S-Modular Games and Power Control in Wireless Networks

Eitan Altman and Zwi Altman

***Abstract*—This note shows how centralized or distributed power control algorithms in wireless communications can be viewed as S-modular games coupled policy sets (coupling is due to the fact that the set of powers of a mobile that satisfy the signal-to-interference ratio constraints depends on powers used by other mobiles). This sheds a new light on convergence properties of existing synchronous and asynchronous algorithms, and allows us to establish new convergence results of power control algorithms. Furthermore, known properties of power control algorithms allow us to extend the theory of S-modular games and obtain conditions for the uniqueness of the equilibrium and convergence of best response algorithms independently of the initial state.**

***Index Terms*—Power control, supermodular games.**

I. Introduction

Noncooperative game theory deals with optimization problems in which several users (players, or agents) have each its own individual utility which it seeks to minimize in a selfish way. Thus, it is typically used to describe competition situations in distributed decision making environment. Power control in wireless network has recently been analyzed within this framework; see [1]–[3], [6], and [7]; utilities are assigned to individual mobiles as a function of the power they consume and the signal-to noise-ratio that they attain, and each mobile is assumed to decide (dynamically) on his own transmission power level so as to maximize his utility. This way to perform power control is an alternative to the classical power control, where power control decisions for mobiles are taken in a centralized way at the base stations. The problem is then typically posed as a constrained minimization problem or simply as a problem of meeting some constraints on the signal-to-interference ratio (SIR) of each mobile.

The aim of this note is to show that even classical centralized power control problems can be modeled as noncooperative games, which further have the S-modular type structure introduced by Topkis in [8]; see also [9] and [12]. These types of noncooperative games have been shown to possess properties that are important in applications: 1) a Nash equilibrium exists; 2) it can be attained using greedy best-response type algorithms; and 3) best response policies are monotone in other players' policies.

Thus, we can make use of the theory of S-games to directly obtain the monotone convergence of distributed power control algorithms [5], [11]. It further allows us to establish convergence for many distributed power control algorithms in the case of discrete available sets of powers; for this setting only, little is known on convergence (see [5] and [10]). Finally, we are able also to obtain new properties of S-modular games from results that have been obtained for power control problems [11]. The idea of using S-modular games in power control already appears in [7], [6]. In [6] it is applied to a utility which is the ratio between the goodput and the power. In [7], it is used in a context related to ours, in which each mobile seeks to minimize the distance between its received SIR and his specified ratio. In both references, however, the role of constraints is not mentioned and the policy sets of the mobiles are taken to be independent of each other. We make use of tools from [12] that allow us to consider a coupled policy set (in which a mobile restricts to policies that make his SIR finite). Our contribution with respect to [6] and [7], and is also in considering discrete available transmission powers.

This note contains two sections, one on S-modular games containing some novel properties, and one on its applications to power control.

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II. S-Modular Games

General models are developed in [8] and [12] for games where the strategy space S_i of player i is a compact sublattice of R^m. (Note that we do not require the entire policy set to be convex, we only require component-wise convexity; in power control problems, the global strategy set is indeed not convex, see, e.g., [4]). By sublattice, we mean that it has the property that for any two elements a, b that are contained in S_i, also $\min(a, b)$ (denoted by $a \wedge b$) and $\max(a, b)$ (denoted by $a \vee b$) are contained there (by $\max(a, b)$ we mean the component-wise max, and similarly with the min). We describe the main results for the case that $m = 1$. We consider N players, and the utility of player i corresponding to the N-dimensional vector of strategies x is given by $f_i(x)$. Let S denote the space of all strategies.

Definition 1: The utility f_i for player i is supermodular if and only if for all $x, y \in S$

$$f_i(x \wedge y) + f_i(x \vee y) \geq f_i(x) + f_i(y).$$

It is submodular if the opposite inequality holds. A game that is either submodular or supermodular is called S-modular game.

If f_i is twice differentiable, then supermodularity is equivalent to

$$\frac{\partial^2 f_i(x)}{\partial x_i \partial x_j} \geq 0$$

for all $x \in S$ and $j \neq i$. Submodularity holds if the opposite inequality holds.

