ERGODICITY, MOMENT STABILITY AND CENTRAL LIMIT THEOREMS OF STATION TIMES IN POLLING SYSTEMS

Eitan ALTMAN
INRIA, BP93
2004 Route des Lucioles
06902 Sophia Antipolis
France

Flos M. SPIEKSMA
Institute of Mathematics & Computer Science, University of Leiden
P.O. Box 9512, 2300RA Leiden
The Netherlands

Abstract

The station times are an important measure of performance in polling systems, and are often used to determine efficiently other performance measures, such as waiting times. In this paper we give sufficient and necessary conditions for the existence of all moments of station times in steady state, for polling systems with Gated and Globally-Gated disciplines. Moreover, we show that the moments converge geometrically fast to the steady state ones under these conditions. We then address the question of the rate of convergence of the sample averages of functions of the station times. We establish the applicability of central limit theorems (CLT) and the law of iterated logarithm (LIL) for all moments of the station times. In particular, we compute explicitly the constants involved in the CLT and LIL for the cycle durations.

Keywords: Polling, Station times, Gated and Globally-Gated disciplines, Geometric Ergodicity, Moment Stability, Central limit theorems, Law of iterated logarithm.

1 INTRODUCTION

Growing attention was given in recent years to establish rigorously sufficient and necessary stability conditions for polling systems. Several different approaches were used to obtain ergodicity as a measure of stability, see [1, 2, 3, 7, 8, 11, 12, 14]. In [1], Altman et al. further present sufficient and necessary conditions for stronger notions of stability, namely the geometric ergodicity and the geometric rate of convergence of the first moment of the process of queues' length, embedded at polling instants (see Tweedie [17, 18] for definitions of these notions of stability).

We consider a cyclic polling system with N infinite capacity queues (stations), served according to the Gated discipline. Hence, in every visit to a queue, the server serves only the customers present at the polling instant. New arrivals to the queue while being attended by the server, will wait for the next visit. We are interested in the stability of station times, which are defined as the total service time given to a queue plus the walking time to the next queue (a precise definition is given in the next section). The interest in these quantities is due to Humblet [9], and Ferguson and Aminetzah [6], who established an efficient way of computing the expected waiting times in different queues that requires the first and second moments of the station times (see also Choudhury and Takagi [5]). Indeed, when station times are used, then the state of the system is described by an N dimension vector of station times; it then follows that in order to determine the expected waiting times, a set of N^2 equations should be solved [5, 6, 9]. A method with a special low complexity of determining the expected waiting time based on the station time approach is presented in [15]. It requires solving a set of $O(N)$ linear equations. If the "buffer occupancy" method is used (see Takagi [16]) then the state of the system is described by an N^2 dimensional vector of queue lengths at different polling instants, and thus N^3 equations should be solved in order to obtain the expected waiting times. Another advantage of working with station times is that in regimes where $n < N$ disjoint groups of queues have each a global gate (see Boxma et al. [4] for the case $n = 1$ and Khamisy et al. [10] for general n) the state of the system can be described by an n-dimensional state of some generalized station times, and the number of equations to obtain the expected waiting times reduces to $n(2n-1)$. In particular, this implies that for $n = 1$ an explicit expression is obtained for the expected waiting times [4]. We shall treat the Globally Gated regime in Section 6.

In Section 3 we obtain sufficient and necessary conditions for geometric ergodicity of the station times, the existence of the first moments in steady state, and the geometric rate of convergence of the first moments to the steady state moments. In Section 4 we obtain sufficient and necessary conditions for the existence of all moments of the station times in steady state, as well as the geometric rate of convergence of the moments to those in steady state. The conditions used in Section 3 are slightly weaker than those in Section 4, which is natural since the results are weaker. However, this is not the only reason for separating the discussion on the stability of the first moment and the stability of all moments. Indeed, in Section 3 we also obtain a bound on the distance between the probabilities at any time and the steady state ones (see Theorems 2 (iv)), which is better than the one that can be obtained in Section 4.

The main tool that we use for establishing conditions for geometric ergodicity is a generalization of Foster's condition due to Tweedie [17], which is required for dealing with the non countable (infinite) state space (of the station times). This involves the construction of so called Lyapunov functions as well as the construction of "small" or "petite" sets. An appropriate choice of the Lyapunov functions implies stability of all moments [13]. Moreover, they allow to obtain estimates on the rate of convergence of the sample averages of the moments of the station times. Indeed, in Section 5 we obtain a central limit theorem (CLT) and a law of iterated logarithm (LIL) for these empirical moments.

Finally, we obtain in Section 6 similar results for the Globally Gated regime, which has recently been introduced by Boxma, Levy and Yechiali [4].

The main results of the paper are presented in Theorems 2, 5, 6, 7 and 8. Theorem 2 establishes the geometric ergodicity of the vector of station times, Theorem 5 establishes the geometric convergence of expected functions of these station times, and, in particular of all moments; Theorem 6 establishes the CLT and LIL, and Theorem 7 specifies these to the cycle time (and presents explicit expressions for the constants that appear in the CLT and LIL). Theorem 8 summarizes these results for the Globally Gated regime.

2 THE MODEL

The polling system has N queues. All queues' capacity are infinite. The server visits the queues in a cyclic order, vz. $0, 1, 2, \cdots, N-1, 0, 1, 2, \cdots$. Without loss of generality, we assume that the server arrives to queue 0 at time zero. Thus, the n-th ($n \geq 0$) queue that the server visits is queue $I(n) = n \bmod N$, which is the remainder of the division of n by N.

Inter-arrival times to queue i are independent and have Poisson distribution with rate λ_i, $i = 0, ..., N-1$. Denote $\lambda = \sum_{i=0}^{N-1} \lambda_i$.

The service times of jobs in queue i, ($i = 0, ..., N-1$), are i.i.d.

random variables distributed as B_i with first moment b_i. Denote $\rho_i \stackrel{\text{def}}{=} \lambda_i b_i$ and $\rho \stackrel{\text{def}}{=} \sum_{i=0}^{N-1} \rho_i$. Denote $b_i^*(s) = \int_0^\infty e^{-st} dF_{B_i}(t)$ the Laplace Stieltjes Transform (LST) of B_i. Let $\lambda_k = \lambda_{I(k)}$, $b_k = b_{I(k)}$ and $\rho_k = \rho_{I(k)}$ for $k \geq N$. We assume without loss of generality that $\rho_i > 0$, $i = 0, 1, ..., N-1$.

