Hints in unification

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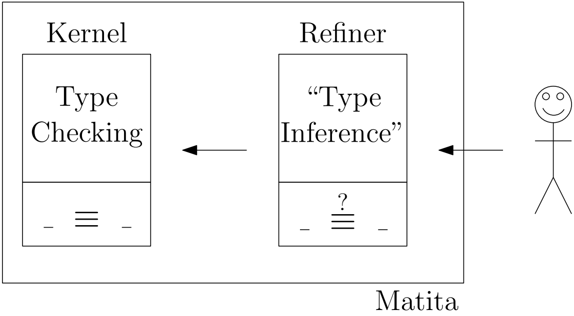
"Hints in Unification", A.Asperti, W. Ricciotti, C. Sacerdoti Coen, E. Tassi. Accepted for publication in the proceedings of TPHOLs 2009

Context of the work

- Matita: ITP based on CIC
- Type-checking v.s. Type-inference
  - conversion v.s. unification



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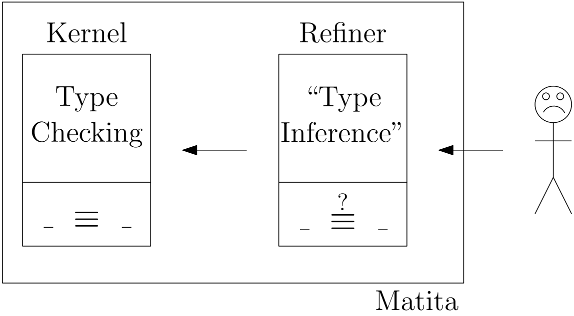

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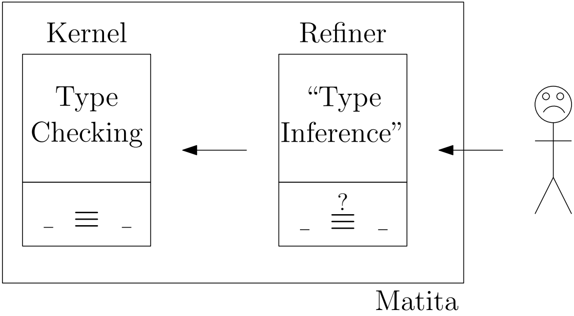

- The user works all the time with the refiner, that is based on an heuristic.
- Can the user customise/drive unification?!

(Ad hoc) mechanisms

Unification is an hard problem in CIC

$$t_1 \quad \stackrel{?}{\equiv} \quad t_2 \qquad \text{find } \sigma \text{ such that} \quad t_1\sigma \equiv t_2\sigma$$

Very common unification problems deserve ad-hoc mechanisms to let the user drive the unification algorithm

- canonical structures
- coercions pullback
- ...

Unification hints are a framework to let the user customise the unification algorithm that generalise these mechanisms (and little more...)

Outline

1. Unification hints by examples
2. (re) implementing ad-hoc mechanisms with hints
3. Unification hints and reification
4. Conclusions

Unification hints — an example

- Hints are pairs of convertible terms
- Unification algorithm $\mathcal{U}\ t_1\ t_2$
- $\mathcal{U}'\ t_1\ t_2 =$
  $\text{try}\ \mathcal{U}\ t_1\ t_2$
  $\text{with Fail} \Rightarrow \text{hints}\ t_1\ t_2$

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$$\frac{(S\ ?_x) + ?_y \stackrel{?}{=} ?_1 + ?_2 \qquad S(x+y) \stackrel{?}{=} S\ (?_x + ?_y)}{?_1 + ?_2 \stackrel{?}{\equiv} S\ (x+y)}\ \text{hint}$$

Unification hint — recursion

What is used to solve $\_ \stackrel{?}{=} \_$? $\mathcal{U}$ or $\mathcal{U}'$?

Another example: $?_1 + ?_2 \stackrel{?}{\equiv} S\ (S\ (x+y))$

$$\frac{\ldots \qquad S\ (S\ (x+y)) \stackrel{?}{=} S\ (?_x + ?_y)}{?_1 + ?_2 \stackrel{?}{\equiv} S\ (S\ (x+y))}\ \text{hint}$$

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Possible (new) source of divergence...

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Some subterms (where $\mathcal{U}'$ may be needed) can confuse indexing

Unification hints

The correct version of the hint:

$$x, y : \mathbb{N}, z := x + y \vdash (S\ x) + y \equiv S\ z$$

Unification hints

The correct version of the hint:

$$\frac{?_z \equiv ?_x + ?_y}{(S\ ?_x) + ?_y \equiv S\ ?_z}\ \text{myhint}$$

Application

$$\frac{(S\ ?_x) + ?_y \stackrel{?}{=} A \qquad B \stackrel{?}{=} S\ ?_z \qquad ?_z \stackrel{?}{\equiv} ?_x + ?_y}{A \stackrel{?}{\equiv} B}\ \text{myhint}$$

Hints for canonical structures

In presence of records like

$$\mathcal{Z} : \mathrm{Group} := \{\mathrm{gcarr} := \mathbb{Z}; \ldots\}$$

The canonical structure mechanism allows the user to specify a canonical solution for $?_g$:

$$\mathrm{gcarr}\ ?_g \stackrel{?}{\equiv} \mathbb{Z} \qquad \Rightarrow \qquad ?_g := \mathcal{Z}$$

