Range of Brownian motion with drift

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Abstract

Let $(B_\delta(t))_{t\geq 0}$ be a Brownian motion starting at 0 with drift $\delta > 0$. Define by induction $S_1 = -\inf_{t\geq 0} B_\delta(t)$, ρ_1 the last time such that $B_\delta(\rho_1) = -S_1$, $S_2 = \sup_{0\leq t\leq \rho_1} B_\delta(t)$, ρ_2 the last time such that $B_\delta(\rho_2) = S_2$ and so on. Setting $A_k = S_k + S_{k+1}\,;\, k \geq 1$, we compute the law of $(A_1, \cdots, A_k)$ and the distribution of $((B_\delta(t+\rho_l) - B_\delta(\rho_l)\,;\, 0 \leq t \leq \rho_{l-1} - \rho_l))_{2\leq l\leq k}$ for any $k \geq 2$, conditionally on $(A_1, \cdots, A_k)$. We determine the law of the range $R_\delta(t)$ of $(B_\delta(s))_{s\geq 0}$ at time t, and the first range time $\theta_\delta(a)$ (i.e. $\theta_\delta(a) = \inf\{t > 0\,;\, R_\delta(t) > a\}$). We also investigate the asymptotic behaviour of $\theta_\delta(a)$ (resp. $R_\delta(t)$) as $a \to \infty$ (resp. $t \to \infty$).

Key words : Range Process, Enlargement of filtration, Brownian motion with drift.

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1 Introduction

1) The range of one-dimensional Markov chains or random walks has been already investigated (see [5] and [4]). Vallois [14] provides a short survey. The aim of this paper is to study the range of a Brownian motion with drift. This process serves as a prototype of transient diffusions.

Let X be a continuous process. The **range** of X, denoted $(R^{(X)}(t))_{t\geq 0}$ is defined by

$$R^{(X)}(t) = \sup_{0\leq u,v\leq t}(X_v - X_u) = \sup_{0\leq u\leq t} X_u - \inf_{0\leq u\leq t} X_u. \tag{1.1}$$

When X is a one-dimensional Brownian motion started at 0, Feller [4] has computed the density function of $R^{(X)}(t)$, using the fact that the joint distribution of $\sup_{0\leq u\leq t} X_u$ and $\inf_{0\leq u\leq t} X_u$ is explicitly known. Unfortunately the result is expressed as the sum of a series, and the result cannot be generalized to diffusions

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since the joint distribution of the maximum and the minimum is in general unknown.

To go further we observe that $t \mapsto R^{(X)}(t)$ is a non-decreasing, continuous function starting at 0. Therefore we can define its right continuous inverse :

$$\theta^{(X)}(a) = \inf \left\{ t \geq 0; R^{(X)}(t) > a \right\}. \tag{1.2}$$

It is equivalent to deal with either $(\theta^{(X)}(a))_{a\geq 0}$ or $(R^{(X)}(t))_{t\geq 0}$, since we have

$$\left\{ R^{(X)}(t) < a \right\} = \left\{ \theta^{(X)}(a) > t \right\}. \tag{1.3}$$

It is more convenient to work with $(\theta^{(X)}(a))_{a\geq 0}$. Assume that $(X_t)_{t\geq 0}$ is a diffusion process and for simplicity $X_0 = 0$. It is proved (Theorem 4, [12]) that the process $(X_t; 0 \leq t \leq \theta^{(X)}(Ua))$ has the same law as $(X_t; 0 \leq t \leq T^{(X)}(aU) \wedge T^{(X)}(a(U-1))$, where $a > 0$, $T^{(X)}(c)$ is the first hitting time of the level c, and U denotes a r.v. uniformly distributed on $[0, 1]$, independent of the underlying process $(X_t)_{t\geq 0}$. This property has been generalized in [10]. The Laplace transform of $T^{(X)}(c) \wedge T^{(X)}(d)$ can be expressed through some eigenfunctions associated with the generator of (X_t). Consequently the previous result gives an analytic expression for the Laplace transform of $\theta^{(X)}(a)$. In the case of Brownian motion, then for any $a > 0$, the Laplace transform of the r.v. $\theta^{(X)}(a)$ can be computed, see [6], [12]. Moreover $(\theta^{(X)}(a))_{a\geq 0}$ has independent increments.

2) Let $(B_\delta(t))_{t\geq 0}$ be Brownian motion with drift δ : $B_\delta(t) = B(t) + \delta\, t$. For simplicity, we note $(R_\delta(t))_{t\geq 0}$ as the range of $(B_\delta(t))_{t\geq 0}$ instead of $(R^{(B_\delta)}(t))_{t\geq 0}$, where $(\theta_\delta(a))_{a\geq 0}$ stands for the right continuous inverse of $(R_\delta(t))_{t\geq 0}$.

Let Σ_δ be the random set :

$$\Sigma_\delta = \{a \geq 0\, ;\, B_\delta(\theta_\delta(a)) B_\delta(\theta_\delta(a-)) < 0\} \tag{1.4}$$

where $\theta_\delta(a-)$ is the left limit of θ_δ at a.

When $\delta = 0$ (recurrent case), it is not possible to enumerate the points in Σ_0 [7, 13].

Suppose $\delta > 0$ (transient case). The process $B_\delta(t)$ drifts to infinity, as $t \to +\infty$. It enables an explicit description of Σ_δ. Let us first introduce the minimum $-S_1$ of $(B_\delta(t))_{t\geq 0}$, and ρ_1 the last time such that $B_\delta(\rho_1) = -S_1$. We next define $(S_k, \rho_k)_{k\geq 2}$ inductively as follows :

$$\begin{cases} S_k &= \sup\limits_{t\in[0,\rho_{k-1}]} (-1)^k B_\delta(t), \\ \rho_k &= \sup\{t \in [0, \rho_{k-1}]\ ,\ B_\delta(t) = (-1)^k\, S_k\}. \end{cases} \tag{1.5}$$

Note that if we set $\rho_0 = \infty$, then (1.5) remains valid for $k = 1$.

Then

$$\Sigma_\delta = \{A_n\, ;\, n \geq 1\} \tag{1.6}$$

where $A_n = S_n + S_{n+1}$.

This leads us to compute the density function of $(S_1, \cdots, S_n)$ and $(A_1, \cdots, A_n)$ (see respectively Proposition 2.1 and Theorem 2.2). It is worth pointing out that the whole trajectory $(B_\delta(t))_{t\geq 0}$ can be express through S_1 and the sequence of processes $(|B_\delta(t+\rho_n) - B_\delta(\rho_n)|\,;\, 0 \leq t \leq \rho_{n-1} - \rho_n)_{n\geq 1}$. The laws of these processes are given in Theorem 2.4 and Proposition 3.1. There are formulated in terms of distribution of non-negative diffusion of the type $(Z^{(\delta)}(t))_{t\geq 0}$, where

$$Z^{(\delta)}(t) = B_t + \delta \int_0^t \coth(\delta Z^{(\delta)}(s))\, ds. \tag{1.7}$$

Heuristically, $(Z^{(\delta)}(t))_{t\geq 0}$ is the process $(B_\delta(t))_{t\geq 0}$ conditioned to be positive, see [16]. Formally taking the limit $\delta \to 0$ in (1.7), we recover the three-dimensional Bessel process starting at 0 which plays a central role in the decomposition of the Brownian motion via the range process [7, 13].

3) In Section 4, we focus on the law of $R_\delta(t)$ (resp. $\theta_\delta(a)$) where $t > 0$ (resp. $a > 0$) is fixed. We compute the two distribution functions of $R_\delta(t)$ and $\theta_\delta(a)$. We partially recover the result of [2]. A path decomposition of $(B_\delta(t)\,;\, 0 \leq t \leq \theta_\delta(a))$ (Proposition 4.1) allows us to determine the Laplace transform of $\theta_\delta(a)$. This in return allows us to obtain the asymptotic behaviour of $\theta_\delta(a)$, $a \to +\infty$: a first result resembles a Law of Large Numbers and a second result is analogous to the Central Limit Theorem.

2 Notations and main results

We retain the notation introduced in the Introduction. Throughout this paper, we assume $\delta > 0$.

Since $B_\delta(t)$ goes to $+\infty$, as $t \to \infty$, and $t \mapsto B_\delta(t)$ is a continuous function, the random times $(\rho_k)_{k\geq 1}$ are well defined. This does not mean that $\{t \in [0, \rho_{k-1}]; B_\delta(t) = (-1)^k S_k\}$ is reduced to the singleton $\{\rho_k\}$. We need to prove this property holds.

We start with the law of $(S_1, \cdots, S_k)\,;\, k \geq 1$.