The time it takes between the instant that the server moves from the kth queue on its path (which is the moment when the server finishes service there, if that queue was non empty upon the arrival of the server to that queue) till the server arrives to the next queue is called the kth *walking time* and is denoted by D_k, $k = 0, 1, 2, ...$ We assume that the walking times are independent, and their distribution depends on k only through $I(k)$. Their LST is denoted by $d_k^*(s)$, and the first moment by d_k. Let $D = \sum_{i=0}^{N-1} D_i$ be a generic random variable distributed as the total walking time in a cycle and denote by d and $d^*(s)$ the expectation and LST of D respectively. The walking times, the inter-arrival times and the service durations are assumed to be mutually independent.

Let $\mathcal{A}_i(T)$ denote the number of arrivals to queue i, $0 \leq i \leq N-1$, during a (possibly random) time interval of length T. Let $\mathcal{B}_k(l)$ be a random variable that represents the total service time of l customers at the k-th visit of the server, $k \geq 1$. Let $\Gamma_k(\cdot) \stackrel{\text{def}}{=} \mathcal{B}_k(\mathcal{A}_{I(k)}(\cdot))$. Thus $\Gamma_k(T)$ is the time needed to serve at the kth queue that is polled all customers that arrive during a time interval T. Let $\tau(k)$ be the kth polling instant to a queue. Thus a queue that is polled at $\tau(k)$ is also polled at $\tau(k+N)$, $\tau(k+2N)$ and so on.

There are two standard ways to describe the evolution of the polling system using embedded Markov chains. The first uses the vector of the number of jobs in queue j at the kth polling instant to a queue ("buffer occupancy" method). We use a second description which is through the *station times* at polling instants [5, 6, 9]; we define the station time θ_k as the (random) time it takes to serve the kth queue that is polled, plus the walking time between that queue and the next queue. We thus consider a Markov chain embedded at time $\tau(k)$, $k = 0, 1, ...$ The *state* at time $\tau(k)$ is then taken to be the following vector of N station times:

$$\boldsymbol{\vartheta}_k = \{\theta_{k-N}, ..., \theta_{k-2}, \theta_{k-1}\}. \tag{1}$$

Thus the lth component of $\boldsymbol{\vartheta}_k$ is given by $\vartheta_k^l = \theta_{k-N+l}, \; l = 0, ..., N-1$. Denote by Θ the state space (i.e. $\Theta = R_+^N$).

3 GEOMETRIC ERGODICITY AND STABILITY OF FIRST MOMENT

The system evolves according to the following dynamics:

$$\theta_k = \Gamma_k \left(\sum_{i=1}^{N} \theta_{k-i} \right) + D_k \tag{2}$$

and thus

$$\boldsymbol{\vartheta}_{k+1} = \left\{ \theta_{k-N+1}, ..., \theta_{k-2}, \theta_{k-1}, \Gamma_k \left(\sum_{i=1}^{N} \theta_{k-i} \right) + D_k \right\},$$

which is a non-homogeneous Markov chain. However, for any $0 \le l < N$, the embedded $\boldsymbol{\vartheta}_{Nk+l}, \; k = 0, 1, ...$ is a homogeneous Markov chain, of which we prove the stability below. l is thus regarded as a reference station (and will often be chosen to be $l = 0$). Note that the values of $N-1$ components of $\boldsymbol{\vartheta}_k$ and of $\boldsymbol{\vartheta}_{k+1}$ overlap. We denote the transition probabilities of the embedded chain by $\mathcal{P}_k$. We shall omit k from the notation when $k = 0$, i.e. we shall understand $\mathcal{P} = \mathcal{P}_0$.

Denote by W_k^l the total amount of work at queue $I(k+l)$ at the kth polling instant $\tau(k)$, $l = 0, ..., N-1$. Let $W_k = \sum_{l=0}^{N-1} W_k^l + D$.

Lemma 1 *The following holds:*

$$\sum_{l=0}^{N-1} w_k^l E[\vartheta_{k+N}^l | \boldsymbol{\vartheta}_k] = E[W_k | \boldsymbol{\vartheta}_k] = \sum_{l=0}^{N-1} \vartheta_k^l \left(\sum_{i=0}^{l} \rho_i \right) + d \tag{3}$$

$$\begin{aligned} \sum_{l=0}^{N-1} w_k^l E[\vartheta_{k+N}^l - \vartheta_k^l | \boldsymbol{\vartheta}_k] &= (\rho - 1) \sum_{l=0}^{N-1} \vartheta_k^l + d \\ &= (\rho - 1) \sum_{l=0}^{N-1} w_k^l \vartheta_k^l + d + (\rho - 1) \sum_{j=1}^{N-1} \left[\rho_{j+k} \sum_{l=0}^{j-1} \vartheta_k^l \right] \end{aligned} \tag{4}$$

where

$$w_k^l \stackrel{\text{def}}{=} 1 - \sum_{j=l+1}^{N-1} \rho_{k+j}, \qquad l = 0, ..., N-1. \tag{5}$$

Proof: We have

$$\begin{aligned} E[\vartheta_{k+N}^l | \boldsymbol{\vartheta}_k] = E[\theta_{k+l} | \boldsymbol{\vartheta}_k] &= E[W_k^l | \boldsymbol{\vartheta}_k] + \rho_{k+l} \sum_{i=0}^{l-1} E[\theta_{k+i+N} | \boldsymbol{\vartheta}_k] + d_{k+l} \\ &= E[W_k^l | \boldsymbol{\vartheta}_k] + \rho_{k+l} \sum_{i=0}^{l-1} E[\vartheta_{k+N}^i | \boldsymbol{\vartheta}_k] + d_{k+l}, \qquad l = 0, ..., N-1, \end{aligned}$$

which implies the first equality in (3).

The total expected amount of work arriving during ϑ_k^l (i.e. during θ_{k+l-N}) is $\rho \vartheta_k^l$, Indeed, let $\mathcal{N}_i$ be the number of arrivals of customers to queue i during ϑ_k^l, $i = 0, 1, ..., N-1$. Let B_i^j, $i = 0, ..., N-1$, $j = 1, 2, ...$ be independent random variables such that B_i^j is distributed like B_i (the service time in queue i). Then $\mathcal{N}_i$ are independent of the service times required by these arrivals, so it follows from Wald's identity that the expected amount of work arriving during ϑ_k^l is

$$\sum_{i=0}^{N-1} E\left[\sum_{j=1}^{\mathcal{N}_i} B_i^j\right] = \sum_{i=0}^{N-1} E[\mathcal{N}_i]b_i = \sum_{i=0}^{N-1}[\lambda_i \vartheta_k^l]b_i = \rho\vartheta_k^l$$

(the second equality follows from the fact that the arrival processes to the queues are Poisson). However, at time $\tau(k)$, the work that arrived during ϑ_k^l to queues $I(k+l+1), ..., I(k+N-1)$ has already left the system, and therefore the expected amount of work arriving during ϑ_k^l that still remains at time $\tau(k)$ is $\vartheta_k^l \sum_{i=0}^{l} \rho_i$. This establishes the second equality in (3). (4) then follows immediately. ■

