta}$ ne charge que les points t tels que $B_\delta(t) = \underline{B_\delta}(t)$ et l'on obtient :

$$\begin{aligned} \mathbb{E}(Y_{T \wedge T_a^\delta}^\rho) &= -2\delta \mathbb{E}\left[\int_0^{T \wedge T_a^\delta} \frac{e^{-2\delta \underline{B_\delta}(t)}}{e^{-2\delta \underline{B_\delta}(t)} - e^{-2\delta a}} d\underline{B_\delta}(t)\right] \\ &= \mathbb{E}\left[A_T\right] = \mathbb{E}\left[\int_0^\infty U_s \, dA_s\right] \end{aligned} \tag{10.32}$$

□

Soit M_t^ρ la partie martingale intervenant dans la décomposition de Y^ρ :

$$M_t^\rho = 2\delta \int_0^t \frac{1}{e^{-2\delta \underline{B_\delta}(s)} - e^{-2\delta a}} e^{-2\delta B_\delta(s)} \, dB(s). \tag{10.33}$$

Démonstration de la proposition 10.3.3

1) Soient f et F des fonctions tests et $\theta = \mathbb{E}\left[f(S) F(B_\delta(s), 0 \leq s \leq \rho)\right]$.
On a :

$$\theta = \mathbb{E}\left[f(-\underline{B_\delta}(\rho)) F(B_\delta(s), 0 \leq s \leq \rho)\right]$$

Posons $U_t = f(-\underline{B_\delta}(t)) F(B_\delta(s), 0 \leq s \leq t)$.

$$\theta = \mathbb{E}\left[U_\rho\right] = \mathbb{E}\left[\int_0^\infty U_t\, dA_t^{(0)}\right]$$

$(U_t,t \geq 0)$ est un processus prévisible donc :

$$\begin{aligned}
\theta &= \mathbb{E}\left[\int_0^\infty U_t\, dA_t\right] \qquad (10.34)\\
&= -2\delta\mathbb{E}\left[\int_0^{T_a^\delta} f(-\underline{B_\delta}(t))F(B_\delta(s),0 \leq s \leq t)\frac{e^{-2\delta\underline{B_\delta}(t)}}{e^{-2\delta\underline{B_\delta}(t)} - e^{-2\delta a}} d\underline{B_\delta}(t)\right]
\end{aligned}$$

Particularisons avec $F = 1$ et effectuons le 'changement de variable' : $x = -\underline{B_\delta}(t)$.

$$\begin{aligned}
\mathbb{E}\left[f(S)\right] &= 2\delta\mathbb{E}\left[\int_0^S f(x)\frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\, dx\right]\\
&= 2\delta\mathbb{E}\left[\int_0^\infty f(x)\frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\mathbb{P}(S \geq x)\, dx\right]
\end{aligned} \qquad (10.35)$$

On en déduit que :

$$\mathbb{P}(S \in dx) = 2\delta\frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\mathbb{P}(S \geq x)\, dx.$$

Donc S admet une densité ϕ et :

$$\phi(x) = 2\delta\frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\int_x^\infty \phi(s)\, ds \qquad (10.36)$$

Soit Φ défini par $\Phi(x) = \int_x^\infty \phi(s)\, ds$.
(10.36) est l'équation différentielle vérifiée par Φ. On la résout sachant que $\Phi(0) = 1$:

$$\Phi(x) = \int_x^\infty \phi(s)\, ds = \frac{1 - e^{-2\delta a}}{e^{2\delta x} - e^{-2\delta a}}.$$

Par dérivation, nous obtenons :

$$\phi(x) = 2\delta(1 - e^{-2\delta a})\frac{e^{2\delta x}}{(e^{2\delta x} - e^{-2\delta a})^2}\mathbb{1}_{\{x \geq 0\}}.$$

Le résultat annoncé sur la densité de S en découle.
Revenons à (10.34) ; $(T_{-x}^\delta,x \geq 0)$ est l'inverse continu à droite de $-\underline{B_\delta}$ donc le

