Game theory

in

Wireless Networking

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OBJECTIVES

• Introduce non-cooperative game notions that are potentially useful in wireless networking:

1. Non-atomic games
2. Wardrop equilibrium,
3. Potential games for infinite player set
4. Replicator dynamics
5. Discrete Potential games, Convergence
6. Constrained games
7. Correlated equilibrium
8. S-modular games

1 Non-atomic games

1.1 Network exampe

Network: a graph $G = (V, \mathcal{C})$ where

- V is a set of nodes

- L a class of directed links

K classes of traffic.

- Class i has a set $\mathbf{P_i}$ of paths, and traffic demand of $\phi^{(i)}$.

- Link cost: $f_l(y_l)$, strictly monotone increasing in the link flow y_l

Definitions

•Strategies: the amount $x_p^{(i)}$ of traffic of class i is sent over path p.

•Link traffic: $y_l = \sum_{p,i} \delta_{lp} x_p^{(i)}$ where $\delta_{lp} = 1$ if l is on the path p.

•Flow constraints: for each class i, $\sum_{p \in \mathbf{P}(i)} x_p^{(i)} = \boldsymbol{\phi}^{(i)}$.

•Let $\gamma_p^{i,i'} = 1$ if $p \in \mathbf{P}(i')$ and $i = i'$, and 0 otherwize.

•Matrix form of flow constraints $\mathbf{\Gamma^T x} = \boldsymbol{\phi}$.

Γ^T has K rows and π columns, where $\pi = \sum_i |P(i)|$.

x is a column vector (size π). The first $P(1)$ elements correspond to the flow over the pathrs of class 1, etc.

1.2 Global optimization

•**Objective:**

$$\min_x \Delta(x) \quad \text{where } \Delta(x) := \frac{1}{D} \sum_l y_l f_l(y_l) \quad D\text{:= total demand.} \tag{1}$$

s.t. (i) flow conservation, (ii) non-negative flows, (iii) y in terms of x.
•Define $t_p^{(k)} = \partial(\Phi\Delta)/\partial x_p^{(k)}$, i.e., class k marginal cost of p, $p \in \Pi^{(k)}$.

•$\mathbf{t} = [t_1^{(1)}, t_2^{(1)}, \ldots, t_1^{(2)}, t_2^{(2)}, \ldots]^{\mathrm{T}}$ is the gradient vector of the function $\Phi\Delta$.

•**Characterization of optimal solution through complementarity:**
$\mathbf{x}$ optimal iff there exist K Lagrange multipliers $\boldsymbol{\alpha}$ such that

$$[\mathbf{t}(\mathbf{x}) - \boldsymbol{\Gamma}\boldsymbol{\alpha}] \cdot \mathbf{x} = 0, \tag{2}$$

$$\mathbf{t}(\mathbf{x}) - \boldsymbol{\Gamma}\boldsymbol{\alpha} \geq 0, \tag{3}$$

$$\boldsymbol{\Gamma}^{\mathrm{T}}\mathbf{x} - \boldsymbol{\phi} = 0, \tag{4}$$

$$\mathbf{x} \geq 0. \tag{5}$$

•Alternative characterisation: Variational inequalities

$\bar{\mathbf{x}}$ is an optimal solution iff

$$\begin{aligned} \mathbf{t}(\bar{\mathbf{x}}) \cdot (\mathbf{x} - \bar{\mathbf{x}}) &\geq 0, \quad \text{for all } \mathbf{x} \\ \text{such that } \mathbf{\Gamma}^{\mathrm{T}}\mathbf{x} &= \boldsymbol{\phi} \text{ and } \mathbf{x} \geq 0. \end{aligned} \tag{6}$$

1.3 Wardrop equilibrium

• Non-atomic setting: large number of non-cooperative players.

• Each player has a negligible influence on others performances.

• $\mathbf{x}$ is a Wardrop equilibrium if each player uses a least costly.
$T_p^{(k)}(\mathbf{x}) :=$ cost of path p for class k user. Equals sum of link costs along p.