Theorem 2 *Assume that $b_i, d_i < \infty$ for all $i = 1, ..., \infty$ and that that $\rho < 1$. Then*
(i) for any $0 \leq l < N$, $\{\boldsymbol{\vartheta}_{l+jN}\}$, $j = 0, 1, ...$ is geometrically ergodic, i.e., there exists some positive constant $\alpha < 1$ and some probability measure π_l on the state space, such that

$$\lim_{n\to\infty} \alpha^{-n} ||\mathcal{P}_{l+nN}(\boldsymbol{\vartheta}, \cdot) - \pi_l(\cdot)|| = 0$$

where $||\cdot||$ denotes total variation of signed measures [17].
(ii) the expectation of all station times exist in steady state,
(iii) the first moments of the station times converge geometrically fast to the steady state first moment.
(iv) Let π denote the steady state distribution on Θ. Define $g : \Theta \to \mathbb{R}$ as $g(x) = 1 + \sum_{l=0}^{N-1} w_k^l x_l$. There exist some $\xi < 1$ and $R > 0$, such that $||\mathcal{P}^n(\boldsymbol{\vartheta}, \cdot) - \pi(\cdot)|| \leq R\xi^n g(\boldsymbol{\vartheta})$ for any $\boldsymbol{\vartheta} \in \Theta$, where $||\cdot||$ is the total variation norm.

In order to prove the above Theorem we need the following definitions and Proposition, for a given a Markov chain (X_n) with state space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and transition kernel $P(x, B)$.

(1) A set $\mathcal{K} \in \mathcal{B}(\mathcal{X})$ is said to be small if there exists some positive measure ϕ on $\mathcal{X}$, such that for any $B \subset \mathcal{X}$ with $\phi(B) > 0$ there exists j such that

$$\inf_{x \in \mathcal{K}} \sum_{n=1}^{j} P^n(x, B) > 0.$$

(2) (X_n) is said to be strongly aperiodic if there exists a set $\mathcal{C} \subset \mathcal{X}$, a probability ν on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ with $\nu(\mathcal{C}) = 1$, and $\delta > 0$ such that

$$L(x, \mathcal{C}) := P(X_n \text{ enters } \mathcal{C} \text{ for some } n \geq 1 | X_0 = x) > 0$$

for all $x \in \mathcal{X}$, and

$$P(x, B) \geq \delta \nu(B), \qquad x \in \mathcal{C}, B \in \mathcal{B}(\mathcal{X}).$$

Proposition 3 *Consider a strongly aperiodic Markov chain X_n on a state space $\mathcal{X}$ with transition probabilities $P : \mathcal{X} \times \mathcal{B}(\mathcal{X}) \to [0, 1]$. Assume that there exists a set $\mathcal{K} \in \mathcal{B}(\mathcal{X})$ and a function $g : \mathcal{X} \to \mathbb{R}$, $g(\cdot) \geq 1$, such that*
(i) there exists some $\epsilon > 0$ such that $E[g(X_{n+1}) - g(X_n) | X_n] \leq -\epsilon g(X_n)$ for $X_n \in \mathcal{K}^c$;
(ii) $E[g(X_{n+1}) | X_n] < \infty$ for $X_n \in \mathcal{K}$;
(iii) $\mathcal{K}$ is a small set.
Then
(a) X_n is geometrically ergodic.
(b) Let π be the steady state distribution and denote by $\hat{X}$ the RV with distribution π. For any function $f : \mathcal{X} \to \mathbb{R}$, such that $0 \leq f \leq ag$ for

some constant $a > 0$,

(b.1) $E[f(\hat{X})] < \infty$;

(b.2) There exist some $\xi < 1$ and $R \geq 0$ such that

$$\left| \int_{y \in \mathcal{X}} f(y) P^n(x, dy) - \int_{y \in \mathcal{X}} f(y) \pi(dy) \right| \leq R \xi^n g(x)$$

for all $x \in \boldsymbol{\mathcal{X}}$.

Proof: (a) is proved in [17], (or in [13], the Corollary to Theorem 6.2, when restricting to $f = g$). (b.1) is proved in [18] and (b.2) follows from [13], the Corollary to Theorem 6.2. In [13] it is required that the set $\mathcal{K}$ be a “petite" set. This is satisfied since any small set is also a petite set. ■

Proof of Theorem 2: We use Proposition 3 (with X_n denoting $\boldsymbol{\vartheta}_{nN+k}$). It follows from (4) that

$$E[g(\boldsymbol{\vartheta}_{k+N}) - g(\boldsymbol{\vartheta}_k) | \boldsymbol{\vartheta}_k] < \infty,$$

and that there exists some real number M_0 and $\epsilon > 0$ such that

$$E[g(\boldsymbol{\vartheta}_{k+N}) - g(\boldsymbol{\vartheta}_k) | \boldsymbol{\vartheta}_k] \;\; \leq \;\; -\epsilon g(\boldsymbol{\vartheta}_k)$$

if $\sum_{l=0}^{N-1} \vartheta_k^l \geq M_0$. This establishes conditions (i) and (ii) of Proposition 3. Consider the set

$$\mathcal{K}(M) = \left\{ \chi : \sum_{l=0}^{N-1} \chi^l < M \right\}, \quad \chi \in \Theta.$$

We shall show that $\mathcal{K}(M)$ is a small set for any $M > 0$, thus establishing condition (iii) as well. Define $\phi_k(B) = P(\{D_k, ..., D_{k+N-1}\} \in B)$. Let C_n denote the time elapsed from the nth polling instant of a queue, till that queue is polled again (the $(n+N)th$ polling instant of a queue). Let S_k

denote the event of no customers present at the kth polling instant $\tau(k)$. Let R_k denote the event of no arrivals during C_k. Choose an arbitrary $\boldsymbol{\vartheta} \in \mathcal{K}(M)$.

