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OPTIMAL STOPPING PROBLEMS FOR SOME MARKOV PROCESSES

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In this paper, we solve explicitly the optimal stopping problem with random discounting and an additive functional as cost of observations for a regular linear diffusion. We also extend the results to the class of one-sided regular Feller processes. This generalizes the result of Beibel and Lerche [*Statist. Sinica* **7** (1997) 93–108] and [*Teor. Veroyatn. Primen.* **45** (2000) 657–669] and Irles and Paulsen [*Sequential Anal.* **23** (2004) 297–316]. Our approach relies on a combination of techniques borrowed from potential theory and stochastic calculus. We illustrate our results by detailing some new examples ranging from linear diffusions to Markov processes of the spectrally negative type.

1. Introduction. Consider a one-dimensional regular diffusion $X = (X_t)_{t\geq 0}$ with state space $E = (l, r)$, an interval of $\mathbb{R}$, defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We denote by $(\mathbb{P}_x)_{x\in E}$ the family of probability measures associated to the process X such that $\mathbb{P}_x(X_0 = x) = 1$, and by $\mathbb{E}_x$ the associated expectation operator. Next, let Σ_∞^X be the family of all stopping times with respect to the filtration $\mathbb{F}(= (\mathcal{F}_t)_{t\geq 0})$. In this paper, we are first concerned with the study of the following optimal stopping problem, for any $x \in E$,

$$\sup_{T\in\Sigma_\infty^X} \mathbb{E}_x[e^{-A_T} g(X_T) - C_T], \tag{1}$$

where g is a nonnegative continuous function on E, $A = (A_t)_{t\geq 0}$ is a continuous additive functional of the form

$$A_t = \int_0^t a(X_s)\,ds \tag{2}$$

with a a continuous function on E such that $a(x) > 0$ for all $x \in E$ and, for any $t \geq 0$,

$$C_t = \int_0^t c(X_s)e^{-A_s}\,ds \tag{3}$$

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with c a nonnegative continuous function on E. We also aim to solve this problem in the case where X is a Feller process of the spectrally negative type, that is, when it has only negative jumps. It is of common use to name g as the reward function, $C = (C_t)_{t\geq 0}$ as the cost of observations and A as the random discount factor. We mention that in the case $A \equiv 0$ and $c \equiv 0$ the problem (1) has been studied by Dynkin [11] and Shiryaev [34] in the general framework of Markov processes. Moreover, the case $c \equiv 0$, that is, without cost of observations, has been intensively studied in the literature for one-dimensional diffusions. In particular, Salminen [32] by means of the Martin boundary, suggested a solution to this problem in terms of the excessive majorant function. More recently, Beibel and Lerche [4, 5], relying on martingales arguments, solve this optimal stopping problem explicitly. We also mention that, by using standard fluctuation theory, Kyprianou and Pistorius [19] offer solution to some optimal stopping problem arising in financial mathematics for appropriate diffusions. A related result on optimal stopping problems for one-dimensional diffusions with discounting has been presented by Dayanik and Karatzas [10]. They characterize excessive functions via generalized concavity and determined the value function as the smallest concave majorant of the reward function. In the former case, the value function is given as the solution of a free boundary value problem associated to a second order differential operator which is the infinitesimal generator of the one-dimensional diffusion X. The term *free*, which comes from the a priori, unknown region where the problem is investigated, forces one to set up an artificial boundary condition of Neumann type to get a well-posed problem. This is the so-called smooth fit principle. All these techniques are well explained in the book of Peskir and Shiryaev [27]. The literature regarding optimal stopping problems associated to diffusion with jumps is more sparse and focused essentially on the study of specific examples. In this vein, we mention the paper of Alili and Kyprianou [1] where the authors deal with the issue of pricing perpetual American put options in a market driven by Lévy processes. We also indicate that Baurdoux [3] solved an optimal stopping problem associated to generalized Ornstein–Uhlenbeck processes of the spectrally negative type.

In this paper, we propose to solve the optimal stopping problem (1) with a cost of observations of the form (3) for one-dimensional regular diffusions. Our strategy can be described as follows. First, by a time change device, we reduce the optimal stopping problem with random discounting to a one with a deterministic discount factor but associated to an appropriate time change diffusion. Then, by an argument of potential theory, we transform the problem (1) to an optimal stopping problem of the same form with a new reward function but without cost of observations. We proceed by using a result of Shiryaev [34] which states that in our context the optimal stopping time is the first exit time of the process from a Borel set. Finally, with this information at hand, we can use a Doob's h-transform technique, with a proper choice of the excessive function, to transform our problem to an optimization problem which has been studied in detail by Beibel and Lerche [5].

The remaining part of the paper is organized as follows. In the next section, we overview the basic facts about one-dimensional diffusions. In Section 3, we state and prove our main result which consists of solving the general optimal stopping problem (1). We also show, in Section 4, how to generalize our result to the class of regular one-sided Feller processes. The last section is devoted to the treatment of new examples ranging from linear diffusions to processes with one sided jumps, such as spectrally negative Lévy processes and self-similar positive Markov processes of the spectrally negative type.

2. Preliminaries. In this part, we provide some well-known facts about linear diffusions which can be found, for instance, in Itô and McKean [15] and in Borodin and Salminen [8]. We recall that $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space. We consider a linear diffusion $X = (X_t)_{t\geq 0}$ with state space E, as the solution to the stochastic differential equation (SDE)

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \tag{4}$$

where $W = (W_t)_{t\geq 0}$ is a one-dimensional Brownian motion. It is supposed that σ and b are continuous and $\sigma(x) > 0$ for all $x \in E$. We assume that X is regular, that is,

$$\mathbb{P}_x(T_y < +\infty) > 0 \qquad \forall x, y \in E,$$

where $T_y = \inf\{t > 0; X_t = y\}$. The transition semigroup $(P_t)_{t\geq 0}$ maps $C_b(E)$, the space of bounded and continuous functions on E, into itself. It follows that X is a Feller process. Moreover, for every $t \geq 0$ and $x \in E$ the corresponding measure $A \longmapsto P_t(x, A)$, with A a Borel set, is absolutely continuous with respect to the speed measure m, a positive σ-finite measure on E. More specifically, we have

$$P_t(x, A) = \int_A p_t(x, y) m(dy),$$

where $p_t(\cdot, \cdot)$ stands for the transition probability density which may be taken to be positive, jointly continuous in all variables and symmetric. The scale function s of X is an increasing continuous function from E to $\mathbb{R}$, satisfying, for any $a \leq x \leq b$,

$$\mathbb{P}_x(T_a < T_b) = \frac{s(b) - s(x)}{s(b) - s(a)}$$

and is given by

$$s'(x) = \exp\left\{-2\int^x \frac{b(z)}{\sigma^2(z)}\,dz\right\}.$$

We also recall that the infinitesimal generator $\mathbf{L}$ of X is the second order differential operator given, for a function $f \in C_c^\infty(E)$, the space of infinitely continuously differentiable functions with compact support, by

$$\mathbf{L}f(x) = \tfrac{1}{2}\sigma^2(x) f''(x) + b(x) f'(x).$$

Next, from the general theory of one-dimensional diffusion (see [15], page 128), the Laplace transform of the first hitting time T_y is expressed, for any $q > 0$, as

$$\mathbb{E}_x[e^{-qT_y}] = \begin{cases} \dfrac{h_q^+(x)}{h_q^+(y)}, & x \le y, \\ \dfrac{h_q^-(x)}{h_q^-(y)}, & x \ge y, \end{cases} \tag{5}$$

where h_q^+ (resp., h_q^-) is the increasing (resp., decreasing) continuous solution to the differential equation

$$\mathbf{L}h(x) = qh(x), \qquad x \in E, \tag{6}$$

with appropriate conditions at the nonsingular boundary points. These functions, h_q^+ and h_q^-, are called the fundamental solutions of the equation (6). They are linearly independent and their Wronskian is defined by

$$W_{h_q^+,h_q^-}(x) = h_q^-(x)\frac{d}{dx}h_q^+(x) - h_q^+(x)\frac{d}{dx}h_q^-(x),$$

and the Wronskian with respect to $d/ds(x)$ denoted by w_q is the constant given by

$$w_q = \frac{W_{h_q^+,h_q^-}(x)}{s'(x)}. \tag{7}$$

Moreover, for any $q > 0$, the Green function or the q-potential density u^q is defined as the Laplace transform of the transition probability density, that is,

$$u^q(x, y) = \int_0^{+\infty} e^{-qt} p_t(x, y)\,dt \qquad \forall x, y \in E.$$

