Closed-Loop Control with Delayed Information

Eitan Altman*and Philippe Nain
INRIA
2004, Route des Lucioles
06565 Valbonne Cedex, France
altman@sophia.inria.fr nain@sophia.inria.fr

Abstract

The theory of stochastic dynamic programming requires that the current state of the system is known to the decision maker. However, this condition is violated in a number of cases, as for instance in high-speed data networks.

In this article, we first show that by enlarging the state space so as to include the last known state as well as all the decisions made during the travel time of the information (e.g., propagation delay in networks), we may reduce a partially observable Markov control model to a completely observable Markov control model.

This result is then applied to a flow control problem with delayed state information pattern. More precisely, we consider a discrete time queueing model with Bernoulli services, where at most one customer may join the queue in every time slot. We consider a general cost formulation that captures standard control problems (e.g., delay versus throughput). At the beginning of slot $t+N$, where N is a fixed integer and $t = 0, 1, 2 \ldots$, the decision maker has to select the probability of having one arrival in the current time slot (it may be either p_1 or p_2, $p_1 \geq p_2$) only on the basis of the queue-length history in $[0, t]$ and on the actions made in $[0, t+N)$. We show that there exists a policy of a threshold type that minimizes a discounted cost criterion over an infinite horizon. The threshold is seen to depend on the N previous actions.

*The work of this author was supported by the France-Israel Scientific Cooperation in Computer Science and Engineering under grant 3321190.

[0]To appear in the Proceedings of the 1992 ACM Sigmetrics — Performance '92 Conference, June 1-5, Newport RI, USA.

1 Introduction

The theory of *stochastic dynamic programming* requires that the current state of the system is known to the decision maker (Ross [8], Schäl [9]). While this assumption may be realistic or may provide good approximations in some cases, it is no longer true, for instance, in modern communication networks. The reason is that the propagation delay in such networks is no longer negligible with respect to the packet transmission time. As a result, the state of the system may have changed considerably by the time it is known by the decision maker. This, in turn, may yield serious problems when closed-loop controls are used.

Control problems with *imperfect* state information have already received much attention in the literature. When the control is *centralized*, the standard assumption in this setting is that the decision maker instead of having perfect knowledge of the state has only access to an observation of this state (see e.g., Bertsekas [1], Hernández-Lerma [3]). When the control is *decentralized* (i.e., there are several decision makers), the so-called Delayed Sharing of State Information (DSSI) pattern provides another instance of imperfect state information structure (Grizzle and Marcus [2], Hsu and Marcus [4], Schoute [10]). In that case, each decision maker informs the others about his observation with a delay. However, in each case the imperfect state information problem is seen to reduce to a perfect state information (or completely observable) problem by enlarging the state space.

The first objective of this work is to introduce a new type of imperfect information structure, the so-called N-Step Delayed State Information (N-SDSI) structure. We assume a discrete-time Markov Control Model (MCM) with a single decision maker. The N-SDSI pattern assumes that the state of the system at time $t \in \mathbb{N} := \{0, 1, 2, \ldots\}$ is not known by the decision maker until time $t + N$, where $N \geq 1$ is an arbitrary integer. However, we assume that all the past actions are known with no delay. Therefore, and as opposed to the DSSI pattern briefly described above, the N-SDSI pattern is such that the decision maker has no information whatsoever on the current state of the system. Alike the previous studies on imperfect state information, we show that this partially observable MCM can be converted into a Completely Observable MCM (CO-MCM) by enlarging the state space. However, because of the simple structure of the N-SDSI pattern the new state of the system remains easy to handle.