We first lower bound $P(S_k|\boldsymbol{\vartheta}_k = \boldsymbol{\vartheta})$ and $P(R_k|S_k, \boldsymbol{\vartheta}_k = \boldsymbol{\vartheta})$. For any subset $\mathcal{S}$ of queues, $\mathcal{S} \in \{0, 1, ..., N-1\}$, we denote $Q_{\mathcal{S}}(T) = \exp\{-\sum_{i\in\mathcal{S}} \lambda_i T\}$ for $T \in \mathbb{R}$. $Q_{\mathcal{S}}(T)$ is the probability of no arrivals during an interval of length T to the queues $\mathcal{S}$ (as the arrivals are Poisson). Let J be the set of all queues. Note that the probability of no arrivals during C_{k-N} given $\boldsymbol{\vartheta}_k$ cannot be written as $Q_J(C_{k-N})$ since C_{k-N} is not independent of the arrival process in that interval.

$$P(S_k|\boldsymbol{\vartheta}_k = \boldsymbol{\vartheta}) = \prod_{i=0}^{N-1} Q_{\{I(k+i)\}}(\sum_{j=i}^{N-1} \vartheta^j) \geq e^{-\lambda M} \tag{6}$$

where the inequality follows since $\boldsymbol{\vartheta} \in \mathcal{K}(M)$.

$$\begin{aligned} &P(R_k|S_k, \boldsymbol{\vartheta}_k = \boldsymbol{\vartheta}) = P(R_k|S_k) \\ &= E[P(\text{no arrivals during } [\tau(k+1), \tau(k+N-1))|S_{k+1}, D_k) Q_J(D_k)] \\ &= E[Prob(\text{no arrivals during } [\tau(k+2), \tau(k+N-1))|S_{k+2}, D_k, D_{k+1}) \\ &\quad \times Q_J(D_k + D_{k+1})] \\ &= ... = E[Q_J(D)] = E\left\{e^{-\lambda D}\right\} \geq e^{-\lambda d}, \end{aligned} \tag{7}$$

where the last inequality follows from Jensen's inequality. We thus obtain

$$\begin{aligned} \mathcal{P}_k(\boldsymbol{\vartheta}, B) &= P(\boldsymbol{\vartheta}_{k+N} \in B|\boldsymbol{\vartheta}_k = \boldsymbol{\vartheta}) \\ &\geq P(\boldsymbol{\vartheta}_{k+N} \in B, S_k, R_k|\boldsymbol{\vartheta}_k = \boldsymbol{\vartheta}) \\ &= P(\boldsymbol{\vartheta}_{k+N} \in B|S_k, R_k, \boldsymbol{\vartheta}_k = \boldsymbol{\vartheta}) P(R_k, S_k|\boldsymbol{\vartheta}_k = \boldsymbol{\vartheta}) \\ &= \phi(B) P(R_k|S_k, \boldsymbol{\vartheta}_k = \boldsymbol{\vartheta}) P(S_k|\boldsymbol{\vartheta}_k = \boldsymbol{\vartheta}) \\ &\geq \phi(B) e^{-\lambda[M+d]}. \end{aligned} \tag{8}$$

Thus, if $\phi(B) > 0$ then $\inf_{\boldsymbol{\vartheta} \in \mathcal{K}(M)} \mathcal{P}_k(\boldsymbol{\vartheta}, B) > 0$, which establishes the smallness of $\mathcal{K}(M)$ for any real number $M > 0$.

Next we establish the strong aperiodicity. Let $R > 0$ be such that $P(D < R) > 0$. We choose

$$\nu(B) = P(\{D_k, ..., D_{k+N-1}\} \in B | D < R).$$

Define $\delta := \exp(-\lambda[R + d])P(D < R)$. Set $\mathcal{C} = \mathcal{K}(R)$. Clearly $L(x, \mathcal{C}) > 0$ and $\nu(\mathcal{C}) = 1$ for any $x \in \mathcal{X}$. Indeed, if $\boldsymbol{\vartheta}_k = x$ (for some arbitrary x) then there exists a strictly positive probability of no arrivals during the time interval $[\tau(k), \tau(k + N)]$. Conditioned on this event, $\boldsymbol{\vartheta}_{k+2N}$ is distributed like $(D_k, D_{k+1}, ..., D_{k+N-1})$. Hence $L(x, \mathcal{C}) > 0$.

Now, for $x \in \mathcal{C}$ we have by (8):

$$\mathcal{P}_k(\boldsymbol{\vartheta}, B) \geq \phi(B)e^{-\lambda[R+d]} \geq \phi(B|D < R)P(D < R)e^{-\lambda[R+d]} = \delta\nu(B)$$

from which we obtain the strong aperiodicity. This establishes the proof of (i).

Consider the function $f : \mathbb{R}^N \to \mathbb{R}$ given by $f(x) = x_i$ for some $i = 0, ...N - 1$. Then (ii) and (iii) follow immediately from Proposition 3 (b). (iv) follow from Proposition 3 (b.2) by choosing $f(x) = 1$. ■

Remark: It is known that $\rho < 1$ is also a necessary condition for the ergodicity of the station times (see [7]). It turns out that it is also a necessary condition for the finiteness of all moments. This follows from the fact that the expected cycle duration is $d/(1 - \rho)$ for $\rho < 1$, and from the fact that the cycle durations are stochastically increasing in λ, see [1].

4 STABILITY OF ALL MOMENTS

Lemma 4 *Assume that $\rho < 1$, $b_l^*(\omega) < \infty$ and $d_l^*(\omega) < \infty$ for some $\omega < 0$, $l = 1, ..., N$. Then there exists some $\epsilon > 0$, $\beta < 1$ and $M_1 > 0$ such that*

$$E\left(\exp\left\{\sum_{i=0}^{N-1} s_k^i \vartheta_{k+N}^i\right\}\Bigg|\,\boldsymbol{\vartheta}_k\right) \leq \beta \exp\left\{\sum_{i=0}^{N-1} s_k^i \vartheta_k^i\right\}$$

for all $\boldsymbol{\vartheta}_k \notin \mathcal{K}(M_1)$ (i.e. for all $\sum_{i=0}^{N-1} w_k^i \vartheta_k^i$ large enough), where $s_k^i = \epsilon w_k^i$, and w_k^i is defined in (5).

Proof: We introduce the following notation:

$$f_k^1(\epsilon) \stackrel{\text{def}}{=} \lambda_{k-1}(b_{k-1}^*(-s_k^{(-1)}) - 1). \tag{9}$$

and we shall understand $s_k^{(-i)} := s_k^{N-i}$ (and $w_k^{(-i)} = w_k^{N-i}$). Note that

$$\exp\left\{f_k^1(\epsilon)\right\} = E\left[\exp\left\{s_k^{(-1)}\Gamma_{k-1}(1)\right\}\right].$$

Indeed, let $F_{B_l^i}$ denote the i-fold convolution of the distribution F_{B_l} of B_l. For any real number T,

$$E\left[e^{\omega\Gamma_l(T)}\right] = \sum_{i=0}^{\infty}\int_0^{\infty} e^{\omega t} dF_{B_l^i}(t) P\{\mathcal{A}_l(T) = i\}$$

$$= \sum_{i=0}^{\infty}[b_l^*(-\omega)]^i P\{\mathcal{A}_l(T) = i\} = \sum_{i=0}^{\infty}\frac{[\lambda_l T b_l^*(-\omega)]^i e^{-\lambda_l T}}{i!} = e^{\lambda_l(b_l^*(-\omega)-1)T}.$$

