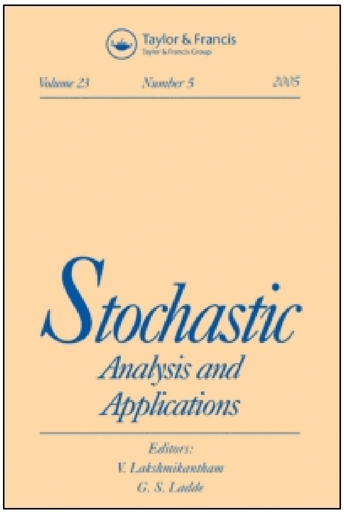

Stochastic Analysis and Applications
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597300

Approximations of a Continuous Time Filter. Application to Optimal Allocation Problems in Finance

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Online Publication Date: 01 March 2009

To cite this Article Martinez, Miguel, Rubenthaler, Sylvain and Tanré, Etienne(2009)'Approximations of a Continuous Time Filter. Application to Optimal Allocation Problems in Finance',Stochastic Analysis and Applications,27:2,270 — 296

To link to this Article: DOI: 10.1080/07362990802678846

URL: http://dx.doi.org/10.1080/07362990802678846

Stochastic Analysis and Applications, 27: 270–296, 2009

ISSN 0736-2994 print/1532-9356 online
DOI: 10.1080/07362990802678846

Approximations of a Continuous Time Filter. Application to Optimal Allocation Problems in Finance

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Abstract: In this article, we study a continuous time optimal filter and its various numerical approximations. This filter arises in an optimal allocation problem in the particular context of a non-stationary economy. We analyse the rates of convergence of the approximations of the filter when the model is misspecified and when the observations can only be made at discrete times. We give bounds that are uniform in time. Numerical results are presented.

Keywords: Applications in optimization; Filtering; Portfolios.

Received July 26, 2006; Accepted September 5, 2008

Financial support by the National Centre of Competence in Research "Financial Valuation and Risk Management" (NCCR FINRISK) is gratefully acknowledged. NCCR-FINRISK is a research program supported by the Swiss National Science Foundation.

A part of this work was done during visits of Etienne Tanré in Chile (supported by CONICYT and INRIA).

The authors would like to thank Pr. Rajna Gibson and Pr. Denis Talay for their support and their the numerous fruitful discussions. They also wish to thank the anonymous referee for his useful suggestions, which contributed greatly to this article.

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AMS Classification: 60G35; 91B28; 46N10.

1. INTRODUCTION: DESCRIPTION OF THE MODEL AND ORGANIZATION OF THE ARTICLE

1.1. Introduction

In this article, we analyse the performances of various approximations of the continuous time filter

$$F_t := \mathbb{P}[\mu(t) = \mu_1 \mid \mathcal{F}_t^X] \tag{1.1}$$

where $(\mu(t))_{t\geq 0}$ is a Markov process which takes only two real values μ_1 and μ_2. Here, $\mathbb{F}^X := (\mathcal{F}_t^X)_{t\geq 0}$ denotes the natural filtration generated by the process

$$X_t = x + \int_0^t \mu(s)ds + \sigma B_t, \tag{1.2}$$

where $(B_t)_{t\geq 0}$ is a one-dimensional Brownian motion independent of the process $(\mu(t))_{t\geq 0}$. In this model, the main point is that we observe only the process $(X_t)_{t\geq 0}$; $(\mu(t))_{t\geq 0}$ is not observed and we want to estimate it. Thanks to [11, 21], the continuous time filter satisfies a stochastic differential equation (see Lemma 2.2). In our case, we can write this SDE as an SDE driven by $(X_t)_{t\geq 0}$ (see (3.1)).

We study numerical approximations of $(F_t)_{t\geq 0}$ in misspecified situations:

a) the process $(X_t)_{t\geq 0}$ is observed at discrete times and all the parameters of the model (μ_1, μ_2 and the jump rates λ_1 and λ_2) are known;
b) the process $(X_t)_{t\geq 0}$ is also observed at discrete times but the parameters (μ_1, μ_2, λ_1 and λ_2) are unknown (we have only access to estimations of these parameters).

First, we use the Euler Scheme associated to (3.1), and we give the rate of convergence of this method. Second, we construct a discrete time approximation for the continuous time filter by using an updating/prediction procedure (see [15] for example). The main result of this article is Theorem 3.4, in which we give the rate of convergence of the second approach, which is better than the Euler Scheme.

As to the applications of our results, we show that these questions are of special interest for the study of allocation problems in finance in the context of a non-stationary economy. Suppose that $(S_t = \exp(X_t))_{t\geq 0}$ defines the evolution of the price of a risky asset which is traded continuously on the market together with some riskless bank account

$(S_t^0)_{t\geq 0}$. At each time t a trader has to invest a part $\pi_t W_t^\pi$ of his/her wealth W_t^π in S and the other part $(1-\pi_t)W_t^\pi$ in S^0. His/her aim is to find the strategy $(\pi_t)_{0\leq t\leq T}$ which maximizes $\mathbb{E}(U(W_T^\pi))$ (where U is a given function called Utility Function). For the Logarithm Utility Function, we can prove that there exists an optimal strategy written in terms of the continuous time filter. Thus, we want to find an approximation of this continuous time filter in a setting where the data is observed at discrete times and the moves (buying and selling assets) can only be made at discrete times. This justifies the study of the case (a). Moreover, optimality is reached under the assumption that the market is perfectly described by our prescribed model: since the parameters of the model are very difficult to know, we study the case (b).

We give numerical results concerning:

- An optimal allocation procedure when the parameters of our mathematical model are perfectly specified and calibrated;
- An allocation procedure in misspecified situations: in this case we have to deal with a mathematical object that corresponds intuitively to some kind of *misspecified filter*.

Our problem is in relation with the rupture detection and can be viewed as a generalization of the one studied in [2–4]. One can quote the reference book [1] on this particular subject. Somehow, the following differences can be found between our work and [1]:

- We work in continuous time and [1] is entirely written for discrete time models;
- We want to detect the changes in the return rate with the objective to maximize our wealth whereas [1] deals with another maximization problem;
- We suppose that the dynamic of the return rate is completely known which is not the case in [1].

We also mention the work of [18] in which numerous interesting theoretical financial results and numerical schemes based on E.M. algorithms are presented in a similar framework. Still:

- The results of [18] are stated in the case where all the parameters of the problem are well specified;
- The numerical schemes are presented without consideration of the discretization error;
- The results are given for the unconstrained problem ($\pi_t \in \mathbb{R}$) whereas we work with constrained strategies ($\pi_t \in [0, 1]$).

Concerning the disorder problem in continuous time, we refer among others to Shiryaev [19, 20]. To our knowledge, the models presented there have only one rupture: a situation which is different from our model.

1.2. Description of the Model and Definition of the Continuous Time Filter

Consider the stochastic process (X_t), solution of the SDE (1.2), where $\mu(t)$ is a continuous-time Markov chain with two states μ_1 and μ_2. The infinitesimal generator matrix G of $\mu(t)$ is:

$$G = \begin{pmatrix} -\lambda_1 & \lambda_1 \\ \lambda_2 & -\lambda_2 \end{pmatrix}.$$

The law of the initial condition $\mu(0)$ is supposed to be known:

$$\mathbb{P}(\mu(0) = \mu_1) = p_0 = 1 - \mathbb{P}(\mu(0) = \mu_2) \tag{1.3}$$

For any process Y, we denote by $\mathbb{F}^Y = (\mathcal{F}_t^Y)_{t\geq 0}$ the filtration generated by Y, that is

$$\mathcal{F}_t^Y = \sigma(Y_s; 0 \leq s \leq t).$$

We define the filter $(F_t)_{t\geq 0}$ to be the optional projection of $(\mathbb{1}_{\mu(t)=\mu_1})_{t\geq 0}$ on $\mathbb{F}^X$. This means that $(F_t)_{t\geq 0}$ is the unique optional process such that (see Revuz and Yor [17], Theorem 5.6, p. 173)

$$\mathbb{E}[\mathbb{1}_{\mu(\tau)=\mu_1}\mathbb{1}_{\tau<\infty} \mid \mathcal{F}_\tau^X] = F_\tau \mathbb{1}_{\tau<\infty} \quad \text{a.s. for every stopping time } \tau.$$

In particular, $F_t = \mathbb{P}(\mu(t) = \mu_1 \mid \mathcal{F}_t^X)$ a.s. for all $t \in [0, T]$.

Remark 1.1. The classical definition of the filter is the conditional law $\mathcal{L}(\mu(t) \mid \mathcal{F}_t^X)$. In our case, the state space has only two elements; our definition is an abuse of notation.

1.3. Organization of the Article

The organization of the article is the following:

In Section 2, we recall some results concerning classical filtering theory and present them in our particular setting. We define the innovation process and present the Kushner–Stratonovich equation satisfied by the filter (see Kurtz and Ocone [11]).

In Section 3, we introduce two approximations of the filter: the Euler Scheme and the Prediction Filter. In Theorem 3.4, which represents the main result of this article, we give new results concerning the rate of convergence (in L^2) toward the continuous time filter. The kind of approximation used for the construction of our Prediction Filter has been investigated in many articles (see, for example, Kushner [12], Di Masi and Runggaldier [7], Picard [16], Florchinger and Le Gland [8], Körezlioğlu and Runggaldier [10], and the references therein). However, to our knowledge, no result can be applied to our model.