'changement de variable' $x = -\underline{B_\delta}(t)$ nous permet d'obtenir :

$$\begin{aligned}
\theta &= 2\delta \int_0^\infty f(x)\mathbb{E}\left[F(B_\delta(s), 0 \leq s \leq T^\delta_{-x})\mathbb{1}_{\{T^\delta_{-x}<T^\delta_a\}}\right] \frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\, dx \\
&= 2\delta \int_0^\infty f(x)\mathbb{E}\left[F(B_\delta(s), 0 \leq s \leq T^\delta_{-x}) | T^\delta_{-x} < T^\delta_a\right] \\
&\qquad \times \frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\, \mathbb{P}(T^\delta_{-x} < T^\delta_a)\, dx \\
&= 2\delta \int_0^\infty f(x)\mathbb{E}\left[F(B_\delta(s), 0 \leq s \leq T^\delta_{-x}) | T^\delta_{-x} < T^\delta_a\right] \\
&\qquad \times \frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\, \frac{1 - e^{-2\delta a}}{e^{2\delta x} - e^{-2\delta a}}\, dx
\end{aligned}$$

On utilise la proposition 10.2.5 et l'on obtient la loi de $(B_\delta(t), t \leq \rho)$ conditionnellement à S.
2) On s'intéresse à présent au processus $(B_\delta(t+\rho) - B_\delta(\rho), 0 \leq t \leq T^\delta_a)$. On note $(\mathcal{F}^\rho_t)_{t\geq 0}$ la plus petite filtration contenant $(\mathcal{F}_t)_{t\geq 0}$ pour laquelle ρ est un temps d'arrêt. On applique le théorème 10.2.1, compte tenu du lemme 10.3.4, de (10.29) et (10.33), on a :

$$\begin{aligned}
\tilde{\chi}(t) &= -\int_0^{t\wedge\rho} \frac{(e^{-2\delta \underline{B_\delta}(s)} - e^{-2\delta a}) 2\delta e^{-2\delta B_\delta(s)}}{(e^{-2\delta B_\delta(s)} - e^{-2\delta a})(e^{-2\delta \underline{B_\delta}(s)} - e^{-2\delta a})} ds \\
&\quad + \int_0^t \mathbb{1}_{\{\rho<s\}} \frac{(e^{-2\delta \underline{B_\delta}(s)} - e^{-2\delta a}) 2\delta e^{-2\delta B_\delta(s)}}{(e^{-2\delta \underline{B_\delta}(s)} - e^{-2\delta B_\delta(s)})(e^{-2\delta \underline{B_\delta}(s)} - e^{-2\delta a})} ds \\
&= -\int_0^{t\wedge\rho} \frac{2\delta e^{-2\delta B_\delta(s)}}{e^{-2\delta B_\delta(s)} - e^{-2\delta a}}\, ds \\
&\quad + \int_0^t \mathbb{1}_{\{\rho<s\}} \frac{2\delta e^{-2\delta B_\delta(s)}}{e^{-2\delta \underline{B_\delta}(s)} - e^{-2\delta B_\delta(s)}}\, ds
\end{aligned} \tag{10.37}$$

On en déduit :

$$\tilde{\chi}(t+\rho) - \tilde{\chi}(\rho) = 2\delta \int_\rho^{\rho+t} \frac{e^{-2\delta B_\delta(s)}}{e^{-2\delta \underline{B_\delta}(s)} - e^{-2\delta B_\delta(s)}}\, ds \tag{10.38}$$

On sait que $(B^2(t) = B(t) - \tilde{\chi}(t))$ est un $\mathcal{F}^\rho$ mouvement brownien.
Soit $Z(t) = B_\delta(\rho + t) - B_\delta(\rho)$. Ce processus vérifie l'équation :

$$Z(t) = \tilde{B^2}(t) + \delta \int_0^t \coth(\delta Z(s))\, ds \tag{10.39}$$

où $\tilde{B^2}(t) = B^2(t+\rho) - B^2(\rho)$ est un $\mathcal{F}^\rho$ mouvement brownien indépendant de $\mathcal{F}^\rho_\rho$.
De plus, $T^\delta_a - \rho = \inf\{t \geq 0, Z(t) = a + b\}$, avec $b = -S$.