• A type k user chooses a path $\hat{p}$ that satisfies

$$T_{\hat{p}}^{(k)}(\mathbf{x}) = \min_{p \in \Pi^{(k)}} T_p^{(k)}(\mathbf{x}) =: A^{(k)} \tag{7}$$

Thus $\mathbf{x}$ is a Wardrop equilibrium if

$$T_p^{(k)}(\mathbf{x}) \geq A^{(k)}, \quad x_p^{(k)} = 0, \tag{8}$$

$$T_p^{(k)}(\mathbf{x}) = A^{(k)}, \quad x_p^{(k)} > 0, \tag{9}$$

and the flow constraintns hold (conservation, nonnegativity). Matrix notation:

$$[\mathbf{T}(\mathbf{x}) - \mathbf{\Gamma A}] \cdot \mathbf{x} = 0, \quad (10)$$
$$\mathbf{T}(\mathbf{x}) - \mathbf{\Gamma A} \geq 0, \quad (11)$$
$$\mathbf{\Gamma}^{\mathrm{T}}\mathbf{x} - \boldsymbol{\phi} = 0, \quad (12)$$
$$\mathbf{x} \geq 0, \quad (13)$$

where $\mathbf{A} = [A^{(1)}, A^{(2)}, \ldots, A^{(K)}]^{\mathrm{T}}$,

•Define the **potential** $G(\mathbf{x}) = \frac{1}{D} \sum_l \int_0^{y_l} f_l(s)ds$. Then

$$T_p^{(k)}(\mathbf{x}) = \frac{\partial}{\partial x_p^{(k)}}(DG(\mathbf{x})).$$

We get the same conditions for optimality of $\mathbf{x}$ in the global optimization problem where link costs are replaced by their integral.

•**Conclusion:** $\mathbf{x}$ is a Wardrop equilibrium if it solves a **global optimization** problem:

$$\text{minimize } G(\mathbf{x}) \qquad \text{s.t. } (12)\text{-}(13).$$

•**Alternative characterisation: Variational inequalities**

$\bar{\mathbf{x}}$ is a Wardrop equilibrium if and only if it is feasible and

$$\begin{aligned} \mathbf{T}(\bar{\mathbf{x}}) \cdot (\mathbf{x} - \bar{\mathbf{x}}) &\geq 0, \quad \text{for all } \mathbf{x} \\ \text{such that } \mathbf{\Gamma}^{\mathrm{T}} \mathbf{x} &= \boldsymbol{\phi} \text{ and } \mathbf{x} \geq 0. \end{aligned}$$

•Again the same form of the global optimization provided that $\mathbf{T}(\bar{\mathbf{x}})$ is interpreted as the gradient of a potential.

1.4 Applications to adhoc networks

•P. Gupta and P. R. Kumar propose a new routing algorithm for Adhoc networks in IEEE CDC, 1997

•No real players, no game.

•This is a shortest path (delay) protocole: each packet that arrives at a mobile is forwarded so as to minimize its delay.

•Reason: to decrease number of out-or-order packets and to minimize resequencing delay

1.5 Limitations of the model (i)

•**Link correlations:** The cost over link ℓ may depend on the flow over other links,

•**Road traffic Examples:** (i) Two way traffic. Congestion in one direction could contribute to congestion in the other direction [Dafermos, Transportation Sc. 1971].

•**Data network traffic:** A congestion of TCP connections in one direction impacts on the flow of ACKs in the opposite direction.

•**Wireless context:** Links are radio channels. They can have mutual interference.

Limitations of the model (ii): multiclass traffic

•(i) Link cost may **depend on the flow of each user** $\{y_l^{(i)}\}$ rather than on the total link flow.

•(ii) Moreover, the link cost may differ from one class to another.

•**Road traffic example:** bicycles, cars and trucks contribute differently to congestion, and may experience congestion differently. [Dafermos, The traffic assignment problem for multiclass-user transportation networks. Transp Sci 1972]

•**Data networks example:** Packets of different size contribute differently to congestion.

•**Example:** Diffserv provides priority to some traffic over other. The priority traffic encounters less congestions, but is more expensive.