In Section 4, we construct Misspecified Filters in continuous and discrete time. These filters take into account the errors concerning a bad specification of the parameters $\mu_1, \mu_2, \lambda_1, \lambda_2$ that appear in the description of the model. In particular, the Misspecified Prediction Filter takes into account all sources of errors (errors on the parameters and errors of discretization). A classical control is given.

In Section 5, we present results concerning the uniform control of the errors of these filters in comparison with the continuous time filter. The main ingredient is the stability of $(F_t)_{t\geq 0}$.

Finally, in Section 6, we present an application of our results in the financial context of an Optimal Allocation Strategy in a non-stationary setting. We present the model and give a formula for the (constrained) optimal allocation policy for the Logarithm Utility function in the continuous time context (see Karatzas and Shreve [9]). In particular, we prove that the constrained optimal strategies depend on the continuous time filter and we show numerical results concerning the asymptotic behavior of the expected wealths.

2. CLASSICAL FILTERING THEORY: THE INNOVATION PROCESS AND THE CONTINUOUS TIME FILTER

2.1. The Innovation Process

Proposition 2.1. *The optional projection of* $(\mu(t))_{t\geq 0}$ *on* $\mathbb{F}^X$ *is*:

$$\mu^{\text{opt}}(t) = \mu_1 F_t + \mu_2(1 - F_t).$$

The following process

$$\overline{B}_t := \frac{1}{\sigma}\left(X_t - \int_0^t \mu^{\text{opt}}(s)ds\right) \tag{2.1}$$

is an $\mathbb{F}^X$*-Brownian motion. It is called the innovation process.*

Proof. From Levy's characterization theorem, it is sufficient to show that $(\overline{B}_t)_{t\geq 0}$ is a continuous local $\mathbb{F}^X$-martingale with

$$\langle \overline{B} \rangle_t = t, \quad t \geq 0, \text{ a.s.}$$

Note that $\overline{B}$ is $\mathbb{F}^X$ adapted (see (2.1)) and continuous because X is. It is also easy to check that

$$\langle \overline{B} \rangle_t = \frac{1}{\sigma^2}\langle X \rangle_t = \langle B \rangle_t = t.$$

Thus, it remains only to prove that $\overline{B}$ is an $\mathbb{F}^X$-martingale.

It is easily seen that $(\mu^{\text{opt}}(t))_{t\geq 0}$ is the optional projection of $(\mu(t))_{t\geq 0}$ on $\mathbb{F}^X$. For all $0 \leq s \leq t$,

$$\begin{aligned}\mathbb{E}[\overline{B}_t - \overline{B}_s \mid \mathscr{F}_s^X] &= \frac{1}{\sigma}\mathbb{E}\left[\int_s^t (\mu(u) - \mu^{\text{opt}}(u))du \mid \mathscr{F}_s^X\right] + \mathbb{E}[B_t - B_s \mid \mathscr{F}_s^X] \\ &= \frac{1}{\sigma}\int_s^t \mathbb{E}[\mu(u) - \mu^{\text{opt}}(u) \mid \mathscr{F}_s^X]du \\ &\quad + \mathbb{E}\{\mathbb{E}[B_t - B_s \mid \mathscr{F}_s^B \vee \mathscr{F}_s^\mu] \mid \mathscr{F}_s^X\}\end{aligned}$$

and from the definition of μ^{opt} and the independence of the σ-algebras $\mathscr{F}_s^B$ and $\mathscr{F}_s^\mu$, we see that

$$\begin{aligned}&\mathbb{E}[\overline{B}_t - \overline{B}_s \mid \mathscr{F}_s^X] \\ &\quad = \frac{1}{\sigma}\int_s^t \mathbb{E}\{[\mu(u) - \mathbb{E}[\mu(u) \mid \mathscr{F}_u^X]] \mid \mathscr{F}_s^X]du + \mathbb{E}\{\mathbb{E}[B_t - B_s \mid \mathscr{F}_s^B]\mid \mathscr{F}_s^X\} = 0.\end{aligned}$$

From the previous equations, we can conclude that $(\overline{B}_t)_{t\geq 0}$ is an $\mathbb{F}^X$-Brownian motion. $\square$

2.2. The Continuous Time Filter

In 1965, Wonham [21] showed that $(F_t)_{t\geq 0}$ satisfies a stochastic differential equation. In our case, due to the definition of $(\mu(t))_{t\geq 0}$ and Kurtz and Ocone [11, p. 90], we have the following lemma.

Lemma 2.2. *The filter satisfies the following Kushner–Stratonovich SDE*:

$$F_t = p_0 + \int_0^t (-\lambda_1 F_s + \lambda_2(1 - F_s))ds + \int_0^t \frac{\mu_1 - \mu_2}{\sigma} F_s(1 - F_s)d\overline{B}_s, \quad (2.2)$$

where p_0 is defined in (1.3).

We precise here the properties of the SDE (2.2).

Lemma 2.3. *The equation* (2.2) *admits a unique strong solution* $(F_t)_{t\geq 0}$. *Moreover, the boundary points* 0 *and* 1 *are entrance-not-exit, that is*: $\forall p_0 \in [0, 1]$, $\forall t > 0$, $F_t \in (0, 1)$.

The proof is based on the Feller test for explosions. See Borodin and Salminen [5, p. 15] for the classification of boundary points.

Remark 2.4.

a. Equation (2.1) gives the decomposition of X in its own filtration $\mathbb{F}^X$

$$dX_t = (\mu_1 F_t + \mu_2(1 - F_t))dt + \sigma d\overline{B}_t. \quad (2.3)$$

b. The filtrations $\mathbb{F}^{\overline{B}}$ and $\mathbb{F}^X$ coincide.

3. FILTER APPROXIMATIONS

We set a time step $\delta > 0$ and we denote the increments of X by $\Delta X_k = X_{(k+1)\delta} - X_{k\delta}$.

3.1. The Euler Scheme

We present here a simple method to estimate the filter F. Thanks to (2.3) and (2.2), we write the dynamics of F as:

$$dF_t = (-\lambda_1 F_t + \lambda_2(1-F_t))dt - \frac{\mu_1 - \mu_2}{\sigma^2} F_t(1-F_t)(\mu_1 F_t + \mu_2(1-F_t))dt + \frac{\mu_1 - \mu_2}{\sigma^2} F_t(1-F_t)dX_t. \tag{3.1}$$

To simplify the notation, we write this SDE in the following way:

$$dF_t = \phi_1(F_t)dt + \phi_2(F_t)dX_t.$$

A naive approach to estimate F is to use a Euler scheme:

$$\begin{cases} \overline{F}^e_0 = F_0 \ (=p_0), \\ \overline{F}^e_{k+1} - \overline{F}^e_k = \phi_1(\overline{F}^e_k)\delta + \phi_2(\overline{F}^e_k)\Delta X_k. \end{cases}$$

The following is a classical result:

$$\mathbb{E}\Big[\sup_{k\delta \le t}(F_{k\delta} - \overline{F}^e_k)^2\Big] \le C_t \delta. \tag{3.2}$$

Remark 3.1.

a. As in Theorem 5.2, we can prove that (3.2) is still available with C independent of t.
b. This procedure does not ensure that $\overline{F}^e_t$ remains in [0, 1]. In practice, we project this scheme on [0, 1].
c. Note that (F_t) remains "naturally" in [0, 1], it is not a reflected process and one cannot use the literature on reflected processes to build a Euler scheme staying in [0, 1].

In the following, we describe another approximation filter (of higher order) based on the filtering theory.

3.2. The Prediction Filter

3.2.1. A Discrete Time Model

We take a Markov chain $(\tilde{\mu}_k)_{k\geq 0}$ taking values in $\{\mu_1, \mu_2\}$ such that $\tilde{\mu}_0 = \mu(0)$ and with transition matrix:

$$\begin{bmatrix} Q(\mu_1, \mu_1) & Q(\mu_1, \mu_2) \\ Q(\mu_2, \mu_1) & Q(\mu_2, \mu_2) \end{bmatrix} = \begin{bmatrix} e^{-\lambda_1 \delta} & 1 - e^{-\lambda_1 \delta} \\ 1 - e^{-\lambda_2 \delta} & e^{-\lambda_2 \delta} \end{bmatrix}$$

We take $(\widetilde{X}_k)_{k\geq 0}$ such as

$$\widetilde{X}_0 = x,$$
$$\widetilde{X}_{k+1} = \widetilde{X}_k + \tilde{\mu}_k \delta + \sigma\sqrt{\delta} U_k$$

where $(U_k)_{k\geq 0}$ are *i.i.d.* variables with law $\mathcal{N}(0, 1)$. The chain $(\tilde{\mu}_k, \widetilde{X}_k)_{k\geq 0}$ may be viewed as an approximation of $(\mu(k\delta), X_{k\delta})_{k\geq 0}$ where $\tilde{\mu}$ is only allowed to jump at the discrete times $k\delta$ (with probabilities near the probabilities that $\mu(t)$ may jump between $k\delta$ and $(k+1)\delta$). We set

$$\begin{aligned} g(y, u) &= \frac{1}{\sigma\sqrt{2\pi\delta}} \exp\left(-\frac{(y - u\delta)^2}{2\sigma^2\delta}\right), \\ \Delta\widetilde{X}_k &= \widetilde{X}_{k+1} - \widetilde{X}_k. \end{aligned} \tag{3.3}$$

Usually, the law $\mathcal{L}(\tilde{\mu}_n \mid \widetilde{X}_0, \ldots, \widetilde{X}_n)$ is called the prediction filter for this discrete time model.

