Performance Bounds and Pathwise Stability for Generalized Vacation and Polling Systems

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Abstract

We consider several models for a generalized vacation or polling system with a general (in particular, non-Poisson) arrival process. Our results are given in the context of an input-output process operating over successive "cycles", in which the service mechanism can be in an "up" mode (processing) or "down" mode (e.g., vacation, walking). Our primary motivation is polling systems, in which there are several queues and the server moves cyclically between them providing some service in queues that it visits. We allow quite general service policies, requiring only that the total amount of work that leaves the system in a "cycle" is at least as great as the amount of work that was present at the beginning of the cycle. This includes the standard gated and exhaustive policies for polling systems, for example, when cycles are defined in the "natural" way. That is, a cycle begins whenever the server arrives to some prespecified queue. The input and output processes are characterized by model-dependent conditions: pathwise bounds on the average rate and the burstiness (Cruz bounds); the existence of long-run average rates; a pathwise generalized Law of the Iterated Logarithm; or exponentially or polynomially bounded tail probabilities of burstiness. In each model we show that these properties are inherited by performance measures such as the workload and output processes and that a sufficient condition for stability (in a sense that depends on the particular model) is that the input rate is smaller than the processing rate while in the up mode.

1 Introduction

We consider a general model for an input-output process, which includes many vacation and polling systems and other queueing systems as special cases. At any point in time, the system is either in the "up" mode, with both the arrival (input) and departure (output) process operating, or the "down" mode, in which the arrival process continues to operate but the departure process may not be operating. Down time may correspond to vacation time or switch-over ("walking") time in a vacation or polling system. The time that the system is empty will also typically be understood to be part of the system down time. The down times may even correspond to operation of the server at some reduced capacity, instead of complete idleness. In that case, the sufficient conditions for stability and the performance bounds that we obtain may turn to be conservative.

A distinguishing feature of our model is that the time axis is divided into a sequence of disjoint time intervals or "cycles" and we make certain assumptions about the behavior of the arrival and departure processes and the down time during each cycle. One example of this type of set-up is a cyclic polling system in which a cycle is defined in the "natural" way. That is, a cycle begins each time the server begins serving the first queue. Another is a vacation model where a cycle begins each time a vacation ends. In addition to vacation and polling systems, our model has other potential applications, including communication systems with time-division multiplexing and flexible manufacturing processes with setups and/or breakdowns.

In general our model allows considerable flexibility in the definition of cycles. Our basic assumption throughout the paper is that the total output during each cycle is at least as great as the quantity in the system at the beginning of the cycle. Examples satisfying this output assumption are polling systems with exhaustive, globally gated, or locally gated service disciplines, with cycles defined as above in the "natural" way. In contrast to most of the literature on polling systems, we allow for the discipline to change over time. Another example is a vacation system in which the server goes on vacation every time the queue becomes empty. In each of our models we also assume that the down time per cycle satisfies certain constraints or growth conditions, and that the traffic intensity (suitably defined in each model) is less than one. It is the interaction of these conditions with the output assumption that essentially constrains the definition of cycles and motivates our derivations of performance bounds and stability.

We examine various types of performance bounds and stability for three special cases of our general model, based on different pathwise characteristics of their arrival and (potential) departure processes. We present the basic model and some general properties in Section 2. The three special cases are discussed in the next three sections.

The first system (Section 3) has bounded down times in each cycle and arrival and (potential) departure processes that satisfy burstiness constraints similar to those of Cruz [12], performance measures, such as the amount of work in the system and the cycle duration. We also show that the output process satisfies a Cruz-type bound. The second system (Section 4) has arrival and potential departure processes with limiting average rates. We show that the system is rate stable (i.e., the quantity in the system is $o(t)$ as t approaches infinity) and give explicit expressions for the limiting average cycle length and fraction of time the system is down. In Section 5 we look at systems whose arrival processes satisfy a pathwise analogue of the stochastic law of the iterated logarithm. We show how all three cases can be analyzed by a unified sample-path approach.

Finally (Section 6) we show how our pathwise results can be applied to stochastic models. In particular we show that the workload and output processes inherit the property of exponentially or polynomially bounded burstiness from the input process. This implies that upper bounds, which are uniform in time, can be computed for the total workload in the system at any moment, for a wide class of stochastic arrival processes, not necessarily stationary nor ergodic. This generalizes many previous stability results for similar stochastic models.

In polling models that have been studied in the literature, it has usually been assumed that the queues are fed by independent Poisson arrival processes. Conditions for ergodicity as a measure of stability were obtained in [4], [7], [10], [16], [17], [23] and [26]. Kroese and Schmidt [20], [21] studied the stability of polling on a graph (i.e. a "continuous" polling model), and Altman and Levy [5] studied the stability of non-cyclic polling in two-dimensional and higher dimensional planes. In [4] and [7] Altman et al. further present sufficient and necessary conditions for stronger notions of stability, namely the geometric ergodicity and the geometric rate of convergence of the moments of several performance measures (embedded at polling instants). Sufficient conditions for Central Limit Theorems and the Law of Iterated Logarithm are given in [7]. In [6], Altman and Liu analyse the stability of the *FDDI* protocol. Conditions for the stability of token rings with spatial reuse were obtained by Szpankowski et al. [18].

All the above references assumed Poisson arrivals and independent walking and service times. This assumption is unrealistic, however, when dealing with many applications, e.g. Local Area Networks using token-ring protocols. The arrival processes there may be quite irregular, highly bursty and correlated. Recently, Altman and Foss [1] obtained sufficient stability conditions for polling systems with a general renewal arrival processes, and Massoulié [22] reported some results on the construction of a stationary regime for general stationary ergodic arrival and service processes.

The analysis in this paper, which is based on pathwise bounds and limits, allows for more general arrival processes and makes it possible either to obtain strict upper bounds on several performance measures, such as waiting times, queue lengths, and workloads, or to derive pathwise stability conditions, using a unified approach (based on the lemmas in Section 2). In a related paper [2], we discuss some special cases of the models presented in this paper, with a focus on their applications to token-ring communication networks.

2 Preliminary Lemmas

Following Stidham and El-Taha [24] (see also El-Taha and Stidham [14], and Borovkov [9] Chapter 2) we consider a non-negative real-valued deterministic process, $Z = \{Z(t), t \geq 0\}$ - an *input-output* process - in which $Z(t) \geq 0$ represents quantity in a system. Specifically, we assume that the state space of Z is $S = R^+$. We assume that $\{Z(t), t \geq 0\}$ is right continuous with left-hand limits, and that

$$Z(t) = Z(0) + A(t) - D(t) \ , \ t \geq 0 \ , \tag{1}$$

where $A(t)$ $(D(t))$ is the cumulative input to (output from) the system in $[0, t]$ and both $\{A(t), t \geq 0\}$ and $\{D(t), t \geq 0\}$ are non-decreasing, right-continuous processes. Thus $Z(t)$ has bounded variation on finite t-intervals. Note that $D(t) \leq Z(0) + A(t)$, since $Z(t) \geq 0$. We shall refer to $Z(t)$ generically as the *work* in the system at time t, with the understanding that it could be some other measure of quantity (e.g., the number of customers in a queue).

Let $u(t)$ $(v(t))$ be the indicator function for "up" ("down") time. That is, $u(t) = 1$ if the system is in the "up" mode and $u(t) = 0$ if it is "down" at time t; $v(t) = 1 - u(t)$. We assume that $u(\cdot)$ (and hence also $v(\cdot)$) is integrable over finite t-intervals. In a cyclic polling system, for example, if "down" time corresponds to the time spent by the server "walking" between queues, then $\int_0^t v(s)ds$ equals the total walking time in $[0, t]$ and $\int_0^t u(s)ds$ equals the total time in $[0, t)$ that the server processes work in the queues.