The second objective is to solve a flow control problem with a N-SDSI pattern (see Stidham [11] for a survey on flow control models). More precisely, we consider a discrete time queueing model with Bernoulli services, where at most one customer may join the queue in every slot. We consider a general cost formulation that captures standard control problems (e.g., delay versus throughput). At the beginning of slot $t + N$, the decision maker has to select the probability of having one arrival in the current time slot (it may be either p_1 or p_2, $p_1 \geq p_2$) only on the basis of the queue-length history in $[0, t]$ and on the actions made in $[0, t + N)$. We show that there exists a policy of a threshold type that minimizes a discounted cost criterion over an infinite horizon. More precisely, we show that there exists a mapping $l : \{p_1, p_2\}^N \to \mathbb{N}$ such that the probability p_1 (resp. p_2) is chosen at time $t + N$ if $x < l(a_1, \ldots, a_N)$ (resp. $x \geq l(a_1, a_2, \ldots, a_N)$), where x is the queue-length at time t and $a_1, \ldots, a_N$ are the actions chosen at time t, $t+1$,..., $t+N-1$, respectively. Monotonicity properties of the mapping $a_1 \to l(a_1, \ldots, a_N)$ are also obtained.

The paper is organized as follows. In Section 2, we briefly recall the basic features of a CO-MCM. Then, the N-SDSI structure is introduced, and we show how the related partially observable MCM problem can be transformed into a CO-MCM. These results are then used in Section 3 to solve the flow control problem with a N-SDSI pattern. Concluding remarks are given in Section 4.

2 Markov Control Model with Delayed Information

Let $\mathbb{R}$ denote the set of all real numbers. Let f be any mapping defined on some set E such that $E = E_1 \times E_2$. With a slight abuse of notation, $f(e_1, e_2)$ will stand for $f(e)$ for any $e = (e_1, e_2) \in E$, $e_i \in E_i$, $i = 1, 2$.

2.1 Completely Observable Markov Control Model

A CO-MCM (see Schäl [9]) is the collection of the following objects: a state space $\mathbf{X}$, an action space $\mathbf{A}$, a transition law q and a cost function $C : \mathbf{X} \times \mathbf{A} \to \mathbb{R}$. In the following, we shall assume that the set $\mathbf{X}$ is denumerable and that the set $\mathbf{A}$ is finite (a more general setting can be found in Hernández-Lerma [3] and in Schäl [9]). We shall also assume without loss of generality that in every state all the actions in $\mathbf{A}$ are available to the decision maker.

Let $\mathcal{P}(\mathbf{X})$ and $\mathcal{P}(\mathbf{A})$ be the set of all probability measures on the Borel σ-algebra of $\mathbf{X}$ and $\mathbf{A}$, respectively.

The transition law q is a transition probability $q : \mathbf{X} \times \mathbf{A} \to \mathcal{P}(\mathbf{X})$, where $q(\bullet \,|\, x, a)$ is the distribution of the next state visited by the system if the system is in state $x \in \mathbf{X}$ and action $a \in \mathbf{A}$ is taken.

A control policy (or simply a policy) u is defined as a sequence of conditional probabilities $u_t : H_t \to \mathcal{P}(\mathbf{A})$, $H_0 := \mathbf{X}$ and $H_{t+1} = H_t \times (\mathbf{A} \times \mathbf{X})$ for all $t \in \mathbb{N}$, such that $u_t(h_t; \bullet)$ assigns probability one to the set $\mathbf{A}$ for all $h_t \in H_t$, $t \in \mathbb{N}$. Let U be the class of all policies.

A *stationary* policy is a policy such that $u_t(h_t; \bullet)$ is concentrated at the point $\alpha(x_t)$ for all $h_t = (x_0, a_0, \ldots, x_{t-1}, a_{t-1}, x_t) \in H_t$, $t \in \mathbb{N}$, where α is a measurable mapping from $\mathbf{X}$ to $\mathbf{A}$. In other words, a stationary policy is a nonrandomized policy that only depends on the current state of the system.

Given some initial distribution p_0 on $\mathbf{X}$ and some transition law q, any policy u defines a probability measure on the product space $(\mathbf{X} \times \mathbf{A})^\infty$ endowed with the product σ-algebra (see Schäl [9]). On this probability space, let us define the random variables X_t and A_t that describe the state of the system at time t and the action taken at time t, respectively, for all $t \in \mathbb{N}$.