Lemma 3.2. *For any function $f : \{\mu_1, \mu_2\} \to \mathbb{R}$, for all $n \geq 0$,*

$$\begin{aligned} &\mathbb{E}(f(\tilde{\mu}_n) \mid \widetilde{X}_0, \ldots, \widetilde{X}_n) \\ &\quad = \frac{\sum_{i_0,\ldots,i_n \in\{1,2\}} f(\mu_{i_n}) \mathbb{P}(\tilde{\mu}_0 = \mu_{i_0}) \prod_{k=0}^{n-1} g(\Delta\widetilde{X}_k, \mu_{i_k}) Q(\mu_{i_k}, \mu_{i_{k+1}})}{\sum_{i_0,\ldots,i_n \in\{1,2\}} \mathbb{P}(\tilde{\mu}_0 = \mu_{i_0}) \prod_{k=0}^{n-1} g(\Delta\widetilde{X}_k, \mu_{i_k}) Q(\mu_{i_k}, \mu_{i_{k+1}})}. \end{aligned} \tag{3.4}$$

If we set

$$\widetilde{F}_n = \mathbb{P}(\tilde{\mu}_n = \mu_1 \mid \widetilde{X}_0, \ldots, \widetilde{X}_n),$$
$$\widetilde{F}'_n = \frac{g(\Delta\widetilde{X}_n, \mu_1)\widetilde{F}_n}{g(\Delta\widetilde{X}_n, \mu_1)\widetilde{F}_n + (1 - \widetilde{F}_n) g(\Delta\widetilde{X}_n, \mu_2)}$$

we then have:

$$\widetilde{F}_{n+1} = \widetilde{F}'_n Q(\mu_1, \mu_1) + (1 - \widetilde{F}'_n) Q(\mu_2, \mu_1).$$

We denote by Θ^n the function such that

$$\widetilde{F}_n = \Theta^n(\widetilde{X}_0, \ldots, \widetilde{X}_n). \tag{3.5}$$

3.2.2. Construction of the Prediction Filter

Let us now describe in details the evolution of a classical discrete approximation $(\overline{F}_k)_{k\geq 0}$ of $(F_{k\delta})_{k\geq 0}$. We will call this approximation the *prediction filter* in the rest of the article. We set $\overline{F}_0 = F_0$ $(=p_0)$ and for all k, $\overline{G}_k = 1 - \overline{F}_k$. We define our approximation recursively by:

First Step: Updating

$$\begin{pmatrix} \overline{F}_k \\ \overline{G}_k \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{F}'_k = \overline{F}_k \frac{1}{\sigma\sqrt{2\pi\delta}} \exp - \frac{(\Delta X_k - \mu_1\delta)^2}{2\sigma^2\delta} \\ \overline{G}'_k = \overline{G}_k \frac{1}{\sigma\sqrt{2\pi\delta}} \exp - \frac{(\Delta X_k - \mu_2\delta)^2}{2\sigma^2\delta} \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{F}''_k = \frac{\overline{F}'_k}{\overline{F}'_k + \overline{G}'_k} \\ \overline{G}''_k = \frac{\overline{G}'_k}{\overline{F}'_k + \overline{G}'_k} \end{pmatrix}$$

Second Step: Prediction

$$\begin{pmatrix} \overline{F}''_k \\ \overline{G}''_k = 1 - \overline{F}_k \end{pmatrix} \longrightarrow \begin{pmatrix} \overline{F}_{k+1} = \overline{F}''_k e^{-\lambda_1\delta} + \overline{G}''_k(1 - e^{-\lambda_2\delta}) \\ \overline{G}_{k+1} = \overline{G}''_k e^{-\lambda_2\delta} + \overline{F}''_k(1 - e^{-\lambda_1\delta}) \end{pmatrix}$$

Remark 3.3.

1) Note that $\forall n$, $\overline{F}_n \in [0, 1]$
2) We have $\overline{F}_n = \Theta^n(X_0, \ldots, X_n)$. Note that $\overline{F}_n$ is constructed exactly in the same way as $\widetilde{F}_n$ (see (3.5), $\widetilde{F}_n = \Theta^n(\widetilde{X}_0, \ldots, \widetilde{X}_n)$). We will prove later that this discrete filter approximates $F_{n\delta}$.

3.2.3. Convergence of the Prediction Filter

Theorem 3.4. *For all $N \in \mathbb{N}$, there exists a constant $C_{N\delta}$ (depending continuously on $N\delta$ and the parameters of the problem) such that for any X_0, F_0:*

$$\mathbb{E}\Big[\sup_{0\leq k\leq N} (F_{k\delta} - \overline{F}_k)^2\Big] \leq C_{N\delta}\delta^2. \tag{3.6}$$

There exists a continuous-time extension $(\overline{\overline{F}}_t)_{t\geq 0}$ of $(\overline{F}_k)_{k\in\mathbb{N}}$ (such that $\forall k$, $\overline{\overline{F}}_{k\delta} = \overline{F}_k$) defined below in Equation (A.7) and for all t_0:

$$\mathbb{E}\Big[\sup_{0\leq t\leq t_0} (\overline{\overline{F}}_t - F_t)^2\Big] \leq C_{t_0}\delta^2$$

where C_{t_0} depends only on t_0 and the parameters of the problem.

The complete proof can be found in Appendix A. The idea is the following. We write:

$$\overline{F}''_k = \frac{\overline{F}_k}{\overline{F}_k + (1 - \overline{F}_k)\exp \frac{(\mu_2 - \mu_1)(2\Delta X_k - (\mu_1 + \mu_2)\delta)}{2\sigma^2}}.$$

We introduce a class of "negligible" processes.

We denote by

- $\mathcal{R}$ the set of sequences of random variables $(R_k)_{k\geq 0}$ such that:

$$\sup_k \mathbb{E}(|R_k|^2) \leq C\delta^4. \tag{3.7}$$

- $\mathcal{D}$ the set of sequences of *r.v.* $(D_k)_{k\geq 0}$ such that:

$$\begin{cases} D_k \text{ is } \mathcal{F}^X_{(k+1)\delta} \text{ measurable;} \\ \mathbb{E}(D_k \mid \mathcal{F}^X_{k\delta}) = 0; \\ \mathbb{E}(D_k^2) \leq C\delta^3. \end{cases} \tag{3.8}$$

Then a careful limited development shows that:

$$\begin{aligned} \overline{F}_{k+1} + \lambda_2\delta + R_k = \overline{F}_k - \overline{F}_k(\lambda_1+\lambda_2)\delta + \lambda_2\delta \\ - \overline{F}_k(1-\overline{F}_k)\Big[\alpha\delta + \beta\Delta X_k + \frac{\beta^2}{2}\Delta X_k^2 + \alpha\beta\delta\Delta X_k \\ + \frac{\beta^3}{6}\Delta X_k^3 - \beta(\lambda_1+\lambda_2)\delta\Delta X_k\Big] \\ + \overline{F}_k(1-\overline{F}_k)^2(\beta^2\Delta X_k^2 + 2\alpha\beta\delta\Delta X_k + \beta^3\Delta X_k^3) \\ - \overline{F}_k(1-\overline{F}_k)^3\beta^3\Delta X_k^3 + R_k \end{aligned} \tag{3.9}$$

with $(R_k)_{k\in\mathbb{N}} \in \mathcal{R}$. And we then show that this is equivalent to a Milstein scheme.

4. THE MISSPECIFIED FILTERS: DEFINITION AND CONTROL OF THE ERROR

4.1. Introduction

In this section we consider the case where the coefficients $\mu_1, \mu_2, \lambda_1, \lambda_2$ are unknown. As to applications, it seems not reasonable to assume that the parameters of the underlying model are perfectly known. In [6], the authors construct consistent estimators via an E.M. algorithm in order to estimate the coefficients that lead the dynamics of $(X_t)_{t\geq 0}$. We also mention the work of Sass and Haussmann [18], here the authors use an E.M. algorithm and calibrate their model with a pre-computation procedure.

In this article, we adopt the following point of view: let $\bar{\mu}_1, \bar{\mu}_2, \bar{\lambda}_1, \bar{\lambda}_2$ denote the results of an estimating procedure (that we do not detail here) for $\mu_1, \mu_2, \lambda_1, \lambda_2$.

We will focus on the consequences of taking misspecified parameters when these are plugged in our algorithm to approximate the continuous time filter.

4.2. Definition

As in (3.1), we may define the *misspecified continuous filter*, solution of:

$$\begin{aligned}\widehat{F}_t = F_0 &+ \int_0^t \frac{\bar{\mu}_1 - \bar{\mu}_2}{\sigma^2}\widehat{F}_s(1-\widehat{F}_s)dX_s + \int_0^t (-\bar{\lambda}_1\widehat{F}_s + \bar{\lambda}_2(1-\widehat{F}_s))ds \\ &- \int_0^t \frac{\bar{\mu}_1 - \bar{\mu}_2}{\sigma^2}(\bar{\mu}_1\widehat{F}_s + \bar{\mu}_2(1-\widehat{F}_s))\widehat{F}_s(1-\widehat{F}_s)ds \\ &+ \int_0^t \frac{\bar{\mu}_1 - \bar{\mu}_2}{2}\widehat{F}_s(1-\widehat{F}_s)ds. \end{aligned} \tag{4.1}$$

It is the filter one can compute with the available observations $(X_t)_{t\geq 0}$ and with the wrong coefficients.