In general we shall use α and δ to denote the arrival rate and the departure rate, respectively, in some sense, the exact meaning of which will depend on the context. Define $\rho = \alpha/\delta$. We define the "burstiness" of each process during a time interval, $[s, t)$ by:

$$B^A_{s,t} \;:=\; A(t) - A(s) - \alpha(t-s) \tag{2}$$

$$B^D_{s,t} \;:=\; D(t) - D(s) - \delta \int_s^t u(\tau)\, d\tau \tag{3}$$

Thus, $B^A_{s,t}$ is the difference between the actual input and the input that would have occurred if the input process had operated at its "average" rate throughout the interval. Similarly, $B^D_{s,t}$ is the difference between the actual output and the output that would have occurred if the output process had operated at its "average" rate throughout the up time in the interval (and at rate 0 during the down time).

We shall assume that there is a sequence of time points, $\{t_n, n \geq 0\}$, that defines *cycles* for the system, with $0 = t_0 \leq t_1 \leq t_2 \leq \cdots$. We interpret t_n to be the time point at which the n^{th} cycle ends, for $n \geq 1$. We define quantities corresponding to each cycle. Let $T_n := t_n - t_{n-1}$, the duration of the n^{th} cycle. Let $U_n := \int_{t_{n-1}}^{t_n} u(s)\, ds$, and let $V_n := \int_{t_{n-1}}^{t_n} v(s)\, ds$, the total up and down time, respectively, during the n^{th} cycle. Define $W_n := Z(t_n)$, the work in the system at the end of the n^{th} cycle. Let $B^A_n := B^A_{t_{n-1},t_n}$, the burstiness of the arrival process during the n^{th} cycle, and let $B^D_n := B^D_{t_{n-1},t_n}$, the burstiness of the departure process during the n^{th} cycle. To avoid technical difficulties, we shall assume that $t_n \to \infty$ as $n \to \infty$, which is the case, for example, when the down times, V_n, $n \geq 1$, are bounded below by a positive constant.

Motivated by gated and exhaustive polling systems, we shall make the following assumption throughout this paper:

> **Assumption.** For all $n \geq 1$, $D(t_n) - D(t_{n-1}) \geq W_{n-1}$. That is, the output in each cycle is at least as great as the work in the system at the beginning of the cycle.

Remark 1. Till now, our definitions of cycles and down times have been very vague. In the following sections, other assumptions on the down times during a cycle (bounds on their duration or on their average duration) will make these notions more precise, and more related to what we understand by "cycles" and down time in applications. Note that without any further restrictions, the above Assumption is relatively innocuous. In fact, in the general setting that we have assumed up until this point – simply an input-output process, $\{Z(t), t \geq 0\}$, with an imbedded non-decreasing sequence of time points, $\{t_n, n \geq 0\}$ – we are free to define the cycles so that the Assumption is trivially satisfied, so long as $D(t) \to \infty$ as $t \to \infty$. (For example, define the sequence, $\{t_n, n \geq 0\}$, recursively by $t_n := \inf\{t : D(t) - D(t_{n-1}) \geq W_{n-1}\}$.) As we have indicated, however, in each of our three applications the definition of the cycles is implicitly constrained by the bounds or growth conditions that we impose on the down time, V_n, in the n^{th} cycle. These constraints are inspired by our motivating examples of vacation and polling systems, in which a cycle begins when the server begins serving a particular customer class. For additional observations and a counterexample when the constraints are not satisfied, see Remark 6 at the end of Section 4.

The following example illustrates these issues and motivates our first result (Lemma 1 below).

Example 1^o**.** *A Globally Gated Polling System.* In a globally gated polling system a single server attends m queues, labeled $i = 1, \ldots, m$, in sequence, serving at each queue all work that was present at the beginning of the cycle and then moving on to the next queue. In this setting, a cycle begins each time the server arrives and begins service at queue 1. The server spends a certain amount of down or *walking* time moving from queue i to queue $i+1$. (We identify queue $m+1$ with queue 1.) The total walking time in cycle n is V_n. For simplicity, assume that $V_n = V$, $n \geq 1$. If the departures were at a uniform rate, δ, then the duration of the n^{th} cycle would be $V + W_{n-1}/\delta$. If arrivals were at a uniform rate, α, then W_n would equal $\alpha V + \rho W_{n-1}$, and therefore also equal $\rho^n W_0 + \alpha V \sum_{i=1}^{n} \rho^{n-i}$. Now suppose arrivals and departures are at these uniform rates, except for a single arrival burst of size X in the third cycle. Then for $n > 3$, $W_n = \rho^n W_0 + \alpha V \sum_{i=1}^{n} \rho^{n-i} + \rho^{n-3} X$.

The following is a similar result for our more general setup.

Lemma 1 *Suppose that $\rho = \alpha/\delta < 1$, and let N be a non-negative integer. Then*

$$W_n \leq \rho^{n-N} W_N + \alpha \sum_{i=N+1}^{n} \rho^{n-i} V_i + \sum_{i=N+1}^{n} \rho^{n-i} B_i^A - \rho \sum_{i=N+1}^{n} \rho^{n-i} B_i^D \ , \ n \geq N+1 \ . \tag{4}$$

Proof. First note that

$$\begin{aligned} W_n &= W_{n-1} + A(t_n) - A(t_{n-1}) - [D(t_n) - D(t_{n-1})] \\ &= W_{n-1} + \alpha T_n + B_n^A - [\delta(T_n - V_n) + B_n^D] \ , \end{aligned}$$

and hence

$$\delta(T_n - V_n) + B_n^D - W_{n-1} = \alpha T_n + B_n^A - W_n \ .$$

By our assumption that the output during the cycle must be no smaller than the work at the beginning of the cycle, both sides of this equation are non-negative. Noting $\rho < 1$, we subtract ρ times the left side from the right side and rearrange them to obtain

$$W_n \leq \rho W_{n-1} + \alpha V_n + B_n^A - \rho B_n^D \; . \tag{5}$$

Beginning with W_N and iteratively substituting for W_{n-1} in (5) leads to (4). ∎

Next we bound the sum of the "discounted" bursts in equation (4) by the sum of the corresponding "un–discounted" bursts when $\rho_i < 1$. This result, which is a consequence of the following lemma, is a key tool in establishing the bounds for all the models considered in the following sections.

Lemma 2 *Suppose we have two sequences of real numbers, $\{B_i\}_{i=1}^n$ and $\{\rho_i\}_{i=1}^n$, such that $0 \leq \rho_1 \leq \cdots \leq \rho_n$. Then*

$$\rho_1 B_1 + \cdots + \rho_n B_n \leq \rho_n \max\{0, B_n, B_n + B_{n-1}, \ldots, B_n + \cdots + B_1\} \; . \tag{6}$$

Proof. The proof is by induction on n. Suppose (6) holds for n. Since the right hand side is non–negative, we can replace ρ_n with ρ_{n+1} on the right hand side, and add $\rho_{n+1} B_{n+1}$ to both sides, thus obtaining

$$\begin{aligned}
&\rho_1 B_1 + \cdots + \rho_n B_n + \rho_{n+1} B_{n+1} \\
&\quad \leq \quad \rho_{n+1} \left(B_{n+1} + \max\{0,\, B_n,\, B_n + B_{n-1}, \ldots,\, B_n + \cdots + B_1\} \right) \\
&\quad = \quad \rho_{n+1} \max\{B_{n+1},\, B_{n+1} + B_n,\, B_{n+1} + B_n + B_{n-1}, \ldots,\, B_{n+1} + B_n + \cdots + B_1\} \\
&\quad \leq \quad \rho_{n+1} \max\{0,\, B_{n+1},\, B_{n+1} + B_n,\, \ldots,\, B_{n+1} + \cdots + B_1\} \; ,
\end{aligned}$$

which establishes (6) for $n + 1$. ∎

Lemma 3 *Suppose that $\rho = \alpha/\delta < 1$, and let N be a non–negative integer. Suppose also that $V_i \leq V$ whenever $i > N$. Then, for $n \geq N + 1$,*

$$\begin{aligned}
W_n \quad \leq \quad & \rho^{n-N} W_N + \frac{\alpha(1 - \rho^{n-N})V}{1 - \rho} + \max\{0, B_n^A, B_n^A + B_{n-1}^A, \ldots, B_n^A + \cdots + B_{N+1}^A\} \\
& + \rho \; \max\{0, (-B_n^D), -(B_n^D + B_{n-1}^D), \ldots, -(B_n^D + \cdots + B_{N+1}^D)\} \; ;
\end{aligned} \tag{7}$$

