Paradoxes in Performance Optimization of Distributed Systems

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Abstract—**Reflecting the way in which a distributed system is used, we have different levels of distribution in the decisions on performance optimization for the distributed system, ranging from the completely centralized to the completely distributed. We identify such paradoxes that adding capacity to the system may degrade the performance of all users in the optimum of some intermediately distributed decision scheme. We characterize the cases where such paradoxical performance degradation occurs. We show that in some situations the magnitude of such performance degradation can be unlimitedly large. We discuss the above paradoxes on the models of distributed computer systems, and show numerical and theoretical results.**

Keywords– Performance optimization, distributed computer system, load balancing, Braess paradox, Nash equilibrium, Wardrop equilibrium.

I. Introduction

We can consider systems that consist of a finite number of facilities and arriving threads or flows of infinitely many customers to be served by the facilities. For example, distributed computer systems have continuous arrivals of infinitely many jobs or transactions to be processed by computers; communication networks have flows of infinitely many packets or calls to pass through communication links; transportation flow networks have incoming threads of infinitely many vehicles to drive through roads, *etc.* We may have various issues on optimization, depending on the degree of distribution of the decisions. Among them, we have three typical optima corresponding to three typical decision schemes:

(A) [Completely centralized decision scheme]: All users, jobs, packets, or vehicles are unified into one group that has only one decision maker. The decision maker seeks to optimize a single performance measure such as the total cost over all users or the mean response time of the entire system over all jobs, packets, or vehicles. The corresponding solution concept is called the system optimum, overall optimum, cooperative optimum or social optimum. We call it the *overall optimum* here. This may reflect the situation where the entire system is used by a single unified organization.

(B) [Completely distributed decision scheme]: Each of infinitely many individuals, users, jobs, *etc.*, optimizes its own cost or expected response time, independently of the others. In this optimized situation each of infinitely many individuals cannot receive any further benefit by changing its own decision. It is further assumed that the decision of a single individual has a negligible impact on the performance of other individuals. The corresponding solution concept is called the individual optimum, Wardrop equilibrium, or user optimum (by some people). We call it the *individual optimum* or *Wardrop equilibrium* here.

(C) [Intermediately distributed decision scheme]: Infinitely many users, jobs, packets, or vehicles are classified into a finite number ($N(>1)$) of groups, each of which has its own decision maker and is regarded as one player, user, or class. Each decision maker optimizes non-cooperatively its own cost or the expected response time over only the jobs of the class. The decision of a single decision maker of a class has a non-negligible impact on the performance of other classes. In this optimized situation each of a finite number of users, classes, or players cannot receive any further benefit by changing its decision. The corresponding solution concept is called the class optimum, Nash non-cooperative equilibrium, or user optimum (by some other people). We call it the *class optimum* or *Nash equilibrium* here. This may reflect the situation where the system is shared by a finite number of mutually independent organizations each of which is totally unified. We may have different levels in the intermediately distributed optimization.

Note that (C) is reduced to (A) when the number of players reduces to 1 ($N = 1$) and approaches (B) when the number of players becomes infinitely many ($N \to \infty$)

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[6].

Intuitively, we can think that the total processing capacity of a system will increase when the capacity of a part of the system increases, and so we expect improvements in performance objectives accordingly in that case. The famous Braess paradox tells us that this is not always the case; *i.e.,* increased capacity of a part of the system may sometimes lead to the degradation in the benefits of all users in a Wardrop equilibrium [3], [4]. We can expect that in the Nash equilibrium, where players seek their own optimization non-cooperatively, the similar type of paradox occurs for N large enough, whenever it occurs for the Wardrop equilibrium ($N \to \infty$). Indeed, Korilis et al., [16], [17], found examples wherein the Braess-like paradox appears in a Nash equilibrium where all user classes are identical in the same topology for which the original Braess paradox (for the Wardrop equilibrium) was in fact obtained. In [10], we presented the existence of a paradox similar to Braess's that appears in the class optimum (Nash equilibrium) but does not occur in the Wardrop equilibrium in the same environment.

In the models studied in the subsequent sections, each node (or processor) may forward, through a communication means, jobs to be processed by the other nodes. In Section II, we show a number of numerical examples where such a paradox appears in the class optimum but not in the Wardrop equilibrium. We present some tendencies or properties that can be seen from those numerical results. It is surprising that in those numerical results, for the class optimum, the Braess-like paradox appears most strongly in symmetrical cases.

In Section III, we focus our investigation on the analytical study of a model with symmetrical nodes. It is intuitively clear that in the overall optimum, no forwarding of jobs should occur. In the individual optimum, no forwarding of jobs occurs, neither. In the class optimum, no forwarding of jobs occurs only for some parameter setting. For some other parameter setting, however, mutual forwarding does occur in the class optimum, and the mean response time for each class can be unlimitedly many times larger than in the case of no mutual forwarding. On the other hand, the ratio of performance degradation decreases and finally disappears as the number of classes increases. These situations look quite paradoxical and surprising, although we know the existence of the prisoners' dilemma, and although it has been already shown that Nash equilibria, even with smooth payoff functions, are generally Pareto-inefficient [5].

II. Numerical Results on Asymmetric Models

In order to give some insight into the problem, we present here a part of the numerical results we obtained for simple asymmetrical models.

A. The Model and Assumptions

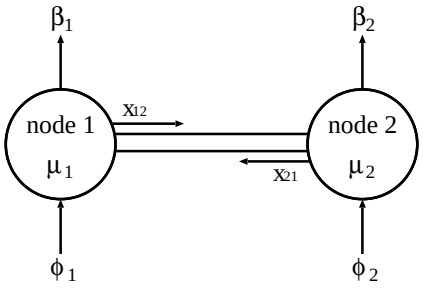

Fig. 1. The system model examined in Section II.

We examine here a model consisting of two nodes (hosts) and a communication means that connects both nodes. Nodes are numbered 1 and 2. Each node consists of a single server whose service times are independent and identically and exponentially distributed with mean μ_i^{-1} ($i = 1,2$). We classify jobs arriving at node i into class i, $i = 1,2$. Node i has external Poisson arrival with rate ϕ_i, out of which the rate x_{ii} of jobs are processed at node i. The rate x_{ij} ($i \neq j$) of jobs are forwarded through the communication means to the other node j to be processed there, and the results of those jobs are returned back through the communication means to node i. We thus have $x_{ii} + x_{ij} = \phi_i$ ($i \neq j$), $x_{ij} \geq 0$, $i,j = 1,2$. We denote the vector $(x_{11}, x_{12}, x_{21}, x_{22})$ by $\boldsymbol{x}$, the set of $\boldsymbol{x}$'s that satisfy the constraints by $\boldsymbol{C}$, and let $\Phi = \phi_1 + \phi_2$. Within these constraints, a set of values of x_{ij} ($i,j = 1,2$) are chosen to achieve optimization. The load on node i is $x_{ii} + x_{ji}$ ($i \neq j$) and is denoted by β_i. The expected processing (including queueing) time of a job that is processed at node i, is $1/(\mu_i - \beta_i)$ for $\beta_i < \mu_i$, and otherwise infinite.