Using (2.3), we can rewrite (4.1) in another form:

$$\begin{aligned}\widehat{F}_t = F_0 &+ \int_0^t \frac{\bar{\mu}_1 - \bar{\mu}_2}{\sigma}\widehat{F}_s(1-\widehat{F}_s)d\overline{B}_s + \int_0^t (-\bar{\lambda}_1\widehat{F}_s + \bar{\lambda}_2(1-\widehat{F}_s))ds \\ &+ \int_0^t \frac{\bar{\mu}_1 - \bar{\mu}_2}{\sigma^2}\{(\mu_1 F_s + \mu_2(1-F_s)) - (\bar{\mu}_1\widehat{F}_s + \bar{\mu}_2(1-\widehat{F}_s))\}\widehat{F}_s(1-\widehat{F}_s)ds.\end{aligned}$$

Like in Section 2.2, we prove that the previous equation has a unique strong solution. Furthermore, this solution takes values in (0, 1).

Remark 4.1 (Parameter σ)**.** In the above computations we assume that we know the exact value of σ. In fact σ can be well estimated in a short period of time (provided we have enough data). This is not the case for the other parameters.

Nevertheless, we could write an erroneous $\bar{\sigma}$ in the definition of the misspecified prediction filter and produce estimations in the same way as above. Here we made the choice not to do so in order to have more readable computations.

4.3. Control of the Error

Lemma 4.2. *We have, for all* X_0, $F_0 \in (0, 1)$ *and* $t_0 \geq 0$:

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\leq t\leq t_0} |\widehat{F}_t - F_t|^2\Big] \\ &\quad \leq C(t_0+1)\exp(C(t_0+1)t_0)\sup_{i=1,2}(|\lambda_i - \bar{\lambda}_i|^2 + |\mu_i - \bar{\mu}_i|^2),\end{aligned}$$

where C *depends* (*continuously*) *only on the parameters* λ_1, λ_2, μ_1, μ_2 *and* σ.

The proof can be found in Appendix B. It is based on classical estimates.

4.4. The Misspecified Prediction Filter

We define the misspecified prediction filter $(\widehat{\widehat{F}}_k)_{k\geq 0}$ by induction, taking $\widehat{\widehat{F}}_0 = F_0$ and $\forall k$, $\widehat{\widehat{G}}_k = 1 - \widehat{\widehat{F}}_k$:

First Step: Updating

$$\begin{pmatrix} \widehat{\widehat{F}}_k \\ \widehat{\widehat{G}}_k \end{pmatrix} \longrightarrow \begin{pmatrix} \widehat{\widehat{F}}'_k = \widehat{\widehat{F}}_k \frac{1}{\sigma\sqrt{2\pi\delta}} \exp - \frac{(\Delta X_k - \bar{\mu}_1\delta)^2}{2\sigma^2\delta} \\ \widehat{\widehat{G}}'_k = \widehat{\widehat{G}}_k \frac{1}{\sigma\sqrt{2\pi\delta}} \exp - \frac{(\Delta X_k - \bar{\mu}_2\delta)^2}{2\sigma^2\delta} \end{pmatrix} \longrightarrow \begin{pmatrix} \widehat{\widehat{F}}''_k = \frac{\widehat{\widehat{F}}'_k}{\widehat{\widehat{F}}'_k + \widehat{\widehat{G}}'_k} \\ \widehat{\widehat{G}}''_k = \frac{\widehat{\widehat{G}}'_k}{\widehat{\widehat{F}}'_k + \widehat{\widehat{G}}'_k} \end{pmatrix}$$

Second Step: Prediction

$$\begin{pmatrix} \widehat{\widehat{F}}''_k \\ \widehat{\widehat{G}}''_k = 1 - \widehat{\widehat{F}}_k \end{pmatrix} \longrightarrow \begin{pmatrix} \widehat{\widehat{F}}_{k+1} = \widehat{\widehat{F}}''_k e^{-\bar{\lambda}_1\delta} + \widehat{\widehat{G}}''_k (1 - e^{-\bar{\lambda}_2\delta}) \\ \widehat{\widehat{G}}_{k+1} = \widehat{\widehat{G}}''_k e^{-\bar{\lambda}_2\delta} + \widehat{\widehat{F}}''_k (1 - e^{-\bar{\lambda}_1\delta}) \end{pmatrix}$$

Note that $\forall k$, $\widehat{\widehat{F}}_k \in [0, 1]$.

The following lemma can be proved exactly like the Theorem 3.4 and so we do not write its proof.

Lemma 4.3. *For all $N \in \mathbb{N}$, there exists a constant $C_{N\delta}$ (depending continuously on $N\delta$, λ_1, λ_2, μ_1, μ_2, $\bar{\lambda}_1$, $\bar{\lambda}_2$, $\bar{\mu}_1$, $\bar{\mu}_2$ and σ) such that for any X_0, F_0:*

$$\mathbb{E}\left[\sup_{0\leq k\leq N} (\widehat{F}_{k\delta} - \widehat{\widehat{F}}_k)^2 \right] \leq C_{N\delta}\delta^2.$$

5. UNIFORM CONVERGENCE OF THE FILTERS

5.1. A General Result of Approximation

For all $0 \leq s \leq t$ and $y \in (0, 1)$, we denote by $P_{s,t}y$ the value, at time t, of the solution of (2.2) whose value in s is equal to y. The operator P is a stochastic flow. For all t, we have $P_{0,t}F_0 = F_t$.

For any stochastic flow $\widetilde{P}$, we will use the following conventions:

- $\forall t$, $\widetilde{P}_{t,t} = Id$
- if $s > t$, $\widetilde{P}_{s,t} = Id$.

We have the following classical uniform control in time.

Proposition 5.1. *Suppose we have a stochastic flow $(\widetilde{P}_{s,t})_{0\leq s\leq t}$ and $\epsilon > 0$ such that $\forall k \in \mathbb{N}$, $\forall y \in [0, 1]$*

- *$\forall s$, $(\widetilde{P}_{s,t}y)_{t\geq s}$ is a Markov process*
- *$(\widetilde{P}_{s,t}y)_{t\geq s}$ is $(\mathcal{F}^X_t)_{t\geq s}$ adapted*

- $\forall t, \omega$, $\mathbb{E}(\sup_{s\in[0,1]} |P_{t,t+s}y - \widetilde{P}_{t,t+s}y| \,|\mathbb{F}_t^X) \leq \epsilon$
  then

$$\sup_{t\geq 0} \mathbb{E}(|F_t - \widetilde{P}_{0,t}F_0|) \leq \frac{2\epsilon}{1-e^{-\lambda_1-\lambda_2}}.$$

Sketch of the Proof. We begin by showing that P is a contracting flow in some sense. We have, $\forall x, x' \in [0, 1]$ and $0 \leq s \leq t$:

$$\mathbb{E}(|P_{s,t}x - P_{s,t}x'|) \leq e^{-(\lambda_1+\lambda_2)(t-s)}|x - x'|. \tag{5.1}$$

And the following decomposition will give the result:

$$F_t - \widetilde{P}_{0,t}F_0 = \sum_{k=0}^{\lfloor t\rfloor -1} (P_{k+1,t}P_{k,k+1}\widetilde{P}_{0,k}F_0 - P_{k+1,t}\widetilde{P}_{k,k+1}\widetilde{P}_{0,k}F_0) + P_{\lfloor t\rfloor,t}\widetilde{P}_{0,\lfloor t\rfloor}F_0 - \widetilde{P}_{0,t}F_0. \qquad \square$$

5.2. Convergence of Our Filters

As a corollary of Theorem 3.4, Lemmas 4.2, 4.3, and Proposition 5.1, we can state the following theorem:

Theorem 5.2. *We have the following uniform bounds*

$$\sup_{t\geq 0} \mathbb{E}(|F_{\lfloor t/\delta\rfloor\delta} - \overline{F}_{\lfloor t/\delta\rfloor}|) \leq C\delta. \tag{5.2}$$

$$\sup_{t\geq 0} \mathbb{E}(|F_t - \widehat{F}_t|) \leq C\left(\sup_{i=1,2}(|\lambda_i - \bar{\lambda}_i| + |\mu_i - \bar{\mu}_i|)\right). \tag{5.3}$$

$$\sup_{t\geq 0} \mathbb{E}(|F_{\lfloor t/\delta\rfloor\delta} - \widehat{\overline{\overline{F}}}_{\lfloor t/\delta\rfloor}|) \leq C\left(\delta + \sup_{i=1,2}(|\lambda_i - \bar{\lambda}_i| + |\mu_i - \bar{\mu}_i|)\right), \tag{5.4}$$

where C depends continuously on the parameters.

Proof. Equation (5.3) is a direct corollary of Lemma 4.2 and Proposition 5.1. By Theorem 3.4 and Proposition 5.1: $\forall t$, $\mathbb{E}(|F_t - \overline{\overline{F}}_t|) \leq C\delta$. For all k, $\overline{\overline{F}}_{k\delta} = \overline{F}_k$ and so, by (A.7), we have (5.2).