In particular, we have

$$u^q(x, y) = \begin{cases} w_q^{-1}h_q^+(x)h_q^-(y), & x \le y, \\ w_q^{-1}h_q^+(y)h_q^-(x), & x \ge y. \end{cases} \tag{8}$$

We say that the process X is recurrent if and only if $\lim_{q\to 0} u^q(x, y) = \infty$, for all $x, y \in E$, which is equivalent to $\mathbb{P}_x(T_y < \infty) = 1$, for all $x, y \in E$. A diffusion which is not recurrent is called transient. In this case, the potential u defined as

$$u(x, y) = \lim_{q\to 0} u^q(x, y)$$

is finite for all $x, y \in E$. Finally, we mention that if X is transient with $\lim_{t\to\infty} X_t = r$ then for any $x \le y$

$$\begin{aligned} u(x, y) &= \int_0^{+\infty} p_t(x, y)\,dt, \\ &= s(r) - s(y). \end{aligned}$$

3. Optimal stopping problem for linear diffusions. Our aim is now to find the value of the function $\mathcal{V}^A_{g,c}$ defined as the solution of the optimal stopping problem (1), that is,

$$\mathcal{V}^A_{g,c}(x) = \sup_{T \in \Sigma^X_\infty} \mathbb{E}_x[e^{-A_T} g(X_T) - C_T],$$

where $A = (A_t)_{t\geq 0}$ is a continuous additive functional of the form (2) and the cost of observations $C_t = \int_0^t c(X_s)e^{-A_s}\, ds$ with c and g nonnegative continuous functions. Our main result is stated in Theorem 3.9 below. It consists of reducing the optimal stopping problem (1) into a new one which can be described as follows. On the one hand, it has a modified reward function but without both cost of observations and discounting factor. On the other hand, it is associated to a new diffusion obtained from the original one by a random time change and by a Doob's h-transform. It turns out that solving this latter optimal stopping problem amounts to finding the solution of an optimization problem which has been studied by Beibel and Lerche [5]. More precisely, our approach can then be split into the following three steps.

(1) First, we time change X by the inverse of the continuous increasing functional A and we use the well-known fact that in our context the two processes have identical hitting time distributions. Hence, we may consider without loss of generality the problem (1) with linear discounting, that is, $A_t = qt$ for some constant $q > 0$.

(2) Then, we characterize the potential associated to the functional C and we show how to reduce our problem to an optimal stopping problem without cost of observations but involving a new reward function.

(3) Finally, we borrow an idea of Williams [35] and Pitman and Yor [28] for constructing conditioned diffusions by the method of h-transform. We transform the problem described in item (2) to an optimal stopping problem without discounting factor which has been solved by Shiryaev [34].

3.1. *Time change for nonnegative additive functional.* We start our program by considering that A is a nonnegative continuous additive functional of X of the form (2) and we assume, without loss of generality, that $A_\infty = \infty$ $\mathbb{P}_x$ a.s. Since A is a continuous increasing function, it admits an inverse functional which we denote by V and given, for all $t \geq 0$, by

$$\begin{aligned} V_t &= \inf\{s \geq 0; A_s > t\}, \\ &= \int_0^t \frac{1}{a(Y_s)}\, ds, \end{aligned}$$

where $Y_t = X_{V_t}$ for any $t \geq 0$. Moreover, if X is the solution to the SDE (4), then it follows, from the Itô's formula, that Y is the unique solution to the SDE

$$dY_t = \left(\frac{b}{a}\right)(Y_t)\, dt + \left(\frac{\sigma}{\sqrt{a}}\right)(Y_t)\, d\hat{W}_t,$$

where $\hat{W}$ is a Brownian motion with respect to the filtration $\mathbb{F}^Y = (\mathcal{F}_{V_t})_{t\geq 0}$. The process Y remains a linear diffusion with respect to the filtration $\mathbb{F}^Y$. In particular, it is a Feller process (see, e.g., Lamperti [20]) and its infinitesimal generator $\mathbf{L}^Y$ takes the form

$$\mathbf{L}^Y f(x) = \frac{1}{a(x)}\mathbf{L} f(x) \tag{9}$$

for a smooth function f on E. Next, we consider an open interval $B \subset E$ and denote by T_B^Y the first exit time of the process Y from B, it is plain that we have the following identity:

$$T_B^Y = A_{T_B^X} \qquad \text{a.s.}$$

We are now ready to state the following.

LEMMA 3.1. *For any $x \in E$, we have, with the obvious notation,*

$$\begin{aligned} &\sup_{T\in\Sigma_\infty^X} \mathbb{E}_x[e^{-A_T} g(X_T) - C_T] \\ &\qquad = \sup_{T\in\Sigma_\infty^Y} \mathbb{E}_x\left[e^{-T} g(Y_T) - \int_0^T \left(\frac{c}{a}\right)(Y_s)e^{-s}\,ds\right], \end{aligned} \tag{10}$$

where Y is characterized by its infinitesimal generator (9).

PROOF. From [31], Section III.21, page 277, we have that for every $\mathbb{F}^X$-stopping time T, A_T is an $\mathbb{F}^Y$-stopping time. Thus, we obtain that

$$\sup_{T\in\Sigma_\infty^X} \mathbb{E}_x[e^{-A_T} g(X_T) - C_T] = \sup_{S\in\Sigma_\infty^Y} \mathbb{E}_x\left[e^{-S} g(X_{V_S}) - \int_0^{V_S} c(X_v)e^{-A_v}\,dv\right].$$

The proof follows by performing the change of variable $u = A_v$ in the integral on the right-hand side of the previous identity. □

Consequently, in the sequel we can assume, without loss of generality, that the additive functional A is linear, that is, $A_t = qt$ for some $q > 0$ and the cost of observations is $C_t = \int_0^t c(X_s)e^{-qs}\,ds$.

3.2. *Get rid of the cost of observations.* Let us now introduce the function δ defined, for any $x \in E$, by

$$\delta(x) = \mathbb{E}_x[C_\infty]. \tag{11}$$

In the following, we provide an expression of δ in terms of the characteristics of X and give some conditions under which it is continuous and finite.