Il reste à remarquer que Z est une solution forte de (10.39) pour terminer la preuve. □

Nous allons donner un résultat annexe ici qui sera utile pour calculer la loi de $(S_1, S_2, \cdots, S_k)$:

Corollaire 10.3.5 *Soit S défini par :*

$$S = - \inf_{t \in [0, T_a^\delta]} B_\delta(t). \tag{10.25}$$

Conditionnellement à $\{S \leq z\}$, S a pour densité :

$$\frac{\delta \operatorname{sh} \delta a \operatorname{sh} \delta(z+a)}{\operatorname{sh} \delta z} \frac{1}{\operatorname{sh}^2 \delta(x+a)} \mathbb{1}_{\{0 \leq x \leq z\}} \tag{10.40}$$

Démonstration On utilise le résultat de la proposition 10.3.3, S a pour densité la fonction φ donnée par (10.5) :

$$\varphi(x) = \delta e^{\delta a} \operatorname{sh} \delta a \frac{1}{\operatorname{sh}^2 \delta(x+a)} \mathbb{1}_{\{x \geq 0\}}.$$

Soit $z > 0$, soit $\varphi_z(x)$ la densité de S, sachant $\{S < z\}$; par définition :

$$\varphi_z(x) = \frac{\varphi(x)}{\displaystyle\int_0^z \varphi(y)\, dy} \mathbb{1}_{\{x \leq z\}}. \tag{10.41}$$

$$\begin{aligned}
\int_0^z \varphi(y)\, dy &= \delta e^{\delta a} \operatorname{sh} \delta a \int_0^z \frac{1}{\operatorname{sh}^2 \delta(y+a)}\, dy \\
&= -e^{\delta a} \operatorname{sh} \delta a \left[\frac{\operatorname{ch} \delta(y+a)}{\operatorname{sh} \delta(y+a)} \right]_{y=0}^{z} \\
&= -e^{\delta a} \operatorname{sh} \delta a \left(\frac{\operatorname{ch} \delta(z+a)}{\operatorname{sh} \delta(z+a)} - \frac{\operatorname{ch} \delta a}{\operatorname{sh} \delta a} \right) \\
&= \frac{e^{\delta a}}{\operatorname{sh} \delta(z+a)} (\operatorname{ch} \delta a \operatorname{sh} \delta(z+a) - \operatorname{ch} \delta(z+a) \operatorname{sh} \delta a) \\
\int_0^z \varphi(y)\, dy &= \frac{e^{\delta a}}{\operatorname{sh} \delta(z+a)} \operatorname{sh} \delta z \qquad (10.42)
\end{aligned}$$

Ce qui termine la preuve. □

10.3.3 Preuve du théorème 10.1.1

Comme annoncé, les résultats du théorème 10.1.1 pour $k = 2$ sont une conséquence immédiate des propositions 10.3.1 et 10.3.2 et de la symétrie du mouvement brownien.

Pour obtenir les résultats pour $k \geq 3$, on raisonne par récurrence en appliquant le corollaire 10.3.5.