By Lemma 4.3, we have for all $k \leq 2/\delta$,

$$\mathbb{E}(|\widehat{F}_{k\delta} - \widehat{\overline{\overline{F}}}_k|) \leq C\delta.$$

So, by Lemma 4.2, we have for all $k \leq 2/\delta$,

$$\mathbb{E}(|F_{k\delta} - \widehat{\overline{\overline{F}}}_k|) \leq C\left(\delta + \sup_{i=1,2}(|\lambda_i - \bar{\lambda}_i| + |\mu_i - \bar{\mu}_i|)\right).$$

Then, by (2.2), for all $t \leq 1$,

$$\mathbb{E}(|F_t - \widehat{F}_t|) \leq C\left(\sup_{i=1,2}(|\lambda_i - \bar{\lambda}_i| + |\mu_i - \bar{\mu}_i|)\right).$$

And so (5.4) comes from Proposition 5.1. □

Remark 5.3. In fact, we can define $\overline{F}$ and $\widehat{\overline{F}}$ in continuous time in the same way as in Subsection 3.2.2 (just replace δ by $(t - \lfloor t/\delta \rfloor \delta)$ in the last *updating* and *prediction* steps): this construction ensures that the results of Theorem 5.2 remain valid in continuous time.

6. APPLICATION: OPTIMAL PORTFOLIO ALLOCATION STRATEGY

6.1. Presentation of the Problem

In this section, we describe an application of our method to a problem arising in financial mathematics. Consider two assets (a bank account and a risky asset) that are traded continuously. The price of the bank account evolves according to:

$$\frac{dS_t^0}{S_t^0} = r\,dt. \tag{6.1}$$

The price of the risky asset evolves according to the following SDE:

$$\frac{dS_t}{S_t} = \left(\mu(t) + \frac{\sigma^2}{2}\right)dt + \sigma\,dB_t, \tag{6.2}$$

where $\mu(t)$ is defined in Subsection 1.2.

Remark 6.1. The process $X_t = \log(S_t)$ satisfies equation (1.2).

Our aim is to compute the optimal strategy of a trader who perfectly knows all the parameters μ_1, μ_2, λ_1, λ_2, and σ.

Let π_t denote the proportion of the trader's wealth invested in the stock S at time t; the remaining proportion $1 - \pi_t$ is invested in the bond S^0. For a given non random initial capital $x > 0$, let $W_{\cdot}^{x,\pi}$ denote the wealth process corresponding to the portfolio $\pi_{\cdot}$. This wealth process is the solution of the following equation:

$$\begin{cases} \dfrac{dW_t^{x,\pi}}{W_t^{x,\pi}} = \pi_t \dfrac{dS_t}{S_t} + (1 - \pi_t)\dfrac{dS_t^0}{S_t^0} \quad 0 \leq t \leq T, \\ W_0^{x,\pi} = x. \end{cases}$$

The set of *admissible* portfolios is defined by:

$$\mathcal{A}(x) := \{\pi_{.}\ \mathbb{F}^X - \text{progressively measurable process with values in } [0,1] \text{ s.t. } W_0^{x,\pi} = x\}$$

Remark 6.2. The constraint $\pi \in [0, 1]$ means that the investor is not allowed to borrow.

The investor's objective is to maximize the expectation of the wealth utility function at the terminal time T (we will consider only the Logarithm Utility); he/she has to solve the following constrained optimization problem:

$$\mathcal{P} : V^*(x) := \sup_{(\pi_t)_{0\le t\le T}\in\mathcal{A}(x)} \mathbb{E}[\log(W_T^{x,\pi})].$$

We denote by $\text{proj}_{[0,1]}$ the projection on the interval $[0, 1]$, that is $\text{proj}_{[0,1]}(x) = x$ if $0 \le x \le 1$, $\text{proj}_{[0,1]}(x) = 0$ if $x \le 0$ and $\text{proj}_{[0,1]}(x) = 1$ if $x \ge 1$.

In this particular context, we can write the optimal allocation strategy:

$$\pi_t^* = \text{proj}_{[0,1]}\left\{\frac{\mu^{\text{opt}}(t) - r}{\sigma^2}\right\},$$

See for instance [9] for a general proof and [13] for more details in this particular example and for some extensions. Sass and Haussmann [18] studied the same model without the constraint for the portfolio to stay in [0, 1].

6.2. Implementable Strategy

The optimal allocation strategy is of the form $\pi_t^* = q^*(F_t)$. In practice, we are only able to act on the allocation of our wealth at discrete times and we may not know exactly $\lambda_1, \lambda_2, \mu_1, \mu_2$.

Suppose we approximate $(F_t)_{t\ge0}$ by a misspecified prediction filter $(\widehat{\overline{F}}_k)_{k\ge0}$ based on a time discretization interval $\delta_1 > 0$ and on "wrong" coefficients $\bar\lambda_1, \bar\lambda_2, \bar\mu_1, \bar\mu_2$ (see Remark 4.1 concerning the reasons that allow us to assume that we perfectly know the value of the parameter σ).

Suppose also that we change the allocation of our wealth at instants $0, \delta_2, 2\delta_2, \ldots$ with $\delta_2 = m\delta_1$. We denote by $\widehat{\overline{W}}_t^x$ the wealth at time t (starting from x) obtained when we replace the optimal strategy $\pi_t^* = q^*(F_t)$ by $\hat\pi_{k\delta_2}^* = q^*(\widehat{\overline{F}}_{k\delta_2})$ in the dynamics.

In [13], we control the error generated by this scheme, for all $T \geq 1$

$$\mathbb{E}\left(\left|\frac{1}{T}\log(W_T^{*,x}) - \frac{1}{T}\log(\widehat{\overline{W}}_t^x)\right|\right) \leq C\left(\delta_1 + \delta_2 + \sup_{i=1,2}(|\mu_i - \bar{\mu}_i| + |\lambda_i - \bar{\lambda}_i|)\right).$$

The above inequality is obtained as follows. The function q^* is Lipschitz and the process μ^{opt} is bounded. By Theorem 5.2, we obtain:

$$\begin{aligned}
&\mathbb{E}\left(\left|\frac{1}{T}\log(W_T^{*,x}) - \frac{1}{T}\log(\widehat{\overline{W}}_t^x)\right|\right) \\
&\leq \frac{C}{T}\int_0^T \mathbb{E}(|\widehat{\overline{F}}_{\lfloor t/\delta_2\rfloor} - F_t|)dt \\
&\leq \frac{C}{T}\int_0^T \mathbb{E}(|\widehat{\overline{F}}_{\lfloor t/\delta_2\rfloor} - F_{\lfloor t/\delta_2\rfloor}|) + \mathbb{E}(|F_{\lfloor t/\delta_2} - F_t|)dt \\
&\leq C\left(\delta_1 + \delta_2 + \sup_{i=1,2}(|\mu_i - \bar{\mu}_i| + |\lambda_i - \bar{\lambda}_i|)\right).
\end{aligned}$$

6.3. Numerical Experiments

6.3.1. Introduction

In this numerical section, we illustrate the performances of the previous strategies. We also compare them to a strategy which does not need any mathematical model: a technical analysis strategy based on the moving average indicator (see [4] for more details).

6.3.2. The Moving Average

At each discrete time, the trader computes the moving average of the prices:

$$M_t^{(\Delta)} = \frac{1}{\Delta}\int_{t-\Delta}^t S_u du. \tag{6.3}$$

- If the price is larger than the moving average, the trader estimates that the price is in an increasing period: He/she buys the risky asset.
- If the price is smaller than the moving average, the trader estimates that the price is in a decreasing period: He/she sells the risky asset.

His/her strategy can be summed up as:

$$\pi_t^{\text{MA}} = \mathbb{1}_{\{S_t > M_t^{(\Delta)}\}}.$$

In this section, the values of the parameters are given in Table 1.

Table 1. Values of the parameters

μ_1	μ_2	λ_1	λ_2	σ	r
−0.1	0.1	1.0	1.0	0.15	0.0

6.3.3. A Nominal Trajectory

In Figure 1, we display a typical trajectory with the parameters given in Table 1.

6.3.4. Comparison of Performances

In Figure 2, we present the performances of traders that use

1) The optimal allocation strategy;
2) The allocation strategy using our estimation of the filter;
3) The allocation strategy using Euler's approximation of the filter;
4) The moving average indicator with a window of 0.5 year.

We can remark that it is difficult to differentiate between the performances of the second and the third traders.

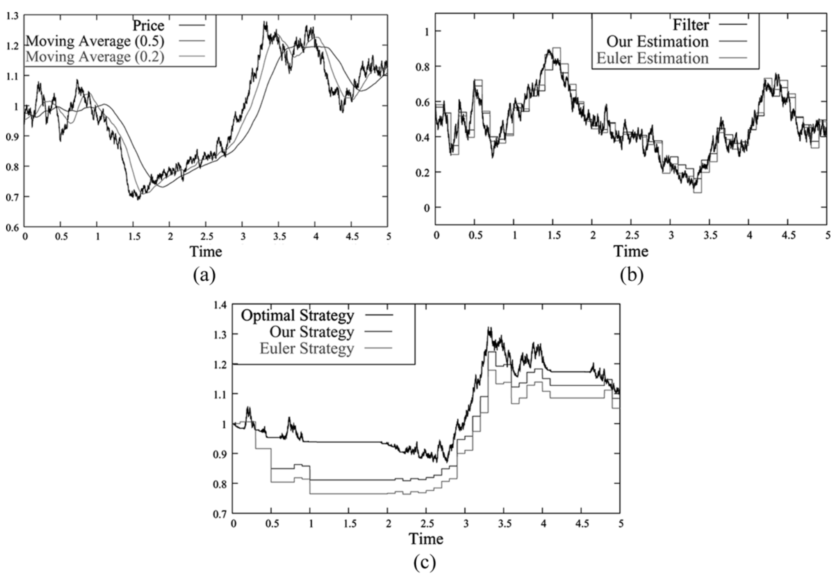

Figure 1. A nominal trajectory. (a) prices and moving averages; (b) exact filter, our estimation, estimation with Euler scheme; and (c) wealths with optimal strategy, with our estimation, with Euler scheme.