$$T_n \leq \frac{\rho^{n-N} W_N}{\alpha(1-\rho)} + \frac{(1-\rho^{n-N})V}{(1-\rho^2)} + \frac{1}{\delta(1-\rho)} \max\{0, B_n^A, B_n^A + B_{n-1}^A, \ldots, B_n^A + \cdots + B_N^A\}$$

$$+ \frac{1}{\delta(1-\rho)} \max\{0, -(B_n^D), -(B_n^D + B_{n-1}^D), \ldots, -(B_n^D + \cdots + B_N^D)\} \; ; \qquad (8)$$

and for $t_n < t \leq t_{n+1}$,

$$Z(t) \leq \rho^{n-N} W_N + \frac{\alpha(2-\rho-\rho^{n-N})V}{1-\rho}$$

$$+ \max\{0, B_{t_n,t}^A, B_{t_n,t}^A + B_n^A, \ldots, B_{t_n,t}^A + B_n^A + \cdots + B_{N+1}^A\}$$

$$+ \rho \max\{0, -(B_{t_n,t}^D), -(B_{t_n,t}^D + B_n^D), \ldots, -(B_{t_n,t}^D + B_n^D + \cdots + B_{N+1}^D)\} \; . \qquad (9)$$

Proof. For (7), we bound the terms of (4). Since the down times are bounded by V,

$$\alpha \sum_{i=N+1}^{n} \rho^{n-i} V_n \leq \alpha \sum_{i=N+1}^{n} \rho^{n-i} V = \frac{\alpha(1-\rho^{n-N})V}{1-\rho} \; . \qquad (10)$$

The bounds on the other terms follow from Lemma 2.

To show (8), note that it follows from (3) that

$$\begin{aligned} \delta T_n &= D(t_n) - D(t_{n-1}) + \delta V_n - B_n^D \\ &\leq W_{n-1} + \alpha T_n + B_n^A + \delta V_n - B_n^D \; , \end{aligned}$$

since the output during a cycle can be no more than the work at the beginning of the cycle plus the input during the cycle. Thus,

$$\begin{aligned} \alpha(1-\rho)T_n &\leq \rho W_{n-1} + \rho B_n^A - \rho B_n^D + \alpha V_n \qquad (11) \\ &\leq \rho^{n-N} W_N + \alpha \sum_{i=N+1}^{n} \rho^{n-i} V_i + \rho \left(\sum_{i=N+1}^{n-1} \rho^{n-1-i} B_i^A + B_n^A \right) - \rho \sum_{i=N+1}^{n} \rho^{n-i} B_i^D \; , \end{aligned}$$

where we have used Lemma 1 to bound W_{n-1}. The result then follows using Lemma 2 and (10).

To prove (9), first note that

$$\begin{aligned} D(t) - D(t_n) &= \delta(t-t_n) - \delta \int_{t_n}^{t} v(s) ds + B_{t_n,t}^D \\ &\geq \delta(t-t_n) - \delta V_{n+1} + B_{t_n,t}^D \end{aligned}$$

for $t \geq t_n$, which implies that

$$\begin{aligned} Z(t) &= W_n + \alpha(t - t_n) + B^A_{t_n,t} - (D(t) - D(t_n)) \\ &\leq W_n + \rho\left(D(t) - D(t_n) + \delta V_{n+1} - B^D_{t_n,t})\right) + B^A_{t_n,t} - (D(t) - D(t_n)) \\ &= W_n + \alpha V_{n+1} + B^A_{t_n,t} - \rho B^D_{t_n,t} - (1-\rho)(D(t) - D(t_n)) \\ &\leq W_n + \alpha V_{n+1} + B^A_{t_n,t} - \rho B^D_{t_n,t} \ . \end{aligned} \tag{12}$$

Using inequality (4) for W_n then yields

$$\begin{aligned} Z(t) \ \leq \ & \rho^{n-N} W_N + \alpha\left(\sum_{i=N+1}^{n} \rho^{n-i} V_i + V_{n+1}\right) \\ & + \left(\sum_{i=N+1}^{n} \rho^{n-i} B^A_i + B^A_{t_n,t}\right) - \rho\left(\sum_{i=N+1}^{n} \rho^{n-i} B^D_i + B^D_{t_n,t}\right) \ . \end{aligned} \tag{13}$$

Bounding the terms on the right-hand side of (13) using Lemma 2 and (10) leads to the desired result. ∎

3 Linear Burstiness Bounds

For this section, in addition to the assumptions of Sections 1 and 2, we assume that the input and output processes have bounds that are a special case of those studied by Cruz [12], [13]. Specifically, we assume that

$$A(t) - A(s) \ \leq \ \alpha(t-s) + \sigma_A \ , \ 0 \leq s < t \ , \tag{14}$$

$$D(t) - D(s) \ \geq \ \delta\left(\int_s^t u(\tau)\, d\tau\right) - \sigma_D \ , \ 0 \leq s < t \ , \tag{15}$$

where σ_A and σ_D are non-negative constants. In words, for both the input and output processes, the burstiness in any interval is bounded by a constant independent of the length of the interval. We also assume that the down time during any cycle is bounded by a constant: $V_n \leq V$, for all $n \geq 1$.

Theorem 1 *Assume $\rho < 1$ and $W_0 = 0$. Then*

$$\begin{aligned} W_n &\le \frac{\alpha V}{1-\rho} + \sigma_A + \rho\sigma_D \ , \ n \ge 1 \\ T_n &\le \frac{V}{(1-\rho)^2} + \frac{\sigma_A + \sigma_D}{\delta(1-\rho)} \ , \ n \ge 1 \\ Z(t) &\le \frac{\alpha(2-\rho)V}{1-\rho} + \sigma_A + \rho\sigma_D \ , \ t \ge 0 \ . \end{aligned}$$

Proof. It follows from (2) and (14) that $B^A_{s,t} \le \sigma_A$, and from (3) and (15) that $-B^D_{s,t} \le \sigma_D$, for arbitrary $s < t$. Then in Lemma 3 one can replace all the $\max\{\cdots\}$ terms by the appropriate term, σ_A or σ_D, replace the W_N terms by 0, and remove the ρ^{n-N} terms to increase the right-hand sides. ∎

Remark 2. It is known that the average cycle time in certain stationary polling systems is given by $\bar{V}/(1-\rho)$, where $\bar{V}$ is the average down (walking) time per cycle. (See, e.g., [4]. A sample-path proof in a general setting is given in Theorem 4 in the next section.) The difference between this expression and the bound obtained in Theorem 1 can be large for ρ close to 1. The following questions arise: (1) Is the bound tight? (2) Can it be improved by giving more information on our system? (3) Is the condition $\rho < 1$ necessary ?

It turns out that the difference between the bound given for T_n in Theorem 1 and $\bar{V}/(1-\rho)$ is due to the weak assumptions that we have made regarding the polling regime. (Recall that we require only that all the work present in the beginning of a cycle should leave during that cycle.) The following example shows that indeed our bound is tight, in the sense that any upper bound has to be at least $\bar{V}/(1-\rho)^2$.

Consider a polling system with a single queue, an arrival process with a constant rate of $\rho < 1$, a departure process with a constant rate of 1, and constant walking times equal to V. Assume that the polling uses the following gated discipline for a very long time: when the server leaves the queue a cycle begins; only customers present at the beginning of a cycle are served in the current cycle (cf. the example of a globally gated discipline in the previous section). Since $T_{n+1} = \rho T_n + V$, it follows that the average cycle time converges to $V/(1-\rho)$. Fix $\epsilon > 0$ and let m be such that $T_m > V/(1-\rho) - \epsilon$. Then the work in the queue at the end of the m^{th} cycle is greater than $\rho[V/(1-\rho) - \epsilon]$. Assume now that at the $m + 1^{st}$ cycle the polling discipline changes to exhaustive, i.e. the server remains at the queue until it is empty. At the beginning of the $m+1^{st}$ cycle, the amount of work in the system satisfies $Z(t_m) > \rho[V/(1-\rho)-\epsilon]$. When the server arrives to the queue, the amount of work in the queue is $Z(t_m)+\rho V$. So the time to empty the queue is

$$\frac{Z(t_m)+\rho V}{1-\rho} > \rho\left(\frac{V}{1-\rho}-\epsilon+V\right)(1-\rho)^{-1} ,$$

so that

$$\begin{aligned} T_{m+1} &> \rho\left(\frac{V}{1-\rho}-\epsilon+V\right)(1-\rho)^{-1}+V \\ &= \frac{V}{(1-\rho)^2}-\frac{\rho\epsilon}{1-\rho} . \end{aligned}$$

Hence the answer to question (1) is “yes": any upper bound has to be at least $V/(1-\rho)^2$.