We consider the following communication means that consists of two two-way communication lines 1 and 2 *. The two-way line i is used for forwarding of jobs that arrive at node i and for sending back the processed results of these jobs. We refer to the expected time length of forwarding and sending back a job as the *mean communication delay*, which is assumed to be

$$G_i(x_{ij}) = \frac{t}{1 - x_{ij}t} , \qquad (1)$$

if $x_{ij}t < 1$, and otherwise infinite. This holds, for example, in the case where each communication channel is modeled by a processor sharing server with service rate $1/t$; *i.e.,* the mean communication (without queueing) time is t, and thus, the capacity of each communication line is $1/t$.

We refer to the length of time between the instant when a job arrives at a node and the instant when it leaves the node, where it has arrived, after all processing and communication (if any) are over, as *the response time for the job* arriving at the node.

The expected response time of a job that arrives at node

*We also examined the case where one two-way communication line is used by both nodes, and obtained similar results. See [7] for more details.

i (*i.e.,* of class i) is thus

$$T_i(\boldsymbol{x}) = \frac{1}{\phi_i} \sum_p x_{ip} T_{i \to p}(\boldsymbol{x}),$$

where

$$T_{i \to i}(\boldsymbol{x}) = \frac{1}{\mu_i - \beta_i} ,$$

$$T_{i \to j}(\boldsymbol{x}) \quad = \quad \frac{1}{\mu_j - \beta_j} + \frac{t}{1 - x_{ij} t} \quad (i \neq j).$$

The above expressions hold, again, only for positive values of denominators, and are otherwise infinite.

The overall expected response time of a job that arrives at the system is then

$$T(\boldsymbol{x}) = \frac{1}{\Phi} \sum_i \phi_i T_i.$$

We examine the three concepts of optima: overall, individual, and class optima.

(A) The overall optimum is given by such $\bar{\boldsymbol{x}}$ as satisfies the following,

$$T(\bar{\boldsymbol{x}}) = \min T(\boldsymbol{x}) \quad \text{with respect to} \quad \boldsymbol{x} \in \boldsymbol{C}. \tag{2}$$

(B) The individual optimum is given by such $\hat{\boldsymbol{x}}$ as satisfies the following for all i,

$$T_i(\hat{\boldsymbol{x}}) = \min\{T_{i \to i}(\hat{\boldsymbol{x}}), T_{i \to j}(\hat{\boldsymbol{x}})\} \ (i \neq j) \text{ such that } \hat{\boldsymbol{x}} \in \boldsymbol{C}. \tag{3}$$

(C) The class optimum is given by such $\tilde{\boldsymbol{x}}$ as satisfies the following for all i,

$$\tilde{T}_i = T_i(\tilde{\boldsymbol{x}}) = \min_{x_{ii}, x_{ij}} T_i(\tilde{\boldsymbol{x}}_{-(ii,ij)}; x_{ii}, x_{ij}),$$
$$\text{such that } (\tilde{\boldsymbol{x}}_{-(ii,ij)}; x_{ii}, x_{ij}) \in \boldsymbol{C}. \tag{4}$$

where $(\tilde{\boldsymbol{x}}_{-(ii,ij)}; x_{ii}, x_{ij})$ denotes the 4-component vector in which the elements corresponding to $\tilde{x}_{ii}$ and $\tilde{x}_{ij}$ have been replaced, respectively, by x_{ii} and x_{ij}.

The existence and uniqueness of the overall, individual, and class optima for this model are established in [8], [9].

Remark 1 *Note that, there is no mutual forwarding in overall and individual optima. That is, in overall and individual optima, either one of x_{ij} $(i \neq j)$ must be zero as shown in [18] and Section 2.2.2 of [12]. Thus, when one of x_{ij} $(i \neq j)$ say x_{ij} is non-zero, $T_i(\boldsymbol{x})$ decreases and $T_j(\boldsymbol{x})$ increases with the change of t as shown in Theorems 2.5 and 2.7 of [12]. Consequently no Braess-type paradox occurs for individual and overall optima.*

Let us define three types of symmetries with respect to the system parameter setting.

[**Individual symmetry**] If the following condition holds

$$\frac{1}{\mu_1 - \phi_1} = \frac{1}{\mu_2 - \phi_2}, \tag{5}$$

it can be proved from definition (3) that at the individual optimum there is no forwarding between the two servers for any value of the communication line capacity $1/t$ (see, *e.g.,* [12]). We say that we have an *individual symmetry* property between the two servers in this case.

[**Overall symmetry**] If we have

$$\frac{\mu_1}{(\mu_1 - \phi_1)^2} = \frac{\mu_2}{(\mu_2 - \phi_2)^2} , \tag{6}$$

then there is no forwarding at the overall optimum. This can be easily seen from the Kuhn-Tucker condition (see *e.g.,* [21]) for the overall optimization problem (2). We say that we have an *overall symmetry* property between the two servers in this case.

[**Complete symmetry**] If both conditions (5) and (6) hold or equivalently if $\mu_1 = \mu_2$ and $\phi_1 = \phi_2$, then we say that we have a *complete symmetry* between the two servers. In the complete symmetry, no forwarding of jobs occurs, both in the overall and individual optima.

B. Numerical Experiments

In the following numerical results, we first show some cases where complete symmetry holds. In these experiments we change the service rates of the servers while keeping the equalities between the two service rates and between the two arrival rates.

Secondly, we introduce some asymmetry. We start from a set of data such that the complete symmetry property holds.

- First, we fix μ_2 and ϕ_2, change μ_1 and ϕ_1 in such a way that the overall symmetry, *i.e.,* condition (6), is preserved.
- Then, starting again from the complete symmetric case, we change both μ_1 and ϕ_1 in such a way that the property of individual symmetry, *i.e.,* condition (5), is preserved.
- Finally, starting from the complete symmetric case, we fix μ_2, ϕ_1 and ϕ_2, and let μ_1 change. In this case none of the property of symmetry holds any more.

For each set of data, *i.e.,* for each vector $(\mu_1, \mu_2, \phi_1, \phi_2)$ we can find some value t^∞ (depending upon the set of data) of the mean communication time (excluding queueing time as defined with respect to (1)) such that the communication line is not used any more at equilibrium if the mean communication time is larger than t^∞. For each set of data $(\mu_1, \mu_2, \phi_1, \phi_2)$, we increase the mean communication time from 0 to t^∞. For each t we compute the class optimum (Nash equilibrium). In Fig. 2, In this case, $0.8 < t^\infty < 0.9$.