Next, we introduce constraints on the policies. Player i aims to restrict his policy to a subset of policies within S_i that depends on x_{-i} where x_{-i} means the components of x corresponding to all users other than i. We denote this subclass by $S_i(x_{-i})$, $i = 1, 2, \ldots, N$.

Definition 2: A joint policy x^* is a (constrained) Nash equilibrium if for each player i

$$x_i^* \in arg\mathbf{opt}_{x_i \in S_i\left(x_{-i}^*\right)} f_i\left(x_i, x_{-i}^*\right).$$

$arg\mathbf{opt}$ stands for argmax when players maximize and for argmin when they minimize.

Monotonicity of optimizers: The following important property was shown to hold in [8], [12]. Let f be a supermodular function. Then the maximizer with respect to x_i is increasing in x_{-i}, i.e. with respect to each x_j, $j \neq i$. More precisely, define the best response

$$\mathrm{BR}_i^*(x_{-i}) = \max\left[\arg\mathbf{max}_{x_i \in S_i(x_{-i})} f(x_i, x_{-i})\right].$$

Then, $x_{-i} \leq x'_{-i}$ implies $\mathrm{BR}_i^*(x_{-i}) \leq \mathrm{BR}_i^*(x'_{-i})$ if $S_i(x_{-i})$ does not depend on x_{-i} (or if it satisfies the ascending property defined later). The "max" before the brackets is taken in order to select the largest maximizer, but the property also holds for the smallest maximizer. A corresponding property for the minimizers also holds for submodular functions.

Definition 3 (Monotonicity of Sublattices): Let A and B be sublattices. We say that $A \prec B$ if for any $a \in A$ and $b \in B$, $a \wedge b \in A$ and $a \vee b \in B$.

Next, we introduce some properties on the policy spaces.

Monotonicity of the constraint policy sets If

$$x_{-i} \leq x'_{-i} \Longrightarrow S_i(x_{-i}) \prec S_i(x'_{-i}).$$

then the set S_i possess the **ascending property**. We define similarly the **descending property** when the relation is reversed.

Lower semicontinuity of $S_i(\cdot)$ We say that the point to set mapping $S_i(\cdot)$ (that maps elements x_{-i} to $S_i(x_{-i})$) is lower semicontinuous if for every $x_{-i}^k \rightarrow x_{-i}^*$ and $x_{-i}^* \in S_i(x_{-i}^*)$, there exist $\{x_i^k\}$ s.t. $x_i^k \in S_i(x_{-1}^k)$ for each k, and $x_i^k \rightarrow x_i^*$.

Existence of equilibria and convergence of greedy algorithms: Consider an n-player game. References [12, Alg. 1] and [8, Alg. I] consider a greedy round robin scheme where at some infinite strictly increasing sequence of time instants T_k, players update their strategies using each the best response to the strategies of the others. Player l updates at times T_k with $k = mN + l, m = 1, 2, 3, \ldots$.

More generally, one can consider the following algorithm.

Definition 4 General Updating Algorithm (GUA): There are N infinite increasing sequences $\{T_k^i\}, i = 1, \ldots, N, k = 1, 2, 3, \ldots$. Player i uses at time T_k^i the best response policy to the policies used by all other players just before T_k^i. This scheme includes in particular parallel updates (when $\{T_k^i\}$ do not depend on i).

Remark 1: If at some time an action $\mathbf{x}$ is used and user i then updates applying the best response to x_{-i}, all we require is that its best response be feasible, i.e., in $S_i(x_{-i})$. Once this user updated his strategy, the strategies of one or more of the other users need not be anymore feasible.

Theorem 1: Assume that for all $i = 1, \ldots, N, S_i(\cdot)$ are compact for all values of their argument, and are lower semicontinuous in its argument. Assume either of the following.

- The game is supermodular, players maximize, and $S_i(\cdot)$ are ascending.
- The game is submodular, players minimize, and $S_i(\cdot)$ are descending.