LEMMA 3.2. *For any $q > 0$ and $x \in E$, we have*

$$\delta(x) = w_q^{-1}\left(h_q^-(x)\int_l^x h_q^+(y)c(y)m(dy) + h_q^+(x)\int_x^r h_q^-(y)c(y)m(dy)\right),$$

where w_q stands for the Wronskian of h_q^- and h_q^+ with respect to the scale function s, as defined in (7) and m is the speed measure of X. Moreover, if c satisfies the integrability condition for any $x \in E$,

$$\int_l^r h_q^-(x \vee y)h_q^+(x \wedge y)c(y)m(dy) < \infty, \tag{12}$$

then δ is continuous and finite on E.

PROOF. By using Fubini's theorem, we obtain that

$$\begin{aligned}\delta(x) &= \int_0^\infty e^{-qs}\mathbb{E}_x[c(X_s)]\,ds = \int_E u^q(x,y)c(y)m(dy)\\ &= w_q^{-1}\int_E h_q^-(x \vee y)h_q^+(x \wedge y)c(y)m(dy),\end{aligned}$$

where we have used the identity (8). The proof of the claims follows readily. □

REMARK 3.3. We note that if $q = 0$ and X is transient with $\lim_{t\to\infty} X_t = r$, then δ is given by

$$\delta(x) = \int_E \bigl(s(r) - s(y)\bigr)c(y)m(dy). \tag{13}$$

In this case, we mention that Khoshnevisan, Salminen and Yor [16] identify the law of the perpetual integral functional C_∞ of a transient diffusion as the law of the first hitting time of a random time change diffusion.

We are now ready to state the following.

LEMMA 3.4. *If δ is finite on E then, for any $x \in E$, we have*

$$\sup_{T\in\Sigma_\infty^X} \mathbb{E}_x[e^{-qT}g(X_T) - C_T] = \sup_{T\in\Sigma_\infty^X} \mathbb{E}_x\bigl[e^{-qT}\bigl(g(X_T) + \delta(X_T)\bigr)\bigr] - \delta(x).$$

PROOF. Note that for any $\mathbb{F}$-stopping time T, we have the identity in law

$$C_\infty \overset{(d)}{=} C_T + e^{-qT}C_\infty \circ \theta_T,$$

where $(\theta_t)_{t\geq 0}$ stands for the shift operator, that is, for any $t, s \geq 0$, $\theta_t w(s) = w(t + s)$. Since δ is finite, the strong Markov property yields

$$\sup_{T\in\Sigma_\infty^X} \mathbb{E}_x[e^{-qT}g(X_T) - C_T] = \sup_{T\in\Sigma_\infty^X} \mathbb{E}_x\bigl[e^{-qT}\bigl(g(X_T) + \delta(X_T)\bigr)\bigr] - \delta(x).$$

□

A nice consequence of the previous result is that the general optimal stopping problem (1) is equivalent to an optimal stopping problem without cost of observations. Before stating our next result, let us introduce a few further notation. Let $\mathcal{O}$ denote all the open subsets of E containing the starting point x of X. Let $\Sigma_{\mathcal{O}}$ be the class of stopping times of the form $T_B = \inf\{t > 0; X_t \notin B\}$ where $B \in \mathcal{O}$.

LEMMA 3.5. *Let f be a continuous nonnegative function. Then, for any $q > 0$ and $x \in E$,*

$$\sup_{T \in \Sigma_\infty^X} \mathbb{E}_x[e^{-qT} f(X_T)] = \sup_{T_B \in \Sigma_{\mathcal{O}}} \mathbb{E}_x[e^{-qT_B} f(X_{T_B})]. \tag{14}$$

PROOF. Let $\hat{X}$ be defined by

$$\hat{X}_t = \begin{cases} X_t, & \text{if } t < \mathbf{e}_q, \\ \partial, & \text{if } t \geq \mathbf{e}_q, \end{cases}$$

where $\mathbf{e}_q$ is an exponential variable of parameter $q > 0$ taken independent of $\mathbb{F}$ and ∂ is a cemetery state. Note that $\hat{X}$ is always transient and clearly with a function f as above and using the convention $f(\partial) = 0$, we have, for any $x \in E$

$$\mathbb{E}_x[f(\hat{X}_t)] = \mathbb{E}_x[e^{-qt} f(X_t)].$$

Therefore, there is a one-to-one correspondence between the excessive functions for $\hat{X}$ and the q-excessive ones for X (see Definition 3.6 below). Moreover, the q-excessive functions of the Feller process X are lower semi-continuous ([13], Theorem 2.1). Then, from Shiryaev [34], Corollary 3, page 129, we obtain that

$$\sup_{T \in \Sigma_\infty^{\hat{X}}} \mathbb{E}_x[f(\hat{X}_T)] = \sup_{T_B \in \Sigma_{\mathcal{O}}} \mathbb{E}_x[f(\hat{X}_{T_B})].$$

Hence,

$$\sup_{T \in \Sigma_\infty^X} \mathbb{E}_x[e^{-qT} f(X_T)] = \sup_{T_B \in \Sigma_{\mathcal{O}}} \mathbb{E}_x[e^{-qT_B} f(X_{T_B})]. \qquad \square$$

3.3. *Doob's h-transform.* Our aim in this part is to show how to transform an optimal stopping problem with discounting factor to an optimal stopping problem without discounting. To this end, we recall some basic facts on excessive functions and Doob's h-transform and we refer to the book of Borodin and Salminen [8], Section II.5, pages 32–35.

DEFINITION 3.6. A nonnegative measurable function $h : E \mapsto \mathbb{R} \cup \{\infty\}$ is called q-excessive, $q \geq 0$, for the process X if the following two statements hold true: for any $x \in E$,

(i) $e^{-qt}\mathbb{E}_x[h(X_t)] \leq h(x)$, $t > 0$,
(ii) $\lim_{t \searrow 0} e^{-qt}\mathbb{E}_x[h(X_t)] = h(x)$.

A q-excessive function is called q-invariant if for any $x \in E$ and $t \geq 0$ we have

$$e^{-qt}\mathbb{E}_x[h(X_t)] = h(x).$$

A function h is q-excessive (resp., q-invariant) if and only if the process $e^{-qt}h(X_t)$ is a positive supermartingale (resp., martingale). For every $y \in (l, r)$ the functions $x \mapsto u^q(x, y)$, $x \mapsto h_q^+(x)$ and $x \mapsto h_q^-(x)$ are q-excessive. These functions are *minimal* in the sense that any other arbitrary nontrivial q-excessive function h can be expressed as a linear combination of them.

DEFINITION 3.7. Let $q \geq 0$ and h be a q-excessive function. For any $x \in E$ such that $0 < h(x) < +\infty$, and $t \geq 0$, we define the new probability measure $\mathbb{P}_x^h$ as

$$d\mathbb{P}_x^h = e^{-qt}\frac{h(X_t)}{h(x)}\,d\mathbb{P}_x \qquad \text{on } \mathcal{F}_t.$$

The process X under the probability measure $\mathbb{P}_x^h$ is called the Doob's h-transform (or q-excessive transform) of X. It is also a regular diffusion process and thus a Feller process.