Proposition 2.1 *Suppose $\delta > 0$. Let $(S_k)_{k\geq 1}$ be defined by* (1.5). *Then*

1. *The law of S_1 is exponential of parameter 2δ (i.e. with density function $2\delta e^{-2\delta x} \mathbb{1}_{\{x\geq 0\}}$).*
2. *Conditionally on $\{S_1 = x_1\}$, S_2 has density function :*

$$\delta e^{\delta x_1} \operatorname{sh}(\delta x_1) \frac{1}{\operatorname{sh}^2(\delta(x + x_1))} \mathbb{1}_{\{x\geq 0\}}. \tag{2.1}$$

3. *For every $k \geq 3$, conditionally on $\{S_1 = x_1, \ldots, S_{k-1} = x_{k-1}\}$, S_k has density function :*

$$\delta \frac{\operatorname{sh} \delta x_{k-1} \operatorname{sh} \delta(x_{k-1} + x_{k-2})}{\operatorname{sh} \delta x_{k-2}} \frac{1}{\operatorname{sh}^2 \delta(x_{k-1} + x)} \mathbb{1}_{\{0\leq x\leq x_{k-2}\}}.$$

Proof The proof of Proposition 2.1 is given in Section 3. □

Note that $x_k + x_{k-1}$ appears in the conditional density function of S_k. This leads us to introduce

$$A_k = S_k + S_{k+1} \quad ; \quad k \geq 1. \tag{2.2}$$

$(A_k)_{k\geq 1}$ is the sequence of maximal ranges associated with $(B_\delta(t))_{t\geq 0}$.

Theorem 2.2 *Suppose $\delta > 0$. Then*

1. *$(A_1, \cdots, A_n)$ has a density function given by*

$$\frac{\delta^n}{2}\left[\prod_{k=1}^{n-1} \frac{e^{(-1)^k \delta a_k}}{\mathrm{sh}(\delta a_k)}\right] \frac{e^{(-1)^n 2\delta a_n} - 1 + (-1)^{n+1} 2\delta a_n}{\mathrm{sh}^2\, \delta a_n} \mathbb{1}_{\{0\leq a_n \leq \cdots \leq a_1\}}. \tag{2.3}$$

2. *Let : $\psi : \mathbb{R} \to (-\infty, 1)$*

$$\psi(x) = \frac{e^x - 1 - x}{e^x - 1} \quad ; \quad x \neq 0 \quad \textit{and} \quad \psi(0) = 0.$$

ψ is one-to-one from $\mathbb{R}$ to $(-\infty, 1)$ and we have

$$\left(\psi(2\delta A_1), \frac{\psi(-2\delta A_2)}{\psi(-2\delta A_1)}, \ldots, \frac{\psi((-1)^{n+1} 2\delta A_n)}{\psi((-1)^{n+1} 2\delta A_{n-1})}\right) \stackrel{d}{=} (U_1, U_2, \ldots, U_n) \tag{2.4}$$

where $U_1, \cdots, U_n$ are i.i.d. r.v.'s, uniformly distributed on $[0, 1]$.

Proof The proof is given in Section 3 □

Remark 2.3 1) Recall that Σ_δ is defined by (1.4). In [13], it is proved that if $\delta = 0$, then Σ_0 is a one-dimensional Poisson point process (P.p.p.) with characteristic measure $\nu(da) = \frac{1}{a}\mathbb{1}_{\{a>0\}}da$. Concerning P.p.p. we refer to [9] (chapter XII).

If $\delta > 0$, we claim that Σ_δ is not a P.p.p.. Using (2.4), after lengthy calculations we can show

$$\mathbb{E}\left[\sum_{n\geq 1} f(A_n)\right] = \delta^2 \int_0^\infty f(x) \frac{x}{\mathrm{sh}^2(\delta x)} dx, \tag{2.5}$$

for any positive Borel function f.

Suppose that Σ_δ is a P.p.p. with characteristic measure ν_δ. Then (2.5) implies that :

$$\nu_\delta(dx) = \frac{\delta^2 x}{\mathrm{sh}^2(\delta x)} \mathbb{1}_{\{x>0\}} dx. \tag{2.6}$$

A straightforward calculation gives :

$$\nu_\delta\left([b,+\infty)\right) = \delta b \coth(\delta b) - \log(\mathrm{sh}(\delta b)) - \log 2 \ , \ b > 0. \tag{2.7}$$

Consequently, $N = \sum_{n\geq 1} \mathbb{1}_{\{A_n \geq b\}}$ is a Poisson random variable with parameter $\nu_\delta\left([b,+\infty)\right)$.

But $n \mapsto A_n$ is a decreasing sequence, hence

$$\mathbb{P}(N=0) = \mathbb{P}(A_1 < b) = \Psi(2\delta b).$$

This generates a contradiction since

$$\mathbb{P}(N=0) = \exp(-\nu_\delta([b,+\infty))) \neq \Psi(2\delta b).$$

2) Due to (2.3), it is easy to check that $(A_{2n-1}, A_{2n}; n \geq 1)$ is a Markov chain, which takes its values in $\left\{(x,y) \in \mathbb{R}^2; 0 < y < x\right\}$, with transition probability density function :

$$\mathbb{P}\left((u,v);(x,y)\right) = \frac{4\delta^2}{\Psi(2\delta v)} \frac{1}{e^{2\delta x}-1} \frac{e^{2\delta y}}{e^{2\delta y}-1} \Psi(2\delta y) \mathbb{1}_{\{0<y<x<v\}}. \tag{2.8}$$

Note that this quantity dœs not depend on u.

3) If we set

$$X_n = \psi\left((-1)^{n+1} A_n\right) \ ; \ n \geq 1,$$

then $(X_n)_{n\geq 1}$ is a Markov chain on $(-\infty, 1)$, whose initial distribution is uniform on $[0,1]$, with transition probability kernel

$$K(x,f) = \mathbb{E}\left[f(U\psi(-\psi^{-1}(x)))\right], \tag{2.9}$$

where U denotes a *r.v.* uniformly distributed on $[0,1]$.

We now give the law of the sequence of processes $((B_\delta(t+\rho_k) - B_\delta(\rho_k)\,;\, t \in [0, \rho_{k-1} - \rho_k]))_{k\geq 1}$. The result uses of the process $Z^{(\delta)}$ defined by (1.7).

Theorem 2.4 *Let $\delta > 0$ and $k \geq 2$.*
Conditionally on $S_1 = x_1, \ldots, S_k = x_k$,

1) $(B_\delta(t)\,;\, 0 \leq t \leq \rho_k)$, $(B_\delta(t+\rho_1) - B_\delta(\rho_1)\,;\, t \geq 0)$, $(B_\delta(t+\rho_2) - B_\delta(\rho_2)\,;\, 0 \leq t \leq \rho_1 - \rho_2)$, ..., $(B_\delta(t+\rho_k) - B_\delta(\rho_k)\,;\, 0 \leq t \leq \rho_{k-1} - \rho_k)$ are independent ;

2) $(B_\delta(t)\,;\, 0 \leq t \leq \rho_k)$ has the law of Brownian motion with drift $(-1)^k\delta$ stopped at the first hitting time of level $(-1)^k x_k$, and conditioned not to hit $(-1)^{k+1}x_{k-1}$;

3) For any $2 \leq l \leq k$, $(|B_\delta(t+\rho_l) - B_\delta(\rho_l)|\,;\, 0 \leq t \leq \rho_{l-1} - \rho_l)$ is a process with the same law as $Z^{(\delta)}$ stopped at the first moment it reaches the level $x_l + x_{l-1}$.

3 Proofs of Theorems 2.2, 2.4 and Proposition 2.1

We retain the notation of Sections 1 and 2.

3.1 Proof of Theorem 2.2

1) Formula (2.3) is a direct consequence of Proposition 2.1. Indeed, let F be a test function. We have :

$$\begin{aligned}\mathbb{E}\left[F(A_1,\dots,A_n)\right] &= \mathbb{E}\left[F(S_1+S_2,\dots,S_n+S_{n+1})\right]\\ &= \int_{\mathbb{R}^{n+1}} F(x_1+x_2,\dots,x_n+x_{n+1})2\delta^{n+1}\frac{e^{-\delta x_1}\operatorname{sh}\delta x_n}{\prod_{k=1}^{n-1}\operatorname{sh}\delta(x_{k+1}+x_k)}\\ &\times\frac{1}{\operatorname{sh}^2\delta(x_{n+1}+x_n)}\mathbb{1}_{\{x_1\geq x_3\geq\dots\geq 0\}}\mathbb{1}_{\{x_2\geq x_4\geq\dots\geq 0\}}dx_1\dots dx_{n+1}.\end{aligned}$$

We use the following change of variables :

$$\begin{cases} x_1 &= x_1\\ a_1 &= x_1+x_2\\ \dots & \\ a_n &= x_n+x_{n+1}.\end{cases}$$

Then,

$$\begin{aligned} x_k-x_{k+2} &= a_k-a_{k+1}\,;\ 1\leq k\leq n-1\\ x_n &= \alpha+(-1)^{n+1}x_1\\ x_{n+1} &= a_n-(\alpha+(-1)^{n+1}x_1)\\ \text{where}\qquad \alpha &= a_{n-1}-a_{n-2}+\dots+(-1)^n a_1.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbb{E}\left[F(A_1,\dots,A_n)\right] = \int_{\mathbb{R}^n} F(a_1,\dots,a_n)&\frac{2\delta^n}{\left(\prod_{k=1}^{n-1}\operatorname{sh}(\delta a_k)\right)\operatorname{sh}^2\delta a_n}\\ &\times\mathbb{1}_{\{0\leq a_n\leq a_{n-1}\leq\dots\leq a_1\}}I(\alpha)da_1\dots da_n,\end{aligned}$$

where

$$I(\alpha) = \int_0^\infty \delta e^{-\delta x_1}\operatorname{sh}\delta(\alpha+(-1)^{n+1}x_1)\mathbb{1}_{\{0\leq\alpha+(-1)^{n+1}x_1\leq a_n\}}\,dx_1.$$

Suppose for instance that n is even. Setting $y=\alpha-x_1$, we get

$$\begin{aligned} I(\alpha) &= e^{-\delta\alpha}\delta\int_0^{a_n} e^{\delta y}\operatorname{sh}\delta y\,dy\\ &= \frac{e^{-\delta\alpha}}{4}\left(e^{2\delta a_n}-1-2\delta a_n\right).\end{aligned}$$

Then (2.3) follows immediately.