Potential game with continuouos players set

•The networking game is said to be a potential game if there is a continuously differentiable function $G : \mathcal{X} \rightarrow R$ such that for all i and ℓ,

$$\frac{\partial G(\mathbf{x})}{\partial x_\ell^i} = f_\ell^i(\mathbf{x})$$

•the Wardrop equilibrium is obtained by maximizing the potential

Conditions for existence of a Potential

•Assume that the matrix

$$\left[\frac{\partial f_l^j(\mathbf{x})}{\partial x_k^i}\right]$$

is **symmetric** and **positive definite**. f_l^j is the cost of link l for class j.

•A potential $G(\mathbf{x})$ exists, it is the sum (over the entries corresponding to links and users) of the line integral

$$G(\mathbf{x}) = \sum_{l,j} \int_0^{\mathbf{x}} f_l^j(\mathbf{s}) d(\mathbf{s})$$

which is path independent.

•There is a "unique" Wardrop equilibrium (if the link costs are monotone).

•Major research problem: what to do when there is no potential.

Limitations of the model (iii): Non-additive costs.

•Example: **loss networks** work with El-Aazouzi and Abramov.

•$\mathcal{C}$ resources. Resource c has R_c capacity units (integer).

•There are N classes of calls ($\mathcal{N} = \{1, 2, .., N\}$),

•Associated with class n are

- Arrival rate λ_n, and an average holding time μ_n^{-1},
- Bandwidth requirement, b_n integer units.
- Route $r_n \subseteq \mathcal{C}$.

Denote $\rho_n = \lambda_n / \mu_n$ the class n workload.

Blocking probabilities

- Let $\mathcal{N}_c$ the subset of classes that use resource c,

$$\mathcal{N}_c = \{n \in \mathcal{N} : c \in r_n\}. \tag{14}$$

- Let m_n the number of calls of class n in the system, and

$$\mathbf{m} = (m_1, m_2, ..., m_N).$$

- The state space is $\mathcal{X} = \Big\{\mathbf{m} : \sum_{n \in \mathcal{N}_c} b_n m_n \le R_c, \quad c \in \mathcal{C}\Big\}$.
- Let $\mathcal{X}_n$ the subset of states for which there is an available bandwidth for another arrival of a class-n call:

$$\mathcal{X}_n = \Big\{\mathbf{m} \in \mathcal{X} : \sum_{i \in \mathcal{N}_c} b_i m_i \le R_c - b_n, c \in r_n\Big\}. \tag{15}$$

The steady state distribution is

$$\mathbf{P}\{\mathbf{X} = \mathbf{m}\} = \frac{1}{G} \prod_{n=1}^{N} \frac{\rho_n^{m_n}}{m_n!}, \qquad \mathbf{m} \in \mathcal{X}, \tag{16}$$

where

$$G = \sum_{\mathbf{m} \in \mathcal{X}} \prod_{n=1}^{N} \frac{\rho_n^{m_n}}{m_n!}. \tag{17}$$

The probability of blocking of a class-n call is

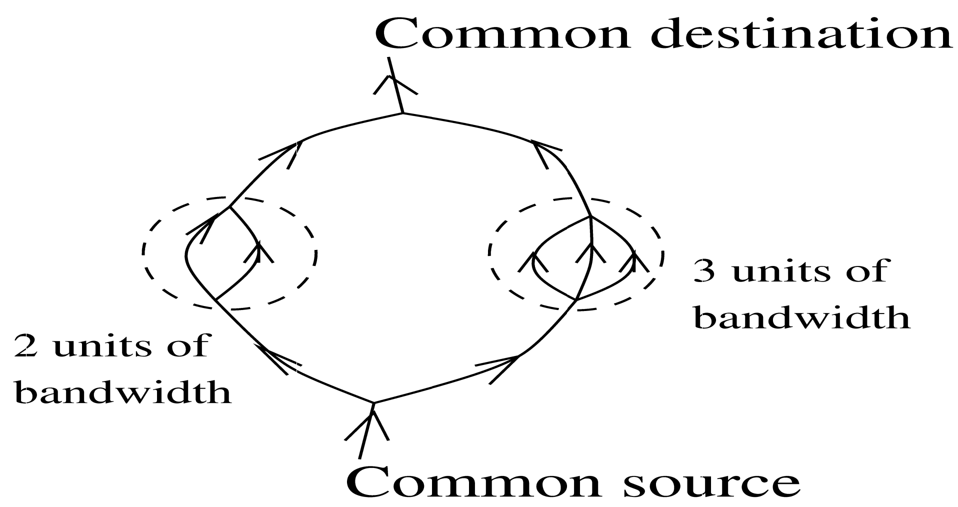
$$B_n = 1 - \frac{\sum_{\mathbf{m} \in \mathcal{X}_n} \prod_{i=1}^{N} \rho_i^{m_i} / m_i!}{\sum_{\mathbf{m} \in \mathcal{X}} \prod_{i=1}^{N} \rho_i^{m_i} / m_i!}. \tag{18}$$