We define recursively for $i = 2, ..., N$

$$f_k^i(\epsilon) \stackrel{\text{def}}{=} \lambda_{k-i}\left[b_{k-i}^*\left(-s_k^{(-i)} - \sum_{l=1}^{i-1} f_k^l(\epsilon)\right) - 1\right].$$

(The recursion begins with (9) for $i=1$.) Hence $f_k^l(0)=1$ for all l, and

$$\frac{d}{d\epsilon} f_k^1(\epsilon) \;=\; \lambda_{k-1} \frac{d}{d\epsilon} b_k^1(-s_k^{(-1)}),$$

so that $(f_k^1)'(0) := \frac{d}{d\epsilon} f_k^1(\epsilon)\big|_{\epsilon=0} = \rho_{k-1} w_k^{(-1)}$. Similarly,

$$\begin{aligned}\frac{d}{d\epsilon} f_k^i(\epsilon) \;&=\; \lambda_{k-i} \frac{d}{d\epsilon} b_{k-i}^* \left(-s_k^{(-i)} - \sum_{l=1}^{i-1} f_k^l(\epsilon)\right) \\ &=\; \lambda_{k-i} \frac{d}{dt} b_{k-i}^*(t)\Big|_{t=\left(-s_k^{(-i)} - \sum_{l=1}^{i-1} f_k^l(\epsilon)\right)} \left(-w_k^{(-i)} - \sum_{l=1}^{i-1} \frac{d}{d\epsilon} f_k^l(\epsilon)\right),\end{aligned}$$

so that

$$(f_k^i)'(0) = \rho_{k-i}\left[w_k^{(-i)} + \sum_{l=1}^{i-1} (f_k^i)'(0)\right]. \tag{10}$$

Since, by (5), $w_k^{(-i)} = 1 - \sum_{l=1}^{i-1} \rho_{k-l}$, it follows by induction from (10), that

$$(f_k^i)'(0) \le \rho_{k-i}. \tag{11}$$

We have

$$\begin{aligned}&E\left(\exp\left\{s_k^{N-1}\theta_{k+N-1}\right\}\Big|\, \theta_{k-1}, ..., \theta_{k+N-2}\right) \\ &\;=\; E\left(\exp\left\{s_k^{(-1)}[\Gamma_{k+N-1}\left(\sum_{i=0}^{N-1} \theta_{k-1+i}\right) + D_{k+N-1}]\right\}\Bigg|\, \theta_{k-1}, ..., \theta_{k+N-2}\right) \\ &\;=\; \exp\left\{f_k^1(\epsilon) \sum_{i=0}^{N-1} \theta_{k-1+i}\right\} d_{k-1}^*(-s_k^{(-1)}).\end{aligned}$$

Similarly,

$$E\left(\exp\left\{\sum_{i=N-2}^{N-1} s_k^i \theta_{k+i}\right\}\Bigg|\, \theta_{k-2}, ..., \theta_{k+N-3}\right)$$

$$
\begin{aligned}
&= E\left(\exp\left\{s_k^{(-2)}\theta_{k+N-2}\right\}\exp\left\{s_k^{(-1)}\theta_{k+N-1}\right\}\Big|\,\theta_{k-2},...,\theta_{k+N-3}\right)\\
&= E\left(\exp\left\{s_k^{(-2)}\theta_{k+N-2}\right\}\exp\left\{f_k^1(\epsilon)\theta_{k-1}\right\}\right.\\
&\quad\left.\times\ \exp\left\{f_k^1(\epsilon)\sum_{i=0}^{N-2}\theta_{k+i}\right\}\Big|\,\theta_{k-2},...,\theta_{k+N-3}\right)d_{k-1}^*(-s_k^{(-1)})\\
&= \exp\left\{f_k^1(\epsilon)\theta_{k-1}\right\}\exp\left\{f_k^1(\epsilon)\sum_{i=0}^{N-3}\theta_{k+i}\right\}\\
&\quad\times E\left(\exp\left\{[s_k^{(-2)}+f_k^1(\epsilon)]\theta_{k+N-2}\right\}\Big|\,\theta_{k-2},...,\theta_{k+N-3}\right)\times d_{k-1}^*(-s_k^{(-1)})\\
&= \exp\left\{f_k^1(\epsilon)\theta_{k-1}\right\}\exp\left\{f_k^1(\epsilon)\sum_{i=0}^{N-3}\theta_{k+i}\right\}\exp\left\{f_k^2(\epsilon)\sum_{i=0}^{N-3}\theta_{k+i}\right\}\\
&\quad\times d_{k-2}^*(-s_k^{(-2)}-f_k^1(\epsilon))\cdot d_{k-1}^*(-s_k^{(-1)})\\
&= \exp\left\{(f_k^1(\epsilon)+f_k^2(\epsilon))\theta_{k-1}\right\}\exp\left\{f_k^2(\epsilon)\theta_{k-2}\right\}\\
&\quad\times\exp\left\{[f_k^1(\epsilon)+f_k^2(\epsilon)]\sum_{i=0}^{N-3}\theta_{k+i}\right\}d_{k-2}^*(-s_k^{(-2)}-f_k^1(\epsilon))\cdot d_{k-1}^*(-s_k^{(-1)}).
\end{aligned}
$$

Continuing this, we finally obtain

$$
E\left(\exp\left\{\sum_{i=0}^{N-1}s_k^i\theta_{k+i}\right\}\Big|\,\theta_{k-N},...,\theta_{k-1}\right)=\zeta\cdot\prod_{l=1}^{N}\exp\left\{\left(\sum_{i=l}^{N}f_k^i(\epsilon)\right)\theta_{k-l}\right\}
$$

where

$$
\zeta=\prod_{m=1}^{N}d_{k-m}^*\left(-s_k^{(-m)}-\sum_{n=1}^{m-1}f_k^n(\epsilon)\right).
$$

Consequently

$$
\begin{aligned}
&\frac{E\left(\exp\left\{\sum_{i=0}^{N-1}s_k^i\theta_{k+i}\right\}\Big|\,\theta_{k-N},...,\theta_{k-1}\right)}{\exp\left\{\sum_{i=0}^{N-1}s_k^i\theta_{k-N+i}\right\}}\\
&= \zeta\cdot\prod_{l=1}^{N}\exp\left\{\left(\sum_{i=l}^{N}f_k^i(\epsilon)-s_k^{(-l)}\right)\theta_{k-l}\right\}.
\end{aligned}
$$

A sufficient condition for the Lemma to hold, is that for all $\epsilon > 0$ sufficiently close to 0, $g_l(\epsilon) := \exp\left\{\sum_{i=l}^{N} f_k^i(\epsilon) - s_k^{(-l)}\right\} < 1$. As $g_l(0) = 1$, it is sufficient to show that $g_l'(0) < 0$. This holds indeed, since

$$g_l'(0) = g_l(\epsilon)\left(\sum_{i=l}^{N}(f_k^i)'(\epsilon) - w_k^{(-l)}\right)\Bigg|_{\epsilon=0} = \sum_{i=l}^{N}(f_k^i)'(0) - w_k^{(-l)} = \rho - 1,$$

$l = 1, ..., N$, where the last equality follows from (11). ■

Under the conditions of Theorem 2, for each $l = 1, ..., N$, $\boldsymbol{\vartheta}_{l+jN}$ weakly converges to some RV which we denote by $\hat{\boldsymbol{\vartheta}}_l$. The following Theorem establishes the finiteness of all joint moments, as well as their geometric rate of convergence.