2) The density function h_1 of A_1 is

$$h_1(x) = \frac{\delta}{2\,\mathrm{sh}^2\,\delta x}\left(e^{-2\delta x} - 1 + 2\delta x\right).$$

Let H_1 denote the distribution function of A_1

$$H_1(x) = \int_0^x h_1(y)\,dy.$$

We have

$$\int_0^x \frac{e^{2y} - 1 - 2y}{\mathrm{sh}^2\,y} dy = \frac{e^x}{\mathrm{sh}\,x}\left(e^{-2x} - 1 + 2x\right). \tag{3.1}$$

Consequently

$$H_1(x) = \psi(2\delta x). \tag{3.2}$$

So

$$A_1 \overset{d}{=} \frac{1}{2\delta}\psi^{-1}(U_1),$$

where U_1 is a uniform random variable on $[0,1]$.

We now compute the density $h_2^{(x)}$ of A_2, conditionally on $A_1 = x$.

$$\begin{aligned} h_2^{(x)}(y) &= \frac{\delta^2}{2}\frac{e^{2\delta y} - 1 - 2\delta y}{\mathrm{sh}^2\,\delta y}\frac{e^{-\delta x}}{\mathrm{sh}\,\delta x}\mathbb{1}_{\{0\le y\le x\}}\frac{2\,\mathrm{sh}^2\,\delta x}{\delta}\frac{1}{e^{-2\delta x} - 1 + 2\delta x} \\ &= \delta\frac{e^{-\delta x}\,\mathrm{sh}\,\delta x}{e^{-2\delta x} - 1 + 2\delta x}\frac{e^{2\delta y} - 1 - 2\delta y}{\mathrm{sh}^2\,\delta y}\mathbb{1}_{\{0\le y\le x\}}. \end{aligned}$$

Let $H_2^{(x)}$ be the associated distribution function. Then

$$H_2^{(x)}(y) = \frac{e^{-\delta x}\,\mathrm{sh}\,\delta x}{e^{-2\delta x} - 1 + 2\delta x}\delta\int_0^y \frac{e^{2\delta t} - 1 - 2\delta t}{\mathrm{sh}^2\,\delta t}dt.$$

Relation (3.1) implies that :

$$\begin{aligned} H_2^{(x)}(y) &= \frac{e^{-\delta x}\,\mathrm{sh}\,\delta x}{e^{-2\delta x} - 1 + 2\delta x}\frac{e^{\delta y}}{\mathrm{sh}\,\delta y}\left(e^{-2\delta y} - 1 + 2\delta y\right) \\ &= \frac{\psi(-2\delta y)}{\psi(-2\delta x)}. \end{aligned}$$

As a result, we have proved that $\left(\psi(2\delta A_1), \dfrac{\psi(-2\delta A_2)}{\psi(-2\delta A_1)}\right)$ is distributed as (U_1, U_2), where U_1 and U_2 are two independent r.v.'s, uniformly distributed on $[0,1]$.

Reasoning by induction, the identity (2.4) may be proved by the same way via (2.3).

3.2 Proofs of Theorem 2.4 and Proposition 2.1

In our approach, Theorem 2.4 and Proposition 2.1 are a direct consequence of Proposition 3.2 stated below. Therefore, we first focus on this key result, and we prove Theorem 2.4 and Proposition 2.1 at the end of this subsection.

The description of the laws of $(B_\delta(t)\,;\,0 \le t \le \rho_1)$ and $(B_\delta(t) - B_\delta(\rho_1)\,;\,t \ge \rho_1)$ is given by the well-known theorem of Williams [16].

Proposition 3.1 **([16])**

1) The law of S_1 is exponential with parameter 2δ (i.e. its density function is $2\delta e^{-2\delta x}\mathbb{1}_{\{x\ge 0\}}$).

2) Conditionally on $S_1 = x_1$,

a. $(B_\delta(t)\,;\,t \le \rho_1)$ and $(B_\delta(t+\rho_1)-B_\delta(\rho_1)\,;\,t \ge 0)$ are independent processes

b. $(B_\delta(t)\,;\,t \le \rho_1)$ is a process with the same law as a $B_{-\delta}$ stopped at its first hitting time of $-x_1$

c. $(B_\delta(t+\rho_1) - B_\delta(\rho_1)\,;\,t \ge 0)$ is a (positive) process distributed as $Z^{(\delta)}$.

To obtain the decomposition of the Brownian motion with drift given in Theorem 2.4, Proposition 3.1 leads us to study $(B_{-\delta}(t)\,;\,0 \le t \le T_{-a}^{-\delta})$ conditionally on $\max_{0\le t\le T_{-a}^{-\delta}} B_{-\delta}(t)$ where :

$$T_{-a}^{-\delta} = \inf\{t \ge 0\ ,\ B_{-\delta}(t) < -a\}\ \ ,\ a > 0. \tag{3.3}$$

Since $(-B_{-\delta}(t)\,;\,0 \le t \le T_{-a}^{-\delta})$ and $(B_\delta(t)\,;\,0 \le t \le T_a^\delta)$ have the same law, it is equivalent to determine the distribution of $(B_\delta(t)\,;\,0 \le t \le T_a^\delta)$ conditionally on

$$S = -\inf_{t\in[0,T_a^\delta]} B_\delta(t), \tag{3.4}$$

T_a^δ being the first hitting time of level a :

$$T_a^\delta = \inf\{t \ge 0\ ,\ B_\delta(t) > a\}\ \ ,\ a > 0.$$

Let ρ be the random time :

$$\rho = \sup\{t \in [0, T_a^\delta], B_\delta(t) = -S\}. \tag{3.5}$$

Time ρ plays a central role in our approach, shows the following proposition

Proposition 3.2 *Let S and ρ be r.v.'s defined by* (3.4) *and* (3.5)*. Then,*

1) S has a density function φ

$$\varphi(x) = \delta e^{\delta a}\,\mathrm{sh}\,\delta a \frac{1}{\mathrm{sh}^2\,\delta(x+a)}\mathbb{1}_{\{x\ge 0\}}. \tag{3.6}$$

2) Conditionally on $S = b$,

a. $(B_\delta(t)\,;\,0 \le t \le \rho)$ and $(B_\delta(t+\rho) - B_\delta(\rho)\,;\,0 \le t \le T_a^\delta - \rho)$ are independent,

b. $(B_\delta(t)\,;\,0 \le t \le \rho)$ is a process with the same law as a Brownian motion with drift $-\delta$, stopped at its first hitting time of level $-b$ and conditioned to stay less than a,

c. $(B_\delta(t+\rho) - B_\delta(\rho)\,;\,0 \le t \le T_a^\delta - \rho)$ is a (positive) process distributed as the process $Z^{(\delta)}$, stopped at its first hitting time of level $a + b$

Before starting the proof of Proposition 3.2, we explain our approach.

Let $\mathbb{F}$ be the natural filtration generated by $(B(t)_{t\ge 0}$. Unfortunately the random time ρ is not a stopping time. However ρ is a last exit time : $\rho = \sup\{t \in [0, T_a^\delta]; B_\delta(t) = \sup_{0\le u\le t} B_\delta(u)\}$. This leads us to apply the theory of enlargement of filtrations (See Protter [8] Ch. VI).

Let $\mathbb{F}^\rho$ be the smallest right-continuous filtration containing $\mathbb{F}$ (i.e. $\mathcal{F}_t \subset \mathcal{F}_t^\rho$, for any $t \ge 0$) such that ρ is a $(\mathcal{F}_t^\rho)$-stopping time. Let :

$$Y_t^\rho = \mathbb{P}\left(\rho \le t | \mathcal{F}_t\right), \tag{3.7}$$

$(Y_t^\rho; t \ge 0)$ is the optional projection of $\left(\mathbb{1}_{\{\rho\le t\}}; t \ge 0\right)$ (cf [8] p 371).
It is easy to check that $(Y_t^\rho)_{t\ge 0}$ is an $\mathbb{F}$-submartingale. Let :

$$Y_t^\rho = M_t^\rho + A_t^\rho, \tag{3.8}$$

be its Doob-Meyer decomposition, where $(M_t^\rho; t \ge 0)$ denotes the martingale part, $(A_t^\rho; t \ge 0)$ is a non-decreasing and adapted process such that $A_0^\rho = 0$. The processes $(Y_t^\rho; t \ge 0)$ and $(A_t^\rho; t \ge 0)$ will be given in Lemma 3.3.
To determine the law of $\big(B_\delta(s); s \le \rho\big)$, we need the following result.
Let $(U_t; t \ge 0)$ be a non-negative and $\mathbb{F}$-adapted process then ([8] p 371) :

$$\mathbb{E}[U_\rho] = \mathbb{E}\left[\int_0^\infty U_t dA_t^\rho\right]. \tag{3.9}$$