By declaring the following hint, the user obtains the same result

$$\frac{}{\mathrm{gcarr}\ \mathcal{Z} \equiv \mathbb{Z}}\ \text{hint-for-group-}\mathcal{Z}$$

Coercions pullback

The user inputs:

$$x * (y + z)$$

Arguments of $*$ must have the same type:

$$\mathrm{gcarr}\ ?_g \stackrel{?}{\equiv} \mathrm{mcarr}\ ?_m$$

$$\begin{tikzcd}
 & \boxed{\text{ring}} \arrow[dl, "\text{r_group}"'] \arrow[dr, "\text{r_monoid}"] & \\
\boxed{\text{group}} \arrow[dr, "\text{gcarr}"'] & & \boxed{\text{monoid}} \arrow[dl, "\text{mcarr}"] \\
 & \text{Type} &
\end{tikzcd}$$

The solution is the pullback of gcarr and mcarr.

$$?_g := \mathrm{r\_group}\ ?_r$$

$$?_m := \mathrm{r\_monoid}\ ?_r$$

The graph must be coherent!

Hints for pullbacks

Problems have many similar forms:

- $f_1 \; ?_1 \overset{?}{\equiv} g_1 \; ?_2$
- $f_1 \; (f_3 \; ?_1) \overset{?}{\equiv} g_1 \; ?_2$
- ...

All diamonds must be declared as hints, but smaller ones should be preferred.

$$\frac{}{f_1 \; (f_2 \;\; ?_c) \equiv g_1 \; (g_2 \;\; ?_c)}\;\text{hint-1}$$

$$\frac{}{f_1 \; (f_3 \;\; ?_d) \equiv g_1 \; (g_3 \;\; ?_d)}\;\text{hint-2}$$

$$\begin{tikzcd}
 & D \arrow[d, "h"] \arrow[ddl, "f_3"'] \arrow[ddr, "g_3"] & \\
 & C \arrow[dl, "f_2"] \arrow[dr, "g_2"'] & \\
A \arrow[dr, "f_1"'] & & B \arrow[dl, "g_1"] \\
 & T &
\end{tikzcd}$$

Reflexive tactics — reification with hints

When we build a reflexive tactic, we need a syntactic representation of the input that cannot be obtained inside the calculus (usually built in $\mathcal{L}$-tac, OCaml, ...)

```
inductive Expr : Type :=
  | Eunit  : Expr
  | Emult  : Expr → Expr → Expr
  | Einv   : Expr → Expr
  | Evar   : ℕ → Expr.
```

let rec $[\![e : \mathrm{Expr};\ \Gamma : \mathrm{list}\ (\mathrm{gcarr}\ \mathrm{g})]\!]_{(\mathrm{g:group})}$ on e : gcarr g :=

  match e **with**

  [Eunit $\Rightarrow 1_g$

  | Emult x y $\Rightarrow [\![x;\ \Gamma]\!]_g * [\![y;\ \Gamma]\!]_g$

  | Einv x $\Rightarrow [\![x;\ \Gamma]\!]_g^{-1}$

  | Evar n $\Rightarrow \Gamma(n)$].

Reflexive tactics — reification with hints

The unification problem for reification with sharing is:

$$[\![?_1;\ ?_2]\!]_{?_3} \stackrel{?}{\equiv} x * (x^{-1} * y)$$

Unification hints follows:

$$\frac{?_m \equiv [\![?_x;\ ?_\Gamma]\!] \qquad ?_n \equiv [\![?_y;\ ?_\Gamma]\!]}{[\![\mathrm{Emult}\ ?_x\ ?_y;\ ?_\Gamma]\!]_{?_g} \equiv ?_m * ?_n}\ \text{h-times}$$

$$\frac{}{[\![\mathrm{Eunit};\ ?_\Gamma]\!]_{?_g} \equiv 1}\ \text{h-unit}$$

$$\frac{?_o \equiv [\![?_z;\ ?_\Gamma]\!]_{?_g}}{[\![\mathrm{Einv}\ ?_z;\ ?_\Gamma]\!]_{?_g} \equiv ?_o^{-1}}\ \text{h-inv}$$

Reflexive tactics — reification with hints

The tricky part is the set of hints to reify variables

$$\frac{}{[\![\mathrm{Evar}\ 0;\ ?_r :: ?_\ominus]\!]_{?_g} \equiv ?_r}\ \text{h-var-base}$$

$$\frac{?_q \equiv [\![\mathrm{Evar}\ ?_p;\ ?_\Delta]\!]_{?_g}}{[\![\mathrm{Evar}\ (S\ ?_p);\ ?_s :: ?_\Delta]\!]_{?_g} \equiv ?_q}\ \text{h-var-rec}$$

Conclusion

- Unification hints are a general framework to let the user drive the unification algorithm
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- (not so) unexpectedly they can drive the unification algorithm in performing reification
- Future works: pragmatic study of divergence.

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Thanks for your attention!