10.4 Quelques applications

10.4.1 Descriptions des densités

Nous allons donner ici une interprétation probabiliste des amplitudes :

$$A_k = S_k + S_{k+1} \tag{10.43}$$

Dans cette première proposition, nous allons donner la densité du vecteur $(A_1, \ldots ,A_n)$ pour tout $n \in \mathbb{N}^*$

Proposition 10.4.1 $(A_1, \ldots ,A_n)$ *a pour densité :*

$$\delta^n \left[\prod_{k=1}^{n-1} \frac{e^{(-1)^k \delta a_k}}{\operatorname{sh}(\delta a_k)}\right] \frac{e^{(-1)^n 2\delta a_n} - 1 + (-1)^{n+1} 2\delta a_n}{\operatorname{sh}^2 \delta a_n} \mathbb{1}_{\{0 \leq a_n \leq \cdots \leq a_1\}} \tag{10.44}$$

Démonstration Soit F une fonction test.

$$\begin{aligned}
\mathbb{E}\left[F(A_1, \ldots ,A_n)\right] &= \mathbb{E}\left[F(S_1 + S_2, \ldots ,S_n + S_{n+1})\right] \\
&= \int F(x_1 + x_2, \ldots ,x_n + x_{n+1}) \delta^{n+1} 2 e^{-2\delta x_1} \\
e^{\delta x_1} \frac{\operatorname{sh}(\delta x_1)}{\operatorname{sh}^2 \delta(x_2 + x_1)} &\prod_{k=3}^{n+1} \frac{\operatorname{sh} \delta x_{k-1} \operatorname{sh} \delta(x_{k-1} + x_{k-2})}{\operatorname{sh} \delta x_{k-2} \operatorname{sh}^2 \delta(x_k + x_{k-1})} \mathbb{1}_{\{0 \leq x_k \leq x_{k-2}\}} dx_1 \ldots dx_{n+1} \\
&= \int F(x_1 + x_2, \ldots ,x_n + x_{n+1}) 2\delta^{n+1} \frac{e^{-\delta x_1} \operatorname{sh} \delta x_n}{\prod_{k=1}^{n-1} \operatorname{sh} \delta(x_{k+1} + x_k)} \\
&\quad \frac{1}{\operatorname{sh}^2 \delta(x_{n+1} + x_n)} \mathbb{1}_{\{x_1 \geq x_3 \geq \cdots \geq 0\}} \mathbb{1}_{\{x_2 \geq x_4 \geq \cdots \geq 0\}} dx_1 \ldots dx_{n+1}
\end{aligned}$$

On effectue maintenant le changement de variable :

$$\begin{aligned}
x_1 &= x_1 \\
a_1 &= x_1 + x_2 \\
&\ldots \\
a_n &= x_n + x_{n+1}
\end{aligned} \tag{10.45}$$

$$\begin{aligned}
\mathbb{E}\left[F(A_1, \ldots ,A_n)\right] &= \int F(a_1, \ldots ,a_n) 2\delta^{n+1} e^{-\delta x_1} \\
&\quad \frac{\operatorname{sh} \delta(a_{n-1} - a_{n-2} + \cdots + (-1)^n a_1 + (-1)^{n+1} x_1)}{\left(\prod_{k=1}^{n-1} \operatorname{sh} \delta a_k\right) \operatorname{sh}^2 \delta a_n} \\
&\quad \mathbb{1}_{\{0 \leq a_n \leq a_{n-1} \leq \cdots \leq a_1\}} \mathbb{1}_{\{a_1 - a_2 + \cdots + (-1)^{n+1} a_n \leq x_1 \leq a_1 - a_2 + \cdots + (-1)^n a_{n-1}\}} \\
&\quad dx_1 da_1 \ldots da_n
\end{aligned}$$

Reste alors à calculer la quantité suivante :

$$\begin{aligned}
I &= \int_{a_1-a_2+\dots+(-1)^{n+1}a_n}^{a_1-a_2+\dots+(-1)^n a_{n-1}} e^{-\delta x_1} \operatorname{sh}\delta(a_{n-1}-a_{n-2}+\dots+(-1)^n a_1+(-1)^{n+1}x_1)\,dx_1 \\
&= e^{-\delta a_1}e^{\delta a_2}\dots e^{(-1)^{n-1}a_{n-1}} \int_0^{a_n} e^{\delta y}\operatorname{sh}\delta y\,dy \\
&= e^{-\delta a_1}e^{\delta a_2}\dots e^{(-1)^{n-1}a_{n-1}} \frac{1}{2\delta}\frac{e^{(-1)^n 2\delta a_n}-1+(-1)^{n+1}2\delta a_n}{\operatorname{sh}^2\delta a_n}
\end{aligned}$$