Note that $S = -\min_{0\le s\le\rho} B_\delta(s)$. Taking $U_t = F\left(B_\delta(s); s \le t\right) f(-\min_{0\le s\le t} B_\delta(s))$, where F and f are measurable and non negative, we get :

$$\mathbb{E}\left[F\left(B_\delta(s); s \le \rho\right) f(S)\right] = \mathbb{E}\left[\int_0^\infty F\left(B_\delta(s); s \le t\right) f(-\min_{0\le s\le t} B_\delta(s)) dA_t^\rho\right].$$

Since $(A_t^\rho; t \ge 0)$ is explicitly known, previous identity allows us to determine the law of $\big(B_\delta(s); s \le \rho\big)$ conditionally on S.
To obtain the law of $(B_\delta(t+\rho) - B_\delta(\rho)\,;\,0 \le t \le T_a^\delta - \rho)$, we use ([8] Theorem 18, p 375) the following property :

$$\overline{B}(t) = B(t) - \chi(t) \;\; \text{is a } (\mathcal{F}_t^\rho) - \text{Brownian motion} \tag{3.10}$$

where :

$$\chi(t) = -\int_0^{t\wedge\rho} \frac{1}{1-Y_s^\rho} d\langle B, M^\rho\rangle_s + \int_0^t \mathbb{1}_{\{\rho<s\}} \frac{1}{Y_s^\rho} d\langle B, M^\rho\rangle_s . \qquad (3.11)$$

This allows us to prove that $(B_\delta(t+\rho) - B_\delta(\rho)\,;\, 0 \le t \le T_a^\delta - \rho)$ solves a stochastic differential equation of the type (1.7).

Lemma 3.3 *Let $(Y_t^\rho)_{t\ge 0}$ be the process defined by (3.7).*

1. *We have :*

$$Y_t^\rho = \frac{e^{2\delta \underline{B_\delta}(t\wedge T_a^\delta)} - e^{-2\delta B_\delta(t\wedge T_a^\delta)}}{e^{2\delta \underline{B_\delta}(t\wedge T_a^\delta)} - e^{-2\delta a}} \;;\, t \ge 0, \qquad (3.12)$$

where $\underline{B_\delta}(t) = -\inf_{s\le t} B_\delta(s)$.

2. *$(Y_t^\rho)_{t\ge 0}$ is an $\mathbb{F}$ sub-martingale with Doob-Meyer decomposition (3.8) and*

$$A_t^\rho = \ln\left(\frac{e^{2\delta \underline{B_\delta}(t\wedge T_a^\delta)} - e^{-2\delta a}}{1 - e^{-2\delta a}}\right) \qquad (3.13)$$

$$M_t^\rho = 2\delta \int_0^{t\wedge T_a^\delta} \frac{1}{e^{2\delta \underline{B_\delta}(s)} - e^{-2\delta a}} e^{-2\delta B_\delta(s)}\, dB(s). \qquad (3.14)$$

Proof a) Recall a classical result concerning hitting times of Brownian motion with drift (see for instance Borodin and Salminen [1] formula 2.1.2 p. 295) :

$$\mathbb{P}\left(T_b^\delta < T_a^\delta \,\middle|\, B_\delta(0) = x\right) = \frac{e^{-2\delta x} - e^{-2\delta a}}{e^{-2\delta b} - e^{-2\delta a}}, \quad \text{for } x \text{ between } a \text{ and } b. \qquad (3.15)$$

b) Let $t > 0$ fixed. We have :

$$\mathbb{1}_{\{\rho\le t\}} = \mathbb{1}_{\{\rho\le t\le T_a^\delta\}} + \mathbb{1}_{\{T_a^\delta\le t\}}. \qquad (3.16)$$

If $t \le T_a^\delta$, $\rho \le t$ means that after the time t, B_δ hits a before $-\underline{B_\delta}(t)$. Consequently :

$$\mathbb{P}(\rho \le t \le T_a^\delta | \mathcal{F}_t) = \frac{e^{2\delta \underline{B_\delta}(t)} - e^{-2\delta B_\delta(t)}}{e^{2\delta \underline{B_\delta}(t)} - e^{-2\delta a}} \mathbb{1}_{\{t\le T_a^\delta\}}. \qquad (3.17)$$

Let $\widetilde{Y}^\rho$ denote the process :

$$\widetilde{Y}_t^\rho = \frac{e^{2\delta \underline{B_\delta}(t)} - e^{-2\delta B_\delta(t)}}{e^{2\delta \underline{B_\delta}(t)} - e^{-2\delta a}} \text{ for } t \ge 0. \qquad (3.18)$$

Note that $\widetilde{Y}_{T_a^\delta}^\rho = 1$, therefore : $Y_t^\rho = \widetilde{Y}_{t\wedge T_a^\delta}^\rho$.

c) We know that $e^{-2\delta B_\delta(t)} = e^{-2\delta B(t) - 2\delta^2 t}$ is a martingale. Consequently, using the classical rules of stochastic calculus, we get :

$$\begin{aligned} d\widetilde{Y}_t^\rho &= \frac{1}{e^{2\delta \underline{B_\delta}(t)} - e^{-2\delta a}} e^{-2\delta B_\delta(t)} 2\delta dB(t) \\ &+ \frac{e^{-2\delta B_\delta(t)} - e^{-2\delta a}}{\left(e^{2\delta \underline{B_\delta}(t)} - e^{-2\delta a}\right)^2} 2\delta e^{2\delta \underline{B_\delta}(t)} d\underline{B_\delta}(t). \end{aligned} \qquad (3.19)$$

Since the support of the random measure $d\underline{B_\delta}$ is included in $\{t \geq 0\,;\, B_\delta(t) = -\underline{B_\delta}(t)\}$, then (3.13) and (3.14) follow immediately. □

The following result will be useful below.

Proposition 3.4 *Let $x > 0$ and $y > 0$. Then $(B_\delta(t)\,;\, 0 \leq t \leq T_x^\delta)$ conditioned by $\{T_x^\delta < T_{-y}^\delta\}$ is distributed as $(B_{-\delta}(t)\,;\, 0 \leq t \leq T_x^{-\delta})$ conditioned by $\{T_x^{-\delta} < T_{-y}^{-\delta}\}$.*

Proof Let F be a test function. We have :

$$\begin{aligned} A &= \mathbb{E}\left[F\left(B(t) + \delta t\,;\, 0 \leq t \leq T_x^\delta\right) | T_x^\delta < T_{-y}^\delta\right] \\ &= \mathbb{E}\left[F\left(B(t) + \delta t\,;\, , 0 \leq t \leq T_x^\delta\right) 1\!\!1_{\{T_x^\delta < T_{-y}^\delta\}}\right] \frac{1}{\mathbb{P}(T_x^\delta < T_{-y}^\delta)}. \end{aligned}$$

The r.v. $F\left(B(t) + \delta t\,;\, 0 \leq t \leq T_x^\delta\right) 1\!\!1_{\{T_x^\delta < T_{-y}^\delta\}}$ being $\mathcal{F}_{T_x^\delta}$ measurable, and $T_x^\delta < \infty$ a.s., Girsanov's theorem and (3.15) imply :

$$\begin{aligned} A &= \mathbb{E}\left[F\left(B(t), 0 \leq t \leq T_x^0\right) 1\!\!1_{\{T_x^0 < T_{-y}^0\}} \exp(\delta B(T_x^0) - \frac{\delta^2}{2} T_x^0)\right] \frac{e^{-2\delta x} - e^{2\delta y}}{1 - e^{2\delta y}}, \\ &= \mathbb{E}\left[F\left(B(t), 0 \leq t \leq T_x^0\right) 1\!\!1_{\{T_x^0 < T_{-y}^0\}} \exp(-\frac{\delta^2}{2} T_x^0)\right] \frac{e^{-\delta x} - e^{2\delta y + \delta x}}{1 - e^{2\delta y}}. \end{aligned}$$

Replacing δ by $-\delta$, we obtain similarly :

$$\begin{aligned} \tilde{A} &= \mathbb{E}\left[F\left(B(t) - \delta t, 0 \leq t \leq T_x^{-\delta}\right) | T_x^{-\delta} < T_{-y}^{-\delta}\right] \\ &= \mathbb{E}\left[F\left(B(t), 0 \leq t \leq T_x^0\right) 1\!\!1_{\{T_x^0 < T_{-y}^0\}} \exp(-\delta B(T_x^0) - \frac{\delta^2}{2} T_x^0)\right] \\ &\quad \times \frac{e^{2\delta x} - e^{-2\delta y}}{1 - e^{-2\delta y}} \\ &= \mathbb{E}\left[F\left(B(t), 0 \leq t \leq T_x^0\right) 1\!\!1_{\{T_x^0 < T_{-y}^0\}} \exp(-\frac{\delta^2}{2} T_x^0)\right] \frac{e^{\delta x} - e^{-2\delta y - \delta x}}{1 - e^{-2\delta y}}. \end{aligned}$$

□

We are now able to describe the law of $(B_\delta(t); 0 \leq t \leq \rho)$ conditionally on S.