Algorithm: We use the following best reply algorithm to compute these Nash equilibria.

For some fixed parameters $\phi_1, \phi_2, \mu_1, \mu_2, t$,

– Initialize $\boldsymbol{x}^0 = (x_{11}^0, x_{12}^0, x_{21}^0, x_{22}^0) \in \boldsymbol{C}$.

– Define $\boldsymbol{x}^n$ as

$$\begin{cases} (x_{11}^n, x_{12}^n) = \arg \min_{(x_{11}, x_{12})} T_1(x_{11}, x_{12}, x_{21}^{n-1}, x_{22}^{n-1}) , \\ (x_{21}^n, x_{22}^n) = \arg \min_{(x_{21}, x_{22})} T_2(x_{11}^n, x_{12}^n, x_{21}, x_{22}). \end{cases}$$

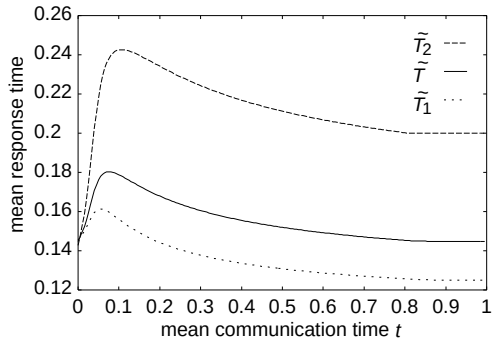

Fig. 2. Mean response times at class optimum, for $(\mu_1, \mu_2, \phi_1, \phi_2) = (64, 25, 56, 20)$: Overall symmetry.

When this algorithm converges, it gives the optimal solution.

We focus our attention on the degradation, *i.e.,* the Braess-like paradox, that may occur when adding the communication line. To this aim we say that a Braess-like paradox occurs if the following holds:

$$\begin{array}{l} D_1(t_1,t_2) > 0 \quad \text{and} \quad D_2(t_1,t_2) > 0 \\ \text{for some} \;\; t_1, t_2 \quad \text{such that } 0 < t_1 < t_2, \end{array} \tag{7}$$

where $D_i(t_1,t_2) = \dfrac{\tilde{T}_i(t_1) - \tilde{T}_i(t_2)}{\tilde{T}_i(t_2)}$, and $\tilde{T}_i(t)$ denotes the mean response time for class i jobs, computed at the (unique) Nash equilibrium, for a fixed value of the parameter t.

For simplicity, we only consider the case where $t_2 = t_\infty$, *i.e.,* the system has no communication means, and we denote $D_i(t, t^\infty)$ by $\Delta_i(t)$.

We define $\Omega(\mu_1, \mu_2, \phi_1, \phi_2)$, the *worst ratio of performance degradation* in the paradox, as:

$$\Omega(\mu_1, \mu_2, \phi_1, \phi_2) = \max_t \min\{\Delta_1(t), \Delta_2(t)\}. \tag{8}$$

Denote the argument of the previous maximum by t_0.

Notice that we can define a *weaker paradox.* We can think of a kind of local paradox, such that we can observe for some interval of t, the overall mean response time $\tilde{T}(t) = (1/\Phi)(\phi_1\tilde{T}_1(t) + \phi_2\tilde{T}_2(t))$ decreases in t although $\tilde{T}_i(t)$, for some i, may increase. Clearly, we can see a weaker paradox on the overall mean response time also in Fig 2. This weaker paradox may occur even if (7) does not hold, and has been already discussed in [11]. Weaker paradoxes appear also for individual optimization (see [12]).

C. Results and Discussion

We show typical numerical examples, with changing parameter values from $\mu_1 = \mu_2 = 25$ and $\phi_1 = \phi_2 = 20$ for each of four directions: complete, overall, individual, and no symmetry maintained.

Complete symmetry maintained: In the following table, we show the effect of changing $\mu_1 = \mu_2$ from 25 (keeping $\phi_1 = \phi_2 = 20$ fixed) with complete symmetry maintained. "No P." means the situation where no paradox is seen for any interval of t.

μ	20.1	22	25	45	65	100	200
$\Delta_i(t_0)$ (%)	35	30	25	9	5	3	No P.
Ω(%)	35	30	25	9	5	3	No P.

Overall symmetry maintained: In the following table, we show the effect of changing μ_1 and ϕ_1 from 25 and 20, respectively (keeping $\mu_2 = 25$ and $\phi_2 = 20$ fixed) with overall symmetry maintained, *i.e.,* such that $\dfrac{\mu_1}{(\mu_1 - \phi_1)^2} = \dfrac{\mu_2}{(\mu_2 - \phi_2)^2} = 1$.

μ_1	4	9	25	64	121	400	40000
ϕ_1	2	6	20	56	110	380	39800
$\Delta_1(t_0)$ (%)	3	14	25	24	19	10	0.1
$\Delta_2(t_0)$ (%)	10	19	25	21	19	15	10
Ω (%)	3	14	25	21	19	10	0.1

Individual symmetry maintained: In the following table, we show the effect of changing μ_1 and ϕ_1 from 25 and 20, respectively (keeping $\mu_2 = 25$ and $\phi_2 = 20$ fixed) with individual symmetry maintained, *i.e.,* such that $\dfrac{1}{\mu_1 - \phi_1} = \dfrac{1}{\mu_2 - \phi_2} = \dfrac{1}{5}$.

μ_1	6	15	25	65	105	125	505
ϕ_1	1	10	25	60	100	120	500
$\Delta_1(t_0)$(%)	41	32	25	9	2	1	No P.
$\Delta_2(t_0)$(%)	-8	12	25	50	62	66	
Ω(%)	-8	12	25	9	2	1	No P.

No symmetry maintained: In the following table, we show the effect of changing μ_1 from 25 (keeping $\mu_2 = 25$ and $\phi_1 = \phi_2 = 20$ fixed), *i.e.,* with no symmetry maintained.

μ_1	20.5	23	25	27	31	33
ϕ_1	20	20	20	20	20	20
Δ_1 (t_0)(%)	No P.	2	25	36	32	No P.
Δ_2 (t_0)(%)		42	25	8	-14	
Ω (%)	No P.	2	25	8	-14	No P.

Overall tendency In these examples, we see that the worst ratio of performance degradation in the paradox is largest in the complete symmetry case with the arrival rate being closest to the service rate, *i.e.,* the case where $(\mu_1, \mu_2, \phi_1, \phi_2) = (20.1, 20.1, 20, 20)$. The degradation decreases rapidly as the parameter setting departs the above-mentioned symmetric case without keeping any kind of symmetries. It decreases slowly with individual symmetry maintained where no forwarding of jobs occurs in the individual optimum. It decreases more slowly with overall symmetry maintained where no forwarding of jobs occurs in the overall optimum. It looks quite strange that in these symmetrical cases, the chances of Braess-like paradox are greater than non-symmetrical situations and that *the paradox appears most strongly in complete symmetry.*

III. Analytical Results on Symmetric Models

Motivated by the insight gained through the numerical studies given in Section II, we now present analytical results on more general symmetrical models.