Then, the following hold.

i) An equilibrium exists.

ii) If each player i initially uses its lowest policy in S_i, or if each user i uses initially its largest policy in S_i, then the GUA converges monotonically to an equilibrium (that may depend on the initial state). Monotonicity is in the same direction for all players: the sequences of strategies either all increase or all decrease.

iii) If we start with a feasible policy $\mathbf{x}$, then the sequence of best responses monotonically converges to an equilibrium: it monotonically decreases in all components in the case of minimizing in a submodular game, and monotonically increases in the case of maximizing in a supermodular game.

Proof: The proof of i) and ii) is a direct extension of the proof of [12, Th. 2.3]. Item iii) directly follows from the monotonicity of the best response. For example, in a submodular game, for any $i = 1, \ldots, N$, $\mathrm{BR}_i(x) \leq x_i$ since x_i is already feasible. Due to the descending property, if we replace x_j by $\mathrm{BR}_j(x)$ for some set $j \in J$ then $x_k, k \notin J$ remains feasible, so by induction we get a monotone nonincreasing sequence that converges to some limit. We could now apply part ii) of the theorem by restricting the game to policies that are not greater than the initial condition x. For supermodular games we obtain the proof similarly. ■

Remark 2: In general, in an S-modular game (as before) the set S^* of equilibria need not be a singleton. However, the following points hold.

i) There exists a unique dominating element in S^* (componentwise). Indeed, consider a submodular game. If $x_j \in S^*, j = 1, 2$, then also $x' := x^1 \wedge x^2$ is feasible (due to the descending property). Using theorem 1 iii), we conclude that there is an equilibrium which is no larger than x'.

ii) The unique minimum equilibrium $x^{\min}$ is attainable using GUA (for any order of updates) starting at the minimal elements of S_i. Indeed, for a submodular game for example, at each iteration of best responses, the new actions are no greater than the best responses to $x^{\min}$ which is again $x^{\min}$, so their limit, which is an equilibrium according to Theorem 1 ii), is bounded by $x^{\min}$.

Motivated by [11] (power control context), we define the scalability property.

Definition 5: In case all players minimize their cost functions, the best responses have the scalability property if for all $\alpha > 1$ and $i = 1, \ldots, N, \alpha \mathrm{BR}_i(x_{-i}) > \mathrm{BR}_i(\alpha x_{-i})$. In case they maximize, the definition holds with the opposite inequality.

The following establishes conditions for uniqueness of Nash equilibrium and general convergence properties of GUA.

Theorem 2: Assume the sets S_i are convex (continuous action spaces), that that the conditions of Theorem 1 and the scalability property hold and that there exists some feasible x (i.e., such that $x_i \in S_i(x_{-i})$ for $i = 1, \ldots, N$). Further assume that for each $i = 1, \ldots, N, \mathrm{BR}_i(x_{-i}) > 0$ for all $x_j \in S_j, j \neq i$. Then, i) the Nash equilibrium is unique and ii) the convergence of GUA holds for any initial policy.

Proof: The proof of the Theorem is based on the monotonicity of the best response together with the scalability property, exactly as in the proof of [11, Ths. 1 and 2]. ■

III. Convergence of Power Control Algorithms

Yates introduces five uplink power control problems with the following common structure. There are N users, M base stations and a common radio channel. User j transmits at a power level p_j. Let h_{kj} be the gain of user j to base k, so the received signal of users j at base k is $p_j h_{kj}$. The interference experienced by mobile j at that base is $\sum_{i \neq j} h_{ki} p_i + \sigma_k$, where σ_k is the receiver noise at base k. The SIR of user j at base k when transmission powers are given by the vector $\mathbf{p}$ is given by $p_j \mu_{kj}(\mathbf{p})$ where

$$\mu_{kj}(\mathbf{p}) = \frac{h_{kj}}{\sum_{i \neq j} h_{ki} p_i + \sigma_k}.$$

User j requires a signal to noise interference of at least γ_j.