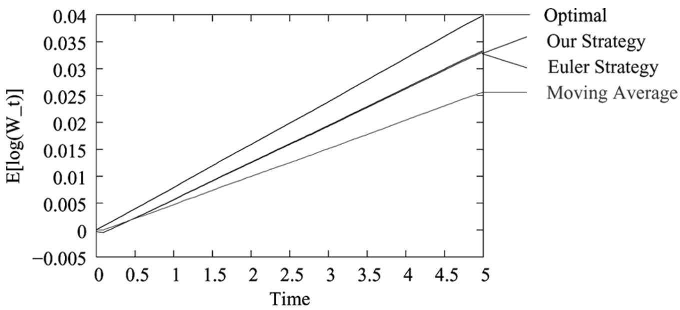

Figure 2. Comparison of performances.

6.3.5. Comparison of Performances with Errors on the Parameters

In Figure 3, we present the performances of traders that use

1) The optimal allocation strategy;
2) The allocation strategy using our estimation of the filter with errors on the parameters;
3) The allocation strategy using the Euler's approximation of the filter with errors on the parameters;
4) The moving average indicator with a window of 0.5 year.

The misspecified parameters are given in Table 2. In this study, we do not use any estimation procedure. We suppose that the trader has his/her own estimation procedure that we do not describe here.

We can observe that, for this particular choice of parameters, the performances of the trader using the moving average indicator is between the performances of the trader using our estimation of the filter (with the calibration errors) and those using a Euler scheme. In other words, our estimation of the filter is more robust to calibration than the Euler scheme.

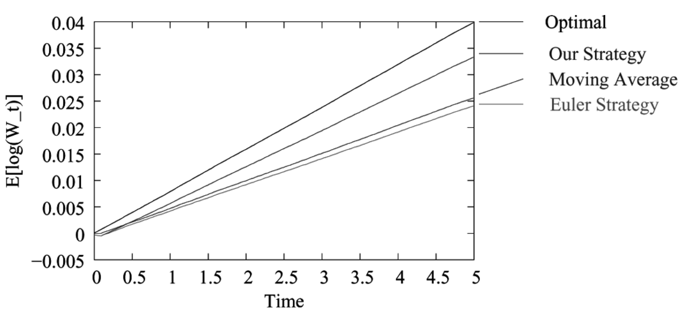

Figure 3. Comparison of performances with errors on the parameters.

Table 2. Estimated values of the parameters

$\bar{\mu}_1$	$\bar{\mu}_2$	$\bar{\lambda}_1$	$\bar{\lambda}_2$
−0.2	0.2	2.0	2.0

APPENDIX A: PROOF OF THEOREM 3.4

In the following, we will write C in the place of some constant depending on the parameters $\mu_1, \mu_2, \sigma, \lambda_1, \lambda_2$. This constant may change from line to line. We suppose also that $\delta \leq 1$. We set, for all integer $k \geq 0$ $\overline{\overline{F}}_{k\delta} = \overline{F}_k$ and $\eta(t) = \delta\lfloor \frac{t}{\delta} \rfloor$.

We have the following properties:

(P1) If $(R_k)_{k\geq 0} \in \mathscr{R}$ and $(R'_k)_{k\geq 0} \in \mathscr{R}$ then $(R_k + R'_k)_{k\geq 0} \in \mathscr{R}$.
(P2) If $(R_k)_{k\geq 0} \in \mathscr{R}$ and $(R'_k)_{k\geq 0}$ are such that $\sup_k |R'_k| \leq C$ *a.s.* then $(R_k R'_k)_{k\geq 0} \in \mathscr{R}$.

By (2.3), we have:

$$\Delta X_k = \int_{k\delta}^{(k+1)\delta} \mu^{\text{opt}}(s)ds + \sigma \Delta \overline{B}_k \tag{A.4}$$

where

$$\Delta \overline{B}_k = \overline{B}_{(k+1)\delta} - \overline{B}_{k\delta}. \tag{A.5}$$

From (A.4), we can easily note that the following sequences are in $\mathscr{R}$: $(\delta^2)_{k\geq 0}$, $(\delta \Delta X_k^2)_{k\geq 0}$, $(\delta^2 \Delta X_k)_{k\geq 0}$, $(\delta^3)_{k\geq 0}$, $(\Delta X_k^4)_{k\geq 0}$.

In the following, we will write $(R_k)_{k\geq 0}$ for a sequence in $\mathscr{R}$ and $(D_k)_{k\geq 0}$ for a sequence in $\mathscr{D}$. These sequences may change from line to line.

Here is now a technical lemma:

Lemma A.1. *We have the following decomposition*:

$$\Delta X_k^2 = \sigma^2 \Delta \overline{B}_k^2 + D_k + R_k.$$

Proof.

$$\begin{aligned}\Delta X_k^2 &= 2\int_{k\delta}^{(k+1)\delta} \left(\int_{k\delta}^{s} \mu_1 F_u + \mu_2(1 - F_u)du \right)(\mu_1 F_s + \mu_2(1 - F_s))ds \\ &\quad + 2\sigma \int_{k\delta}^{(k+1)\delta} (\overline{B}_s - \overline{B}_{k\delta})(\mu_1 F_s + \mu_2(1 - F_s))ds \\ &\quad + \sigma^2(\overline{B}_{(k+1)\delta} - \overline{B}_{k\delta})^2 + 2\sigma \int_{k\delta}^{(k+1)\delta} \int_{k\delta}^{s} \mu_1 F_u + \mu_2(1 - F_u)du\, d\overline{B}_s\end{aligned}$$

Obviously, the first term is in $\mathscr{R}$.

By integration by part, the sum of the second and the last term is equal to:

$$
\begin{aligned}
A &= 2\sigma\Delta\overline{B}_k \int_{k\delta}^{(k+1)\delta} \mu_1 F_u + \mu_2(1-F_u)du \\
&= 2\sigma\Delta\overline{B}_k \delta[\mu_1 F_{k\delta} + \mu_2(1-F_{k\delta})] + 2\sigma\Delta\overline{B}_k(\mu_1-\mu_2)\int_{k\delta}^{(k+1)\delta}(F_s - F_{k\delta})ds \\
&= D_k + R_k.
\end{aligned}
$$

□

Proof of Theorem 3.4.

First Step

We will show in this step that the sequence $(\overline{F}_k)_{k\geq 0}$ is the Euler–Milstein scheme associated to $(F_t)_{t\geq 0}$ up to negligible terms. We have:

$$
\overline{F}''_k = \frac{\overline{F}_k}{\overline{F}_k + (1-\overline{F}_k)\underbrace{\exp\frac{(\mu_2-\mu_1)(2\Delta X_k - (\mu_1+\mu_2)\delta)}{2\sigma^2}}_{=:A}}.
$$

Let us set: $\alpha := \frac{(\mu_1+\mu_2)(\mu_1-\mu_2)}{2\sigma^2}$ and $\beta := \frac{\mu_2-\mu_1}{\sigma^2}$. We have:

$$
\begin{aligned}
\overline{F}''_k &= \frac{\overline{F}_k}{1-(1-A)(1-\overline{F}_k)} \\
&= \overline{F}_k(1 + (1-A)(1-\overline{F}_k) + (1-A)^2(1-\overline{F}_k)^2 \\
&\quad + (1-A)^3(1-\overline{F}_k)^3 + R_k^{(1)})
\end{aligned}
\tag{A.6}
$$

$$
\text{with } R_k^{(1)} = \int_0^{(1-A)(1-\overline{F}_k)} 4((1-A)(1-\overline{F}_k)-s)^3 \frac{1}{(1-s)^5}ds.
$$

We will prove that $(R_k^{(1)})_k \in \mathcal{R}$. First, we have

$$
|R_k^{(1)}| \leq \begin{cases} 4|1-A|^4|1-F_k|^4 & \text{if } 1-A<0 \\ \dfrac{4|1-A|^4|1-F_k|^4}{(1-(1-A)(1-\overline{F}_k))^5} & \text{if } 0\leq 1-A\leq 1. \end{cases}
$$

And thus,

$$
\begin{aligned}
|R_k^{(1)}| &\leq 4|1-A|^4 \sup(1, A^{-5}). \\
A &= \exp(\beta\Delta X_k + \alpha\delta) =: e^U.
\end{aligned}
$$

Due to the classical inequality $|e^U - 1| \leq |U|e^{|U|}$, the following inequality is satisfied:

$$\begin{aligned}
|A-1| &\leq C(|\Delta X_k| + \delta)e^{C(|\Delta X_k|+\delta)}.\\
\mathbb{E}(|R_k^{(1)}|^2) &\leq \mathbb{E}[C|1-A|^8 \sup(1, A^{-10})] \leq \mathbb{E}[C(|\Delta X_k| + \delta)^8 e^{C(|\Delta X_k|+\delta)}]\\
&\leq \mathbb{E}[C(|\Delta \overline{B}_k| + \delta)^8 e^{C(|\Delta \overline{B}_k|+\delta)}] \leq C(\delta)^4 \text{ for } \delta \text{ small enough.}
\end{aligned}$$