A standard objective in this setting is to solve the following control problem (e.g., see Ross [8]):

($\mathbf{P}_\beta$): Minimize $V_\beta(x,u)$ over U for all $x \in \mathbf{X}$, where

$$V_\beta(x,u) := E^u \left[\sum_{t \in \mathbb{N}} \beta^t \, C(X_t, A_t) \,|\, X_0 = x \right],$$

$u \in U$, where β is a discount factor in $(0,1)$. The quantity $V_\beta(x,u)$ is the expected β-discounted cost incurred over an infinite horizon. Let $V_\beta(x) := \inf_{u \in U} V_\beta(x,u)$.

The solution to the problem $\mathbf{P}_\beta$ may be obtained by using the following basic result of the CO-MCM theory (cf. Ross [8]):

Theorem 2.1 *Let $\mathcal{H}$ be the set of all real-valued function on $\mathbf{X}$. For every $\beta \in (0,1)$, V_β satisfies the Dynamic Programming (DP) equation*

$$V_\beta(x) = T_\beta \, V_\beta(x) \tag{1}$$

for all $x \in \mathbf{X}$, where the DP operator $T_\beta : \mathcal{H} \to \mathcal{H}$ is given by

$$T_\beta f(\bullet) := \min_{a \in \mathbf{A}} \left\{ C(\bullet, a) + \beta \sum_{x' \in \mathbf{X}} f(x') \, q(x' \,|\, \bullet, a) \right\}. \tag{2}$$

Further, the stationary policy which selects an action minimizing the right-hand side of (1) for all $x \in \mathbf{X}$ solves the problem $\mathbf{P}_\beta$.

2.2 The N-SDSI Pattern

We now consider the case when the state of the system at time t is not known to the decision maker until time $t+N$ for all $t \in \mathbb{N}$, where N is a given integer. However, we will still assume that all past actions are known without delay at any decision epoch.

It is worth noting that this setting differs from the imperfect information setting usually encountered in the literature (see Bertsekas [1, Chapter 3] and Hernández-Lerma [3, Chapter 4]), where at each stage t the decision maker receives *some observation* about the current state of the system, that may be corrupted by some random disturbance.

Alike to the standard imperfect information setting we shall show in this section that, by enlarging the state space so as to include the last known state and the successive actions taken during the delay period, the control problem with the N-SDSI pattern can be reformulated into the CO-MCM framework.

Define $\mathbf{Y} := \mathbf{X} \times \mathbf{A}^N$. We first introduce the notion of N-SDSI policies.

Definition 2.1 *A N-SDSI-policy u is defined as a sequence of transition probabilities $u_t : K_t \to \mathcal{P}(\mathbf{A})$ with $K_N := \mathbf{Y}$ and $K_{t+1} = K_t \times (\mathbf{A} \times \mathbf{X})$ for all $t > N$, such that $u_t(k_t; \bullet)$ assigns probability one to the set $\mathbf{A}$ for any $k_t \in K_t$, $t = N, N+1, \ldots$*

Let U_N be the set of all N-SDSI-policies.

A natural objective is then to minimize over U_N the cost criterion

$$V_\beta^N(y,u) := E^u \left[\sum_{t \geq N} \beta^{t-N} \, C'(Z_{t-N}, A_t) \,|\, Z_0 = y \right], \tag{3}$$

for all $y \in \mathbf{Y}$, where

$$Z_t := (X_t, A_t, A_{t+1}, \ldots, A_{t+N-1}), \quad t \in \mathbb{N},$$

and where C' is a measurable mapping from $\mathbf{Y} \times \mathbf{A}$ to $\mathbb{R}$. In (3) E_u is the expectation operator associated with the probability space defined by a policy $u \in U_N$, given some initial distribution p on $\mathbf{Y}$ and the transition law q.

Let us now show that the solution to the non CO-MCM problem (3) may be obtained by solving a related CO-MCM problem.