Next, we use an idea of Williams [35] and Pitman and Yor [28] for constructing conditioned diffusions by the method of h-transform by means of the Laplace transform of first passage times. In fact, we slightly generalize their methodology by considering as q-excessive function the Laplace transform of the first exit time of an open set by X. To this end, let $B \in \mathcal{O}$ and we recall that we denote by T_B the first exit time from B by X, that is,

$$T_B = \inf\{t > 0; X_t \notin B\}.$$

For any $x \in E$, we write the Laplace transform of the stopping time T_B as

$$\phi^B(x) = \mathbb{E}_x[e^{-qT_B}].$$

Thus, without loss of generality, the continuity of X allows us to restrict $\mathcal{O}$ to open intervals (a, b) for some $l \leq a < b \leq r$. It is well known that the function ϕ^B is solution to the following Sturm–Liouville boundary value problem

$$\begin{cases} \mathbf{L}u(x) = qu(x), & x \in (a, b), \\ u(a) = u(b) = 1. \end{cases}$$

In other words, ϕ^B can be written as a linear combination of the fundamental solutions h_q^- and h_q^+. Setting

$$h^B(y) = \frac{\phi^B(y)}{\phi^B(x)},$$

it is plain that the mapping h^B is a q-excessive function for X. Thus, as in the Definition 3.7, we define the probability measure $\mathbb{P}_x^{h^B}$ as

$$d\mathbb{P}_x^{h^B} = e^{-qT_B} h^B(X_{T_B})\, d\mathbb{P}_x \qquad \text{on } \mathcal{F}_{T_B^+}. \tag{15}$$

The diffusion X under the family of probability measures $\mathbb{P}^{h^B} = (\mathbb{P}_x^{h^B})_{x\in E}$ is transient with

$$\mathbb{P}_x^{h^B}(X_{\xi^-} \in \partial B) = 1, \tag{16}$$

except if $q = 0$, where ξ stands for the lifetime of X under $\mathbb{P}^{h^B}$. Clearly, the probability $\mathbb{P}_x^{h^B}(\xi < \infty)$ is either 1 or 0 for all x. Moreover, since X is solution to the SDE (4), then under the probability $\mathbb{P}^{h^B}$, the diffusion X can be characterized as the solution of the SDE

$$dX_t = \big(b(X_t) + \log'(h^B(X_t))\sigma^2(X_t)\big)\, dt + \sigma(X_t)\, d\tilde{W}_t,$$

where $\tilde{W}$ is a standard Brownian motion under the new probability $\mathbb{P}_x^{h^B}$.

REMARK 3.8. As explained in [28], we take $\mathbb{P}_x^{h^B}$ to be defined by the requirement that for each $x \in B$ the process X runs up to the time T_B has the same law under $\mathbb{P}_x^{h^B}$ as it does under $\mathbb{P}_x$ conditional on $T_B < \mathbf{e}_q$, where $\mathbf{e}_q$ is an independent exponentially distributed random variable with parameter $q > 0$.

We are now ready to state and prove the main theorem of this section.

THEOREM 3.9. *If δ is finite then solving the problem* (1) *amounts to solving the following optimal stopping problem*:

$$\sup_{T_B \in \Sigma_{\mathcal{O}}} \mathbb{E}_x^{h^B}\left[\frac{g(X_{T_B}) + \delta(X_{T_B})}{h^B(X_{T_B})}\right], \tag{17}$$

where the probability $\mathbb{P}_x^{h^B}$ is defined in (15). *If there exists an open interval B^* of E such that*

$$\frac{g(u^*) + \delta(u^*)}{h^{B^*}(u^*)} = \sup_{B\in\mathcal{O}, u\in\partial B} \frac{g(u) + \delta(u)}{h^B(u)} \qquad \textit{where } u^* \in \partial B^*, |u^*| < \infty, \tag{18}$$

then, the value function of (17) *is given by* (18) *with the optimal stopping time T_{B^*}.*

REMARK 3.10. (1) We mention that the optimization problem (18) has been studied in detail by Beibel and Lerche [5]. We refer to their paper for more precise information concerning its solution for all possible choices of the reward function g. We also point out that, in the specific case $a = 0$ and X is a standard Brownian motion, a similar optimization problem was studied by Graversen and Peskir [12].

(2) If B^* is a bounded interval, that is, $B^* = (u_1^*, u_2^*)$, (18) reads

$$\frac{g(u_1^*) + \delta(u_1^*)}{h^{B^*}(u_1^*)} = \frac{g(u_2^*) + \delta(u_2^*)}{h^{B^*}(u_2^*)} = \sup_{B \in \mathcal{O}} \sup_{u \in \partial B} \frac{g(u) + \delta(u)}{h^B(u)}.$$

PROOF OF THEOREM 3.9. First, we deal with the additive functional C. Since δ is finite, we have, from Lemma 3.4, that solving the problem (1) is equivalent to solving the following optimal stopping problem without cost of observations:

$$\sup_{T \in \Sigma_\infty^X} \mathbb{E}_x\big[e^{-qT}\big(g(X_T) + \delta(X_T)\big)\big].$$

Then, from Lemma 3.5, we deduce that

$$\sup_{T \in \Sigma_\infty^X} \mathbb{E}_x\big[e^{-qT}\big(g(X_T) + \delta(X_T)\big)\big] = \sup_{T_B \in \Sigma_{\mathcal{O}}} \mathbb{E}_x\big[e^{-qT_B}\big(g(X_{T_B}) + \delta(X_{T_B})\big)\big].$$

Next, we use the Doob's h-transform device. Let $\mathbb{P}_x^{h^B}$ be the probability measure defined in (15), then we have

$$\sup_{T_B \in \Sigma_{\mathcal{O}}} \mathbb{E}_x\big[e^{-qT_B}\big(g(X_{T_B}) + \delta(X_{T_B})\big)\big] = \sup_{T_B \in \Sigma_{\mathcal{O}}} \mathbb{E}_x^{h^B}\left[\frac{g(X_{T_B}) + \delta(X_{T_B})}{h^B(X_{T_B})}\right],$$

which completes the proof of the first assertion. Finally, since X under $\mathbb{P}_x^{h^B}$ is transient, it is stated above that $T_B < \infty$ a.s., the value function of the last optimal stopping problem is the solution to the following optimization problem:

$$\sup_{B \in \mathcal{O}} \sup_{u \in \partial B} \frac{g(u) + \delta(u)}{h^B(u)}. \qquad \square$$

4. Extension to one-sided regular Feller processes. Let us now consider X to be the càdlàg modification of a one-sided regular Feller process defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and taking values in an interval $E \subset \mathbb{R}$. It means that X is a regular Feller process having jumps only in one direction which we assume, without loss of generality, to be of the spectrally negative type. That is, X does not have positive jumps; $\mathbb{P}_x(\sup_{t \geq 0}(X_t - X_{t-}) > 0) = 0, \forall x \in E$. For sake of simplicity, we also assume that the process X has infinite lifetime. We wish to extend the results of the previous section to this class of stochastic processes. In comparison to the diffusion case, the difficulty is that we do not have, in general, any information on the excessive functions for this class of Markov processes. Indeed, the infinitesimal generator associated to X is an integro-differential linear operator for which there does not exist general results regarding the solutions to the boundary value problem (6). Nevertheless, as explained in the following, the one-sided feature of X allows us to identify increasing excessive functions.

PROPOSITION 4.1. *For any $q > 0$, there exists a unique increasing left-continuous function $h_q^+ : E \mapsto [0, \infty]$, such that, for any $x, y \in E$ with $x \le y$,*

$$\mathbb{E}_x[e^{-qT_y}] = \frac{h_q^+(x)}{h_q^+(y)}.$$

In the case X is recurrent, the function $x \mapsto h_q^+(x)$ is continuous.