We now develop A:

$$A = 1 + \alpha\delta + \beta\Delta X_k + \frac{1}{2}(\alpha\delta + \beta\Delta X_k)^2 + \frac{1}{3!}(\alpha\delta + \beta\Delta X_k)^3 + R_k^{(2)}$$

$$\text{with } R_k^{(2)} = \int_0^{\alpha\delta+\beta\Delta X_k} \frac{(\alpha\delta + \beta\Delta X_k - s)^3}{3!} e^s\, ds.$$

We prove also that $(R_k^{(2)})_k \in \mathscr{R}$.

$$\mathbb{E}(|R_k^{(2)}|^2) \leq \mathbb{E}\{C(\delta + |\Delta X_k|)^8 e^{C(\delta+|\Delta X_k|)}\} \leq C(\delta)^4.$$

Hence:

$$\begin{aligned}
A - 1 &= \alpha\delta + \beta\Delta X_k + \frac{\beta^2}{2}\Delta X_k^2 + \alpha\beta\delta\Delta X_k + \frac{\beta^3}{6}\Delta X_k^3 + R_k.\\
(A-1)^2 &= 2\alpha\beta\delta\Delta X_k + \beta^2(\Delta X_k)^2 + \beta^3(\Delta X_k)^3 + R_k\\
(A-1)^3 &= \beta^3(\Delta X_k)^3 + R_k.
\end{aligned}$$

Getting back to (A.6), we have:

$$\begin{aligned}
\overline{F}_k'' &= \overline{F}_k - \overline{F}_k(1-\overline{F}_k)\left(\alpha\delta + \beta\Delta X_k + \frac{\beta^2}{2}\Delta X_k^2 + \alpha\beta\delta\Delta X_k + \frac{\beta^3}{6}\Delta X_k^3\right)\\
&\quad + \overline{F}_k(1-\overline{F}_k)^2(\beta^2\Delta X_k^2 + 2\alpha\beta\delta\Delta X_k + \beta^3\Delta X_k^3)\\
&\quad - \overline{F}_k(1-\overline{F}_k)^3\beta^3\Delta X_k^3 + R_k.\\
\overline{F}_{k+1} &= \overline{F}_k'' e^{-\lambda_1\delta} + (1-\overline{F}_k'')(1-e^{-\lambda_2\delta})\\
&= \overline{F}_k''(1-\lambda_1\delta - \lambda_2\delta) + \lambda_2\delta + R_k\\
&= \overline{F}_k - \overline{F}_k(\lambda_1+\lambda_2)\delta + \lambda_2\delta - \overline{F}_k(1-\overline{F}_k)\\
&\quad \times \left[\alpha\delta + \beta\Delta X_k + \frac{\beta^2}{2}\Delta X_k^2 + \alpha\beta\delta\Delta X_k + \frac{\beta^3}{6}\Delta X_k^3 - \beta(\lambda_1+\lambda_2)\delta\Delta X_k\right]\\
&\quad + \overline{F}_k(1-\overline{F}_k)^2(\beta^2\Delta X_k^2 + 2\alpha\beta\delta\Delta X_k + \beta^3\Delta X_k^3)\\
&\quad - \overline{F}_k(1-\overline{F}_k)^3\beta^3\Delta X_k^3 + R_k.
\end{aligned}$$

By (A.4), we have

$$\delta\Delta X_k = \sigma\delta\Delta\overline{B}_k + R_k, \quad \Delta X_k^3 = \sigma^3\Delta\overline{B}_k^3 + R_k.$$

Thanks to Lemma A.1, we have:

$$\Delta X_k^2 = \sigma^2 \Delta \overline{B}_k^2 + D_k + R_k$$

where $(D_k)_{k\geq 0} \in \mathscr{D}$.

$$\begin{aligned}\overline{F}_{k+1} &= \overline{F}_k - \overline{F}_k(\lambda_1+\lambda_2)\delta + \lambda_2\delta \\ &\quad + \delta\overline{F}_k(1-\overline{F}_k)\left[-\alpha - \frac{\sigma^2\beta^2}{2} + \sigma^2\beta^2(1-\overline{F}_k)\right] - \beta\Delta X_k\overline{F}_k(1-\overline{F}_k) \\ &\quad + (\sigma^2\Delta\overline{B}_k^2 - \sigma^2\delta)\overline{F}_k(1-\overline{F}_k)\left[-\frac{\beta^2}{2} + \beta^2(1-\overline{F}_k)\right] + \widetilde{D}_k + A_k\Delta\overline{B}_k^3 + R_k\end{aligned}$$

where $(\widetilde{D}_k)_{k\geq 0} \in \mathscr{D}$ and $(A_k)_{k\geq 0}$ is some bounded sequence such that A_k is $\mathscr{F}_{k\delta}^X$-measurable. Thus:

$$\begin{aligned}\overline{F}_{k+1} &= \overline{F}_k - \lambda_1\overline{F}_k\delta + \lambda_2\delta(1-\overline{F}_k) \\ &\quad + \delta\overline{F}_k(1-\overline{F}_k)\frac{(\mu_2-\mu_1)}{\sigma^2}[\mu_1\overline{F}_k + \mu_2(1-\overline{F}_k)] \\ &\quad + \frac{(\mu_1-\mu_2)}{\sigma^2}\overline{F}_k(1-\overline{F}_k)\Delta X_k \\ &\quad + (\Delta\overline{B}_k^2 - \delta)\overline{F}_k(1-\overline{F}_k)\frac{(\mu_2-\mu_1)^2}{2\sigma^2}(1-2\overline{F}_k) + \widetilde{D}_k + A_k\Delta\overline{B}_k^3 + R_k.\end{aligned}$$

Second Step

In this step, we reduce the problem to the Euler–Milstein method. We set $\forall x \in \mathbb{R}$:

$$\begin{aligned} u(x) &:= \mu_1 x + \mu_2(1-x); \\ v(x) &:= -\lambda_1 x + \lambda_2(1-x) + x(1-x)\frac{\mu_2-\mu_1}{\sigma^2}(\mu_1 x + \mu_2(1-x)); \\ w(x) &:= \frac{\mu_1-\mu_2}{\sigma^2}x(1-x). \end{aligned}$$

We extend $(\overline{F}_k)_{k\in\mathbb{N}}$ to a continuous time process $(\overline{\overline{F}}_t)_{t\geq 0}$ such that $\forall k$, $\overline{\overline{F}}_{k\delta} = \overline{F}_k$:

$$\begin{aligned}\overline{\overline{F}}_t &= \overline{\overline{F}}_0 + \int_0^t v(\overline{\overline{F}}_{\eta(s)})ds + \int_0^t w(\overline{\overline{F}}_{\eta(s)})dX_s \\ &\quad + \int_0^t \sigma^2(ww')(\overline{\overline{F}}_{\eta(s)})(\overline{B}_s - \overline{B}_{\eta(s)})d\overline{B}_s + \sum_{0\leq k\leq \lfloor t/\delta\rfloor}(\widetilde{D}_k + A_k\Delta\overline{B}_k^3 + R_k)\end{aligned}$$

$$
\begin{aligned}
&= \overline{\overline{F}}_0 + \int_0^t v(\overline{\overline{F}}_{\eta(s)}) + w(\overline{\overline{F}}_{\eta(s)})u(F_s)ds + \int_0^t \sigma w(\overline{\overline{F}}_{\eta(s)})d\overline{B}_s \\
&\quad + \int_0^t \sigma^2 (ww')(\overline{\overline{F}}_{\eta(s)})(\overline{B}_s - \overline{B}_{\eta(s)})d\overline{B}_s + \sum_{0\le k\le \lfloor t/\delta \rfloor} (\widetilde{D}_k + A_k \Delta \overline{B}_k^3 + R_k)
\end{aligned} \tag{A.7}
$$

where we have used (2.3) in the second equality. For all t, we define

$$
\begin{aligned}
\overline{\overline{F}}'_t := F_0 &+ \int_0^t v(\overline{\overline{F}}'_{\eta(s)}) + w(\overline{\overline{F}}'_{\eta(s)})u(F_s)ds \\
&+ \int_0^t \sigma w(\overline{\overline{F}}'_{\eta(s)})d\overline{B}_s + \int_0^t \sigma^2(ww')(\overline{\overline{F}}'_{\eta(s)})(\overline{B}_s - \overline{B}_{\eta(s)})d\overline{B}_s.
\end{aligned}
$$

This is the Milstein scheme associated to (3.1). We have for all t_0:

$$
\begin{aligned}
&\sup_{0\le t\le t_0} (\overline{\overline{F}}_t - \overline{\overline{F}}'_t)^2 \\
&\quad \le Ct_0 \int_0^{t_0} (v(\overline{\overline{F}}_{\eta(s)}) - v(\overline{\overline{F}}'_{\eta(s)}) + u(F_s)(w(\overline{\overline{F}}_{\eta(s)}) - w(\overline{\overline{F}}'_{\eta(s)})))^2 ds \\
&\qquad + C \sup_{0\le t\le t_0} \left(\int_0^t (w(\overline{\overline{F}}_{\eta(s)}) - w(\overline{\overline{F}}'_{\eta(s)}))\sigma d\overline{B}_s \right)^2 \\
&\qquad + C \sup_{0\le t\le t_0} \left(\int_0^t ((ww')(\overline{\overline{F}}_{\eta(s)}) - (ww')(\overline{\overline{F}}'_{\eta(s)}))\sigma^2 (\overline{B}_s - \overline{B}_{\eta(s)})d\overline{B}_s \right)^2 \\
&\qquad + C \sup_{0\le t\le t_0} \left(\sum_{0\le k\le \lfloor t/\delta \rfloor} \widetilde{D}_k + A_k \Delta \overline{B}_k^3 + R_k \right)^2.
\end{aligned} \tag{A.8}
$$