This CO-MCM is constructed as follows (see Section 2.1). Let $\mathbf{Y}$, $\mathbf{A}$, q', C' and U' be its state space, action space, transition law, cost function and set of policies, respectively, where

$$\begin{aligned} q'(y' \,|\, y; a) := \mathbf{1}\{(a'_1, a'_2, \ldots, a'_N) = \\ (a_2, \ldots, a_N, a)\} \; q(x' \,|\, x; a_1), \end{aligned} \tag{4}$$

for all $y = (x, a_1, a_2, \ldots, a_N) \in \mathbf{Y}$, $y' = (x', a'_1, a'_2, \ldots, a'_N) \in \mathbf{Y}$, $a \in \mathbf{A}$.

Given the initial distribution p on $\mathbf{Y}$ and the transition law q' (or, equivalently, see (4), given the transition law q), any policy $u \in U'$ defines a probability measure on $(\mathbf{Y} \times \mathbf{A})^\infty$ endowed with the product σ-algebra. Let Y_t and A'_t be random variables describing the state of this new system at time t and the action taken at time t, respectively, for all $t \in \mathbb{N}$.

For any vector $k_t \in \mathbf{Y} \times (\mathbf{A} \times \mathbf{X})^t$, $t \in \mathbb{N}$, such that

$$k_t = (x_0, a_0, \ldots, a_{N-1}, a_N, x_1, \ldots, a_{N+t-1}, x_t),$$

define the mappings $g_t : \mathbf{Y} \times (\mathbf{A} \times \mathbf{X})^t \to \mathbf{Y} \times (\mathbf{A} \times \mathbf{Y})^t$ as

$$g_t(k_t) = (y_0, a_0, \ldots, y_{t-1}, a_{t-1}, y_t),$$

with $y_i = (x_i, a_i, a_{i+1}, \ldots, a_{i+N-1})$ for $i = 0, 1, \ldots, t$.

The following lemma holds:

Lemma 2.1 *Let $u' = \{u'_t\}_{t \in \mathbb{N}} \in U'$ and $u = \{u_{t+N}\}_{t \in \mathbb{N}} \in U_N$ be such that*

$$u'_t(\bullet \mid g_t(k_t)) = u'_{t+N}(\bullet \mid k_t), \tag{5}$$

for all $k_t \in \mathbf{Y} \times (\mathbf{A} \times \mathbf{X})^t$, $t \in \mathbb{N}$. Then, the processes $\{Y_t, A'_t; t \in \mathbb{N}\}$ and $\{Z_t, A_{t+N}; t \in \mathbb{N}\}$ have the same distribution under policies u' and u, respectively.

Proof. The proof is left to the reader (hint: show by induction in t that $\{Y_s, A'_s; 0 \le s \le t\}$ and $\{Z_s, A_{s+N}; 0 \le s \le t \in \mathbb{N}\}$ have the same distribution under policies u' and u, respectively, for all $t \in \mathbb{N}$). ∎

Applying Theorem 2.1 to the CO-MCM $\{Y_t, t \in \mathbb{N}\}$, it is seen that there exists a stationary policy $u^* = \{u^*_t\}_{\in \mathbb{N}}$ in U' that solves the following problem:

$(\mathbf{Q}_\beta)$: Minimize $J_\beta(y, u)$ over U' for all $y \in \mathbf{Y}$, where

$$J_\beta(y, u) := E^u \left[\sum_{t \in \mathbb{N}} \beta^t \, C'(Y_t, A'_t) \mid Y_0 = y \right], \tag{6}$$

$u \in U'$. Let $J_\beta(y) := \inf_{u \in U'} J_\beta(y, u)$.

Let α^* be a measurable mapping from $\mathbf{Y}$ to $\mathbf{A}$ such that

$$u^*_t(\bullet \mid y_0, a_0, \ldots, y_t) = \mathbf{1}(\alpha^*(y_t) = \bullet), \quad t \in \mathbb{N}. \tag{7}$$

We have the following proposition:

Proposition 2.1 *The policy $v^* = \{v^*_{t+N}\}_{t \in \mathbb{N}} \in U_N$ such that*

$$v^*_t(\bullet \mid k_t) = \mathbf{1}(\alpha^*(x_t, a_t, \ldots, a_{t+N-1}) = \bullet), \tag{8}$$

for all $k_t \in \mathbf{Y} \times (\mathbf{A} \times \mathbf{X})^t$ with

$$k_t = (x_0, a_0, \ldots, a_{N-1}, a_N, x_1, \ldots, a_{t+N-1}, x_t),$$

minimizes $V^N_\beta(y, u)$ over U_N, for all $y \in \mathbf{Y}$.