PROOF. As a consequence of the regularity assumption, it is well known (see, e.g., [7]) that for each singleton $\{y\} \in E$, X admits a local time at y, which we denote by $L^y = (L_t^y)_{t \ge 0}$. The continuous additive functional L^y is determined by its q-potential, u_q^y, which is finite for any $q > 0$ and given by

$$u_q^y(x) = \mathbb{E}_x\left[\int_0^\infty e^{-qt}\, dL_t^y\right].$$

From the definition of L^y and the strong Markov property, we obtain the identity (see [7], Chapter V.3)

$$\mathbb{E}_x[e^{-qT_y}] = \frac{u_q^y(x)}{u_q^y(y)}, \qquad x, y \in E.$$

Next, following Itô and McKean [15] or Pitman and Yor [28], for instance, we write $\phi_q(x, y) = \mathbb{E}_x[e^{-qT_y}]$ and for a fixed $z_0 \in E$, we define

$$h_q^+(y) = \begin{cases} \phi_q(y, z_0), & y \le z_0, \\ 1/\phi_q(z_0, y), & y > z_0. \end{cases}$$

Next, using the fact that X has no positive jumps, we get for any $x \le z \le y$, and by means of the strong Markov property,

$$\phi_q(x, y) = \phi_q(x, z)\phi_q(z, y).$$

Thus, from the identity

$$\phi_q(x, z) = h_q^+(x)/h_q^+(z)$$

we deduce that the choice of the reference point z_0 affects h_q^+ only by a constant factor. The monotonicity of the mapping h_q^+ follows readily from its definition and the absence of positive jumps for X. We recall that $x \mapsto \mathbb{E}_x[e^{-qT_y}]$ is a q-excessive function (see, e.g., [7], page 74). By linearity, the mapping $x \mapsto h_q^+(x)$ is also q-excessive and thus finely continuous. So, the Feller property of X implies that the increasing excessive function h_q^+ is lower semi-continuous (see [13]) and hence, left-continuous. Then, the claim of the last assertion is a straightforward consequence of the fact that if X is also recurrent then the fine topology coincides with the initial topology of E (see, e.g., [2], page 243). □

We point out that, in Patie and Vigon [26], Proposition 4.1 is extended to a larger class of homogenous Markov processes with only negative jumps. Following the proof of Theorem 3.9, and observing that $h^{(-\infty,u)}(u) = h_q^+(u)/h_q^+(x)$, the proof of the theorem below goes through verbatim.

THEOREM 4.2. *With the notation used in Theorem 3.9 we assume that*

$$D^* := \sup_{u\in E} \frac{g(u)+\delta(u)}{h_q^+(u)} = \sup_{u\geq x, u\in E} \frac{g(u)+\delta(u)}{h_q^+(u)} < \infty.$$

Then, for any $x \in E$

$$\sup_{T\in\Sigma_\infty^X} \mathbb{E}_x[e^{-qT} g(X_T) - C_T] = D^* h_q^+(x) - \delta(x).$$

If there exists a point $u^* \geq x$ such that $D^* = \frac{g(u^*)+\delta(u^*)}{h_q^+(u^*)}$, then the optimal stopping time of the problem (1) *is given by*

$$T_{u^*} = \inf\{t > 0; X_t \geq u^*\}.$$

REMARK 4.3. Since Lemma 3.1 applies also in this more general framework, one could have also considered a random discounting factor in the previous result.

5. Examples.

We now illustrate our methodology by presenting the solutions to some new optimal stopping problems. We consider both the diffusion case and also the case when the processes are of the spectrally negative type.

5.1. *An optimal stopping problem with cost of observations for one-sided Lévy processes.* Let $Z = (Z_t)_{t\geq 0}$ be a spectrally negative Lévy process starting from $x \in \mathbb{R}$, that is, a process with stationary and independent increments having only negative jumps. Plainly, the law of Z is characterized by ψ, the Laplace exponent of Z_1, which admits the following Lévy–Khintchine representation, for any $u \geq 0$,

$$\psi(u) = \frac{1}{2}\sigma^2 u^2 + bu + \int_{-\infty}^0 (e^{ux} - 1 - ux\mathbb{1}_{\{-1<x\leq 0\}})\nu(dx), \tag{19}$$

where $b \in \mathbb{R}$ and $\sigma \geq 0$ and the measure ν is such that $\int_{-\infty}^0 (1 \wedge y^2)\nu(dy) < \infty$. Next, recalling that ψ is a convex function on $[0,\infty)$ with $\lim_{u\to\infty} \psi(u) = +\infty$, we denote by θ the nonnegative largest root of the equation $\psi(u) = 0$. We also mention that being continuous and increasing on $[\theta,\infty)$, ψ has a well-defined inverse function $\phi : [0,\infty) \to [\theta,\infty)$ which is also continuous and increasing. We refer to the excellent monographs of Bertoin [6] and Kyprianou [18] for background on Lévy processes. We now consider a perpetual American option in a market driven by Z where the agent takes into account some costs from hedging, which might come from some transaction costs or liquidity issues. We assume that the agent hedges continuously and that he chooses the cost of observations

$c(x) = \exp(\gamma x)$ to be of exponential form. The payoff at time $t > 0$ of such a product can be written as

$$e^{-qt} g(e^{Z_t}) - \int_0^t e^{\gamma Z_s} e^{-qs}\, ds, \tag{20}$$

where $q > 0$ is the risk-free rate, $\gamma > -1$ and g is a smooth function. Next, assuming that $q = \psi(1)$ it is easy to check that one may choose the probability $\mathbb{P}_x$ as the risk neutral probability measure. Therefore, in the sequel, we suppose that the characteristics b, σ and ν are chosen such that

$$q = \psi(1). \tag{21}$$

5.1.1. *The Brownian motion with drift case.* We start with the case where $Z_t = W_t^{(b)} = bt + W_t$ is a Brownian motion with drift b starting from 0 and the reward function g is defined, for any $0 \le L \le K$, by

$$g(x) = \begin{cases} L - x, & \text{if } x \le L, \\ 0, & \text{if } L \le x \le K, \\ x - K, & \text{if } x \ge K. \end{cases} \tag{22}$$

This function corresponds to the payoff function of a strangle option which is a combination of a put with exercise price L and a call with exercise price K. In this case, the condition (21) is fulfilled if $q = \frac{1}{2} + b$. We want to compute the constant

$$V_W = \sup_{T \in \Sigma_\infty} \mathbb{E}_0\left[e^{-qT} g\big(e^{W_T^{(b)}}\big) - \int_0^T e^{\gamma W_s^{(b)}} e^{-qs}\, ds\right].$$

The case without cost of observations, that is, $c \equiv 0$, has already been studied by Beibel and Lerche [4]. Next, it is well known that the functions h_q^+ and h_q^- defined in (5) are given by

$$h_q^+(x) = D_1 e^{\alpha_1 x}, \qquad h_q^-(x) = D_2 e^{\alpha_2 x},$$

where $\alpha_1 = -b + \sqrt{2q + b^2}$ and $\alpha_2 = -b - \sqrt{2q + b^2}$, and D_1, D_2 are positive real numbers. The function δ, defined in (11), is finite if $q > b\gamma + \frac{\gamma^2}{2}$ and is given by

$$\delta(x) = \frac{e^{\gamma x}}{q - \gamma b - \gamma^2/2}, \qquad x \in \mathbb{R}.$$