The five different power control problems introduced in [11] differ according to which base station(s) each mobile connect. Yet all have in common the fact that there is a given constraint on the power to noise interference ratio, which implies that the transmission power of each mobile j should be larger than or equal to some level $I_j(\mathbf{p})$ that depends on the transmission power of all mobiles $\mathbf{p}$.

The schemes considered there are as follows.

1) Fixed assignment: mobile j is assigned to base a_j. The constraint has the form

$$p_j \geq I^{\mathrm{FA}}(\mathbf{p}) = \frac{\gamma_j}{\mu_{a_j,j}(\mathbf{p})}.$$

2) Minimum power assignment: the mobile is connected to the base for which the signal to interference is maximum. The constraint is given by

$$p_j \geq I^{\mathrm{MPA}}(\mathbf{p}) = \min_k \frac{\gamma_j}{\mu_{k,j}(\mathbf{p})}.$$

3) Macro diversity: each mobile is connected to all base stations. The constraint is

$$p_j \geq I^{\mathrm{MD}}(\mathbf{p}) = \frac{\gamma_j}{\sum_k \mu_{k,j}(\mathbf{p})}.$$

4) Limited diversity: mobile j is connected a set K_j of stations. The constraint is

$$p_j \geq I^{\mathrm{MD}}(\mathbf{p}) = \frac{\gamma_j}{\sum_{k \in K_j} \mu_{k,j}(\mathbf{p})}.$$

5) Multiple connection reception: mobile j is required to maintain an acceptable SIR γ_j at d_j distinct base stations. The constraint $I_j^{\mathrm{MC}}(\mathbf{p})$ is given as the d_jth smallest value of the set $\gamma_j/\mu_{k,j}(\mathbf{p})$.

In all five schemes, it is shown in [11] that I satisfies the following properties.

- *Positivity*: $I(\mathbf{p}) > 0$.
- *Monotonicity*: If $\mathbf{p} \geq \mathbf{p}'$ then $I(\mathbf{p}) \geq I(\mathbf{p}')$.
- *Scalability*: For $a > 1$, $aI(\mathbf{p}) \geq I(a\mathbf{p})$.

In order to use tools from submodular game setting, we make the following observations.

i) Although power control decisions are often taken centrally at base stations, they are computed so as to minimize the power of each mobile separately while satisfying all signal to interference constraints. Thus, the centralized problem is indeed a game one: there are several utilities (the powers) each corresponding to another mobile, and minimization (subject to constraints) are performed for each mobile (although centrally). Our framework is, moreover, useful for the challenging distributed power control setting in which the game setting is more evident.

ii) The goal of the power control is to obtain $\mathbf{p}^*$ such that for all $i = 1, \ldots, N$

$$p_i^* = \min\left\{p_i \in S_i \ s.t.\ p_i \geq I_i(\mathbf{p}^*)\right\}. \tag{1}$$

This agrees with the definition of the constrained Nash equilibrium. In the case of convex power sets (continuous sets of powers), this simplifies (as in [5] and [11]) to finding a fixed point of $\mathbf{p} = I(\mathbf{p})$. We assume here that S_i is unbounded (there are no maximum power restrictions, although we shall show below that one can restrict indeed to compact (bounded) sets S_i. Unless otherwise stated, we take S_i to be the real positive numbers.

Theorem 3: Assume that there is a feasible solution $\mathbf{p}'$ to $\mathbf{p} \geq I(\mathbf{p})$. Then

i) there exists a fixed point to the equation $\mathbf{p} = I(\mathbf{p})$;

ii) if the sets S_i are discrete, then GUA converges for initial powers of all users corresponding to the lowest or to the largest powers;

iii) if the sets S_i are convex (continuous action spaces) then there is a unique fixed point to the equation $p = I(p)$. Moreover, GUA converges for any initial policy.