The fact that the bounds were shown to be tight implies that the condition $\rho < 1$ is a necessary condition for stability: we see that arbitrarily large workloads and cycle times can be obtained as $\rho \to 1$. A partial answer to question (2) is given by Altman and Kofman [3], who consider polling systems with a finite number of queues. The bounds are improved by restricting to fixed polling disciplines (e.g. gated or exhaustive), and then exploiting specific characteristics of the polling regime. However, this requires a case-by-case analysis of different polling regimes. The methodology used in [3] to improve the bounds involves inductive arguments to estimate different quantities at each time that the server arrives to a queue (rather than at each time that a cycle begins).

Finally we show that the output from the system also satisfies a Cruz-type bound (of the same type as (14)) with the same average rate as the input average rate.

Theorem 2 *Under the assumptions of Theorem 1, for all* $0 \le t < t'$,

$$D(t')-D(t) \le \alpha(t'-t)+\frac{\alpha(2-\rho)V}{1-\rho}+\sigma_A+\rho\sigma_D .$$

Proof. For $0 \le t < t'$, we have

$$D(t')-D(t) \le Z(t)+A(t')-A(t) = Z(t)+\alpha(t'-t)+B^A_{t,t'} . \tag{16}$$

Suppose $t_n \leq t < t_{n+1}$, where $n \geq 0$. Using (13), we obtain

$$D(t') - D(t) \leq \alpha(t'-t) + \frac{\alpha V}{1-\rho} + \alpha V \quad + \quad \left(\sum_{i=1}^{n} \rho^{n-i} B_i^A + B_{t_n,t}^A + B_{t,t'}^A\right)$$
$$- \quad \rho\left(\sum_{i=1}^{n} \rho^{n-i} B_i^D + B_{t_n,t}^D\right) \,.$$

The desired result then follows, again using Lemma 2. ∎

Remark 3. The bounds obtained in this section may be useful in the analysis and design of communication systems. The characterization of the input process in terms of the average rate and burstiness is typical of the traffic in a communication network at the output of a spacer or a leaky bucket. The bound for the workload in the system can be used to obtain upper bounds on the size of the buffers required so as to guarantee no losses. The strict bound on the cycle time is also quite desirable in communication applications, especially in order to guarantee some service quality for synchronous traffic (such as voice or video). Indeed, several token-ring protocols possess some distributed mechanism to enforce a strict upper bound on the cycle time; an example is the *FDDI* protocol. The fact that an upper bound for the cycle times is obtained for the Cruz-type arrival process suggests that previous distributed mechanisms to enforce such constraints can be replaced by shaping the input flows to a token ring using leaky buckets. Moreover, the fact that the output process satisfies a Cruz-type bound may be used in analysing networks where the output from one token ring can be the input to another element of the network. Applications to token rings are discussed in more detail in [3] and [2].

4 Limiting Average Rates

In this section we consider systems in which the input and output processes have limiting average rates. Specifically, in the same spirit as in Stidham and El-Taha [24], we make the following assumptions:

$$\lim_{t\to\infty} \frac{A(t)}{t} \quad = \quad \alpha \,, \tag{17}$$

$$\lim_{t\to\infty} \frac{\int_0^t u(s)\, dD(s)}{\int_0^t u(s)\, ds} \quad = \quad \delta \,. \tag{18}$$

We interpret δ as the sample-path version of the conditional departure rate, given that the system is "up". We also assume that

$$\lim_{n\to\infty} \frac{V_n}{t_n} = 0; . \tag{19}$$

Theorem 3 *Assume (17), (18), (19), and $\rho = \alpha/\delta < 1$. Then*

$$\lim_{n\to\infty} \frac{W_n}{t_n} = 0 \tag{20}$$

$$\lim_{n\to\infty} \frac{T_n}{t_n} = 0 \tag{21}$$

$$\lim_{t\to\infty} \frac{Z(t)}{t} = 0 \tag{22}$$

Proof. Let $\epsilon > 0$ be given and suppose N is large enough so that for all $i \geq N$ and $t \geq t_N$,

$$\begin{array}{rcl} (\alpha-\epsilon)t \leq & A(t) & \leq (\alpha+\epsilon)t \ , \\ (\delta-\epsilon)\int_0^t u(s)\,ds \leq & \int_0^t u(s)\,dD(s) & \leq (\delta+\epsilon)\int_0^t u(s)\,ds \ , \\ & V_i & \leq \epsilon t_i \ , \end{array}$$

Then for $n > N$ and $t_N \leq s < t \leq t_n$,

$$A(t) - A(s) \leq \alpha t + \epsilon t - \alpha s + \epsilon s \leq \alpha(t-s) + 2\epsilon t_n \ ,$$

so that $B^A_{s,t} \leq 2\epsilon t_n$; similarly,

$$D(t) - D(s) \geq \int_s^t u(x)\,dD(x) \geq \delta(\int_s^t u(x)\,dx) - 2\epsilon t_n \ ,$$

so that $-B^D_{s,t} \leq 2\epsilon t_n$. Replacing the $\max\{\cdots\}$ terms in Lemma 3 with $2\epsilon t_n$ (in the case of (7) and (8)) or $2\epsilon t_{n+1}$ (in the case of (9)), and replacing V with ϵt_n or ϵt_{n+1}, and dividing by t_n or t_{n+1} (as the case may be) gives

$$\frac{W_n}{t_n} \leq \frac{\rho^{n-N} W_N}{t_n} + \left[\frac{\alpha(1-\rho^{n-N})}{1-\rho} + 2(1-\rho)\right]\epsilon$$

$$
\begin{aligned}
\frac{T_n}{t_n} &\leq \frac{\rho^{n-N} W_n}{t_n \alpha(1-\rho)} + \left[\frac{(1-\rho^{n-N})}{(1-\rho)^2} + \frac{4}{\delta(1-\rho)} \right] \epsilon \\
\frac{Z(t)}{t_{n+1}} &\leq \frac{\rho_{n-N} W_N}{t_{n+1}} + \left[\frac{\alpha(2-\rho^{n-N})}{1-\rho} + 2(1-\rho) \right] \epsilon \ \ , \text{ for } t_n \leq t \leq t_{n+1}
\end{aligned}
$$

Taking limits as $n \to \infty$, (20) and (21) follow. Define $N(t) := \max\{n : t_n < t\}$. Then

$$
\begin{aligned}
\frac{Z(t)}{t} &= \frac{Z(t)}{t_{N(t)+1}} \cdot \frac{t_{N(t)+1}}{t} \\
&\leq \frac{Z(t)}{t_{N(t)+1}} \cdot \frac{t_{N(t)+1}}{t_{N(t)}} ,
\end{aligned}
$$

and thus (22) follows from (21). ∎

Remark 4. The condition (22) is a kind of pathwise stability condition, called *rate stability* by El-Taha and Stidham [14] (see also Stidham and El-Taha [24]). Given $\lim_{t\to\infty} A(t)/t = \alpha < \infty$, it is equivalent to equality of the input and output rates:

$$
\lim_{t\to\infty} \frac{D(t)}{t} = \lim_{t\to\infty} \frac{A(t)}{t} = \alpha .
$$

As an elementary consequence, we obtain the following result for the long-run fraction of time that the system is down.

