Nonuniform Coercions via Unification Hints

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TYPES 2010 — 15 October 2010 — Warsaw

Context of this work

- Interactive theorem prover Matita (CIC)
- Formalization of formal topology (Algebraic Structures)

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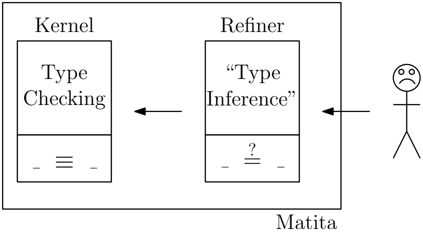

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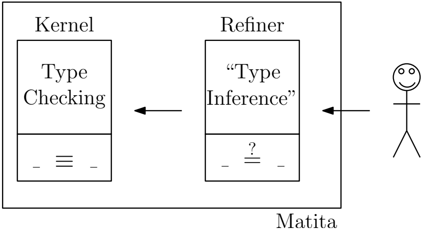

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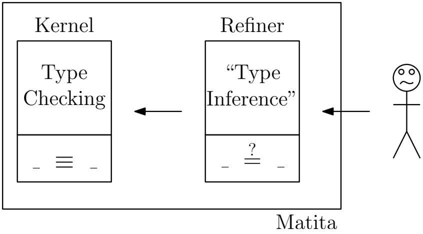

- Unification made user-extensible (Unification Hints)
- In some corner cases the system is unable to exploit the knowledge given by hints

Example

record Group : Type := { carr : Type, $\_*\_$: ...}
definition $\mathcal{Z}$: Group := $\langle$ Z, +, 0, ...$\rangle$.
lemma mulg1: $\forall$ G:Group, $\forall$ a:carr G. $a * 1 = a$.
lemma cardG_gt0 : $\forall$ G : Group, $0 < |G|$.

check (mulg1 $?_G$ 2).

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Error: Z has type Type but it's context expects a term of type Group. The unification problem Type $\stackrel{?}{=}$ Group has no solution.

Outline

1. Coercions
    - Nonuniform coercions
    - Examples
2. Implementation
    - Ingredients
    - Declaring nonuniform coercions
    - Reusing existing hints
3. Conclusions

Type inference and coercions

- These problems have to be addressed by type inference

$$\Gamma \vdash t : T \rightsquigarrow t' : T'$$

- Looks like coercions could solve these typing errors

$$\frac{}{\Gamma \vdash x : N \rightsquigarrow \qquad : Z}$$

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- but (uniform) coercions are type theoretic functions whose insertion is type driven.

$$\frac{(\lambda\_.\mathcal{Z},(\mathit{Type},\mathit{Group})) \in \Delta \quad \Gamma \vdash (\lambda\_.\mathcal{Z})\ Z : \mathit{Group}}{\Gamma \vdash Z : \mathit{Type} \rightsquigarrow (\lambda\_.\mathcal{Z})\ Z : \mathit{Group}}$$

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Nonuniform coercions

$$\Delta = \left\{ \Gamma_1 \vdash \begin{array}{ccc} S_1 & \to & T_1 \\ s_1 & \mapsto & t_1 \end{array} \quad \ldots \quad \Gamma_n \vdash \begin{array}{ccc} S_n & \to & T_n \\ s_n & \mapsto & t_n \end{array} \right\}$$

where

$$\Gamma_i \vdash s_i : S_i \qquad \Gamma_i \vdash t_i : T_i$$

Inserting a nonuniform coercion works as follows:

$$\frac{}{\Gamma \vdash s : S \rightsquigarrow \quad : T}$$

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Nonuniform coercions: examples

Uniform coercions

$$x : N \;\vdash\; \begin{array}{ccc} N & \rightarrow & Z \\ x & \mapsto & k\ x \end{array}$$

Nonuniform coercions

$$\vdash\; \begin{array}{ccc} Type & \rightarrow & Group \\ Z & \mapsto & \mathcal{Z} \end{array}$$

$$\vdash\; \begin{array}{ccc} Type & \rightarrow & Group \\ Q & \mapsto & \mathcal{Q} \end{array}$$

Cheap implementation: ingredient #1

Unification hints:

$$\Gamma \vdash \frac{\overrightarrow{?_x} \;:=\; \overrightarrow{H}}{P \;\equiv\; Q}\ \text{myhint}$$

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Examples:

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$$G, H : Group \vdash \frac{\begin{array}{rcl} ?_A & := & carr\ \ G \\ ?_B & := & carr\ \ H \\ ?_X & := & product\_group\ \ G\ \ H \end{array}}{?_A \;\times\; ?_B \;\equiv\; carr\ ?_X}$$

Cheap implementation: ingredient #1 (cont.)

Note that hints define "equivalence classes" of constants, thus approximated indexing for fast retrieval must take them into account.

$$\frac{(k,(N,Z)) \in \Delta \quad \Gamma \vdash k\ s : Z \quad Z \stackrel{?}{=} carr\ \mathcal{Z}}{\Gamma \vdash s : N \leadsto k\ s : carr\ \mathcal{Z}}$$

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Cheap implementation: ingredient #2

Uniform coercion loosely indexed:

$$\frac{(\mathit{result}, (*, \mathit{target})) \in \Delta \qquad \Gamma \vdash \mathit{result}\ s : \mathit{target} \qquad \mathit{target} \stackrel{?}{=} T}{\Gamma \vdash x : S \rightsquigarrow \mathit{result}\ s : T}$$

Note that T and *target* can be in the same equivalence class.

Encoding nonuniform coercions

record solution (S : Type) (s : S) : Type :={
  target : Type; *(* T *)*
  result : target *(* t *)*
}.

coercion result : ∀S:Type.∀s:S.∀sol:solution S s. target S s sol
on s : ? $\rightarrowtail$ target ???.

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$$s \rightsquigarrow result\ ?\ s\ ?_{sol}$$

Declaring nonuniform coercions

$$\vdash \begin{array}{lcl} Type & \rightarrow & Group \\ Z & \mapsto & \mathcal{Z} \end{array}$$

$$\frac{}{\Gamma \vdash Z : Type \rightsquigarrow \qquad\qquad : Group}$$

Declaring nonuniform coercions

$$\vdash \begin{array}{lll} Type & \to & Group \\ Z & \mapsto & \mathcal{Z} \end{array}$$

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We declare the following hint:

$$\vdash \frac{?_{sol}\ := \ mk\_solution\ Type\ Z\ Group\ \mathcal{Z}}{target\ \ Type\ Z\ ?_{sol}\ \equiv\ Group}$$

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Declaring nonuniform coercions (the right way)

This is unsatisfactory, we need one new hint per coercion

$$\vdash \frac{?_{sol} \;:=\; mk\_solution\ Type\ Z\ Group\ \mathcal{Z}}{target\ Type\ Z\ ?_{sol} \;\equiv\; Group}$$

Moreover, the system is already aware that

$$\Gamma \vdash \frac{?_G \;:=\; \mathcal{Z}}{Z \;\equiv\; carr\ ?_G}$$

We need only this hint:

$$G : Group \vdash \frac{\begin{array}{lcl} ?_Z & := & carr\ G \\ ?_{sol} & := & mk\_solution\ Type\ ?_Z\ Group\ G \end{array}}{target\ Type\ ?_Z\ ?_{sol} \;\equiv\; Group}$$

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Further research:

- Notion of coherence (sanity check on Δ as a whole)
- Notion of composition for nonuniform coercions

Thanks

Thanks for your attention!