Proof of Proposition 3.2 1) Let f and F be two non-negative test functions and $\Delta = \mathbb{E}\left[f(S) F(B_\delta(s), 0 \leq s \leq \rho)\right]$.
We have :

$$\Delta = \mathbb{E}\left[f(\underline{B_\delta}(\rho)) F(B_\delta(s), 0 \leq s \leq \rho)\right].$$

Setting $U_t = f(\underline{B_\delta}(t)) F(B_\delta(s), 0 \leq s \leq t)$, we have : $\Delta = \mathbb{E}\left[U_\rho\right]$. Since $(U_t, t \geq 0)$ is a predictable process, property (3.9) implies that :

$$\Delta = \mathbb{E}\left[\int_0^\infty U_t\, dA_t^\rho\right].$$

Using (3.13), we obtain :

$$\Delta = 2\delta\mathbb{E}\left[\int_0^{T_a^\delta} f(\underline{B_\delta}(t))F(B_\delta(s), 0 \le s \le t)\frac{e^{2\delta\underline{B_\delta}(t)}}{e^{2\delta\underline{B_\delta}(t)} - e^{-2\delta a}}d\underline{B_\delta}(t)\right].$$

In particular, if $F = 1$, the *change of variable* : "$x = \underline{B_\delta}(t)$" yields

$$\begin{aligned}\mathbb{E}\left[f(S)\right] &= 2\delta\mathbb{E}\left[\int_0^S f(x)\frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\, dx\right] \\ &= 2\delta\mathbb{E}\left[\int_0^\infty f(x)\frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\mathbb{P}(S \ge x)\, dx\right].\end{aligned} \tag{3.20}$$

As a result :

$$\mathbb{P}(S \in dx) = 2\delta\frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\mathbb{P}(S \ge x)\, dx.$$

Hence, S has a density function φ and :

$$\varphi(x) = 2\delta\frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\int_x^\infty \varphi(s)\, ds. \tag{3.21}$$

(3.21) may be written as a linear differential equation with respect to $\Phi(x) = \int_x^\infty \varphi(s)\, ds$. It is easy to solve explicitly since $\Phi(0) = 1$:

$$\Phi(x) = \int_x^\infty \varphi(s)\, ds = \frac{1 - e^{-2\delta a}}{e^{2\delta x} - e^{-2\delta a}}. \tag{3.22}$$

This implies (3.6).

2) We now determine the law of $(B_\delta(t)\,;\, 0 \le t \le \rho)$. $(T_{-x}^\delta\,;\, x \ge 0)$ being the right continuous inverse of $(\underline{B_\delta}(t))_{t\ge 0}$. We have :

$$\begin{aligned}\Delta &= 2\delta\int_0^\infty f(x)\mathbb{E}\left[F(B_\delta(s), 0 \le s \le T_{-x}^\delta)1\!\!1_{\{T_{-x}^\delta < T_a^\delta\}}\right]\frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\, dx \\ &= 2\delta\int_0^\infty f(x)\mathbb{E}\left[F(B_\delta(s), 0 \le s \le T_{-x}^\delta)|T_{-x}^\delta < T_a^\delta\right] \\ &\qquad \times\frac{e^{2\delta x}}{e^{2\delta x} - e^{-2\delta a}}\,\mathbb{P}(T_{-x}^\delta < T_a^\delta)\, dx \\ &= \int_0^\infty f(x)\mathbb{E}\left[F(B_\delta(s), 0 \le s \le T_{-x}^\delta)|T_{-x}^\delta < T_a^\delta\right]\varphi(x)\, dx.\end{aligned}$$

The law of $(B_\delta(t)\,;\, 0 \le t \le \rho)$, conditionally on S, follows from Proposition 3.4.

3) It remains to determine the law of $(B_\delta(t + \rho) - B_\delta(\rho), 0 \le t \le T_a^\delta - \rho)$ and to prove that this process is independent of $(B_\delta(t); 0 \le t \le \rho)$.

Let $\mathbb{F}^\rho$ denote the smallest filtration including $\mathbb{F}$ for which ρ is a stopping time. ρ is an honest time since, for any $t > 0$, on $\{\rho < t\}$, ρ coincides with a $\mathcal{F}_t$-measurable r.v. (see [8] p 370). That allows us to use (3.10). Combining (3.11), (3.14) with (3.12), we obtain :

$$\begin{aligned}\chi(t) &= -\int_0^{t\wedge\rho} \frac{(e^{2\delta \underline{B_\delta}(s)} - e^{-2\delta a})2\delta e^{-2\delta B_\delta(s)}}{(e^{-2\delta B_\delta(s)} - e^{-2\delta a})(e^{2\delta \underline{B_\delta}(s)} - e^{-2\delta a})} ds \\ &+ \int_0^{t\wedge T_a^\delta} \mathbb{1}_{\{\rho<s\}} \frac{(e^{2\delta \underline{B_\delta}(s)} - e^{-2\delta a})2\delta e^{-2\delta B_\delta(s)}}{(e^{2\delta \underline{B_\delta}(s)} - e^{-2\delta B_\delta(s)})(e^{2\delta \underline{B_\delta}(s)} - e^{-2\delta a})} ds,\end{aligned}$$

$$\chi(t) = -\int_0^{t\wedge\rho} \frac{2\delta e^{-2\delta B_\delta(s)}}{e^{-2\delta B_\delta(s)} - e^{-2\delta a}} ds + \int_0^{t\wedge T_a^\delta} \mathbb{1}_{\{\rho<s\}} \frac{2\delta e^{-2\delta B_\delta(s)}}{e^{2\delta \underline{B_\delta}(s)} - e^{-2\delta B_\delta(s)}} ds.$$

Since $\underline{B_\delta}(s) = S$, then for any $\rho \le s \le T_a^\delta$, we have :

$$\chi(t+\rho) - \chi(\rho) = 2\delta \int_\rho^{(t+\rho)\wedge T_a^\delta} \frac{e^{-2\delta B_\delta(s)}}{e^{2\delta S} - e^{-2\delta B_\delta(s)}} ds;\ 0 \le t \le T_a^\delta - \rho.$$

Let $(W(t))$ and $\left(\widehat{Z}(t)\right)$ be the processes

$$W(t) = \bar{B}(t+\rho) - \bar{B}(\rho)\ ;\ t \ge 0, \quad \widehat{Z}(t) = B_\delta(\rho+t) - B_\delta(\rho) = B_\delta(\rho+t) + S\ ;\ 0 \le t \le T_a^\delta - \rho.$$

According to (3.10), for any $t \in [0, T_a^\delta - \rho]$ we have :

$$\begin{aligned}\widehat{Z}(t) &= W(t) + \delta t + 2\delta \int_0^t \frac{e^{-2\delta B_\delta(s+\rho)}}{e^{2\delta S} - e^{-2\delta B_\delta(s+\rho)}} ds. \\ &= W(t) + \delta \int_0^t \coth(\delta \widehat{Z}(s)) ds.\end{aligned}$$

We know that $\left(\bar{B}(t)\right)_{t\ge0}$ is an $\left(\mathcal{F}^\rho_{\rho+t}\ ;\ t \ge 0\right)$- Brownian motion. Therefore $(W(t))_{t\ge0}$ is independent of $\mathcal{F}^\rho_\rho$. Since $S = -B_\delta(\rho)$ and $(B_\delta(t); 0 \le t \le \rho)$ are $\mathcal{F}^\rho_\rho$-measurable, $(W(t))_{t\ge0}$ is independent of S and $(B_\delta(t); 0 \le t \le \rho)$. Furthermore, $T_a^\delta - \rho = \inf\{t \ge 0\,;\, \widehat{Z}(t) = a + b\}$, with $b = S$. To prove parts 1 and 3 of Proposition 3.2, we use the fact that the stochastic differential equation (1.7) admits a unique non negative strong solution. This property may be proved as for Bessel processes (applying the Yamada-Watanabe Theorem to the square of $Z^{(\delta)}$). □

Remark 3.5 1) Conditionally on $\{S \le z\}$, the density of S is :

$$\frac{\delta \operatorname{sh} \delta a \operatorname{sh} \delta(z+a)}{\operatorname{sh} \delta z} \frac{1}{\operatorname{sh}^2 \delta(x+a)} \mathbb{1}_{\{0\le x\le z\}}. \tag{3.23}$$

Formula (3.23) is a direct consequence of (3.6) and (3.22).

2) We can determine directly the distribution function of S since

$$\mathbb{P}(S > y) = \mathbb{P}(T^{\delta}_{-y} < T^{\delta}_{a}) = \frac{e^{-2\delta a} - 1}{e^{-2\delta a} - e^{2\delta y}} = \Phi(x),$$

where Φ is defined by (3.22).