Next, we introduce the function

$$G_p(x) = \frac{g(x) + \delta(x)}{p e^{\alpha_1 x} + (1-p) e^{\alpha_2 x}},$$

which, according to Theorem 3.9, gives the solution to our problem. Then, if $q > \gamma b + \frac{\gamma^2}{2}$ and $b > -1$ then $\alpha_1 > 1$ and $\alpha_2 < -1$. Thus, we verify easily the following inequalities:

$$\sup_{x \le 0}\big(e^{-\alpha_1 x}\big(\delta(x) + g(x)\big)\big) > \sup_{x \ge 0}\big(e^{-\alpha_1 x}\big(\delta(x) + g(x)\big)\big) > 0 \tag{23}$$

and

$$\sup_{x\geq 0}\bigl(e^{-\alpha_2 x}\bigl(\delta(x)+g(x)\bigr)\bigr)>\sup_{x\leq 0}\bigl(e^{-\alpha_2 x}\bigl(\delta(x)+g(x)\bigr)\bigr)>0. \tag{24}$$

Note that

$$\sup_{x\geq 0} G_p(x)=\sup_{x\geq 0;\delta(x)+g(x)>0} G_p(x)=\Bigl(\inf_{x\geq 0;\delta(x)+g(x)>0}\frac{pe^{\alpha_1 x}+(1-p)e^{\alpha_2 x}}{g(x)+\delta(x)}\Bigr)^{-1}$$

and

$$\sup_{x\leq 0} G_p(x)=\sup_{x\leq 0;\delta(x)+g(x)>0} G_p(x)=\Bigl(\inf_{x\leq 0;\delta(x)+g(x)>0}\frac{pe^{\alpha_1 x}+(1-p)e^{\alpha_2 x}}{g(x)+\delta(x)}\Bigr)^{-1}.$$

Then for all $p\in(0,1)$

$$0<\sup_{x\geq 0;\delta(x)+g(x)>0} G_p(x)\leq \frac{1}{p}\sup_{x\geq 0}\bigl(e^{-\alpha_1 x}\bigl(g(x)+\delta(x)\bigr)\bigr)<+\infty$$

and

$$0<\sup_{x\leq 0;\delta(x)+g(x)>0} G_p(x)\leq \frac{1}{1-p}\sup_{x\leq 0}\bigl(e^{-\alpha_2 x}\bigl(g(x)+\delta(x)\bigr)\bigr)<+\infty.$$

We assume that $\log(L)\leq 0=W_0^b\leq\log(K)$, $q>1/2+b$ and $b>-1$. Then (23) and (24) hold and, as in Beibel and Lerche [4], Lemma 1, page 98, there exists a number $p^*\in(0,1)$ such that

$$\sup_{x\geq 0} G_{p^*}(x)=\sup_{x\leq 0} G_{p^*}(x).$$

Let x_1, x_2 and p^* be solutions with $x_1>\log K$, $x_2<\log L$ and $p^*\in(0,1)$ of the following system:

$$\begin{cases}\dfrac{(q-\gamma b-\gamma^2/2)(e^{x_1}-K)+e^{-x_1}}{p^*e^{\alpha_1 x_1}+(1-p^*)e^{\alpha_2 x_1}}=\dfrac{(q-\gamma b-\gamma^2/2)(L-e^{x_2})+e^{-x_2}}{p^*e^{\alpha_1 x_2}+(1-p^*)e^{\alpha_2 x_2}},\\ \dfrac{e^{x_2}(\gamma b+\gamma^2/2-q)-e^{-x_2}}{(q-\gamma b-\gamma^2/2)(L-e^{x_2})+e^{-x_2}}=\dfrac{p^*\alpha_1 e^{\alpha_1 x_2}+(1-p^*)\alpha_2 e^{\alpha_2 x_2}}{p^*e^{\alpha_1 x_2}+(1-p^*)e^{\alpha_2 x_2}},\\ \dfrac{e^{x_1}(q-\gamma b-\gamma^2/2)-e^{-x_1}}{(e^{x_1}-K)(q-\gamma b-\gamma^2/2)+e^{-x_1}}=\dfrac{p^*\alpha_1 e^{\alpha_1 x_1}+(1-p^*)\alpha_2 e^{\alpha_2 x_1}}{p^*e^{\alpha_1 x_1}+(1-p^*)e^{\alpha_2 x_1}}.\end{cases} \tag{25}$$

Let

$$M^*=\frac{(q-\gamma b-\gamma^2/2)(e^{x_1}-K)+e^{-x_1}}{p^*e^{\alpha_1 x_1}+(1-p^*)e^{\alpha_2 x_1}}=\frac{(q-\gamma b-\gamma^2/2)(L-e^{x_2})+e^{x_2}}{p^*e^{\alpha_1 x_2}+(1-p^*)e^{\alpha_2 x_2}}.$$

Then,

$$\sup_{T\in\Sigma_\infty^X}\mathbb{E}_0\Bigl[e^{-qT}g\bigl(W_T^{(b)}\bigr)-\int_0^T c\bigl(W_s^{(b)}\bigr)e^{-qs}\,ds\Bigr]=M^*-\frac{1}{q-\gamma b-\gamma^2/2}$$

and the optimal stopping time is

$$T_{(x_1,x_2)}=\inf\{t>0;\ W_t^{(b)}\notin(x_1,x_2)\}.$$

5.1.2. *The spectrally negative Lévy case.* In the second example, we consider as the reward function

$$g(x) = (x - K)^+.$$

We write, for any $x \in E$,

$$\mathcal{V}_Z(x) = \sup_{T \in \Sigma_\infty} \mathbb{E}_x\left[e^{-qT}(e^{Z_T} - K)^+ - \alpha \int_0^T e^{\gamma Z_s} e^{-qs}\, ds\right], \tag{26}$$

where $\alpha > 0$ and we recall that we choose $q = \psi(1)$.

PROPOSITION 5.1. *We assume that* $p_\gamma = \psi(1) - \psi(\gamma) > 0$ *and*

$$x < x^* = \frac{1}{\gamma} \log\left(\frac{p_\gamma K}{(1-\gamma)\alpha}\right). \tag{27}$$

Then

$$\mathcal{V}_Z(x) = e^{x - x^*}\left((e^{x^*} - K)^+ + \alpha \frac{e^{\gamma x^*}}{p_\gamma}\right) - \alpha \frac{e^{\gamma x}}{p_\gamma}$$

and the optimal stopping time is $T_{x^*} = \inf\{t > 0; Z_t \geq x^*\}$.

PROOF. First, by means of Fubini's theorem and using the fact that $p_\gamma > 0$, we easily get that, for any $x \in E$,

$$\delta(x) = \alpha \mathbb{E}_x \int_0^\infty e^{-qt} e^{\gamma Z_t}\, dt = \alpha e^{\gamma x} \int_0^\infty e^{-qt + t\psi(\gamma)}\, dt = \alpha \frac{e^{\gamma x}}{p_\gamma}.$$

Then, from Lemma 3.4, we deduce that

$$V_Z(x) = \sup_{T \in \Sigma_\infty} \mathbb{E}_x\left[e^{-qT}\left((e^{Z_T} - K)^+ + \delta(Z_T)\right)\right] - \delta(x).$$

Next, we recall from Bertoin [6], pages 189 and 190, that, for any $x \leq y$,

$$\mathbb{E}_x[e^{-rT_y}] = e^{-\phi(r)(y-x)}, \qquad r \geq 0.$$

Then, writing

$$G(u) = \frac{(e^u - K)^+ + \delta(u)}{e^{u-x}},$$

we have that

$$G'(u) = e^{x-u}\left(K \mathbb{1}_{\{u \geq \log K\}} + \frac{\alpha(\gamma - 1)}{p_\gamma} e^{\gamma u}\right).$$