A. The Model and Assumptions

We consider a system with m nodes (host computers or processors) connected with a communication means. Jobs that arrive at each node i, $i = 1, 2, \cdots, m$, are classified into n types k, $k = 1, 2, \cdots, n$. Consequently, we have mn different job classes R_{ik}. Each class R_{ik} is distinguished by the node i at which its jobs arrive and by the type k of the jobs. We call such a class *local class*, or simply *class*.

We assume that each node has identical arrival and processing capacity. That is, the system has multiple identical nodes. Jobs of type k arrive at each node with node-independent rate ϕ_k. We denote the total arrival rate to the node by ϕ $(= \sum_k \phi_k)$, and we have the time scale whereby $\phi = 1$.

We also consider what we call *global class* J_k that consists in the collection of local class R_{ik}, *i.e.*, $J_k = \bigcup_i R_{ik}$. J_k thus consists of all jobs of type k. Whereas, for local class R_{ik}, all the jobs arrive at the same node i, the arrivals of the jobs of global class J_k are equally distributed over all nodes i.

The average processing (service) time (without queueing delays) of a type k job at any node is $1/\mu_k$ and is, in particular, node-independent. We denote ϕ_k/μ_k by ρ_k and $\rho = \sum_k \rho_k$.

Out of type k jobs arriving at node i, the ratio x_{ijk} of jobs is forwarded upon arrival through the communication means to another node j $(\neq i)$ to be processed there. The remaining ratio $x_{iik} = 1 - \sum_{j(\neq i)} x_{ijk}$ is processed at node i. Thus $\sum_j x_{ijk} = 1$. That is, the rate $\phi_k x_{ijk}$ of type k jobs that arrive at node i is forwarded through the communication means to node j, while the rate $\phi_k x_{iik}$ of local-class R_{ik} jobs is processed at the arrival node i. We have $0 \leq x_{ijk} \leq 1$, for all i, j, k. Within these constraints, a set of values of $\boldsymbol{x}_{ik}$,

$$\boldsymbol{x}_{ik} = (x_{i1k}, \cdots, x_{imk}), i = 1, 2, \cdots, m, k = 1, 2, \cdots, n,$$

are chosen to achieve optimization. The m-dimensional vector $\boldsymbol{x}_{ik}$ is called the 'local-class R_{ik} strategy.'

We define a global-class J_k strategy as the mm-dimensional vector

$$\boldsymbol{x}_k = (\boldsymbol{x}_{1k}, \boldsymbol{x}_{2k}, \cdots, \boldsymbol{x}_{mk}).$$

We will also denote, by an mmn-dimensional vector $\boldsymbol{x}$, the vector of strategies concerning all job classes,

$$\begin{aligned} \boldsymbol{x} &= (\boldsymbol{x}_{11}, \boldsymbol{x}_{12}, \cdots, \boldsymbol{x}_{1n}, \boldsymbol{x}_{21}, \cdots, \boldsymbol{x}_{2n}, \cdots, \boldsymbol{x}_{m1}, \cdots, \boldsymbol{x}_{mn}), \\ &\text{or } \boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n). \end{aligned}$$

We call $\boldsymbol{x}$ the *strategy profile*.

For a strategy profile $\boldsymbol{x}$, the load β_i on node i is

$$\beta_i = \beta_i(\boldsymbol{x}) = \sum_{j,k} \rho_k x_{jik}. \tag{9}$$

The contribution $\beta_i^{(k)}$ on the load of node i by the global class k jobs is

$$\beta_i^{(k)} = \beta_i^{(k)}(\boldsymbol{x}) = \sum_j \rho_k x_{jik}, \tag{10}$$

and clearly $\beta_i = \beta_i^{(1)} + \beta_i^{(2)} + \cdots + \beta_i^{(n)}$.

Again we denote by $\boldsymbol{C}$ the set of $\boldsymbol{x}$'s that satisfy the constraints (*i.e.*, $\sum_l x_{ilk} = 1, x_{ijk} \geq 0$, for all i, j, k). Note that $\boldsymbol{C}$ is a compact set.

We have the following assumptions:

Assumption Π1 *We assume that the expected processing (including queueing) time of a type k job that is processed at node i (or the cost function at node i), is a strictly increasing, strictly convex and continuously differentiable function of β_i, denoted by $\mu_k^{-1} D(\beta_i)$ for all i, k.*

Assumption Π2 *We assume that the mean communication delay (including queueing delay) or the cost for forwarding type k jobs arriving at node i to node j $(i \neq j)$, denoted by $G_{ijk}(\boldsymbol{x})$, is a positive, non-decreasing, convex and continuously differentiable function of $\boldsymbol{x}$. We assume that $G_{iik}(\boldsymbol{x}) = 0$.*

Example 1 We may consider the following simple functions for the mean processing time and the mean communication delay. For the mean processing time:

$$1/\mu_k D(\beta_i) = \frac{1/\mu_k}{1 - \beta_i} \text{ for } \beta_i < 1, \text{ otherwise it is infinite.} \tag{11}$$

For the mean communication delay:

$$G_{ijk}(\boldsymbol{x}) = t. \tag{12}$$

Equation (11) holds, *e.g.*, if we have a simple assumption of the external time-invariant Poisson arrival for each local class, and the mean service time (without queueing delays) for each type k jobs is μ_k^{-1} at each node i. The service discipline is processor sharing or preemptive-resume last-come first-served. When $\mu_k = \mu$ for all k and when no forwarding of jobs occurs, the mean processing time is, simply, $1/(\mu - 1)$.

Equation (12) holds, *e.g.*, in the case where one communication line is provided separately for sending jobs from one node to another. The line (ij) is used for forwarding a job that arrives at node i to node j $(\neq i)$. The expected communication time of a job arriving at node i and being processed at node j $(\neq i)$ is expressed simply as t, *i.e.*, independent of the traffic and of the job class, with no queueing delay. □

We refer to the length of time between the instant when a job arrives at a node and the instant when it leaves one of

the nodes after all processing and communication, if any, are over as *the response time* for the job. The expected response time of a local-class R_{ik} job that arrives at node i, $T_{ik}(\boldsymbol{x})$, is expressed as,

$$T_{ik}(\boldsymbol{x}) = \sum_j x_{ijk} T_{ijk}(\boldsymbol{x}), \tag{13}$$

where

$$T_{iik}(\boldsymbol{x}) = \mu_k^{-1} D(\beta_i(\boldsymbol{x})), \quad \text{and} \tag{14}$$

$$T_{ijk}(\boldsymbol{x}) = \mu_k^{-1} D(\beta_j(\boldsymbol{x})) + G_{ijk}(\boldsymbol{x}), \text{ for } j \neq i. \tag{15}$$

The expected response time of a global-class J_k jobs is

$$T_k(\boldsymbol{x}) = \frac{1}{m} \sum_i T_{ik}(\boldsymbol{x}). \tag{16}$$

The overall expected response time of a job that arrives at the system is

$$T(\boldsymbol{x}) \quad = \quad \sum_k \phi_k T_k(\boldsymbol{x}) = \frac{1}{m} \sum_{i,k} \phi_k T_{ik}(\boldsymbol{x}). \tag{17}$$

Again we consider different decision schemes.