Ainsi, nous obtenons :

$$\mathbb{E}\left[F(A_1,\dots,A_n)\right] = \int F(a_1,\dots,a_n)\delta^n \left[\prod_{k=1}^{n-1}\frac{e^{(-1)^k\delta a_k}}{\operatorname{sh}(\delta a_k)}\right] \tag{10.46}$$
$$\frac{e^{(-1)^n 2\delta a_n}-1+(-1)^{n+1}2\delta a_n}{\operatorname{sh}^2\delta a_n}\mathbb{1}_{\{0\leq a_n\leq\dots\leq a_1\}}\,da_1\dots da_n$$

□

Donnons maintenant une représentation probabiliste de ses variables aléatoires :

Proposition 10.4.2

$$\left(\psi(2\delta A_1),\frac{\psi(-2\delta A_2)}{\psi(-2\delta A_1)},\dots,\frac{\psi((-1)^{n+1}2\delta A_n)}{\psi((-1)^{n+1}2\delta A_{n-1})}\right) = (U_1,U_2,\dots,U_n) \tag{10.47}$$

où $U_1,U_2,\dots,U_n$ sont des variables aléatoires indépendantes uniformes sur $[0,1]$,

$$\psi(x) = \frac{e^x-1-x}{e^x-1} \tag{10.48}$$

Démonstration On connaît la densité h_1 de A_1 par rapport à la mesure de Lebesgue :

$$h_1(x) = \frac{\delta}{2\operatorname{sh}^2\delta x}\left(e^{-2\delta x}-1+2\delta x\right) \tag{10.49}$$

Donnons ici le lemme calculatoire suivant :

Lemme 10.4.3

$$\int_0^x \frac{e^{2y}-1-2y}{\operatorname{sh}^2 y}dy = \frac{e^x}{\operatorname{sh} x}\left(e^{-2x}-1+2x\right) \tag{10.50}$$

Notons H_1 la fonction de répartition de A_1, on a :

$$H_1(x) = \frac{\delta}{2}\int_0^x \frac{e^{-2\delta u}-1+2\delta u}{\operatorname{sh}^2\delta u}du$$

On utilise le lemme, et on obtient :

$$\begin{aligned} H_1(x) &= \frac{\delta}{2\,\mathrm{sh}(-\delta x)} \frac{e^{-\delta x}}{-\delta} \left(e^{2\delta x} - 1 - 2\delta x\right) \\ &= \frac{e^{2\delta x} - 1 - 2\delta x}{e^{2\delta x} - 1} \\ &= \psi(2\delta x). \end{aligned} \tag{10.51}$$

On montre aisément que ψ est bijective et l'on obtient ainsi :

$$A_1 \sim \frac{1}{2\delta}\psi^{-1}(U_1) \tag{10.52}$$

où U_1 est une variable aléatoire uniforme sur [0,1].
On calcule ici la densité $h_2^{(x)}$ de A_2 sachant $A_1 = x$

$$\begin{aligned} h_2^{(x)}(y) &= \frac{\delta^2}{2} \frac{e^{2\delta y} - 1 - 2\delta y}{\mathrm{sh}^2\,\delta y} \frac{e^{-\delta x}}{\mathrm{sh}\,\delta x} \mathbb{1}_{\{0 \leq y \leq x\}} \frac{2\,\mathrm{sh}^2\,\delta x}{\delta} \frac{1}{e^{-2\delta x} - 1 + 2\delta x} \\ &= \delta \frac{e^{-\delta x}\,\mathrm{sh}\,\delta x}{e^{-2\delta x} - 1 + 2\delta x} \frac{e^{2\delta y} - 1 - 2\delta y}{\mathrm{sh}^2\,\delta y} \mathbb{1}_{\{0 \leq y \leq x\}} \end{aligned} \tag{10.53}$$