Proof. Let u be an arbitrary policy in U_N. Let $u' \in U'$ be such that (5) holds for all $t \in N$. Then, by Lemma 2.1, (3), (6),

$$J_\beta(y, u') = V^N_\beta(y, u),$$

for all $y \in \mathbf{Y}$.

On the other hand,

$$J_\beta(y, u^*) \le J_\beta(y, u'),$$

for all $y \in \mathbf{Y}$, since u^* solves the problem $\mathbf{Q}_\beta$.

It remains to show that $J_\beta(y, u^*) = V^N_\beta(y, v^*)$ for all $y \in \mathbf{Y}$. This result again follows from Lemma 2.1 since it is easily seen from (7) and (8) that policies u^* and v^* satisfy (5). ∎

It other words, we have shown that a policy that minimizes the cost $V^N_\beta(\bullet, u)$ over U_N may be obtained by solving the problem $\mathbf{Q}_\beta$.

To conclude this section, let us briefly discuss the problem of minimizing the β-discounted cost over all stages $t \in \mathbb{N}$.

Assume that the initial distribution p_0 is known. Then, once J_β is known, the following optimization problem may be addressed:

$$\min_{i_0, i_1, \ldots, i_{N-1}} \left\{ \sum_{x \in \mathbb{N}} p_0(x) \left[\sum_{t=0}^{N-1} \beta^t \, E[\tau(X_t, A_t) \right.\right.$$

$$| X_0 = x, A_j = i_j, 0 \leq j \leq N-1]$$
$$+ \quad \beta^N J_\beta(x, i_0, \ldots, i_{N-1}) \Big] \Big\},$$

where τ is a measurable mapping from $\mathbf{Y}$ to $\mathbb{R}$.

This optimization problem will not be investigated in this article. In the remainder of this paper, we will assume that the initial distribution p_0 is not known.

3 A Flow Control Problem with Delayed Information

The results in Section 2.1 are illustrated through the study of a simple flow control problem with the N-SDSI pattern. Throughout this section β is a fixed number in $(0,1)$ and N a fixed integer.

3.1 The Queueing Model

Considered is a discrete-time single-server queueing model. The service times form a sequence of i.i.d. random variables distributed according to a *geometric* distribution with known parameter $0 \leq b \leq 1$. We assume that at most one customer may join the system in every time slot. This arrival (if any) is assumed to occur at the beginning of the time slot (synchronized arrivals). At the beginning of each time slot, the decision maker chooses in the set $\mathbf{A} := \{p_1, p_2\}$, $0 \leq p_2 \leq p_1 \leq 1$, the probability of having one arrival in this time slot. Therefore, if action p_i is chosen at time $t \in \mathbb{N}$ then a customer will enter the system at time t with the probability p_i, $i = 1, 2$. We further assume that a customer that enters an empty system may leave the system (with the probability b) at the end of this same time slot.

Let $X_t \in \mathbf{X}$, $\mathbf{X} = \mathbb{N}$, denote the number of customers in the system at time t, $t \in \mathbb{N}$. We assume that the state of the system is known with a delay of N units of time or, equivalently, we assume that the N-SDSI pattern is used.

Our objective is to minimize, over U_N, the cost criterion $V_\beta^N(\bullet, u)$ defined in (3) in the case when

$$C'(y, a) := c(x) + \gamma\, a_1, \tag{9}$$

for all $y = (x, a_1, \ldots, a_N) \in \mathbf{Y}$, $a \in \mathbf{A}$. In (9), $c(x)$ is any real-valued *nondecreasing convex* function on $\mathbb{N}$ and γ is an arbitrary real constant (by convention, a_1 will correspond to the action taken N units of time ago). The choice of the cost function C' is discussed at the end of this section.