Proofs of Theorem 2.4 and Proposition 2.1
The processes $(-B_\delta(t))_{t\geq 0}$ and $(B_{-\delta}(t))_{t\geq 0}$ have the same law. Consequently Proposition 3.2 admits a version where $(B_\delta(t)\,;\,0 \leq t \leq T^{\delta}_{a})$ (resp. S) is replaced by $(B_{-\delta}(t)\,;\,0 \leq t \leq T^{-\delta}_{-a})$ (resp. $\sup_{0\leq t\leq T^{-\delta}_{-a}} B_{-\delta}(t)$). Then Theorem 2.4 and Proposition 2.1 may be proved by induction on k. □

4 First range time, range at a fixed time, asymptotic results

In this section, we focus on B_δ stopped at its first range time $\theta_\delta(a)$. The first result concerns the law of $(B_\delta(t)\,;\,0 \leq t \leq \theta_\delta(a))$. We determine the joint law of $(B_\delta(t)\,;\,0 \leq t \leq \tilde{\theta_\delta}(a))$ and $(B_\delta(t+\tilde{\theta_\delta}(a)) - B_\delta(\tilde{\theta_\delta}(a))\,;\,0 \leq \theta_\delta(a) - \tilde{\theta_\delta}(a))$ where

$$\tilde{\theta_\delta}(a) = \begin{cases} \sup\{t \leq \theta_\delta(a)\,;\,B_\delta(t) = \inf_{0\leq u\leq \theta_\delta(a)} B_\delta(u)\} \text{ if } B_\delta(\theta_\delta(a)) > 0 \\ \\ \sup\{t \leq \theta_\delta(a)\,;\,B_\delta(t) = \sup_{0\leq u\leq \theta_\delta(a)} B_\delta(u)\} \text{ otherwise .} \end{cases} \tag{4.1}$$

Proposition 4.1 *1. The r.v.* $B_\delta(\tilde{\theta_\delta}(a))$ *has density function* f *:*

$$\begin{aligned} f(u) &= \frac{2\delta}{(1 - e^{-2\delta a})^2}(e^{2\delta u} - e^{-2\delta a})\mathbb{1}_{\{-a\leq u<0\}} \\ &\quad + \frac{2\delta}{(e^{2\delta a} - 1)^2}(e^{2\delta a} - e^{2\delta u})\mathbb{1}_{\{0\leq u\leq a\}}. \end{aligned} \tag{4.2}$$

2. Conditionally on $\{B_\delta(\tilde{\theta_\delta}(a)) = u\}$,

a. the processes $(B_\delta(s)\,;\,0 \leq s \leq \tilde{\theta_\delta}(a))$ *and* $(B_\delta(s + \tilde{\theta_\delta}(a)) - u\,;\,0 \leq s \leq \theta(a) - \tilde{\theta_\delta}(a))$ *are independent.*

b. $(B_\delta(s)\,;\,0 \leq s \leq \tilde{\theta_\delta}(a))$ *has the same law as a Brownian motion with drift* δ *stopped at its first hitting time at level* u*, conditioned by hitting* u *before* $u - sign(u)a$*.*

c. $(|B_\delta(s + \tilde{\theta_\delta}(a)) - u|\,;\,0 \leq s \leq \theta(a) - \tilde{\theta_\delta}(a))$ *has the same law as the process* $Z^{(\delta)}$ *(1.7), stopped at its first hitting time of level* a*.*

Vallois [12] has obtained a similar result when $\delta = 0$.

As for the proof of Proposition 3.2, a straightforward approach may be developed using enlargement of filtrations, since $\tilde{\theta_\delta}(a)$ is an honest time. More precisely, the optional projection of $\left(\mathbb{1}_{\{\tilde{\theta_\delta}(a) \le t\}}; t \ge 0\right)$ may be decomposed in the following way :

$$\mathbb{P}\left(\tilde{\theta_\delta}(a) \le t | \mathcal{F}_t\right) = M_t + A_t,$$

where $(M_t; t \ge 0)$ is a martingale and :

$$A_t = \frac{2\delta e^{-\delta a}}{e^{\delta a} - e^{-\delta a}} \overline{B_\delta}(t \wedge \theta(a)) - \frac{2\delta e^{\delta a}}{e^{\delta a} - e^{-\delta a}} \underline{B_\delta}(t \wedge \theta(a)). \tag{4.3}$$

The details are left to the reader.

Proposition 4.1 allows us to determine the Laplace transform of $\theta_\delta(a)$. Let $\sigma_\delta(a)$ be the stopping time

$$\sigma_\delta(a) = \inf\left\{s \ge 0 \ , \ Z^{(\delta)}(s) > a\right\}. \tag{4.4}$$

Proposition 4.2 *The Laplace transforms of $\tilde{\theta_\delta}(a)$, $\sigma_\delta(a)$ and $\theta_\delta(a)$ are given by :*

$$\mathbb{E}\left[e^{-\lambda \tilde{\theta_\delta}(a)}\right] = \frac{\delta}{\lambda}\Bigg(\sqrt{2\lambda + \delta^2}\coth(a\sqrt{2\lambda+\delta^2})\coth(\delta a) - \delta - \frac{\sqrt{2\lambda+\delta^2}}{\mathrm{sh}(a\sqrt{2\lambda+\delta^2})\,\mathrm{sh}(\delta a)}\Bigg), \tag{4.5}$$

$$\mathbb{E}\left[e^{-\lambda \sigma_\delta(a)}\right] = \frac{\sqrt{2\lambda+\delta^2}}{\delta}\frac{\mathrm{sh}(\delta a)}{\mathrm{sh}(a\sqrt{2\lambda+\delta^2})}, \tag{4.6}$$

$$\mathbb{E}\left[e^{-\lambda \theta(a)}\right] = \frac{\sqrt{2\lambda+\delta^2}}{\lambda}\Bigg[\frac{\sqrt{2\lambda+\delta^2}\,\mathrm{ch}(a\sqrt{2\lambda+\delta^2})\ \mathrm{ch}(\delta a)}{\mathrm{sh}^2(a\sqrt{2\lambda+\delta^2})} - \frac{\delta\,\mathrm{sh}(\delta a)}{\mathrm{sh}(a\sqrt{2\lambda+\delta^2})} - \frac{\sqrt{2\lambda+\delta^2}}{\mathrm{sh}^2(a\sqrt{2\lambda+\delta^2})}\Bigg] \tag{4.7}$$

where $\lambda \ge 0$.

Remark 4.3 1) In [15], using approximation by random walks, it is proved :

$$\mathbb{E}\left[\theta_\delta(a)\right] = \frac{a^2}{2} f(\delta a),$$

where

$$f(x) = \frac{1}{x^2}(x - x\coth x + 1)(x + x\coth x - 1)\,;\ x > 0.$$

Moreover f is decreasing, in particular :

$$\mathbb{E}\left[\theta_\delta(a)\right] \leq \mathbb{E}\left[\theta_0(a)\right] = \frac{a^2}{2}.$$

The variance of $\theta_\delta(a)$ has been computed

$$\mathrm{Var}\left[\theta_\delta(a)\right] = \frac{a^4}{12} g_1(\delta a) g_2(\delta a),$$

with

$$g_1(x) = \frac{3\,\mathrm{sh}^2 x - x^2}{x^2\,\mathrm{sh}^2 x}\ ;\ g_2(x) = \frac{x^2\coth^2 x + 4x\coth x - 5 - x^2}{x^2}\ ;\ x > 0.$$

Moreover,

$$\mathrm{Var}\left(\theta_\delta(a)\right) \leq \mathrm{Var}\left(\theta_0(a)\right) = \frac{a^4}{12}.$$

2) It is possible to prove Propositions 4.1 and 4.2 using Theorems 2.2, 2.4 and the identity : $\Omega = \cup_{n\geq 0}\left\{\rho_{n+1} \leq \tilde{\theta}_\delta(a) < \rho_n\right\}$, where the sequence $(\rho_n)_{n\geq 0}$ is defined at the beginning of Section 2. It is however easier and shorter to prove directly Propositions 4.1 and 4.2.

Proof of Proposition 4.2

We give only the main ideas of the proof; the details are left to the reader.

Let f_λ be the function :

$$f_\lambda(x) = \frac{\delta}{\sqrt{2\lambda+\delta^2}}\frac{\mathrm{sh}(x\sqrt{2\lambda+\delta^2})}{\mathrm{sh}(\delta x)}\ ;\ x > 0.$$

It is easy to check that f_λ is an eigenfunction of the infinitesimal generator associated with $Z^{(\delta)}$:

$$\frac{1}{2}f_\lambda''(x) + \delta\coth(\delta x) f_\lambda'(x) = \lambda f_\lambda(x)\ ;\ x > 0.$$

The function f_λ being locally bounded on $[0, +\infty)$, then

$$\mathbb{E}_0\left[\exp(-\lambda\sigma_\delta(a))\right] = \frac{f_\lambda(0)}{f_\lambda(a)}.$$

This proves (4.6).

Recall (cf formulas 2.1.4 and 2.2.4 p. 295 of [1])

$$\mathbb{E}\left[e^{-\lambda T_u^\delta} | T_u^\delta < T_{u+a}^\delta\right] = \frac{\mathrm{sh}((a+u)\sqrt{2\lambda+\delta^2})}{\mathrm{sh}(a\sqrt{2\lambda+\delta^2})}\frac{\mathrm{sh}(\delta a)}{\mathrm{sh}(\delta(a+u))}, \quad (4.8)$$

$$\mathbb{E}\left[e^{-\lambda T_u^\delta} | T_u^\delta < T_{u-a}^\delta\right] = \frac{\mathrm{sh}((a-u)\sqrt{2\lambda+\delta^2})}{\mathrm{sh}(a\sqrt{2\lambda+\delta^2})}\frac{\mathrm{sh}(\delta a)}{\mathrm{sh}(\delta(a-u))}. \quad (4.9)$$

(4.5) is a direct consequence of (3.15) and Proposition 4.1. □

By inversion of the Laplace transform of $\theta_\delta(a)$, the authors in [2] have computed the probability density function of this r.v.