Proof: We first note that the feasibility of $\mathbf{p}'$ implies also that $a\mathbf{p}'$ is feasible for all $a \geq 1$. This follows from the scalability property of I. Consider an initial strategy $\mathbf{p}_0$, not necessarily feasible. Choose some $a > 1$ such that $\mathbf{p}_0 < a\mathbf{p}'$. Consider now the following submodular game:

- the actions of player j is the power $x_j = p_j$;
- the utility of player j is simply $f(p_j) = p_j$;
- the constraint policy set of player j is given by

$$S_j(p_{-j}) = \left\{p_j : p_j \geq I_j(p),\ 0 \leq p_j \leq ap_j'\right\}.$$

(The set $S_j(p_{-j})$ can thus be considered to be a subset of the compact set $S_j = \{p_j : 0 \leq p_j \leq ap_j'\}$).

We check the conditions of Theorem 1 (and then use also Theorem 2). The cost of every player j is (trivially) submodular (since it depends only on the policy of player j). What makes it then a nontrivial submodular game is the dependence of the set of available transmission powers for player j on the powers of other player. The policy space of each user is indeed a compact sublattice. The monotonicity of I implies the descending property of the policy sets.

We note that for any $\mathbf{p}$ whose components satisfy $0 \leq p_j \leq ap_j'$, for all j, $S_j(p_{-j})$ is nonempty for all j. This is due to the scalability and monotonicity of I.

Obviously, only in the case of continuous strategy spaces we need to check that the point to set functions S_j are lower semicontinuous, which we do next.

Since $S_j(\cdot)$ are nonempty convex sets, it suffices to show that its minimal value, $I_j(\cdot)$, is continuous. Let $\mathbf{p} = \mathbf{p}^1 \vee \mathbf{p}^2$. Choose the smallest $a \geq 1$ such that $\mathbf{p}/a \leq \mathbf{p}^i$, $i = 1, 2$. Since the monotonicity and scalability of I imply

$$aI\left(\frac{\mathbf{p}}{a}\right) \geq I(\mathbf{p}) \geq I\left(\frac{\mathbf{p}}{a}\right)$$

we have

$$\frac{I(\mathbf{p})}{a} \leq I\left(\frac{\mathbf{p}}{a}\right) \leq I(\mathbf{p}^i) \leq I(\mathbf{p}), \qquad i = 1, 2.$$

Hence,

$$\left|I(\mathbf{p}^1) - I(\mathbf{p}^2)\right| \leq I(\mathbf{p})\left(1 - \frac{1}{a}\right).$$

If $\mathbf{p}^2$ tends to $\mathbf{p}^1$ then a tends to 1, which shows the continuity of I. We conclude that the conditions of Theorem 1 holds which establishes the proof. ∎

Remark 3:

i) Another reference that pursued the direction of [11] is [5]. It presents a "canonical" algorithm that converges to a feasible desired power region rather than to a given global min as in Yate's paper. Within this framework, a special discrete valued power control problem is also shown to yield convergence.

ii) A convergence of an algorithm of the type of GUA is established in bert for the "fixed assignment" problem (the first mentioned in this section) for the discrete case.

iii) We have not considered here the case of maximum power constraints. In the presence of such constraints, the set of best response policies might be empty and iterative power control schemes may converge to a point that is not feasible for some mobiles. To handle this case one could formulate an alternative game with no power constraints, but in which the objective of player i is to minimize $f(p_j, I_j(p))$ where f is some increasing function of the absolute value of the difference between its arguments. This approach also leads to S-modular games and to convergence of iterative schemes, as was shown in [7, Sec. 13].

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Local Stabilization of Linear Systems Under Amplitude and Rate Saturating Actuators

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***Abstract*—This note addresses the problem of local stabilization of linear systems subject to control amplitude and rate saturation. Considering the actuator represented by a first-order system subject to input and state saturation, a condition for the stabilization of an *a priori* given set of admissible initial states is formulated from certain saturation nonlinearities representation and quadratic stability results. From this condition, an algorithm based on the iterative solution of linear matrix inequalities-based problems is proposed in order to compute the control law.**

***Index Terms*—Constrained control, control saturation, linear matrix inequality (LMI), stabilization.**

I. Introduction

Physical and technological constraints do not allow that control actuators provide unlimited amplitude signals neither react unlimited fast. The negligence of both amplitude and rate control bounds can be source of limit cycles, parasitic equilibrium points and even instability of the closed-loop system. In particular, the problem of stabilization of linear systems only with amplitude saturation has been exhaustively addressed in the literature (see, among others, [1]–[3] and the references therein). On the other hand, the rate saturation problem has first received a special interest in the aeronautic field, where the tradeoff between high performance requirements and the use of hydraulic servos presenting rate limitations is always present (see, for instance, [4], [5], and the references therein).