Theorem 5 *Assume that $\rho < 1$ and that $b_l^*(\omega) < \infty$ and $d_l^*(\omega) < \infty$ for some $\omega < 0$. Let $f : \mathbb{R}^N \to \mathbb{R}$ any polynomial function, i.e. of the form $f(x) = \sum_{j=1}^{K}\prod_{i=0}^{N-1} x_i^{n(ij)}$, where $n(ij)$ and K are arbitrary (finite) integers. Then for any $0 \le l < N$, $E[f(\boldsymbol{\vartheta}_{l+jN})]$ converges geometrically fast to $E_\pi f(\hat{\boldsymbol{\vartheta}}_l)$, and the latter expression is finite.*

Proof: Let ϵ be as in Lemma 4. Define $g : \Theta \to \mathbb{R}$ as $g(x) = \exp\left\{\sum_{l=0}^{N-1} s_k^l x_l\right\}$, (where s_k^l is defined in Lemma 4). It follows from Lemma 4 that there exists some real number M_f and $\delta > 0$ such that $E[g(\boldsymbol{\vartheta}_{k+N}) - g(\boldsymbol{\vartheta}_k)|\boldsymbol{\vartheta}_k] \le -\delta g(\boldsymbol{\vartheta}_k)$ if $\sum_{l=0}^{N-1} \vartheta^l \ge M_f$, and otherwise $E[g(\boldsymbol{\vartheta}_{k+N}) - g(\boldsymbol{\vartheta}_k)|\boldsymbol{\vartheta}_k] < \infty$. This establishes conditions (i) and (ii) of Proposition 3. The set $\mathcal{K}(M_f)$ is small (see the proof of Theorem 2). The Theorem now follows by applying Proposition 3(b). Note that indeed $f \le ag$ for some $a \ge 0$ since f is polynomial whereas g is exponential. ■

5 CENTRAL LIMIT THEOREM AND LAW OF ITERATED LOGARITHM

In this Section we establish a CLT and LIL for the moments of the station times. The CLT provides the asymptotic distribution of the empirical mean of functionals of the station times; it shows that by a scaling of square root of n we get an asymptotic Normal distribution. The LIL provides asymptotic bounds on the empirical mean of functionals of the station times, which hold for each sample. The scaling factor is then $\sqrt{n \log\log(n)}$. We compute explicitly the constants involved in the CLT and LIL of the cycle times. Let E_π denote the expectation with respect to the probability in steady state (of the Markov chain $\{\boldsymbol{\vartheta}_{k+lN}\}_{l=1}^{\infty}$). For any $h : \Theta \to \mathbb{R}$, let $\tilde{h}(x) := h(x) - E_\pi h(x)$.

Theorem 6 *Assume that* $\rho < 1$ *and that* $b_l^*(\omega) < \infty$ *and* $d_l^*(\omega) < \infty$ *for some* $\omega < 0$. *Let* $h : \mathbb{R}^N \to \mathbb{R}$ *any polynomial function, i.e. of the form* $h(x) = \sum_{j=1}^{K} \prod_{i=0}^{N-1} x_i^{n(ij)}$, *where* $n(ij)$ *and* K *are arbitrary (finite) integers. Then*

$$(\gamma_h)^2 = E_\pi[\tilde{h}(\boldsymbol{\vartheta}_0)]^2 + 2\sum_{i=2}^{\infty} E_\pi[\tilde{h}(\boldsymbol{\vartheta}_N)\tilde{h}(\boldsymbol{\vartheta}_{iN})]. \tag{12}$$

is well defined and finite, and if $(\gamma_h)^2 > 0$ *then*
(i)

$$\frac{\sum_{k=1}^{n} \tilde{h}(\boldsymbol{\vartheta}_{kN})}{\gamma_h \sqrt{n}} \to_d \mathcal{N}(0,1)$$

(ii) The limit infimum and limit supremum of the sum

$$\frac{1}{\gamma_h \sqrt{2n \log\log(n)}} \sum_{k=1}^{n} \tilde{h}(\boldsymbol{\vartheta}_{kN})$$

are respectively -1 *and* 1 *with probability one.*

Proof: Follows from [13] Theorem 9.1. It remains to show that the conditions of this Theorem are satisfied. It follows immediately from Lemma 4 that condition (V5) of this Theorem holds. The petite set required in the Theorem is of the form $\mathcal{K}(M)$ for some $M > 0$. The strong aperiodicity of the Markov chain is established as in the proof of Theorem 2. ■

Next we restrict to the special case where $h(\boldsymbol{\vartheta}) = \sum_{i=0}^{N-1} \vartheta^i$. $C_{k-N} := h(\boldsymbol{\vartheta}_k)$ is thus the duration of the cycle that starts at $\tau(k-N)$. In the next Theorem we obtain the exact value of γ_h for that case. Let $r_{ij} := E_\pi[\tilde{\theta}_i \tilde{\theta}_j],\ i,j = 0,1,...,N-1$, where $\tilde{\theta}_i := \theta_i - E_\pi \theta_i$. The values of r_{ij} were obtained in [6] (see also [5, 9]).

Theorem 7 *Under the assumptions of Theorem 6,*
(i)

$$\frac{\sum_{k=1}^{n}(C_k - c)}{\gamma_h \sqrt{n}} \to_d \mathcal{N}(0,1)$$

(ii) The limit infimum and limit supremum of the sum

$$\frac{1}{\gamma_h \sqrt{2n \log\log(n)}} \sum_{k=1}^{n} [C_k - c]$$

are respectively -1 *and* 1 *with probability one, where*

$$c = \frac{d}{1-\rho}$$

$$(\gamma_h)^2 = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} r_{ij} + \frac{2\sum_{i=0}^{N-1} \sum_{l=0}^{N-1} r_{li} \left(\sum_{j=0}^{i} \rho_j\right)}{1-\rho} \tag{13}$$

and both c *and* γ_h *do not depend on the initial queue that is polled.*

Proof: The proof follows from Theorem 6, and it remains to calculate the constants. It is well known that the expected cycle duration in steady state is $c = d/(1-\rho)$, independent of the initial queue that is polled (see e.g. [16]). It remains to compute γ_h. Fix some integer k. Denote

$$\hat{\theta}_i^k := E[\theta_{k+i}|\boldsymbol{\vartheta}_{k+N}] - E_\pi(\theta_{k+i}).$$