We develop an alternative approach based on the knowledge (c.f. [1], formula 1.15.8 (1) p. 271) of the joint distribution of $(B_\delta(t), R_\delta(t))$. By somewhat lengthy calculations (see Section 5) we determine the probability distribution function of $R_\delta(t)$.

Using relation (1.3), we obtain the probability distribution function of $\theta_\delta(a)$ and the rate of decay of $\mathbb{P}(\theta_\delta(a) > t)$, as $t \to \infty$.

Proposition 4.4 *Let $a > 0$, $t > 0$ and $C_k = k^2\pi^2 + a^2\delta^2$, $k \in \mathbb{N}$. Then :*

$$\mathbb{P}\left(R_\delta(t) < a\right) = \sum_{k=1}^{\infty} \frac{4k^2\pi^2}{C_k^2} \exp\left(-\frac{C_k t}{2a^2}\right) \Bigg\{ \Big(1 - (-1)^k \operatorname{ch}(\delta a)\Big) \times \left(1 + \frac{k^2\pi^2 t}{a^2} - \frac{4a^2\delta^2}{C_k}\right) - (-1)^k a\delta \operatorname{sh}(\delta a) \Bigg\}. \tag{4.10}$$

$$\mathbb{P}\left(\theta_\delta(a) > t\right) \underset{t\to\infty}{\sim} \frac{4\pi^4 \left(1 + \operatorname{ch}(\delta a)\right)}{a^2 \left(\pi^2 + a^2\delta^2\right)} t \exp\left(-\frac{\pi^2 + a^2\delta^2}{2a^2} t\right). \tag{4.11}$$

Remark 4.5 1) The probability distribution function of $R_\delta(t)$ given in [1] is in a different form (formula 1.15.4 (1) p. 270). Our formula (4.10) seems to be simpler.

2) Formula (4.10) has been obtained in [11], using a different approach.

3) Taking the a-derivative in (4.10) gives the density function of $R_\delta(t)$.

4) Relation (1.3) implies that :

$$\mathbb{P}\left(\theta_\delta(a) > t\right) = \sum_{k=1}^{\infty} \frac{4k^2\pi^2}{C_k^2} \exp\left(-\frac{C_k t}{2a^2}\right) \Bigg\{ \Big(1 - (-1)^k \operatorname{ch}(\delta a)\Big) \times \left(1 + \frac{k^2\pi^2 t}{a^2} - \frac{4a^2\delta^2}{C_k}\right) - (-1)^k a\delta \operatorname{sh}(\delta a) \Bigg\}. \tag{4.12}$$

Again, taking the t-derivative in (4.12), we obtain the density function of $\theta_\delta(a)$. However that series expension is more complicated than (4.12) (c.f. also Theorem 9 of [2]) : it is more convenient to use probability distribution function instead of probability density function.

The law of $\theta_\delta(a)$ or $R_\delta(t)$ being complicated, it seems natural to consider the asymptotic behaviour of $\theta_\delta(a)$ (resp. $R_\delta(t)$) as a goes to $+\infty$ (resp. $t \to +\infty$).

Proposition 4.6 *The asymptotic behaviours of $\theta_\delta(a)$ and $R_\delta(t)$ are given by the following :*

$$\frac{\theta_\delta(a)}{a} \xrightarrow[a\to\infty]{a.s.} \frac{1}{\delta}, \tag{4.13}$$

$$\frac{R_\delta(t)}{t} \xrightarrow[t\to\infty]{a.s.} \delta, \tag{4.14}$$

$$\delta^{3/2}\sqrt{a}\left(\frac{\theta_\delta(a)}{a} - \frac{1}{\delta}\right) \xrightarrow[a\to\infty]{d} \mathcal{N}\left(0,1\right), \tag{4.15}$$

$$\sqrt{t}\left(\frac{R_\delta(t)}{t} - \delta\right) \xrightarrow[t\to\infty]{d} \mathcal{N}(0,1). \tag{4.16}$$

Proof 1) We first prove (4.13). By (4.7), we have :

$$\mathbb{E}\left[e^{-\frac{\lambda}{a}\theta_\delta(a)}\right] = \frac{\sqrt{\frac{2\lambda}{a}+\delta^2}}{\frac{\lambda}{a}}\left[\frac{\sqrt{\frac{2\lambda}{a}+\delta^2}\,\mathrm{ch}(a\sqrt{\frac{2\lambda}{a}+\delta^2})\,\mathrm{ch}(\delta a)}{\mathrm{sh}^2(a\sqrt{\frac{2\lambda}{a}+\delta^2})} - \frac{\delta\,\mathrm{sh}(\delta a)}{\mathrm{sh}(a\sqrt{\frac{2\lambda}{a}+\delta^2})} - \frac{\sqrt{\frac{2\lambda}{a}+\delta^2}}{\mathrm{sh}^2(a\sqrt{\frac{2\lambda}{a}+\delta^2})}\right].$$

$$\mathbb{E}\left[e^{-\frac{\lambda\theta_\delta(a)}{a}}\right] = \frac{a\left(\delta+\frac{\lambda}{\delta a}+o\left(\frac{1}{a}\right)\right)}{\lambda}\left[\frac{\left(\delta+\frac{\lambda}{\delta a}+o\left(\frac{1}{a}\right)\right)\mathrm{ch}(\delta a+\frac{\lambda}{\delta}+o(1))\,\mathrm{ch}(\delta a)}{\mathrm{sh}^2(\delta a+\frac{\lambda}{\delta}+o(1))} - \frac{\delta\,\mathrm{sh}(\delta a)}{\mathrm{sh}(\delta a+\frac{\lambda}{\delta}+o(1))} + \frac{\delta+\frac{\lambda}{\delta a}+o\left(\frac{1}{a}\right)}{\mathrm{sh}^2(\delta a+\frac{\lambda}{\delta}+o(1))}\right].$$

The limit of the two first terms in the bracket is easy to determine :

$$\frac{\mathrm{sh}(\delta a)}{\mathrm{sh}(\delta a+\frac{\lambda}{\delta}+o(1))} = \frac{e^{\delta a}-e^{-\delta a}}{e^{\delta a+\frac{\lambda}{\delta}+o(1)}-e^{-\delta a-\frac{\lambda}{\delta}+o(1)}} \xrightarrow[a\to\infty]{} e^{-\frac{\lambda}{\delta}},$$

$$\frac{\mathrm{ch}(\delta a+\frac{\lambda}{\delta}+o(1))\,\mathrm{ch}(\delta a)}{\mathrm{sh}^2(\delta a+\frac{\lambda}{\delta}+o(1))} \xrightarrow[a\to\infty]{} e^{-\frac{\lambda}{\delta}}.$$

As for the third term, it may be neglected (being equivalent to $4\delta\exp(-2\delta a)$).

Hence $\theta_\delta(a)/a$ converges in distribution to the constant $1/\delta$. This implies that $\theta_\delta(a)/a$ converges in probability to $1/\delta$. Since $(R_\delta(t))_{t\geq 0}$ is the right continuous inverse of $(\theta_\delta(a))_{a\geq 0}$, $R_\delta(t)/t$ converges in probability to δ.

$(R_\delta(t))_{t\geq 0}$ is a subadditive process (cf [3], example 6.2 p.320, in the discrete case). The subadditive ergodic theorem implies that $R_\delta(t)/t$ converges a.s., as $t\to\infty$. As a result, $R_\delta(t)/t$ converges a.s. towards δ when $t\to\infty$.

2) To prove (4.15), we use the characteristic function of $\theta_\delta(a)$. This function can be explicitly determined through (4.7) and an analytic continuation argument. In the discrete case (cf proof of Theorem 20 of [14]) a detailed approach is developed.

3) We claim that $\left(\frac{R_\delta(t)}{t} - \delta\right)\sqrt{t}$ converges in distribution to $\mathcal{N}(0,1)$ when $t \to +\infty$.

Let $a > 0$ and $t > 0$ such that $a\sqrt{t} + \delta t > 0$. We have :

$$p = \mathbb{P}\left(\left(\frac{R_\delta(t)}{t} - \delta\right)\sqrt{t} < a\right) = \mathbb{P}\left(R_\delta(t) < s\right),$$

where $s = a\sqrt{t} + \delta t$.