(A) For the completely centralized decision scheme, the strategy of the single decision maker consists in the choice of an optimal mmn-dimensional vector $\bar{\boldsymbol{x}}$ with components $\bar{x}_{ijk}$.

(B) For the completely distributed decision scheme, we denote by $\hat{x}_{ijk}$ the resulting optimal ratio of local-class R_{ik} jobs that choose the node j to be processed. We denote the individually optimal strategy profile, *i.e.,* the vector of components $\hat{x}_{ijk}$, by $\hat{\boldsymbol{x}}$.

(C) For intermediately distributed decision schemes, we consider two slightly different schemes, C-I and C-II.

(C-I) In the first scheme, each local class R_{ik} has its own decision maker (ik). The amount of forwarding local-class R_{ik} jobs is chosen by the corresponding decision maker (ik). The optimal strategy for decision maker (ik), or equivalently local class R_{ik}, is denoted by the m-dimensional vector

$$\tilde{\boldsymbol{x}}_{ik} = (\tilde{x}_{i1k}, \tilde{x}_{i2k}, \cdots, \tilde{x}_{imk}),$$

and an optimal strategy profile, that we will denote by $\tilde{\boldsymbol{x}}$, is the collection of strategies $\tilde{\boldsymbol{x}}_{ik}$. We call this scheme the *intermediately distributed decision scheme I*, and this optimized situation the *local-class optimum.*

(C-II) In the second scheme, a decision maker is associated with each global class J_k, and chooses the amount of local-class R_{ik} jobs to be forwarded, for $i = 1, 2, \cdots, m$. The optimal strategy for the decision maker k is consequently an mm-dimensional vector

$$\check{\boldsymbol{x}}_k = (\check{\boldsymbol{x}}_{1k}, \check{\boldsymbol{x}}_{2k}, \cdots, \check{\boldsymbol{x}}_{mk}).$$

An optimal strategy profile is an mmn-dimensional vector $\check{\boldsymbol{x}} = (\check{\boldsymbol{x}}_1, \check{\boldsymbol{x}}_2, \ldots, \check{\boldsymbol{x}}_n)$. We call this scheme the *intermediately distributed decision scheme II*, and this optimized situation the *global-class optimum.*

B. Results

(A) [Completely centralized decision scheme: Overall optimization] The overall optimum is given by such $\bar{\boldsymbol{x}}$ as satisfies the following (a generalization of (2)),

$$T(\bar{\boldsymbol{x}}) = \min T(\boldsymbol{x}) \quad \text{with respect to} \quad \boldsymbol{x} \in \mathbb{C}. \tag{18}$$

We define $\boldsymbol{x}_{-(iik)}$ to be an $m(m-1)n$-dimensional vector such that all elements x_{iik}, for all i, k, are excluded from the mmn-dimensional vector $\boldsymbol{x}$ whereas all its elements are the same as the remaining $m(m-1)n$ elements of $\boldsymbol{x}$.

Solution: *The solution $\bar{\boldsymbol{x}}$ is unique and given as follows:*

$$\bar{\boldsymbol{x}}_{-(iik)} = \boldsymbol{0}, \quad \textit{i.e., } \bar{x}_{ijk} = 0, \textit{ and } \bar{x}_{iik} = 1,$$

for all $i, j(\neq i), k$. The mean response time is

$$T_k(\bar{\boldsymbol{x}}) = T_{ik}(\bar{\boldsymbol{x}}) = \mu_k^{-1} D(\rho), \textit{ for all } i,k, \quad T(\bar{\boldsymbol{x}}) = \rho D(\rho).$$

Proof: This solution and its uniqueness are clear from the strict convexity assumption on D and the fact that G_{ijk} is positive and non-decreasing. □

(B) [Completely distributed decision scheme: Individual optimization] The individual optimum (*i.e.,* Wardrop equilibrium) is given by such $\hat{\boldsymbol{x}}$ as satisfies the following for all i, k (a generalization of (3)),

$$T_{ik}(\hat{\boldsymbol{x}}) = \min_j \{T_{ijk}(\hat{\boldsymbol{x}})\} \quad \text{and} \quad \hat{\boldsymbol{x}} \in \mathbb{C}. \tag{19}$$

Solution: *The solution $\hat{\boldsymbol{x}}$ is unique and given as follows:*

$$\hat{\boldsymbol{x}}_{-(iik)} = \boldsymbol{0}, \quad \textit{i.e., } \hat{x}_{ijk} = 0, \textit{ and } \hat{x}_{iik} = 1,$$

for all $i, j(\neq i), k$. The mean response time is

$$T_k(\hat{\boldsymbol{x}}) = T_{ik}(\hat{\boldsymbol{x}}) = \mu_k^{-1} D(\rho), \textit{ for all } i,k, \quad T(\bar{\boldsymbol{x}}) = \rho D(\rho).$$

Proof: See [13]. □

We suppose in the following (C-I) scheme and in the subsequent (C-II) scheme that the following assumption holds true:

[Assumption Π3] *We assume that $G_{ijk}(\boldsymbol{x})$ is one of the following functions, where ω_k are constants, $\sigma_k = \phi_k/\omega_k$, and $\underline{G}(x)$ is a non-decreasing, convex, and differentiable function of x such that $\underline{G}(0) = 1$.*

Type G-I: $\quad G_{ijk}(\boldsymbol{x}) = \omega_k^{-1} \underline{G}(\sigma_k x_{ijk})$

(one dedicated line for each combination of a pair of origin and destination nodes, and a class: i.e., $m(m-1)n$ lines in total),

Type G-II(a): $\quad G_{ijk}(\boldsymbol{x}) = \omega_k^{-1} \underline{G}(\sum_{p,q \neq p} \sigma_k x_{pqk})$

(one bus line for each global class:

i.e., n bus lines in total),

Type G-II(b) : $G_{ijk}(\boldsymbol{x}) = \omega_k^{-1}\underline{G}(\sum_{p,q(\neq p),k}\sigma_k x_{pqk})$

(*one common bus line for the entire system: i.e., 1 bus line*).