Corollary 1 *Under the conditions of Theorem 3,*

$$
\liminf_{t\to\infty} \frac{\int_0^t v(s)\, ds}{t} \geq 1 - \rho . \tag{23}
$$

If in addition $\int_0^t v(s)\, dD(s) = 0$ *for all* $t \geq 0$*, then*

$$
\lim_{t\to\infty} \frac{\int_0^t v(s)\, ds}{t} = 1 - \rho . \tag{24}
$$

Proof. We have

$$
\begin{aligned}
Z(0) + A(t) &= Z(t) + D(t) \\
&\geq Z(t) + \int_0^t u(s)\, dD(s) ,
\end{aligned}
$$

so that

$$\frac{Z(0)}{t} + \frac{A(t)}{t} \geq \frac{Z(t)}{t} + \frac{\int_0^t u(s)\, dD(s)}{\int_0^t u(s)\, ds} \cdot \frac{\int_0^t u(s)\, ds}{t} \; .$$

Taking limits and using (22), one obtains

$$\alpha \geq \delta \left(\limsup_{t\to\infty} \frac{\int_0^t u(s)\, ds}{t} \right) \; ,$$

from which (23) follows immediately. When $\int_0^t v(s)\, dD(s) = 0$, we have $D(t) = \int_0^t u(s)\, dD(s)$ for all $t \geq 0$ and hence (24) holds. ∎

In the following theorem we derive an expression for the long-run average cycle length.

Theorem 4 *Under the assumptions of Theorem 3,*

$$\liminf_{n\to\infty} \frac{t_n}{n} \quad \leq \quad (1-\rho)^{-1} \liminf_{n\to\infty} \frac{\sum_{i=1}^n V_i}{n} \; , \tag{25}$$

$$\limsup_{n\to\infty} \frac{t_n}{n} \quad \geq \quad (1-\rho)^{-1} \limsup_{n\to\infty} \frac{\sum_{i=1}^n V_i}{n} \; . \tag{26}$$

If, in addition, $\lim_{n\to\infty} n^{-1} \sum_{i=1}^n V_i = V < \infty$, *then*

$$\lim_{n\to\infty} \frac{t_n}{n} = \frac{V}{1-\rho} \; . \tag{27}$$

Proof. Let n and m be positive integers where $n > m$. Then

$$W_m + A(t_n) - A(t_m) = W_n + D(t_n) - D(t_m) \; ,$$

or, equivalently,

$$W_m + \alpha(t_n - t_m) + B^A_{t_m,t_n} = W_n + \delta(t_n - t_m) + B^D_{t_m,t_n} - \delta \sum_{i=m+1}^{n} V_i \; .$$

Rearranging terms, we have

$$(\delta - \alpha)t_n + W_n - B^A_{t_m,t_n} + B^D_{t_m,t_n} = \delta \sum_{i=1}^{n} V_i + E_m \; , \tag{28}$$

where $E_m := W_m - \delta \sum_{i=1}^{m} V_i + (\delta - \alpha)t_m$. Now let ϵ and N be given as in Theorem 3 and suppose that $n > m \geq N$. Then $|B^A_{t_M,t_n}| \leq 2\epsilon t_n$ and $|B^D_{t_M,t_n}| \leq 2\epsilon t_n$. Moreover, from (20) it follows that $0 \leq W_n/t_n \leq \epsilon$ for sufficiently large n. Using these bounds in (28) and dividing by n, we obtain

$$(\delta - \alpha - 5\epsilon)\frac{t_n}{n} \leq \frac{\delta \sum_{i=1}^{n} V_i}{n} + \frac{E_m}{n} \leq (\delta - \alpha + 5\epsilon)\frac{t_n}{n} \ .$$

Now fix $m \geq N$ and let $n \to \infty$. Since E_m is constant and ϵ was arbitrary, we conclude that

$$(\delta - \alpha) \liminf_{n\to\infty} \frac{t_n}{n} \leq \delta \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} V_i}{n} \leq \delta \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} V_i}{n} \leq (\delta - \alpha) \limsup_{n\to\infty} \frac{t_n}{n} \ ,$$

from which the desired result follows upon dividing by $\delta - \alpha$. ∎

Remark 5. Note that the proof of Theorem 4 did not use our basic assumption that $D(t_n) - D(t_{n-1}) \geq W_{n-1}$, for all $n \geq 1$, except implicitly when we invoked Theorem 3 to conclude that (20) holds, that is,

$$\lim_{n\to\infty} \frac{W_n}{t_n} = 0 \ .$$

In fact the proof demonstrates that Theorem 3 holds for any system in which (17), (18), and (19) hold, with $\alpha < \delta$, provided that (20) also holds. In particular, it suffices that the system be rate stable: $Z(t)/t \to 0$ as $t \to \infty$.

The fact that the average cycle length equals $V/(1-\rho)$ for a large class of polling systems seems to be part of the folklore, but we could not find a previous pathwise proof in the literature. A proof is given in Altman et al [4], but the proof there uses the theory of stationary point processes. The assumptions are in some sense orthogonal to ours in that they assume stationarity but not ergodicity of the processes involved.

Remark 6. The pathwise assumptions in this section and the last are usually easier to verify in polling systems if the cycles are defined in the "natural" way. That is, each cycle ends when the server completes the walking time between the last queue and first queue. Then as long as the walking time between each of the queues is bounded or has a long run average, the total walking time in each cycle will be bounded or have a long run average. The situation in systems where the cycles are not defined in the "natural" way becomes more complicated. For example, as we observed in Remark 1 in Section 2, we can define the cycles so that the output Assumption is trivially satisfied. But then, typically, the bounds or growth conditions on down times per cycle or the constraint that $\rho < 1$ will not be satisfied.

To illustrate, consider a system with two queues and cycles defined trivially as above. The service intervals and walking times are always one time unit. The server can serve two customers per service interval at queue 1, and one customer per service interval at queue 2. All arrivals occur during the walking time between queue 2 and queue 1, with four customers arriving to queue 1 and one customer arriving to queue 2. We define the server to be "up" when it is serving at queue 2. Thus $\rho = 5/8 < 1$. However, it is not difficult to show that for any initial work load, the "artificial" cycles we defined to meet the output assumption will start to double in length each successive cycle after some initial number of cycles. As a result $\{V_n\}$ is unbounded and $V_n/t_n \to 3/8$ as $n \to \infty$. The restrictions on V_n in this section and the last are not met.

This example points up the delicate interplay between the output assumption, the requirement that $\rho < 1$, and the bounds or growth conditions on V_n. In particular, we cannot simply define cycles so that the output condition is met and define up time so that $\rho < 1$, if as a result the constraints on down time per cycle end up being violated.

5 Law of the Iterated Logarithm

In this section we show how our pathwise analysis can be used to characterize the asymptotic behavior of W_n more precisely, in the presence of information about the rate of convergence of $A(t)/t$ to α. To keep the exposition simple we shall confine our attention to the special case of a constant unit output rate while the system is up. That is, we assume that

$$\frac{\int_0^t u(x)\, dD(x)}{\int_0^t u(x)\, dx} = \delta = 1\ ,\ t \geq 0\ . \tag{29}$$

Thus $\rho = \alpha$. (Similar results will hold in the general case.)

Let $h(t)$ be a monotone non-decreasing function of t such that $\lim_{t\to\infty} h(t) = \infty$. We make the following assumptions about $\{A(t), t \geq 0\}$ and $\{V_n, n \geq 1\}$:

$$\limsup_{t\to\infty} \left| \frac{A(t) - \rho t}{h(t)} \right| \leq 1\ , \tag{30}$$

$$\lim_{n\to\infty} \frac{V_n}{h(t_n)} = 0\ . \tag{31}$$

Theorem 5 *Assume (29), (30), (31), and $\rho = \alpha < 1$. Then*

$$\limsup_{n\to\infty} \frac{W_n}{h(t_n)} \le 2 \tag{32}$$

Proof. Let $\epsilon > 0$ be given. It follows from (30) and (31) that there exists an integer N such that, for all $t_N \le s < t \le t_n$,

$$A(t) - A(s) \le \rho(t-s) + 2(1+\epsilon)h(t_n) \ ,$$

and, for all $N+1 \le i \le n$,

$$V_i \le \epsilon h(t_n) \ .$$

It then follows from (4) and Lemma 2 that

$$W_n \le \rho^{n-N} W_N + \left(\frac{\rho\epsilon}{1-\rho} + 2(1+\epsilon)\right) h(t_n) \ ,$$

for all $n \ge N+1$. Since ϵ was arbitrary, the desired result, (32), then follows by dividing both sides by $h(t_n)$ and letting $n \to \infty$. ∎

Theorem 5 applies, for example, to stochastic models in which the input process, $\{A(t), t \ge 0\}$, satisfies a Law of the Iterated Logarithm (*LIL*), in which case (30) holds with probability one with

$$h(t) = \sigma\sqrt{2t \log\log t} \ . \tag{33}$$

6 The Stochastic Case

In the previous three sections we considered a single sample path for which we obtained both performance bounds and stability conditions. In this section we shall discuss stochastic models. We first note that the assumptions on the arrival and departure processes in Section 4 (specifically, (17), (18)) hold for any ergodic arrival and departure processes with probability 1. In that case, Theorems 1 and 4 still hold, where the equalities and inequalities should be interpreted in the almost sure sense. The assumptions in Section 5 also hold in an a.s. sense for many ergodic arrival processes, with h given in (33), and hence Theorem 5 may be interpreted in the a.s. sense for these cases as well.