Note that for $i = 0, ..., N-1$,

$$\hat{\theta}_i^k = \tilde{\theta}_{k+i} = \theta_{k+i} - E_\pi(\theta_{k+i}). \tag{14}$$

We have for any integer $i \geq 0$,

$$E[\theta_{k+N+i}|\boldsymbol{\vartheta}_k] = \rho_{k+i} \sum_{n=0}^{N-1} E[\theta_{k+n+i}|\boldsymbol{\vartheta}_k] + d_{k+i}$$

$$E_\pi[\theta_{k+N+i}] = \rho_{k+i} \sum_{n=0}^{N-1} E_\pi \theta_{k+n+i} + d_{k+i}.$$

Subtracting these equations and summing to an arbitrary R (larger than N), we get

$$\begin{aligned}
\sum_{i=0}^{R} \hat{\theta}_{N+i}^k &= \sum_{i=0}^{R} \rho_{k+i} \sum_{n=0}^{N-1} \hat{\theta}_{n+i}^k \\
&= \sum_{m=0}^{R+N-1} \sum_{i=\max\{0,m-N+1\}}^{\min\{R,m\}} \rho_{k+i} \hat{\theta}_m^k \\
&= \sum_{m=0}^{N-1} \sum_{i=0}^{m} \rho_{k+i} \hat{\theta}_m^k + \rho \sum_{m=N}^{R-1} \hat{\theta}_m^k + \sum_{m=R}^{R+N-1} \sum_{i=m-N+1}^{R} \rho_{k+i} \hat{\theta}_m^k \\
&= \sum_{m=0}^{N-1} \sum_{i=0}^{m} \rho_{k+i} \hat{\theta}_m^k + \left(\rho \sum_{i=0}^{R} \hat{\theta}_{N+i}^k - \rho \sum_{i=0}^{N} \hat{\theta}_{R+i}^k \right) \\
&\quad + \sum_{l=0}^{N-1} \sum_{i=R+l-N+1}^{R} \rho_{k+i} \hat{\theta}_{l+R}^k
\end{aligned}$$

As $R \to \infty$, $\hat{\theta}^k_{l+R}$ goes to zero, for all integers $l \geq 0$ (this easily follows from Theorem 5), and hence we have

$$\sum_{t=0}^{\infty} \hat{\theta}^k_{N+t} = \frac{\sum_{m=0}^{N-1} \hat{\theta}^k_m \sum_{i=0}^{m} \rho_{k+i}}{1-\rho}. \tag{15}$$

For $k = 0$ we get from (12), (14) and (15)

$$\begin{aligned}
(\gamma_h)^2 &= E_\pi[\bar{h}(\boldsymbol{\vartheta}_0)]^2 + 2\sum_{j=2}^{\infty} E_\pi[\bar{h}(\boldsymbol{\vartheta}_N)\bar{h}(\boldsymbol{\vartheta}_{jN})] \\
&= E_\pi\left(\sum_{i=0}^{N-1} \tilde{\theta}_i\right)^2 + 2\sum_{l=0}^{N-1}\sum_{t=0}^{\infty} E_\pi\left(\tilde{\theta}_l \tilde{\theta}_{N+t}\right) \\
&= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} r_{ij} + \frac{2\sum_{l=0}^{N-1}\sum_{m=0}^{N-1} r_{lm}\left(\sum_{i=0}^{m}\rho_i\right)}{1-\rho}.
\end{aligned}$$

Next we show that γ_h does not depend on the initial queue. Assume that the initial queue is $l \neq 0$. Then for $n > N$,

$$\sum_{k=1}^{n} C_k = \sum_{j=l}^{N-1} \theta_j + \sum_{k=0}^{n-1}\left(\sum_{j=0}^{N-1} \theta_{kN+(N-l)+j}\right) - \sum_{j=0}^{l-1} \theta_{(n-1)N+j}.$$

Now, it follows from what we just established, that

$$\sum_{k=0}^{n-1}\left(\sum_{j=0}^{N-1} \theta_{kN+(N-l)+j}\right)$$

satisfies a CLT with constants γ_h and c, since $\sum_{j=0}^{N-1} \theta_{kN+(N-l)+j}$ are cycles that start from queue 0. This implies, that so does $\sum_{k=1}^{n} C_k$, and with the same constant γ_h, because the other terms tend to zero when divided by $\gamma_h\sqrt{2n\log\log(n)}$. ■

6 GLOBALLY GATED REGIME

The results of the previous sections extend to the case of the (cyclic) Globally Gated (GG) service regime, recently introduced by Boxma, Levy and Yechiali [4]. According to this service discipline, there are gates in all queues, which are closed (globally) at the moment the server polls queue 1; during the following cycle only the customers "captured" (present) at the different queues at the start of the cycle, will be served, whereas the others have to wait till the next cycle. This service discipline is known to possess two attractive properties: (i) it brings the polling system closer to the (fair) First Come First Served discipline (as opposed to the regular Gated or Exhaustive disciplines), and (ii) it enables one to obtain closed form results for cycle time, moments of waiting times and other performance measures.

The calculations involved in establishing the stability and rate of convergence turn out to be much simpler than those from the previous sections, required for the Gated discipline.

As the state of the system we use the cycles durations $\{C_k\},\ k = 1, 2, ...$, where a cycle is the time between two consecutive polling instants of queue 1. Denote by $\Theta = \mathbb{R}_+$ the state space, and C the cycle duration in steady state. The moments of C_k in steady state are obtained in [4]; moreover, other performance measures, such as the moments of queues' length and waiting times, are expressed as functions of the moments of the cycle time. The state evolves according to

$$C_{k+1} = \sum_{j=0}^{N-1} \Gamma_j(C_k) + D^{k+1},$$

where D^k is the total walking time in the kth cycle.

Theorem 8 *Assume that* $b_i, d_i < \infty$ *for all* $i = 1, ..., \infty$ *and that that* $\rho < 1$. *Then*

(i) $\{C_j\},\ j = 1, ...$ *is geometrically ergodic,*

(ii) EC_j *converge geometrically fast to* $E_\pi C = d/(1-\rho)$.

(iii) Define $g : \Theta \to \mathbb{R}$ *as* $g(x) = 1 + \rho x$. *There exist some* $\xi < 1$ *and* $R > 0$, *such that*

$$||\mathcal{P}^n(\boldsymbol{\vartheta}, \cdot) - \pi(\cdot)|| \leq R\xi^n g(\boldsymbol{\vartheta})$$

for any $\boldsymbol{\vartheta} \in \Theta$.