Property (1.3) implies that

$$p = \mathbb{P}\left(\theta_\delta(s) > t\right) = \mathbb{P}\left(\left(\frac{\theta_\delta(s)}{s} - \frac{1}{\delta}\right)\sqrt{s} > u(t)\right),$$

where

$$u(t) = \left(\frac{t}{s} - \frac{1}{\delta}\right)\sqrt{s} = -\frac{a}{\delta\sqrt{\delta + \frac{a}{\sqrt{t}}}}.$$

Since $t \to +\infty$ implies $s \to +\infty$, and $u(t) \underset{t\to\infty}{\sim} -\dfrac{a}{\delta^{3/2}}$, (4.16) follows immediately. $\square$

5 Proof of Proposition 4.4

In order to calculate the probability distribution function of $R_\delta(a)$, we perform the following calculations. Let us start with the formula 1.15.8 (1) p. 271 of [1] :

$$\mathbb{P}\left(R_\delta(t) < a\right) = \frac{1}{\sqrt{2\pi t}} e^{-\dfrac{\delta^2 t}{2}} \sum_{k\in\mathbb{Z}} \mu_k,$$

with

$$\mu_k = \int_{-a}^{a} \left(2k + 1 - \frac{2k(a - |z|)(|z| + 2ka)}{t}\right) \exp\left(\delta z - \frac{(|z| + 2ka)^2}{2t}\right) dz.$$

Recalling the Poisson formula

$$\frac{1}{\sqrt{2\pi t}} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{(x + 2ky)^2}{2t}\right) = \frac{1}{2y} \sum_{k\in\mathbb{Z}} \exp\left(-\frac{k^2\pi^2}{2y^2} t\right) \exp\left(\frac{ik\pi x}{y}\right). \quad (5.1)$$

Let us compute the y-derivative and the x-derivative :

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi t}}\sum_{k\in\mathbb{Z}}\frac{2k}{t}(x+2ky)\exp\left(-\frac{(x+2ky)^2}{2t}\right) &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\exp\left(-\frac{k^2\pi^2}{2y^2}t\right)\\
&\quad\times\exp\left(\frac{ik\pi x}{y}\right)\left(\frac{1}{y^2}-\frac{k^2\pi^2 t}{y^4}+\frac{ik\pi x}{y^3}\right),
\end{aligned} \tag{5.2}
$$

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi t}}\sum_{k\in\mathbb{Z}}\left(\frac{x+2ky}{t}\right)\exp\left(-\frac{(x+2ky)^2}{2t}\right) &= -\frac{1}{2y}\sum_{k\in\mathbb{Z}}\exp\left(-\frac{k^2\pi^2}{2y^2}t\right)\\
&\quad\times\exp\left(\frac{ik\pi x}{y}\right)\frac{ik\pi}{y}.
\end{aligned} \tag{5.3}
$$

Replacing x with $|z|$ and y with a in $\frac{t}{y}\times(5.3)+(1-\frac{x}{y})\times(5.1)$, we obtain

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi t}}\sum_{k\in\mathbb{Z}}(2k+1)\exp\left(-\frac{(|z|+2ka)^2}{2t}\right) &= \frac{1}{2}\sum_{k\in\mathbb{Z}}\exp\left(-\frac{k^2\pi^2 t}{2a^2}\right)\\
&\quad\times\exp\left(\frac{ik\pi|z|}{a}\right)\left(-\frac{ik\pi t}{2a^3}-\frac{|z|-a}{a^2}\right).
\end{aligned} \tag{5.4}
$$

Replacing x with $|z|$ and y with a in $-(y-x)\times(5.2)$ and adding (5.4), we obtain $\frac{1}{\sqrt{2\pi t}}\sum_{k\in\mathbb{Z}}\mu_k=\frac{1}{2}\sum_{k\in\mathbb{Z}}I_k$, where

$$
\begin{aligned}
I_k &= \int_{-a}^{a} e^{\delta z}\exp\left(-\frac{k^2\pi^2 t}{2a^2}\right)\exp\left(\frac{ik\pi|z|}{a}\right)\\
&\quad\times\left(-\frac{ik\pi t}{a^3}+(|z|-a)\left(\frac{ik\pi|z|}{a^3}-\frac{k^2\pi^2 t}{a^4}\right)\right)dz.
\end{aligned} \tag{5.5}
$$

Let us introduce :

$$
\begin{aligned}
C_1(\delta) &= \int_0^a \exp\left(\delta z+\frac{ik\pi z}{a}\right)dz=\frac{a}{a\delta+ik\pi}\left[(-1)^k e^{\delta a}-1\right],\\
C_2(\delta) &= \int_0^a (z-a)\exp\left(\delta z+\frac{ik\pi z}{a}\right)dz=\frac{a^2}{a\delta+ik\pi}+\frac{a^2(1-(-1)^k e^{\delta a})}{(a\delta+ik\pi)^2},\\
C_3(\delta) &= \int_0^a z(z-a)\exp\left(\delta z+\frac{ik\pi z}{a}\right)dz\\
&= -\frac{a^3\left(1+(-1)^k e^{\delta a}\right)}{(a\delta+ik\pi)^2}+\frac{2a^3\left((-1)^k e^{\delta a}-1\right)}{(a\delta+ik\pi)^3}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
I_k\exp\left(\frac{k^2\pi^2 t}{2a^2}\right) &= -\frac{ik\pi t}{a^3}(C_1(\delta)+C_1(-\delta))-\frac{k^2\pi^2 t}{a^4}(C_2(\delta)+C_2(-\delta))\\
&\quad+\frac{ik\pi}{a^3}(C_3(\delta)+C_3(-\delta)).
\end{aligned} \tag{5.6}
$$

We decompose $C_i(\delta)+C_i(-\delta)$, $1\le i\le 3$, as follows :

$$\begin{aligned}
C_1(\delta)+C_1(-\delta) &= -aA_1+(-1)^kA_2,\\
C_2(\delta)+C_2(-\delta) &= a^2A_1+a^2A_3-(-1)^ka^2A_4,\\
C_3(\delta)+C_3(-\delta) &= -a^3A_3-(-1)^ka^3A_4-2a^3A_5+(-1)^k2a^3A_6,
\end{aligned}$$

where we have set :

$$\begin{aligned}
A_1 &= \frac{1}{\delta a+ik\pi}+\frac{1}{-\delta a+ik\pi}=\frac{-2ik\pi}{C_k}\\
A_2 &= \frac{e^{\delta a}}{\delta a+ik\pi}+\frac{e^{-\delta a}}{-\delta a+ik\pi}=\frac{2\,\mathrm{sh}(\delta a)a\delta-2\,\mathrm{ch}(\delta a)ik\pi}{C_k}\\
A_3 &= \frac{1}{(\delta a+ik\pi)^2}+\frac{1}{(-\delta a+ik\pi)^2}=\frac{2(a^2\delta^2-k^2\pi^2)}{C_k^2}\\
A_4 &= \frac{e^{\delta a}}{(\delta a+ik\pi)^2}+\frac{e^{-\delta a}}{(-\delta a+ik\pi)^2}\\
&= \frac{2\,\mathrm{ch}(\delta a)(a^2\delta^2-k^2\pi^2)-4ia\delta k\pi\,\mathrm{sh}(\delta a)}{C_k^2}\\
A_5 &= \frac{1}{(\delta a+ik\pi)^3}+\frac{1}{(-\delta a+ik\pi)^3}=\frac{-6\delta^2a^2ik\pi+2ik^3\pi^3}{C_k^3}\\
A_6 &= \frac{e^{\delta a}}{(\delta a+ik\pi)^3}+\frac{e^{-\delta a}}{(-\delta a+ik\pi)^3},\\
&= \frac{(2\delta^3a^3-6k^2\pi^2\delta a)\,\mathrm{sh}(\delta a)+i(-6\delta^2a^2k\pi+2k^3\pi^3)\,\mathrm{ch}(\delta a)}{C_k^3}.
\end{aligned}$$

Coming back to (5.4), we easily obtain successively :

$$\begin{aligned}
I_k\exp\left(\frac{k^2\pi^2t}{2a^2}\right) &= \left(\frac{ik\pi t}{a^2}-\frac{k^2\pi^2t}{a^2}\right)A_1-(-1)^k\frac{ik\pi t}{a^2}A_2\\
&\quad -\left(\frac{k^2\pi^2t}{a^2}+ik\pi\right)A_3+(-1)^k\left(\frac{k^2\pi^2t}{a^2}-ik\pi\right)A_4\\
&\quad -2ik\pi A_5+(-1)^k2ik\pi A_6\\
&= Re\left(I_k\exp\left(\frac{k^2\pi^2t}{2a^2}\right)\right)\\
&= \frac{2k^2\pi^2t}{C_ka^2}-(-1)^k\frac{2k^2\pi^2t\,\mathrm{ch}(\delta a)}{C_ka^2}-2\frac{(a^2\delta^2-k^2\pi^2)k^2\pi^2t}{C_k^2a^2}\\
&\quad +(-1)^k\frac{2\,\mathrm{ch}(\delta a)(a^2\delta^2-k^2\pi^2)k^2\pi^2t-4k^2\pi^2a^3\delta\,\mathrm{sh}(\delta a)}{C_k^2a^2}\\
&\quad +\frac{4k^4\pi^4-12k^2\pi^2a^2\delta^2}{C_k^3}\\
&\quad +(-1)^k\frac{(12\delta^2a^2k^2\pi^2-4k^4\pi^4)\,\mathrm{ch}(\delta a)}{C_k^3}.
\end{aligned}$$

In particular :

$$
\begin{aligned}
(I_k + I_{-k}) \exp\left(\frac{k^2\pi^2 t}{2a^2}\right) &= \frac{k^2\pi^2}{C_k^2}(1 - (-1)^k \operatorname{ch}(\delta a)) \\
&\times \left(\frac{4t(k^2\pi^2 + \delta^2 a^2)}{a^2} - 4\frac{(a^2\delta^2 - k^2\pi^2)t}{a^2} \right. \\
&\left. + \frac{8k^2\pi^2 - 24a^2\delta^2}{C_k} \right) \\
&-(-1)^k 8 \frac{k^2\pi^2}{C_k^2} a\delta \operatorname{sh}(\delta a).
\end{aligned}
$$

This achieves the proof of Proposition 4.4. □

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