Now we consider stochastic arrival processes of the type introduced by Yaron and Sidi [25] and Chang [11] characterized by bounds on the tail distribution. We shall consider a general bound and then specialize to both exponential and polynomial bounds. More precisely, we assume that there exist some constant $\rho > 0$, and a non-increasing function $G : R \to R_+$ such that for all $\sigma > 0$,

$$P\{A(t) - A(s) - \rho(t-s) > \sigma\} \le G(\sigma)\ ,\ 0 \le s < t < \infty\ . \tag{34}$$

We shall further restrict the arrival process and the polling discipline to ensure that (34) holds when replacing t and s by some random times. Specifically, we shall assume that it holds also for t and s chosen as the beginning and end of a cycle. That is, for all $\sigma > 0$,

$$P(B_n^A > \sigma) = P\{A(t_{n+1}) - A(t_n) - \rho(t_{n+1} - t_n) > \sigma) \le G(\sigma)\ ,\ n \ge 0\ . \tag{35}$$

Condition (35) is natural for a quite general class of arrival processes. At the end of this section we show that this includes i.i.d. arrivals each of which brings an amount of work that satisfies some tail condition.

Throughout this section we shall make the following simplifying assumptions: (i) departures occur at a constant unit rate; (ii) the down times are uniformly bounded, i.e. there exists some constant V such that $V_n \le V$ a.s.

We show below that if $\rho < 1$, then these assumptions imply that the workload at beginning of cycles and at arbitrary times, as well as the cycle durations, have similar types of bounds on their distributions. For the case of exponential bound, this will imply that all moments of the workload in the systems are uniformly bounded in time.

Moreover, we show that the departure process has a characterization of the same type as the input process, with the same ρ. This again is important when considering a network that consists of a number of elements, each of which maps input processes of the type (34) into departure processes of the same type (with possibly different constants). Denote

$$\overline{W} := \frac{\rho V}{1-\rho}, \qquad \overline{T} := \frac{V}{(1-\rho)^2}, \qquad \overline{Z} := \frac{\rho(2-\rho)V}{1-\rho}.$$

Theorem 6 *Assume that (35) holds. Assume $\rho < 1$ and $W_0 = 0$. Then the workload at the end of the nth cycle and the duration of the nth cycle satisfy*

$$P\left\{W_n - \overline{W} > \sigma\right\} \ \le\ \sum_{i=0}^{n-1} G(a_i \rho^{-i})\ , \tag{36}$$

$$P\left\{T_n - \overline{T} > \sigma\right\} \ \le\ \sum_{i=0}^{n-1} G\left(\rho(1-\rho) a_i \rho^{-i}\right), \tag{37}$$

for any nonnegative constants $a_0, a_1, ..., a_{n-1}$ *such that* $a_0 + a_1 + ... + a_{n-1} \leq \sigma$.

Proof: It follows from Lemma 1 that for all $n = 0, 1, ...,$ $W_n \leq \overline{W} + \sum_{i=1}^{n} \rho^{n-i} B_i^A$. Hence

$$\begin{aligned} P\left\{W_n > \overline{W} + \sigma\right\} &\leq P\left\{\sum_{i=1}^{n} \rho^{n-i} B_i^A > \sigma\right\} \\ &\leq \sum_{i=1}^{n} P\left\{\rho^{n-i} B_i^A > a_{n-i}\right\} \\ &= \sum_{i=1}^{n} P\left\{B_i^A > a_{n-i}\rho^{-n+i}\right\} \\ &\leq \sum_{i=1}^{n} G\left(a_{n-i}\rho^{-n+i}\right) \ . \end{aligned}$$

Similarly, it follows from (11) that

$$(1-\rho)T_n \leq \frac{V}{1-\rho} + \sum_{i=1}^{n-1} \rho^{n-1-i} B_i^A + B_n^A \leq \frac{V}{1-\rho} + \sum_{i=1}^{n} \rho^{n-1-i} B_i^A \ .$$

Hence,

$$\begin{aligned} P\left\{T_n - \overline{T} > \sigma\right\} &\leq P\left\{\sum_{i=1}^{n} \rho^{n-1-i} B_i^A > \sigma(1-\rho)\right\} \\ &\leq \sum_{i=1}^{n} P\left\{\rho^{n-1-i} B_i^A > a_{n-i}(1-\rho)\right\} \\ &\leq \sum_{i=1}^{n} G\left(\rho(1-\rho)a_{n-i}\rho^{-n+i}\right) \ , \end{aligned}$$

from which the theorem follows. ∎

Corollary 2 *Assume that (35) holds. Assume* $\rho < 1$ *and* $W_0 = 0$. *Choose some arbitrary* $\epsilon > 0$. *Consider an exponential bound on the tail distribution of the arrivals (35):*

$$G(\sigma) = K_0 \exp(-k_0\sigma) \ , \ \sigma \geq 0 \ , \tag{38}$$

where k_0, K_0 *are some positive constants. Then the workload at the end of the nth cycle satisfies*

$$P\left\{W_n - \overline{W} > \sigma + \epsilon\right\} \leq K_1 \exp\{-(\sigma+\epsilon)k_1\} \ , \ n \geq 1 \ , \tag{39}$$

for all $\sigma \geq 0$, where k_1 and K_1 are constants given by

$$k_1 := (1-\rho)^2 k_0 \ , \qquad K_1 := \frac{K_0}{1-\exp(-k_1\epsilon)} .$$

The duration of the nth cycle satisfies

$$P\left\{T_n - \overline{T} > \sigma + \epsilon\right\} \leq K_1' \exp\{-(\sigma+\epsilon)k_1'\} \ , \ n \geq 1 \ , \tag{40}$$

for all $\sigma \geq 0$, where k_1' and K_1' are constants given by

$$k_1' := \rho(1-\rho)^3 k_0 \ , \qquad K_1' := \frac{K_0}{1-\exp(-k_1'\epsilon)} .$$

Proof. The proof is obtained by applying Theorem 6. Set

$$a_i := (i+1)\rho^i d, \qquad \text{where } d := (1-\rho)^2\sigma' \ .$$

Note that $\sum_{i=0}^{\infty} a_i = \sigma'$. We obtain in particular,

$$\begin{aligned} P(W_n - \overline{W} > \sigma') &\leq \sum_{i=0}^{\infty} G(a_i\rho^{-i}) \\ &= K_0 \sum_{i=0}^{\infty} \exp\{-k_0(i+1)d\} \\ &= \frac{K_0}{1-\exp\{-dk_0\}} \exp\{-dk_0\} \ , \end{aligned}$$

for all n. For $\sigma' := \sigma + \epsilon$, we have

$$\begin{aligned} \exp\{-dk_0\} &= \exp\{-(\sigma+\epsilon)k_1\} \\ &\leq \exp\{-\epsilon k_1\} \ , \end{aligned}$$

from which the bound for W_n follows. Similarly.