Since $G'(u) \geq 0$ on $(-\infty, x^*]$ and $G'(u) \leq 0$ otherwise, we deduce from (27) that

$$\sup_{u \in E} G(u) = \sup_{u \geq x} G(u) < \infty.$$

The proof of the claim follows then by applying Theorem 4.2. $\square$

5.2. *Optimal stopping problems associated to self-similar positive Markov processes of the spectrally negative type.* Let $Z = (Z_t)_{t\geq 0}$ again be a spectrally negative Lévy process. We introduce, for any $\alpha > 0$, the process $X = (X_t)_{t\geq 0}$ defined by

$$\log(X_t) = Z_{A_t}, \qquad t \geq 0, \tag{28}$$

where

$$A_t = \inf\left\{s \geq 0;\, \Sigma_s = \int_0^s e^{\alpha Z_r}\, dr > t\right\}.$$

We denote its law by $\mathbb{Q}_x$ when it starts from $x > 0$. Lamperti [21] showed that X is an α-self-similar positive Markov process on $(0, \infty)$, that is, a Feller process which enjoys the following α-self-similarity property, for any $c > 0$,

$$((X_{c^\alpha t})_{t\geq 0}, \mathbb{Q}_x) \overset{(d)}{=} ((cX_t)_{t\geq 0}, \mathbb{Q}_x). \tag{29}$$

It is plain that X is also of the spectrally negative type, in the sense that it has no positive jumps. Next, we recall that the law of Z is characterized by its Laplace exponent, ψ, which is of the form (19). In the sequel, writing θ for the largest root in $[0, \infty)$ of the equation $\psi(u) = 0$, we assume that the following conditions

$$\theta < \alpha \quad \text{and} \quad \lim_{u\to\infty} \frac{\psi(u)}{u} = \infty \tag{30}$$

hold. The first condition secures that the lifetime of X is infinite since in the case $0 < \theta < \alpha$, we consider X to be the unique recurrent extension which hits and leaves 0 continuously (see Rivero [30] for more details). Under the second condition, the paths of the process X are of unbounded variation on any compact interval and the process X is regular. Next, we introduce more notation taken from Patie [25]. Define for any integers n

$$a_n(\psi, \alpha)^{-1} = \prod_{k=1}^{n} \psi(\alpha k), \qquad a_0(\psi, \alpha) = 1,$$

and we introduce the entire function $\mathcal{I}$ which admits the series representation

$$\mathcal{I}_{\psi,\alpha}(z) = \sum_{n=0}^{\infty} a_n(\psi, \alpha) z^n, \qquad z \in \mathbb{C}.$$

It is important to note that whenever θ, the largest root of the equation $\psi(u) = 0$, satisfies $\theta < \alpha$, it follows that all of the coefficients in the definition of $\mathcal{I}_{\psi,\alpha}$ are strictly positive. Then, Patie [25], Theorem 2.1, characterized the Laplace transform of

$$T_a = \inf\{t > 0;\, X_t \geq a\}$$

as follows. Suppose that $0 \le x \le a$. Then, for any $q \ge 0$, we have

$$\mathbb{E}_x[e^{-qT_a}] = \frac{\mathcal{I}_{\psi,\alpha}(qx^\alpha)}{\mathcal{I}_{\psi,\alpha}(qa^\alpha)}. \tag{31}$$

Next, we introduce the Ornstein–Uhlenbeck process associated to X which is defined, for any $t \ge 0$, by

$$U_t = e'_\lambda(-t) X_{e_\chi(t)}, \qquad t \ge 0, \tag{32}$$

where $e_\lambda(t) = \frac{e^{\lambda t}-1}{\lambda}$, $\chi = \alpha\lambda$ and we write $v_\lambda(t) = \frac{\log(1+\lambda t)}{\lambda}$ the continuous increasing inverse function of e_λ. These processes were introduced and studied by Carmona, Petit and Yor [9]. In particular, they proved that U is a Feller process and under the conditions (30), U has also infinite lifetime and is regular. In [23], the author computed the Laplace transform of the first passage times above of U as follows. With the obvious notation, for any $r \ge 0$ and $0 \le x \le a$,

$$\mathbb{E}_x[e^{-rT_a^U}] = \frac{\mathcal{I}_{\psi,\alpha}(r; x^\alpha)}{\mathcal{I}_{\psi,\alpha}(r; a^\alpha)}, \tag{33}$$

where

$$\mathcal{I}_{\psi,\alpha}(q; x) = \sum_{n=0}^{\infty} \frac{\Gamma(q+n)}{\Gamma(q)} a_n(\psi; \alpha) x^n \tag{34}$$

and Γ stands for the gamma function. By means of classical criteria on power series, it is easily seen that, under the second condition in (30), that $\mathcal{I}_{\psi,\alpha}(q; x)$ is an entire function in x and is analytic on the domain $\{q \in \mathbb{C}; \Re(q) > -1\}$. We shall also need the following representation of the function $\mathcal{I}_{\psi,\alpha}(q; x)$

$$\mathcal{I}_{\psi,\alpha}(q; x) = \frac{1}{\Gamma(q)} \int_0^\infty \mathcal{I}_{\psi,\alpha}(rx) e^{-r} r^{q-1}\, dr, \tag{35}$$

which is readily obtained by using the integral representation of the gamma function $\Gamma(q) = \int_0^\infty e^{-r} r^{q-1}\, dr$, $\Re(q) > 0$. We postpone to the end of this section the description of some specific examples of these power series. Let us assume that g is a continuous function and $q, \beta > 0$. We are ready to introduce the following optimal stopping problems:

$$\mathcal{V}_g^X(x) \triangleq \sup_{T \in \Sigma_\infty^X} \mathbb{E}_x[e^{-qT} g(X_T)],$$

$$\mathcal{V}_g^U(x) \triangleq \sup_{T \in \Sigma_\infty} \mathbb{E}_x[e^{-qT} g(U_T)],$$

$$\mathcal{V}_g^{U,\triangle}(x) \triangleq \sup_{T \in \Sigma_\infty} \mathbb{E}_x[e^{-q\triangle_T} g(U_T)], \qquad \triangle_t = \int_0^t U_s^{-\alpha}\, ds,$$

$$\mathcal{V}_g^{S,q}(x) \triangleq \sup_{T \in \Sigma_\infty} \mathbb{E}_x\left[e^{-qT} g\left(\frac{e^{\alpha Z_T}}{1 + \beta \int_0^T e^{\alpha Z_s}\, ds}\right)\right].$$

We note that in the case Z is a Brownian motion with drift, the last optimal stopping problem is intimately connected to the so-called integral option problem studied by Kramkov and Mordetski [17].