On note $H_2^{(x)}$ la fonction de répartition associée :

$$H_2^{(x)}(y) = \frac{e^{-\delta x}\,\mathrm{sh}\,\delta x}{e^{-2\delta x} - 1 + 2\delta x} \delta \int_0^y \frac{e^{2\delta t} - 1 - 2\delta t}{\mathrm{sh}^2\,\delta t} dt. \tag{10.54}$$

On applique à nouveau le lemme et l'on obtient :

$$H_2^{(x)}(y) = \frac{e^{-\delta x}\,\mathrm{sh}\,\delta x}{e^{-2\delta x} - 1 + 2\delta x} \frac{e^{\delta y}}{\mathrm{sh}\,\delta y} \left(e^{-2\delta y} - 1 + 2\delta y\right) \tag{10.55}$$

$$= \frac{\psi(-2\delta y)}{\psi(-2\delta x)}. \tag{10.56}$$

On a ainsi prouvé :

$$\left(\psi(2\delta A_1), \frac{\psi(-2\delta A_2)}{\psi(-2\delta A_1)}\right) \sim (U_1, U_2).$$

La proposition se démontre alors sans difficulté par récurrence. □

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RÉSUMÉ

Cette thèse est composée de trois parties indépendantes.

La première partie est une étude probabiliste des équations de coagulation de Smoluchowski. Une représentation des solutions est établie grâce à des processus de branchement de type Galton-Watson. On montre par ailleurs une correspondance entre les noyaux additif et multiplicatif. Le comportement asymptotique des solutions après renormalisation est également étudié. Enfin, on construit un processus, solution d'une E.D.S. *non-linéaire* gouvernée par un processus de Poisson, dont les marginales temporelles sont solutions des équations de Smoluchowski. Ce processus permet d'obtenir des approximations au moyen d'un système de particules.

Dans la deuxième partie, nous estimons l'erreur commise en remplaçant une diffusion régulière par son approximation obtenue avec le schéma d'Euler pour calculer l'espérance de certaines fonctionnelles irrégulières de la trajectoire de cette diffusion. Nous obtenons notamment la vitesse optimale de convergence dans le cas de l'intégrale d'une fonction seulement mesurable et bornée de la trajectoire.

Dans la troisième partie, nous étudions le processus de l'amplitude d'un mouvement brownien avec dérive non nulle. Nous donnons une décomposition des trajectoires en utilisant les extremums successifs en « remontant » le temps. Les résultats sont obtenus notamment à l'aide de techniques de grossissements de filtrations.

Mots-Clés : Équations de coagulation de Smoluchowski, Processus de branchement, Processus de Poisson, Schéma d'Euler, Équation différentielle stochastique, Amplitude, Mouvement brownien

ABSTRACT

This thesis consists of three independent parts.

The first part is a probabilist study of the Smoluchowski's coagulation equations. A representation of the solutions is established thanks to branching processes of type Galton-Watson. One shows besides a connection between additive and multiplicatif kernels. The asymptotique behavior of the solutions after renormalisation is also studied. Finally, one builds a process, a solution of a non-linear S.D.E. governed by a Poisson process, temporal of which marginal are solutions of the equations of Smoluchowski. This process allows to obtain estimates with a particles system.

In the second part, we estimate the error committed by replacing a regular diffusion by its estimate obtained with Euler's scheme to compute the expectation of some irregular functional of the trajectory of this diffusion. We notably obtain the optimal speed of convergence in the case of the integral of a function only measurable and bounded of the trajectory.

In the third part, we study the process of the range of a brownian motion with drift. We give a decomposition of trajectories by using the successive extremums by "raising" the time. The results are notably obtained by means of techniques of filtrations enlargement.