$$\begin{aligned} P\{T_n - \overline{T} > \sigma'\} &\leq \sum_{i=0}^{\infty} G(\rho(1-\rho)a_i\rho^{-i}) \\ &= K_0 \sum_{i=0}^{\infty} \exp\{-k_0\rho(1-\rho)(i+1)d\} \\ &= \frac{K_0}{1-\exp\{-\rho(1-\rho)dk_0\}} \exp\{-\rho(1-\rho)dk_0\} \ , \end{aligned}$$

for all n. For $\sigma' := \sigma + \epsilon$, we have

$$\begin{aligned} \exp\{-\rho(1-\rho)dk_0\} &= \exp\{-(\sigma+\epsilon)k_1'\} \\ &\leq \exp\{-\epsilon k_1'\} \ , \end{aligned}$$

from which the bound for T_n follows. ∎

Corollary 3 *Assume that (35) holds. Assume $\rho < 1$ and $W_0 = 0$. Consider a polynomial bound on the tail distribution of the arrivals (35):*

$$G(\sigma) = K_0 \sigma^{-m} \ , \tag{41}$$

where K_0, m are positive constants, with $m > 1$. Then the workload at the end of the nth cycle satisfies

$$P\left\{W_n - \overline{W} > \sigma\right\} \leq K_1 \sigma^{-m} \ , \ n \geq 1 \ , \tag{42}$$

for all $\sigma \geq 0$, where K_1 is given by

$$K_1 := K_0 (1-\rho)^{-2m} \sum_{l=1}^{\infty} l^{-m}.$$

The nth cycle duration satisfies

$$P\left\{T_n - \overline{T} > \sigma\right\} \leq K_1' \sigma^{-m} \ , \ n \geq 1 \ , \tag{43}$$

for all $\sigma \geq 0$, where K_1' is given by

$$K_1' := K_0 \rho^{-m} (1-\rho)^{-3m} \sum_{l=1}^{\infty} l^{-m}.$$

Proof. As in Corollary 2, set

$$a_i := (i+1)\rho^i d, \qquad \text{where } d := \sigma(1-\rho)^2 \ .$$

The proof for W_n is again obtained by applying Theorem 6, and

$$G(a_i \rho^{-i}) = G((i+1)d) = K_0 d^{-m} (i+1)^{-m}.$$

The proof for T_n is similar. ∎

Next, we obtain bounds on the amount of work at an arbitrary moment, and on the departure process. Let $n(t)$ denote the (random) number of cycles that started prior to time t. We shall need the following assumption:

$$P(B^A_{t_{n(t)},t} > \sigma) = P\{A(t) - A(t_{n(t)}) - \rho(t - t_{n(t)}) > \sigma) \leq G(\sigma)\ ,\ t \geq 0\ . \tag{44}$$

Theorem 7 *Assume that (35) and (44) hold. Assume $\rho < 1$ and $W_0 = 0$. Then the workload at an arbitrary time point t and the departures during an arbitrary interval$(t, t']$ satisfy*

$$\begin{aligned} P\left\{Z(t) - \overline{Z} > \sigma\right\} &\leq \sum_{l=0}^{\infty} G(a_l \rho^{-l}) + G(a'), \\ P\left\{D(t') - D(t) - \rho(t' - t) > \sigma\right\} &\leq \sum_{l=0}^{\infty} G(a_l \rho^{-l}) + G(a') + G(a''), \end{aligned}$$

for any nonnegative constants $a', a'', a_0, a_1, \ldots$ such that $a' + a'' + \sum_{i=0}^{\infty} a_i \leq \sigma$.

Proof. It follows from (13) that

$$Z(t) \leq \overline{Z} + \sum_{i=1}^{n} \rho^{n-i} B^A_i + B^A_{t_{n(t)},t}.$$

The bound for $Z(t)$ is now obtained as in the proof of Theorem 6. The bound for $D(t') - D(t)$ is obtained similarly by noting that

$$D(t') - D(t) - \rho(t' - t) \leq Z(t) + B^A_{t,t'}.$$

■

As was done in Corollaries 2 and 3 for W_n and T_n, one may also obtain explicit exponential or polynomial bounds on the tail probabilities of $Z(t)$ and on the departure process by using Theorem 7, when such bounds hold for the arrival process.

Sufficient conditions for the burstiness constraint (35)

Since (35) is slightly different from the standard burstiness constraints (34) on the arrivals ([25], [11]), we present sufficient conditions for it to hold.

We first show that for both exponential-type and polynomial-type bounds (35) is satisfied under the following condition. Assume that there is a constant $\rho > 0$, and a non-increasing function $\overline{G}$ such that, for all $\sigma > 0$,

$$P\{A(t_n + t) - A(t_n) \geq \rho \cdot t + \sigma\} \leq \overline{G}(\sigma) \ , \ n \geq 0 \ , \ t \geq 0 \ . \tag{45}$$

Indeed, if (45) holds then, for all $\hat{\rho} > \rho$, $\Delta > 0$, $n \geq 0$, $\sigma \geq 0$,

$$\begin{aligned}
&P\{A(t_{n+1}) - A(t_n) \geq \hat{\rho}(t_{n+1} - t_n) + \sigma\} \\
&= \sum_{l=0}^{\infty} P\{A(t_{n+1}) - A(t_n) \geq \hat{\rho}(t_{n+1} - t_n) + \sigma \ ; \ l\Delta \leq t_{n+1} - t_n \leq (l+1)\Delta\} \\
&\leq \sum_{l=0}^{\infty} P\{A(t_n + (l+1)\Delta) - A(t_n) \geq \rho\Delta(l+1) + \sigma + \Delta(\hat{\rho}l - \rho(l+1))\} \\
&\leq \sum_{l=0}^{\infty} \overline{G}(\sigma + \Delta(\hat{\rho}l - \rho(l+1))) \ .
\end{aligned}$$

Consider first the exponential bound, i.e., assume that there exist $\overline{K}_0$, $\overline{k}_0$ such that

$$\overline{G}(\sigma) = \overline{K}_0 \cdot \exp\{-\overline{k}_0 \cdot \sigma\}.$$

Then we obtain

$$\begin{aligned}
P\{A(t_{n+1}) - A(t_n) \geq \hat{\rho}(t_{n+1} - t_n) + \sigma\} &\leq \sum_{l=0}^{\infty} \overline{K}_0 \exp\{-\overline{k}_0(\sigma + \Delta(\hat{\rho}l - \rho(l+1)))\} \\
&= \overline{C} \cdot \overline{K}_0 \cdot \exp\{-\overline{k}_0 \sigma\} \ ,
\end{aligned}$$

where

$$\overline{C} := \exp\{\Delta\rho\overline{k}_0\}/[1 - \exp\{-\Delta\overline{k}_0(\hat{\rho} - \rho)\}] < \infty \ .$$

For the case of polynomial bounds, i.e.,

$$\overline{G}(\sigma) = C/\sigma^{m+1} \ ,$$

for some $C, m > 0$, we get $G(x) = c/x^m$ (where $c > 0$ is some constant).

Next, we present sufficient conditions for (45). We shall assume that work arrives to the system with "customers". Denote by τ_i the time between the arrival of customer i and customer

$i+1$, $i \geq 1$. Denote by s_i the "work" that customer i brings (i.e., the service time required by it). Without loss of generality, assume that customer 1 arrives to an empty system at time $t = 0$.

Lemma 4 *Assume*

(i) the inter-arrival times of the customers $\{\tau_i\}$ form an i.i.d. sequence with finite positive mean;

(ii.a) the service times $\{s_i\}$ form an i.i.d. sequence; (ii.b) there exists a constant $\lambda > 0$ such that $E\exp\{\lambda s_1\} < \infty$, and the sequences $\{\tau_i\}$ and $\{s_i\}$ are independent;

(iii) $\{t_n\}$ are "independent of the future" in the following sense: for all r, n, an event $B_{n,r} \equiv \{\sum_{j=1}^{r-1} \tau_j < t_n \leq \sum_{j=1}^{r} \tau_j\}$ is independent of the σ-algebra generated by the sequences $\{\{\tau_l\}_{l>r}; \{s_l\}_{l>r}\}$.

Then there exists a constant $\rho > 0$ such that, for all $\sigma > 0$, the bound (45), and hence (35), hold with exponential tail.

If (ii.b) is replaced by the assumption that $Es_1^{m+2} < \infty$ for some integer m, then (45), and hence (35), hold with a polynomial tail: $\overline{G}(x) = C/x^{m+1}$ and $G(x) = c/x^m$, for some constants c and C.