We have:

$$
\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le t_0} \left(\sum_{0\le k\le \lfloor t/\delta \rfloor} \widetilde{D}_k \right)^2 \right) &\le C\mathbb{E}\left(\left(\sum_{0\le k\le \lfloor t_0/\delta \rfloor} \widetilde{D}_k \right)^2 \right) \\
&= C\mathbb{E}\left(\sum_{0\le k,q\le \lfloor t_0/\delta \rfloor} \widetilde{D}_k \widetilde{D}_q \right).
\end{aligned}
$$

Note that we have for $k > q$,

$$
\mathbb{E}(\widetilde{D}_k \widetilde{D}_q) = \mathbb{E}(\mathbb{E}(\widetilde{D}_k \widetilde{D}_q \mid \mathscr{F}^X_{k\delta})) = \mathbb{E}(\widetilde{D}_q \mathbb{E}(\widetilde{D}_k \mid \mathscr{F}^X_{k\delta})) = 0.
$$

So

$$
\mathbb{E}\left(\sup_{0\le t\le t_0} \left(\sum_{0\le k\le \lfloor t/\delta \rfloor} \widetilde{D}_k \right)^2 \right) \le C\mathbb{E}\left(\sum_{0\le k\le \lfloor t_0/\delta \rfloor} (\widetilde{D}_k)^2 \right) \le Ct_0 \delta^2.
$$

In the same way:

$$\mathbb{E}\left(\sup_{0\le t\le t_0}\left(\sum_{0\le k\le\lfloor t/\delta\rfloor} A_k\Delta\overline{B}_k^3\right)^2\right)\le t_0\delta^2.$$

We also have:

$$\mathbb{E}\left(\sup_{0\le t\le t_0}\left(\sum_{0\le k\le\lfloor t/\delta\rfloor} R_k\right)^2\right)\le \mathbb{E}\left(\sum_{0\le k,q\le\lfloor t_0/\delta\rfloor}|R_kR_q|\right)\le Ct_0^2\delta^2.$$

Using the Lipschitz properties of v, w, w' and the Burkholder–Davis–Gundy inequality, we have:

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_0}\left(\int_0^t((ww')(\overline{\overline{F}}_{\eta(s)})-(ww')(\overline{\overline{F}}'_{\eta(s)}))\sigma^2(\overline{B}_s-\overline{B}_{\eta(s)})d\overline{B}_s\right)^2\right)\\
&\le C\int_0^{t_0}\mathbb{E}\left(\left(((ww')(\overline{\overline{F}}_{\eta(s)})-(ww')(\overline{\overline{F}}'_{\eta(s)}))\sigma^2(\overline{B}_s-\overline{B}_{\eta(s)})\right)^2\right)ds\\
&\le C\delta\int_0^{t_0}\mathbb{E}\left(\sup_{0\le u\le s}(\overline{\overline{F}}_{\eta(u)}-\overline{\overline{F}}'_{\eta(u)})^2\right)ds.
\end{aligned}$$

Using again the Lipschitz properties of v, w, w' and the fact that $(u(F_t))_{t\ge 0}$ is bounded in (A.8), we have:

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_0}(\overline{\overline{F}}_t-\overline{\overline{F}}'_t)^2\right)\\
&\quad\le C(t_0+1)\int_0^{t_0}\mathbb{E}\left(\sup_{0\le v\le s}(\overline{\overline{F}}_v-\overline{\overline{F}}'_v)^2\right)ds+C(t_0\delta^2+t_0^2\delta^2).
\end{aligned}$$

We are now in position to apply Gronwall's Lemma and we obtain:

$$\mathbb{E}\left(\sup_{0\le t\le t_0}(\overline{\overline{F}}_t-\overline{\overline{F}}'_t)^2\right)\le C(t_0+t_0^2)\delta^2\exp(C(t_0+1)t_0).$$

The process $(\overline{\overline{F}}'_t)_{t\ge 0}$ is the Euler–Milstein scheme associated to $(F_t)_{t\ge 0}$, so we know by [14] that:

$$\mathbb{E}\left(\sup_{0\le t\le t_0}(\overline{\overline{F}}_t-F_t)^2\right)\le C_{t_0}\delta^2$$

where C_{t_0} is a constant depending only on t_0 and on the parameters of the problem. This implies that for all N:

$$\mathbb{E}\left(\sup_{0\le k\le N}(F_{k\delta}-\overline{F}_k)^2\right)\le C_{N\delta}\delta^2$$

where $C_{N\delta}$ is a constant depending on $N\delta$ and the parameters of the problem. This finishes the proof. □

APPENDIX B: PROOF OF LEMMA 4.2

Proof. From the previous equations we deduce that

$$F_t - \widehat{F}_t = \underbrace{\int_0^t \left\{ \frac{\mu_1 - \mu_2}{\sigma} F_s(1 - F_s) - \frac{\bar{\mu}_1 - \bar{\mu}_2}{\sigma} \widehat{F}_s(1 - \widehat{F}_s) \right\} d\overline{B}_s}_{:= M_t}$$

$$+ \int_0^t \underbrace{\{(-\lambda_1 F_s + \lambda_2(1 - F_s)) - (-\bar{\lambda}_1 \widehat{F}_s + \bar{\lambda}_2(1 - \widehat{F}_s))\}}_{:=\mathrm{err}_1(s)} ds$$

$$- \int_0^t \underbrace{\frac{\bar{\mu}_1 - \bar{\mu}_2}{\sigma^2} \{(\mu_1 F_s + \mu_2(1 - F_s)) - (\bar{\mu}_1 \widehat{F}_s + \bar{\mu}_2(1 - \widehat{F}_s))\} \widehat{F}_s(1 - \widehat{F}_s)(s)}_{:=\mathrm{err}_2} ds$$

In the following, we use C for a constant depending continuously on the parameters and which may change from line to line.

Control of err_1

$$\mathrm{err}_1(t) = -(\lambda_1 + \lambda_2)(F_t - \widehat{F}_t) + \widehat{F}_t(\bar{\lambda}_1 - \lambda_1 + \bar{\lambda}_2 - \lambda_2)$$

from which we deduce that

$$|\mathrm{err}_1(t)| \leq C|F_t - \widehat{F}_t| + |\lambda_2 - \bar{\lambda}_2| + |\lambda_1 - \bar{\lambda}_1|$$

because the filter F is bounded by 1 (since it is a conditional probability).

Control of err_2

The same type of calculations yields

$$|\mathrm{err}_2(t)| \leq C(|F_t - \widehat{F}_t| + |\mu_2 - \bar{\mu}_2| + |\mu_1 - \bar{\mu}_1|)$$

Control of the Martingale Term M

Since F and $\widehat{F}$ are almost surely bounded processes, we find that

$$\begin{aligned}
\mathbb{E}[|M_t|^2] &= \mathbb{E}\left[\int_0^t \left\{ \frac{\mu_1 - \mu_2}{\sigma} F_s(1 - F_s) - \frac{\bar{\mu}_1 - \bar{\mu}_2}{\sigma} \widehat{F}_s(1 - \widehat{F}_s) \right\}^2 ds\right] \\
&\leq C\mathbb{E}\left[\int_0^t \left\{ \frac{\mu_1 - \mu_2}{\sigma} - \frac{\bar{\mu}_1 - \bar{\mu}_2}{\sigma} \right\}^2 ds\right] \\
&\quad + C\mathbb{E}\left[\int_0^t \left\{ \frac{\mu_1 - \mu_2}{\sigma}(F_s - \widehat{F}_s) \right\}^2 ds\right] \\
&\leq C((\mu_1 - \bar{\mu}_1)^2 + (\mu_2 - \bar{\mu}_2)^2) + C\int_0^t \mathbb{E}(F_s - \widehat{F}_s)^2 ds
\end{aligned}$$

Conclusion

From all the previous, and by using the trivial inequality $(i = 1, 2)$ $(\int_0^t \mathrm{err_i}(s)ds)^2 \le t \int_0^t \mathrm{err}_i^2(s)ds$, we find that for all $t \le t_0$:

$$\mathbb{E}\left[\sup_{0\le s\le t}(F_s - \widehat{F}_s)^2\right] \le C(t_0+1)\left(\int_0^t \mathbb{E}\left(\sup_{0\le u\le s}(F_u - \widehat{F}_u)^2\right)ds + \sup_{i=1,2}(|\lambda_i - \bar{\lambda}_i|^2 + |\mu_i - \bar{\mu}_i|^2)\right)$$

We are now in position to apply Gronwall's Lemma, for all t in $[0, t_0]$,

$$\mathbb{E}\left[\sup_{0\le t\le t_0}(F_t - \widehat{F}_t)^2\right] \le C(t_0+1)\exp(C(t_0+1)t_0)\sup_{i=1,2}(|\lambda_i - \bar{\lambda}_i|^2 + |\mu_i - \bar{\mu}_i|^2). \qquad \square$$

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