Remark 2 ω_k^{-1} *can be regarded as the mean communication time (without queueing delays) for forwarding a type k job from the arrival node to another processing node.* $\sigma_k x_{ijk}(j \neq i)$ *is the traffic intensity of the communication line for the local-class* R_{ik} *jobs being forwarded to node j.*

Example 2 We use the same definition (11) for the mean processing time as in Example 1 . We define $G_{ijk}(\boldsymbol{x})$ for the mean communication delay as follows. We assume $\omega_k = \theta$ for all k and thus $\sigma_k = \phi_k/\theta$, and set

$$G_{ijk}(\boldsymbol{x}) = \frac{1/\theta}{1 - \sum_{p,q(\neq p),k}\sigma_k x_{pqk}}$$

$$\text{for} \sum_{p,q(\neq p),k}\sigma_k x_{pqk} < 1, \text{ and otherwise infinite.}$$

This is identical to:

$$G_{ijk}(\boldsymbol{x}) = \frac{1}{\theta - \sum_{p,q(\neq p),k}\phi_k x_{pqk}}$$

$$\text{for} \sum_{p,q(\neq p),k}\phi_k x_{pqk} < \theta, \text{ and otherwise infinite.}$$

This holds, *e.g.*, in the case where one bus-type communication line is provided commonly for all the nodes to be used for forwarding jobs to other nodes in the same way as in Example 1, whereas the transmission time without queueing delay is exponentially distributed with mean θ^{-1} and the scheduling discipline is First-Come-First-Served. Thus, the expected communication time of a job arriving at node i and being processed at node j $(\neq i)$ is expressed as $1/(\theta - \sum_{p,q(\neq p),k}\phi_k x_{pqk})$, *i.e.,* independent of the job class and the origin and destination nodes. □

(C-I)[Intermediately distributed decision scheme I: Local-class optimization] The local-class optimum (or a Nash equilibrium) is given by such $\tilde{\boldsymbol{x}}$ as satisfies the following for all i, k (a generalization of (4)),

$$T_{ik}(\tilde{\boldsymbol{x}}) = \min T_{ik}(\tilde{\boldsymbol{x}}_{-(ik)};\boldsymbol{x}_{ik}) \text{ with respect to } \boldsymbol{x}_{ik}$$
$$\text{such that } (\tilde{\boldsymbol{x}}_{-(ik)};\boldsymbol{x}_{ik}) \in \boldsymbol{C},$$

where $(\tilde{\boldsymbol{x}}_{-(ik)};\boldsymbol{x}_{ik})$ denotes the mmn-dimensional vector in which the elements corresponding to $\tilde{\boldsymbol{x}}_{ik}$ have been replaced by $\boldsymbol{x}_{ik}$.

Let us define $\tilde{g}_{ijk}(\cdot)$ as

$$\tilde{g}_{ijk}(\boldsymbol{x}) = \frac{\partial}{\partial x_{ijk}}\{\phi_k \sum_{p\neq i} x_{ipk}G_{ipk}(\boldsymbol{x})\}. \tag{20}$$

If Assumption II3 holds, for $\boldsymbol{x}$ such that $x_{ijk} = x_k$, for all $i, j(\neq i), k$ we denote

$$G_k(\boldsymbol{x}) = G_{ijk}(\boldsymbol{x}) \text{ and } \tilde{g}_k(\boldsymbol{x}) = \tilde{g}_{ijk}(\boldsymbol{x}).$$

Solution: *We denote* $\Gamma_k = \rho_k^2\sigma_k^{-1}$. *The solution* $\tilde{\boldsymbol{x}}$ *is unique and is given as follows:*

For Types G-I and G-II(a)
(*i*) *For local class* R_{ik} *such that* $\rho_k^2 D'(\rho) \leq \tilde{g}_k(0) = \sigma_k$, *i.e.,* $\Gamma_k D'(\rho) \leq 1$,

$$\tilde{x}_{ijk} = 0, \text{ and } \tilde{x}_{iik} = 1, \text{ for all } i, j(\neq i).$$

The mean response time is

$$T_k(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1}D(\rho), \text{ for all } i,k, \quad T(\tilde{\boldsymbol{x}}) = \rho D(\rho).$$

(*ii*) *For local class* R_{ik} *such that* $\rho_k^2 D'(\rho) > \tilde{g}_k(0) = \sigma_k$, *i.e.,* $\Gamma_k D'(\rho) > 1$, *the solution is given as follows:*

$$\tilde{x}_{ijk} = \tilde{x}_k, \text{ for all } i,\ j(\neq i), \tag{21}$$

where $\tilde{x}_k$ *is the unique solution of*

$$\begin{aligned}&\rho_k^2(1 - m\tilde{x}_k)D'(\rho) = \tilde{g}_k(\tilde{x}_k)\\ &= \sigma_k[\underline{G}(m(m-1)\sigma_k\tilde{x}_k)\\ &+\sigma_k(m-1)\tilde{x}_k\underline{G}'(m(m-1)\sigma_k\tilde{x}_k)].\end{aligned} \tag{22}$$

The mean response time is

$$\begin{aligned}T_k(\tilde{\boldsymbol{x}}) &= T_{ik}(\tilde{\boldsymbol{x}})\\ &= \mu_k^{-1}D(\rho) + (m-1)\tilde{x}_k G_k(\tilde{\boldsymbol{x}}),\\ &\quad \text{for all } i.\end{aligned} \tag{23}$$

For Type G-II(b)

The solution is given as follows. We first change the numbering of k such that

$$\Gamma_1 \geq \Gamma_2 \geq \cdots \geq \Gamma_k \geq \cdots \geq \Gamma_n. \tag{24}$$

The following three situations can occur: We can find K such that

$$\Gamma_K D'(\rho) > 1 \quad \text{and} \quad \Gamma_{K+1}D'(\rho) \leq 1, \tag{25}$$
$$\text{or } \Gamma_n D'(\rho) > 1 \ (\text{i.e., } K = n), \tag{26}$$
$$\text{or } \Gamma_1 D'(\rho) \leq 1 \ (\text{i.e., } K = 0). \tag{27}$$

When (27) holds, we have a unique solution of $\tilde{x}_k = 0$ *for all k. When (25) or (26) holds, we can find a unique solution as follows. Let us define the function* $F_k(X)$ *as*

$$\begin{aligned}F_k(X) &= \{\sum_{l=1}^{k}\frac{\sigma_l[\Gamma_l D'(\rho) - \underline{G}(X)]}{m\Gamma_k D'(\rho) + (m-1)\sigma_l\underline{G}'(X)}\}\\ &\quad -\frac{X}{m(m-1)}.\end{aligned} \tag{28}$$