Group of users

- Consider L groups which split their demands via the networks.
- Group l can use any one of subset $\mathcal{N}^l \subset \mathcal{N}$ of classes. The set $\mathcal{N}^l$ is characterized by a common source and destination as well as a common parameters b_l, μ_l and λ_l.
- **Strategies:** Group l sends a fraction $p_{l,n}$ of its demand via the route r_n.

Non-uniqueness of Wardrop equilibrium

Consider the following example:



- There are two parallel links :
  - The first one, a, has a capacity of 2 bandwidth units
  - The second one , b, has a capacity of 3 bandwidth units

- There are 2 groups:
  - The calls of group I require 1 bandwidth units.
    $\mathcal{N}^1 = \{(I,a),(I,b)\}$, $b_{(I,a)} = b_{(I,b)} = 1$.
  - The calls of group II require 2 bandwidth units.

    $$\mathcal{N}^2 = \{(II,a),(II,b)\},\ b_{(II,a)} = b_{(II,b)} = 2.$$

- Two groups can send traffic through both links.
  We have then 4 classes with the same source and destination :

$$\mathcal{N} = \{(I,a),(I,b),(II,a),(II,b)\},$$

where $l = I, II$ is the group and $j = a, b$ is the link.

Results

- We obtained three different Wardrop equilibria.
- In general, no potential

Parallel links with equal bandwidth requirements

The blocking probability over link i, is given by the Erlang loss formula:

$$B_i(\lambda(i)) = \frac{\lambda(i)^{R_i}/R_i!}{\sum_{j=0}^{R_i} \lambda(i)^j/j!}. \qquad \text{(Erlang B formula)}$$

Let $\Lambda = \sum_{l=1}^{L} \lambda_l$. The game has a potential:

$$G(\lambda) := \sum_{i \in \mathcal{N}} \int_0^{\lambda(i)} B_i(z)dz = -\sum_{i \in \mathcal{N}} \log g_{R_i}(\lambda(i)), \quad \text{where} \quad g_r(x) = \sum_{i=0}^{r} x^i/i! \tag{19}$$

where $\lambda = (\lambda(i), i \in \mathcal{N})$, by solving :

$$\min G(\lambda) \quad s.t. \sum_{i \in \mathcal{N}} \lambda(i) = \Lambda,\ \lambda(i) \geq 0,\ \forall i = 1, ..., N. \tag{20}$$

•Unique Wardrop equilibrium

•We can interpret the Wardrop equilibrium as the proportional fair assignment.

1.6 Convergence [A. Kumar, S. Shakkottai, E.A.]

•For a strategy $\mathbf{y}_n$ of class n, Define $\mathcal{S}_n(\mathbf{y})$ all the pure strategies in its support.

•**Definition [Sandholm]**: The dynamic

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{V}(\mathbf{y})$$

is said to be PC (Positively Correlated) if

$$\sum_{n \in \mathcal{N}} \sum_{p \in \mathcal{S}_n(\mathbf{y})} T_p^i(\mathbf{y}) V_p^i > 0 \quad \text{whenever } V(\mathbf{y}) \neq 0.$$

•If V satisfies PC then all Wardrop equilibria are stationary points.