PROPOSITION 5.2. *Let us write for some function h*

$$a_*(h(\cdot)) = \arg \max_{u \geq x, u \in E} \frac{g(u)}{h(u^\alpha)}.$$

(1) *If $a_* = a_*(\mathcal{I}_{\psi,\alpha}(q\cdot))$ exists and $x < a_*$ then*

$$\mathcal{V}_g^X(x) = \frac{\mathcal{I}_{\psi,\alpha}(qx^\alpha)}{\mathcal{I}_{\psi,\alpha}(qa_*^\alpha)} g(a_*).$$

(2) *If $a_* = a_*(\mathcal{I}_{\psi,\alpha}(\frac{q}{\chi}; \chi\cdot))$ exists and $x < a_*$ then*

$$\mathcal{V}_g^U(x) = \frac{\mathcal{I}_{\psi,\alpha}(q/\alpha; \chi x^\alpha)}{\mathcal{I}_{\psi,\alpha}(q/\alpha; \chi a_*^\alpha)} g(a_*).$$

(3) *If $a_* = a_*(\mathcal{I}_{\psi,\alpha}(\frac{q}{\chi}; \chi\cdot))$ exists and $x < a_*$ then*

$$\mathcal{V}_g^{U,\triangle}(x) = \left(\frac{x}{a_*}\right)^\gamma \frac{\mathcal{I}_{\psi,\alpha}(\gamma/\alpha; \chi x^\alpha)}{\mathcal{I}_{\psi,\alpha}(\gamma/\alpha; \chi a_*^\alpha)} g(a_*).$$

(4) *If $a_* = a_*(\mathcal{I}_{\psi,\alpha}(\frac{q}{\chi}; \chi\cdot))$ exists and $x < a_*$ then*

$$\begin{aligned}\mathcal{V}_g^{S,q}(x) &= \Psi_{\triangle,g}^U(1)\\ &= \frac{\mathcal{I}_{\psi,\alpha}(q/\chi; \chi x^\alpha)}{\mathcal{I}_{\psi,\alpha}(q/\chi; \chi a_*^\alpha)} g(a_*).\end{aligned}$$

In all the above cases the optimal stopping time is given by T_{a_*}.

PROOF. The first item follows readily from the identity (33) and Theorem 4.2. Next, let

$$T_{y,\alpha}^X = \inf\{t > 0; X_t = y(1 + \alpha\lambda t)^{1/\alpha}\}.$$

The Mellin transform of the positive random variable $T_{y,\alpha}^X$ has been computed by Patie (see [23], Theorem 2). However, for sake of completeness, we provide a slightly different proof here which relies on a device introduced by Shepp [33]. In the proof of [25], Theorem 1, it is shown that the mapping $x \mapsto \mathcal{I}_{\psi,\alpha}(rx^\alpha)$ is an r-eigenfunction for the infinitesimal generator of X. Thus, by the Dynkin formula, using the fact that the function $\mathcal{I}_{\psi,\alpha}$ is increasing and applying the dominated convergence theorem, we deduce that

$$\mathbb{E}_x\big[e^{-rT_{y,\alpha}^X} \mathcal{I}_{\psi,\alpha}\big(ry^\alpha(1 + \alpha\lambda T_{y,\alpha}^X)\big)\big] = \mathcal{I}_{\psi,\alpha}(rx^\alpha).$$

Integrating both sides of the previous identity by the measure $e^{-\chi r} r^{q/\chi-1}\,dr$, using Fubini's theorem and the change of variable $u = r(T^X_{y,\alpha} + \chi)$, we get

$$\mathbb{E}_x[(1+\chi T^X_{y,\alpha})^{-q/\chi}] = \frac{\mathcal{I}_{\psi,\alpha}(q/\chi;\chi x^\alpha)}{\mathcal{I}_{\psi,\alpha}(q/\chi;\chi a^\alpha)},$$

where we have used (35). Moreover, from the definition of U (32), we observe the identity

$$T^U_y = v_\chi(T^X_{y,\alpha}) \qquad \text{a.s.} \tag{36}$$

Hence, we obtain that

$$\mathbb{E}_x[e^{-qT^U_a}] = \frac{\mathcal{I}_{\psi,\alpha}(q/\chi;\chi x^\alpha)}{\mathcal{I}_{\psi,\alpha}(q/\chi;\chi a^\alpha)}.$$

We deduce the item (2) by an application of Theorem 4.2. We complete the proof of the proposition from [23], Corollary 3.2. □

REMARK 5.3. We note that the identity (36) combined with the item (2) of the previous proposition allows us to solve the following nonhomogeneous optimal stopping time problem

$$\mathcal{V}^X_{g,\alpha}(x) = \sup_{T\in\Sigma^X_\infty} \mathbb{E}_x\left[(1+2\lambda T)^{-q} g\left(\frac{X_T}{(1+2\lambda T)^{1/\alpha}}\right)\right].$$

Indeed, we easily deduce that

$$\mathcal{V}^X_{g,\alpha}(x) = \frac{\mathcal{I}_{\psi,\alpha}(q/\chi;\chi x^\alpha)}{\mathcal{I}_{\psi,\alpha}(q/\chi;\chi a_*^\alpha)} g(a_*),$$

where a_* is characterized by

$$a_* = \arg\max_{u\geq x, u\in E} \frac{g(u)}{\mathcal{I}_{\psi,\alpha}(q/\chi;\chi u^\alpha)}.$$

In what follows, we provide some examples of the power series and we refer to [25] for the description of additional examples.

The modified Bessel functions. We consider Z to be a Brownian motion with drift $\nu \geq 0$, that is, $\psi(u) = \frac{1}{2}u^2 + \nu u$ and we set $\alpha = 2$. Its associated self-similar process is well known to be a Bessel process of index ν. We have

$$a_n(\psi;2)^{-1} = 2^n n! \frac{\Gamma(n-\nu+1)}{\Gamma(-\nu+1)}, \qquad a_0 = 1.$$

Thus, we get

$$\mathcal{I}_{2,\psi}(x) = (x/2)^{\nu/2}\Gamma(-\nu+1)\mathrm{I}_{-\nu}(\sqrt{2x}),$$

where $\mathrm{I}_\nu(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}$ stands for the modified Bessel function of index ν (see, e.g., [22], Chapter 5) and

$$\mathcal{I}_{2,\psi}(q; x^2) = \Phi\left(q, 1-\nu, \frac{x^2}{2}\right),$$

where $\Phi(q, \nu, x) = \sum_{n=0}^{\infty} \frac{(q)_n}{(\nu)_n n!} x^n$ stands for the confluent hypergeometric function of the first kind (see, e.g., [22], Section 9.9, page 260) and $(q)_n = \frac{\Gamma(q+n)}{\Gamma(n)}$ stands for the Pochhammer symbol.

Some generalized Mittag–Leffler functions. In [24], the author introduced a parametric family of one-sided Lévy processes which are characterized by the following Laplace exponent, for any $1 < \alpha < 2$, and $\gamma > 1 - \alpha$,

$$\psi_\gamma(u) = \bigl((u + \gamma - 1)_\alpha - (\gamma - 1)_\alpha\bigr). \tag{37}$$

Its Lévy measure is absolutely continuous with a density f given by

$$f(y) = C\frac{e^{(\alpha+\gamma-1)y}}{(1-e^y)^{\alpha+1}}, \qquad y < 0,$$

where C is a positive constant. We focus on the case $\gamma = 1$ in (37). We have $\psi_1(u) = \psi(u) = (u)_\alpha$ and

$$a_n(\psi; \alpha)^{-1} = \frac{\Gamma(\alpha(n+1))}{\Gamma(\alpha)}, \qquad a_0 = 1.$$

Thus, the power series can be written as

$$\mathcal{I}_{\psi,\alpha}(x) = \Gamma(\alpha)\mathcal{M}_{\alpha,\alpha}(\alpha x),$$
$$\mathcal{I}_{\psi,\alpha}(q; x) = \Gamma(\alpha)\mathcal{M}^q_{\alpha,\alpha}(\alpha x),$$

where $\mathcal{M}_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n+\beta)}$ [resp., $\mathcal{M}^q_{\alpha,\beta}(x) = \sum_{n=0}^{\infty} \frac{(q)_n x^n}{\Gamma(\alpha n+\beta)}$] stands for the Mittag–Leffler function of parameters $\alpha, \beta > 0$ (resp., of parameters $\alpha, \beta, q > 0$) introduced by Prabhakar [29].

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