We obtain the largest $k = \tilde{k} \leq K$ *and* $X = \tilde{X}_{\tilde{k}} (> 0)$ *that satisfies* $F_{\tilde{k}}(\tilde{X}_{\tilde{k}}) = 0$ *and* $\sigma_{\tilde{k}}[\Gamma_{\tilde{k}} D'(\rho) - \underline{G}(\tilde{X}_{\tilde{k}})] > 0$. *Then by using*

$$\begin{aligned} &\sigma_k[\Gamma_k D'(\rho) - \underline{G}(\tilde{X}_{\tilde{k}})] \\ &= \sigma_k \tilde{x}_k [m\Gamma_k D'(\rho) + (m-1)\sigma_k \underline{G}'(\tilde{X}_{\tilde{k}})], \end{aligned} \quad (29)$$

for $k = 1, 2, \cdots, \tilde{k}$, *we can obtain the unique set of values, such that* $\tilde{x}_k > 0, k = 1, 2, \cdots, \tilde{k}$, *and* $\tilde{x}_{\tilde{k}+1} = \tilde{x}_{\tilde{k}+2} = \cdots = \tilde{x}_n = 0$ *that satisfies the above relation, which is a unique solution. The mean response time is*

$$T_k(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1} D(\rho) + (m-1)\tilde{x}_k G_k(\tilde{\boldsymbol{x}}), \quad \text{for all } i. \quad (30)$$

Proof: See [13]. □

(C-II)[Intermediately distributed decision scheme: Global-class optimization] The global-class optimum (Nash equilibrium for another set of decision makers) is given by such $\check{\boldsymbol{x}}$ as satisfies the following for all i, k,

$$T_k(\check{\boldsymbol{x}}) = \min T_k(\check{\boldsymbol{x}}_{-(k)}; \boldsymbol{x}_k), \text{ with respect to } \boldsymbol{x}_k \text{ such that } (\check{\boldsymbol{x}}_{-(k)}; \boldsymbol{x}_k) \in \boldsymbol{C}. \quad (31)$$

where $(\check{\boldsymbol{x}}_{-(k)}; \boldsymbol{x}_k)$ denotes the mmn-dimensional vector in which the elements corresponding to the coordinates of $\check{\boldsymbol{x}}_k$ have been replaced by the vector $\boldsymbol{x}_k$. We note that

$$\phi_k m T_k(\boldsymbol{x}) = \sum_i \beta_i^{(k)}(\boldsymbol{x}) D(\beta_i(\boldsymbol{x})) + \sum_{i,j \neq i} \phi_k x_{ijk} G_{ijk}(\boldsymbol{x}). \quad (32)$$

Note that we have the assumption Π3 on the function G_{ijk}.

Solution:

$$\check{\boldsymbol{x}}_{-(iik)} = \boldsymbol{0}, \quad \textit{i.e.,}\ \check{x}_{ijk} = 0, \ \textit{and}\ \check{x}_{iik} = 1,$$

for all $i, j(\neq i), k$. *The mean response time is*

$$T_k(\check{\boldsymbol{x}}) = T_{ik}(\check{\boldsymbol{x}}) = \mu_k^{-1} D(\rho), \ \textit{for all } i, k, \quad T(\check{\boldsymbol{x}}) = \rho D(\rho).$$

Proof: See [13]. □

Now we have the following property.

Theorem 1 *Consider a symmetric system that satisfies all the assumptions given in this paper and that employs one optimization decision scheme out of* (A), (B), $(C\text{-}I)$, *and* $(C\text{-}II)$. *The necessary and suffucient condition that a Braess-like paradox occurs is that there exists at least a job type* k *for which* $\Gamma_k D'(\rho) > 1$ *holds. In that case, the paradox occurs only for the job type under the decision scheme* $(C\text{-}I)$.

Remark 3 *We therefore see that the chances of Braess-like paradoxical performance degradation vary from job type to job type on the basis of the values of* Γ_k $(= \rho_k^2/\sigma_k = \phi_k \omega_k / \mu_k^2)$. *That is, the performance for the job types that have larger arrival rate* (ϕ_k), *larger processing time requirement* (μ_k^{-1}), *and smaller communication time requirment* (ω_k^{-1}), *has more chances to be degraded under local-class optimization* $(C\text{-}I)$.

The performance for all job types has more chances to be degraded with a larger value of ρ $(= \sum_k \rho_k)$ *under local-class optimization* $(C\text{-}I)$. *No paradox occurs under other decision schemes.*

IV. Examples

We consider here Example 1 introduced in Subsection III.A. We restrict ourselves to local-class optimization (C-I) where $\mu_k = \mu$, $\phi_k = 1/n$ for all k. We have $\rho = 1/\mu$.

Example 1 In this case we have $\omega_k^{-1} = t$ and $D(\rho) = 1/(1-\rho)$. We therefore note that

$$\Gamma_k D'(\rho) - 1 = \frac{\phi_k \omega_k}{\mu_k^2} D'(\rho) - 1 = \frac{1}{nt(\mu-1)^2} - 1.$$

(i) If $t > 1/\{n(\mu-1)^2\}$, then the solution $\tilde{\boldsymbol{x}}$ is unique and given by

$$\tilde{\boldsymbol{x}}_{-(iik)} = \boldsymbol{0}, \ \textit{i.e.,}\ \tilde{x}_{ijk} = 0, \quad \tilde{x}_{iik} = 1,$$

for all $i, j(\neq i), k$. The mean response time is

$$T(\tilde{\boldsymbol{x}}) = T_{ik}(\tilde{\boldsymbol{x}}) = \mu_k^{-1} D(\rho) = \frac{1}{\mu - 1}, \quad \text{for all } i, k.$$

In particular it is the same for overall, individual, and global-class optima.

(ii) If $t \leq 1/\{n(\mu-1)^2\}$, the solution $\tilde{\boldsymbol{x}}$ is given by

$$\tilde{x}_{ijk} = \frac{1}{m}\{1 - nt(\mu-1)^2\}, \quad (33)$$

$$\tilde{x}_{iik} = \frac{1}{m}\{1 + (m-1)nt(\mu-1)^2\}, \quad (34)$$

for all $i, j(\neq i), k$. The mean response time is, for all i, k,

$$\begin{aligned} T(\tilde{\boldsymbol{x}}) &= T_{ik}(\tilde{\boldsymbol{x}}) \\ &= \frac{1}{\mu - 1} + \frac{m-1}{m} t\{1 - nt(\mu-1)^2\}. \end{aligned} \quad (35)$$

For some parameters (μ, m, n), $\tilde{T} = T(\tilde{\boldsymbol{x}})$ attains its maximum in t (*i.e.,* the worst performance), that we denote $\tilde{T}_{\max}(\mu, m, n)$ for

$$t_{\max} = \frac{1}{2n(\mu-1)^2}. \quad (36)$$

We have

$$\tilde{T}_{\max}(\mu, m, n) = \frac{1}{\mu - 1}\{1 + \frac{m-1}{4mn(\mu-1)}\}. \quad (37)$$