Since the proof is technical, it will be left to the appendix.

Remark 7. Property (iii) is the only one that is related to the specific polling policy that is used. It is known to hold for a very large class of policies, including gated type and exhaustive type policies (see [1]).

Remark 8. We may generalize the arrival process considered in Lemma 4 to a regenerative type of arrival process, and still get the same type of results. Consider some regeneration times $T_1, T_2, \ldots$. Define $\tau_i = T_{i+1} - T_i$. The location of arrivals and the workloads they require in each regenerative period are given by a measure M defined on $([0, \tau_i) \times \mathcal{E})$, where $\mathcal{E}$ is the set of possible values of the workloads. For any Borel sets $B_1 \subset \mathcal{R}$, $B_2 \subset \mathcal{E}$, $M(B_1 \times B_2)$ is the amount of workload that arrives at time points in B_1 with values in B_2. Assume that τ_i, and the amount of work that arrives during a regenerative period both have finite first moments. Under fairly general conditions, this type of arrival process allows, in particular, for K independent i.i.d. arrival streams into K nodes see [15].

7 APPENDIX: Proof of Lemma 4

For all r, n, we have

$$A(t_n+t)-A(t_n)\le s_{r+1}+\sum_{i=1}^{\eta_r(t)} s_{r+i+1}$$

a.s. on the event $B_{n,r}$, where

$$\eta_r(t)=\max\{m\ge 0:\tau_{r+1}+\ldots+\tau_{r+m}\le t\}\,.\,r\ge 0\,.$$

Then

$$\begin{aligned}
&P\{A(t_n+t)-A(t_n)>\rho\cdot t+\sigma\}\\
&= \sum_{r=1}^{\infty} P(B_{n,r})\cdot P\{A(t_n+t)-A(t_n)>\rho\cdot t+\sigma|B_{n,r}\}\\
&\le \sum_{r=1}^{\infty} P(B_{n,r})\cdot P\left\{ s_{r+1}+\sum_{i=1}^{\eta_r(t)} s_{r+i+1}>\rho\cdot t+\sigma \,\middle|\, B_{n,r}\right\}\\
&= \sum_{r=1}^{\infty} P(B_{n,r})\cdot P\left\{ s_{r+1}+\sum_{i=1}^{\eta_r(t)} s_{r+i+1}>\rho\cdot t+\sigma \right\}\\
&= P\left\{ s_1+\sum_{i=1}^{\eta_0(t)} s_{i+1}>\rho\cdot t+\sigma \right\}\ ,
\end{aligned}$$

where the next-to-last equality follows from condition (iii) and the last equality from assumptions (i) and (ii.a). We see that our problem is reduced to estimating the probability

$$P_{t,\sigma}:=P\left\{ s_1+\sum_{i=1}^{\eta(t)} s_{i+1}>\rho\cdot t+\sigma \right\}\ , \tag{46}$$

where $\eta(t)=\eta_0(t)$. Set $a=E\tau_1$, $b=Es_1$. Choose $\rho>b/a$ and set $c=\rho a/b>1$. Fix some constants $\tilde{c}$ and $\tilde{\sigma}$ satisfying $1<\tilde{c}<c$, $\tilde{\sigma}=\alpha\sigma$, where $0<\alpha<1$ is arbitrary. Denote $x=(\tilde{c}/a)t+\tilde{\sigma}$, and let $\lceil x\rceil$ denote the smallest integer greater than or equal to x. We shall decompose (46) into two parts which we shall estimate separately:

$$P_{t,\sigma}\le P_1+P_2,\qquad P_1:=P\left\{\eta(t)\ge\lceil x\rceil\right\}\ ,\qquad P_2:=P\left\{\sum_{i=1}^{\lceil x\rceil} s_i>\rho t+\sigma\right\}\ .$$

Let $T_n := \tau_1 + ... + \tau_n$ and let $\gamma > 0$ be arbitrary. Then

$$\begin{aligned} P_1 &= P\{T_{\lceil x \rceil} \le t\} = P\left\{\exp\{-\gamma T_{\lceil x \rceil}\} \ge \exp\{-\gamma t\}\right\} \\ &\le [\psi(\gamma)]^x \exp(\gamma t) = [\psi(\gamma)]^{\tilde{\sigma}} \left[\psi(\gamma)\exp\{\gamma a/\tilde{c}\}\right]^{(\tilde{c}/a)t} \ , \end{aligned} \tag{47}$$

where $\psi(\gamma) := E\exp\{-\gamma\tau_1\}$, the Laplace-Stieltjes transform of τ_1 evaluated at γ. Now

$$\begin{aligned} \psi(\gamma) &= 1 - \gamma a + o(\gamma) = 1 - \gamma a/\tilde{c} - \gamma a(\tilde{c} - 1)/\tilde{c} + o(\gamma) \\ &\le 1 - \gamma a/\tilde{c} \le \exp\{-\gamma a/\tilde{c}\} \ , \end{aligned}$$

as $\gamma \to 0$. Hence, for $\gamma << 1$, it follows from (47) that

$$P_1 \le \exp\{\log\psi(\gamma)\tilde{\sigma}\} = \exp\{\log\psi(\gamma)\alpha\sigma\} \ ,$$

for all $t \ge 0$. (So far, we have not used any assumption on the tail probability of the service times.)

Next we bound P_2. Let t_0 and ϵ be some strictly positive constants such that

$$(\tilde{c}t_0/a + 1)(b + \epsilon) = \rho t_0 \ .$$

Consider first the case $\tilde{c}t/a + \tilde{\sigma} \ge \tilde{c}t_0/a$. Note that

$$\begin{aligned} \lceil \tilde{c}t/a + \tilde{\sigma} \rceil (b + \epsilon) &\le (\tilde{c}t/a + \tilde{\sigma} + 1)(b + \epsilon) \le \frac{\tilde{c}t/a + \tilde{\sigma}}{\tilde{c}t_0/a}(\tilde{c}t_0/a + 1)(b + \epsilon) \\ &= \frac{\tilde{c}t/a + \tilde{\sigma}}{\tilde{c}t_0/a}\rho t_0 = \rho t + \rho a\tilde{\sigma}/\tilde{c}. \end{aligned}$$

Recall that in the definition of $\tilde{\sigma}$, α was arbitrary. We now choose a specific α so that $\rho a\alpha/\tilde{c} =: q$ is less than 1. Hence $\rho a\tilde{\sigma}/\tilde{c} = q\sigma$. Denote $R := \sup_{n \ge 0}\sum_{i=1}^{n}(s_i - b - \epsilon)$. We have

$$\begin{aligned} P_2 &\le P\left\{\sum_{i=1}^{\lceil \tilde{c}t/a + \tilde{\sigma} \rceil} s_i > \rho t + \sigma\right\} \le P\left\{\sum_{i=1}^{\lceil \tilde{c}t/a + \tilde{\sigma} \rceil}(s_i - b - \epsilon) > (1 - q)\sigma\right\} \\ &\le P\{R > (1 - q)\sigma\} =: Z \ . \end{aligned}$$

Now, it is well known (cf., e.g., Asmussen[8], p. 184) that $Es_1^{m+2} < \infty$, $m \ge 0$, implies that $ER^{m+1} < \infty$, so that $Z \le M\sigma^{-(m+1)}$ for some constant $M < \infty$; likewise, if $E\exp\lambda s_1 < \infty$ for some $\lambda > 0$, then $Z \le \exp\{-M\sigma\}$ for some constant $M > 0$ (cf. Kingman [19]).

For $\tilde{c}t/a + \tilde{\sigma} < \tilde{c}t_0/a$, we have

$$P_2 \leq P\left\{\sum_{i=1}^{\lceil \tilde{c}t_0/a \rceil} s_i > \sigma\right\} \leq \sum_{i=1}^{\lceil \tilde{c}t_0/a \rceil} P\left\{s_i > \frac{\sigma}{\lceil \tilde{c}t_0/a \rceil}\right\} = \lceil \tilde{c}t_0/a \rceil P\left\{s_1 > \frac{\sigma}{\lceil \tilde{c}t_0/a \rceil}\right\} .$$

Hence P_2 has the same tail behavior as s_1. This concludes the proof. ∎

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