(iv) Assume that $b_l^*(\omega) < \infty$ *and* $d_l^*(\omega) < \infty$ *for some* $\omega < 0$. *Then*

(1) for any $k > 0$, $E_\pi C^k < \infty$,

(2) EC_j^k *converges geometrically fast to* $E_\pi C^k$,

(3) For any integer k,

$$(\gamma_k)^2 = E_\pi[C_1 - E_\pi C]^k + 2\sum_{i=2}^{\infty} E_\pi[(C_1 - E_\pi C)^k (C_i - E_\pi C)^k]$$

is well defined and finite, and if $(\gamma_k)^2 > 0$ *then*

$$\frac{\sum_{j=1}^n [(C_j)^k - E_\pi(C)^k]}{\gamma_k \sqrt{n}} \to_d \mathcal{N}(0,1)$$

and the limit infimum and limit supremum of the sum

$$\frac{1}{\gamma_k \sqrt{2n \log\log(n)}} \sum_{j=1}^n [(C_j)^k - E_\pi(C)^k]$$

are respectively -1 *and* 1 *with probability one. For the special case of* $k = 1$, $E_\pi(C) = d/(1-\rho)$ *and*

$$\gamma_1 = \frac{1+\rho}{1-\rho} var_\pi C = \frac{1}{(1-\rho)^2}\left(var^2(D) + \sum_{i=0}^{N-1} \lambda_i b_i^{(2)} E_\pi[C]\right).$$

Proof: The proof is based on the following simple observation. Consider a second polling system with Globally Gated service discipline and only one queue. The arrivals are Poisson with rate $\lambda = \sum_{i=0}^{N-1} \lambda_i$. The service duration B of each customer is chosen at random between RVs that are distributed as $B_0, ..., B_{N-1}$ with probabilities $\lambda_0/\lambda, ..., \lambda_{N-1}/\lambda$. Thus

$$b^*(\omega) = \sum_{i=0}^{N-1} \frac{\lambda_i b_i^*(\omega)}{\lambda}, \qquad b := \sum_{i=0}^{N-1} \frac{\lambda_i b_i}{\lambda}, \qquad \rho = \sum_{i=0}^{N-1} \lambda_i b_i.$$

The walking time is distributed as D. Then the $\{C_k\}$ in this new system have the same (joint) distribution as in the original one. Since there is only one queue, the Globally Gated and the Gated disciplines coincide. Therefore, all the results from the previous sections on the Gated discipline can be applied. Note that since there is only one queue, the cycles and station times are the same. The expression for γ_1 finally follows from (13) since by [4], $E_\pi[C] = d/(1-\rho)$ and

$$E_\pi[C^2] = \frac{1}{1-\rho^2}\left(d^{(2)} + 2d\rho E[C] + \sum_{i=0}^{N-1} \lambda_i b_i^{(2)} E[C]\right).$$

■

ACKNOWLEDGEMENT: The work by the first author was supported by a Grant from the France-Israel Scientific Cooperation in Computer Science and Engineering between the French Ministry of Research and Technology and the Israeli Ministry of Science and Technology, Grant Number 3321190, and by the European Grant BRA-QMIPS of CEC DG XIII. The work by the second author was supported by the Netherlands Organization for Scientific Research N.W.O.

References

[1] E. Altman, P. Konstantopoulos and Z. Liu, "Stability, Monotonicity and Invariant Quantities in General Polling Systems", *Queuing Systems, special issue on Polling Models*, Eds. H. Takagi et O. Boxma, pp. 35-57, 1992.

[2] E. Altman and H. Levy, "Queueing in space", *Advances of Applied Probability* **26**, pp. 1095-1116, 1994.

[3] E. Altman and Z. Liu, "Improving the stability characteristics of asynchronous traffic in FDDI token rings", *Proceeding of the 5th international conference on High Speed Networks*, Chapel Hill, USA, October 1993.

[4] O. J. Boxma, H. Levy and U. Yechiali, "Cyclic Reservation Schemes for Efficient Operation of Multiple-Queue Single-Server Systems", *Annals of Operations Research* **35**, pp. 187-208, 1992.

[5] G. L. Choudhury and H. Takagi, Comments on "Exact Results for Nonsymmetric Token Ring Systems", *IEEE Trans. on Communications* **38** No. 8, pp. 1125-1127, 1990.

[6] M. J. Ferguson and Y. J. Aminetzah, "Exact Results for Nonsymmetric Token Ring Systems", *IEEE Trans. on Communications* **33** No. 3, March 1985.

[7] C. Fricker and M.R. Jaibi, "Monotonicity and stability of periodic polling models", *Queueing Systems* **15**, pp. 211-238, 1994.

[8] L. Greorgiadis and W. Szpankowski "Stability of Token Passing Rings", *Queuing Systems, special issue on Polling Models*, **11**, Eds. H. Takagi et O. Boxma, pp. 7-33, 1992.

[9] Humblet P., "Source coding for communication concentrators", Electron. Syst. Lab., MIT, Cambridge, ESL-R-798, 1978.

[10] A. Khamisy, E. Altman and M. Sidi, "Polling Systems with Synchronization Constraints", *Annals of OR, special issue on Stochastic Modeling of Telecommunication Systems*, Eds. P. Nain et K. W. Ross, pp. 231-267, 1992.

[11] D. P. Kroese and V. Schmidt, "A continuous Polling System with General Service Times", *Annals of Applied Probability* **2**, No. 4, pp. 906-927, 1992.

[12] D. P. Kroese and V. Schmidt, "Single-Server Queues with Spatially Distributed Arrivals", *QUESTA* **17**, No. 1,2, pp. 317-345, 1994.

[13] S. P. Meyn and R. L. Tweedie, "Stability of Markovian processes I: criteria for discrete time chains", *Advances in Applied Probability*, **24** No. 3, pp. 542-574, 1992.

[14] J. A. C. Resing, "Polling systems and multi-type branching processes", *Queueing Systems* **13**, pp. 409-426, 1993.

[15] D. Sarkar and W. I. Zangwill, "Expected waiting time for nonsymmetric cyclic queueing systems – exact results and applications", *Management Science* **35**, No. 12, 1463-1474, 1989.

[16] H. Takagi, *Analysis of Polling Systems*, The MIT Press, 1986.

[17] R. L. Tweedie, "Criteria for rates of convergence of Markov chains, with application to queuing and storage theory", *Probability, Statistics and Analysis*, ed. J.F.C. Kingman and G.E.H. Renter, London Math Soc. Lecture Notes Series 79, Cambridge Univ. Press, pp. 260-276, 1983.

[18] R. L. Tweedie, "The existence of moments for stationary Markov chains", *J. Applied Probability* **20**, pp. 191-196, 1983.