Thus, if we add the communication lines with delay $t_{\max} = 1/\{2n(\mu-1)^2\}$ to the system that has had no

communication means, the mean response time $T_{ik}(\tilde{\boldsymbol{x}})$ for each local class increases in the amount of $\dfrac{m-1}{4mn(\mu-1)^2}$ (*i.e.,* the performance degrades). This is a Braess-like paradox. We define the *worst ratio of the performance degradation* $\Delta(\mu,m,n)$ in the paradox for given μ and n, as

$$\Delta(\mu,m,n)=\frac{\tilde{T}_{\max}(\mu,m,n)-T_0(\mu)}{T_0(\mu)}, \tag{38}$$

where $T_0(\mu)=1/(\mu-1)$ denotes the mean response time of each local-class jobs for given μ when the system has no communication means. We have

$$\Delta(\mu,m,n)=\frac{m-1}{4mn(\mu-1)}. \tag{39}$$

Remark 4 *From the above, we see that no forwarding of jobs occurs in the overall, individual, and global-class optima and in case (i) of the local-class optimum. That is, in those optima, jobs that arrive at each node are processed only at the node, and thereby the system has no performance improvement or degradation due to adding the communication means (which is not used).*

On the other hand, in case (ii) of the local-class optimum, each local class forwards a part of its jobs through the communication means to other nodes for remote processing, and thereby has degradation in its mean response time. The ratio of such degradation can become unlimitedly large *as the total arrival rate approaches the processing capacity of each node, i.e., as* $\mu\simeq 1$. *In Example 1 of the model, we see that, as n and thus the number of local classes* $(m\times n)$ *increase up to infinity with the number of nodes, m, fixed, the ratio of degradation,* $\Delta(\mu,m,n)$, *and the chances of the paradox decrease and finally disappear.*

The effect of m is not so large in the example, wherein, if we increase m from 2 unlimitedly, the worst ratio of performance degradation increases only up to twice.

Numerical Examination of Example 1 We examine Example 1 with $m=5$, *i.e.,* the system with five nodes, and consider the case: $\mu=1.01$. The mean response time is $T_0(\mu)=1/(\mu-1)=100$ in the overall optimum, in the individual optimum (Wardrop equilibrium), and in the case of no communication line and no forwarding of jobs.

Firstly, we consider the case where $n=1$, *i.e.,* the total number of local classes R_{ik} is 5. $T=T_{ik}$ takes its maximum value

$$\tilde{T}(1.01,5,1)=2100 \quad \text{(see (37))},$$

and the worst ratio of the performance degradation $\Delta(\mu,m,n)$ in the paradox is

$$\Delta(1.01,5,1)=20 \ (i.e.,2000\% \text{ degradation}) \text{ (see (38))},$$

when $t=1/\{2(\mu-1)^2\}=5000$ (see (36)). Then

$$\tilde{x}_{ijk}=(1/5)\{1-t(\mu-1)^2\}=1/10 \ (i\neq j) \quad \text{(see (34))}.$$

In this case, $\tilde{x}_{ijk}$ $(=\tilde{x}_k, i\neq j)$ decrease from 1/5 down to 0 as t increases from 0 to 10000 $(=1/(\mu-1)^2)$, and for $t>10000$, no forwarding of jobs occurs.

It is amazing that each local class keeps to forward a part of its jobs equally to the other nodes even though the communication delay for forwarding is much greater than the processing delay at the node at which its jobs arrive.

Then we consider the case where $n=100$, *i.e.,* the total number of local classes is 500.

$T=T_{ik}$ takes its maximum value

$$\tilde{T}(1.01,5,100)=120 \text{ (see (37))},$$

and the worst ratio of the performance degradation $\Delta(\mu,m,n)$ in the paradox is

$$\Delta(1.01,5,100)=0.2 \ (i.e., 20\% \text{ degradation}) \text{ (see (38))},$$

when $t=1/\{2n(\mu-1)^2\}=50$ (see (36)). Then

$$\tilde{x}_{ijk}=(1/5)\{1/n-t(\mu-1)^2\}=1/1000 \ (i\neq j) \text{ (see (34))}.$$

In this case, $\tilde{x}_{ijk}$ $(i\neq j)$ decrease from 1/500 down to 0 as t increases from 0 to 100 $(=1/(\mu-1)^2)$, and for $t>100$, no forwarding of jobs occurs.

Thus we see that the chances of paradoxes and the magnitude in the degradation of the performance in the paradox are greatly reduced from the case $n=1$.

Furthermore we consider other values of μ with $n=1$. For $\mu=1.001$,

$$\Delta(1.001,5,1)=200 \ (i.e., 20000\% \text{ degradation}),$$

and for $\mu=1.00001$,

$$\Delta(1.00001,2,1)=20000 \ (i.e., 2000000\% \text{ degradation}),$$

etc. In this way, we see that the worst ratio of the performance degradation $\Delta(\mu,m,n)$ in the paradox becomes unlimitedly large as μ approaches 1 with $n=1$.

V. Concluding remarks

We have presented, firstly, a number of numerical examples for models of asymmetric systems for which Braess-like paradox occurs, *i.e.,* for which adding capacity to a system leads to the degradation of performance for all users at the class optimum. We have shown the examples in the model of load balancing between two distinct servers. It is amazing that, in the model of asymmetric servers, the paradox in the class optimum keeps to appear in the case of overall symmetry wherein no forwarding of loads occurs in the overall optimum. It is surprising that the paradox appears most strongly in the model of identical servers with each identical arrival rate in the class optimum (Nash equilibrium) while we cannot expect that such a paradox occurs neither in the overall optimum nor

in the individual optimum (Wardrop equilibrium) in such situations.

Then, we have analytically examined the models consisting of symmetrical nodes with identical arrivals to all nodes where forwarding of jobs to the other nodes through communication means with non-zero delays may clearly lead to performance degradation. We have confirmed that in the overall optimization and in the individual optimization (Wardrop equilibrium) such forwarding never occurs. We have shown that, in some parameter setting of the class optimization (Nash equilibrium for one set of decision makers), mutual forwarding of jobs for remote processing through communication means definitely occurs and the ratio of the performance degradation may become unlimitedly large. We have also shown that in the global-class optimization (Nash equilibrium for another set of decision makers) such forwarding never occurs.

For the results on local-class and global-class optima, special assumptions were necessary only for the communication means (*i.e.,* dedicated lines and bus-type connections). It has been quite hard to extend the proofs to more general assumptions. It is not certain whether in some cases of the communication means the optima may still be unique. It has been also difficult for us to analyze asymmetrical models. These are open future problems.

We think that more exhaustive research into these problems is worth pursuing in order to gain insight into the optimal design and QoS (quality of service) management of distributed computer systems, communication networks, etc.

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