Studies addressing the stabilization in the presence of both the amplitude and the rate saturation, as a more generic problem, have started to appear in the last few years. In [6] and [7], the semiglobal stabilization of linear systems with both amplitude and rate constraints is addressed. Considering a low-gain approach (the actuator does not effectively saturate), in [6], solutions to the problem via both state feedback and observer based output feedback are stated. In [7], the notion of an operator for modeling the amplitude and the rate saturation is introduced. Based on this modeling, a low and high gain approach is used for addressing the problem of semiglobal output regulation via both state and dynamic output feedback. In [8], the problem of external L_p-stabilization with internal global stabilization via a scheduled low gain (saturation is avoided) state feedback is addressed. It should be pointed out that, since the objective is the semiglobal or global stabilization, these results can be applied only when the open-loop system is null-controllable (i.e., all the poles are in the closed left half plane).

On the other hand, we can identify some works dealing with local stabilizing solutions (see, among others, [9]–[12]). In [9], a method for designing dynamic output controllers based on a *position type feedback* modeling of the rate saturation and the use of the positive real lemma is proposed. The main objective pursued in that paper is the minimization of a linear quadratic Gaussian criterion. A region of stability (region of attraction) is associated to the closed-loop system. However, it should be pointed out that the size and the shape of this region are not taken into account in the design procedure which can lead to very conservative domains of stability. Furthermore, the controller is computed from the solution of strong coupled equations which, in general, are not simple to solve. A different modeling for the actuator, subject to both rate and amplitude limitations, is considered in [10] and [12]. In these papers, the actuator is modeled by a pure integrator: the control rate appears as the system input and the original control signal becomes a state of the system. The physical meaning of this kind of modeling is not clarified in these papers. Parallel to these works, in [11], the problem of disturbance attenuation in the presence of rate and amplitude actuator saturation is addressed. In that paper, however, no explicitely consideration is made about the region of attraction associated to the controller.

Since we also aim to consider strictly unstable systems, our note figures in the context of local stabilization of linear systems subject to both actuator amplitude and rate saturation. In this case, two objectives are quite natural: the control law should guarantee a certain time-domain performance for the closed-loop system and the associated region of attraction should be as large as possible. Regarding these objectives, a fundamental issue is whether the use of effective saturating control laws can be advantageous or not. In a recent work considering only amplitude saturation [13], it was shown that, at least in some cases, the use of the saturating control laws does not help in obtaining larger regions of stability. It is, however, very important to highlight that no constraints concerning neither the performance, nor the robustness, were taken into account in this analysis. In this case, although the optimal region of stability is obtained with a linear control law, the closed-loop poles associated to this solution can be very close to the imaginary axis, which implies a very slow behavior.

The objective of this note is then to propose a method for computing state feedback saturating control laws, that ensure both asymptotic stability of the closed-loop system with respect to a given set of admissible initial conditions, and a certain degree of time-domain performance in a neighborhood of the origin. We also aim to emphasize the compromise between performance and the size of the region of attraction. As we will see, over performance constraints, the use of saturating control laws can ensure larger regions of stability. As in [9], our approach is based on the modeling of the actuator by a first-order system subject to input and state saturation (position-feedback-type model with speed limitation). Differently, however, from [9] and [11], the objective in the synthesis is explicitely to enlarge the region of attraction. Moreover, the stabilization conditions are based on a mixed polytopic/norm-bounded differential inclusion for modeling the behavior of the closed-loop nonlinear system. Comparing to the polytopic approach used in [11] and

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