¿From the results in Section 2.2, we know that the solution to this control problem is obtained by solving the problem $\mathbf{Q}_\beta$, defined in (6), where the transition law q' is given by (see (4))

$$q'(y' \,|\, y; a) = \begin{cases} \bar{a}_1 b, & \text{if } x \geq 1,\ x' = x - 1; \\ a_1 b + \bar{a}_1 \bar{b}, & \text{if } x \geq 1,\ x' = x; \\ a_1 \bar{b}, & \text{if } x \geq 1,\ x' = x + 1; \\ 1 - a_1 \bar{b}, & \text{if } x = x' = 0; \\ a_1 \bar{b}, & \text{if } x = 0,\ x' = 1; \\ 0, & \text{if } |x - x'| > 1, \end{cases} \tag{10}$$

if $y' = (x', a'_1, \ldots, a'_N) \in \mathbf{Y}$ is such that $(a'_1, \ldots, a'_N) = (a_2, \ldots, a_N, a)$, and zero otherwise.

Let us now comment on the definition of the cost C' (see (9)). If c is nonnegative and if $\gamma < 0$ then the cost C' reduces to a cost frequently encountered in the literature on flow control models with completely observable states (see for instance Stidham [11]). In that case, $c(x)$ can be interpreted as a holding cost per unit time, and γa as a reward related to the acceptance of an incoming customer. Then, the problem $\mathbf{Q}_\beta$ involves a trade-off between low expected response time and low throughput on one hand (e.g., always choose action p_2) and high expected response time and high throughput on the other hand (e.g., always choose action p_1). A similar problem in the case of completely observable states was addressed by Ma and Makowski [6] under the additional requirements that the expected response time must be bounded from above and $p_2 = 0$. Also note that the problem $\mathbf{Q}_\beta$ becomes trivial when c is nonnegative and $\gamma \geq 0$. In that case, the optimal action is always to choose p_2.

The remainder of Section 3 is devoted to identifying a stationary policy in U' that solves the problem $\mathbf{Q}_\beta$ in the case when the cost function and the transition law are given by (9) and (10), respectively.

3.2 Preliminary Results

Further notation are needed at this point.

Define the operator $D : \mathbf{A}^N \to \mathbf{A}^{N-1}$ as $D(\mathbf{a}) = (a_2, \ldots, a_N)$ for all $\mathbf{a} = (a_1, a_2, \ldots, a_N) \in \mathbf{A}^N$.

Let $\mathcal{K}$ be the set of all real-valued functions on $\mathbf{Y}$. Let $S_A : \mathcal{K} \times \mathbf{Y} \to \mathbb{R}$ be the operator defined as

$$S_A(f, y) := \sum_{y' \in \mathbf{Y}} f(y')\, q'(y' \,|\, y; A), \tag{11}$$

for all $f \in \mathcal{K}$, $y \in \mathbf{Y}$, $A \in \mathbf{A}$.

It is easily seen from (11), (10) that

$$\begin{aligned} S_A(f, 0, \mathbf{a}) &:= (1 - a_1\bar{b})\, f(0, D(\mathbf{a}), A) \\ &\quad + a_1\bar{b} f(1, D(\mathbf{a}), A); \qquad (12) \\ S_A(f, x, \mathbf{a}) &:= \bar{a}_1 b f(x-1, D(\mathbf{a}), A) \\ &\quad + (a_1 b + \bar{a}_1\bar{b}) f(x, D(\mathbf{a}), A) \\ &\quad + a_1\bar{b} f(x+1, D(\mathbf{a}), A), \qquad (13) \end{aligned}$$

for all $x \geq 1$, $\mathbf{a} = (a_1, a_2, \ldots, a_N) \in \mathbf{A}^N$, $A \in \mathbf{A}$.

Last, let $T_\beta : \mathcal{K} \to \mathcal{K}$ be the DP operator associated with the problem $\mathbf{Q}_\beta$. It is easily seen from (2), (9) and (11) that

$$\begin{aligned} T_\beta f(y) &= c(x) + \gamma\, a_1 \\ &\quad + \beta \min\{S_{p_1}(f, x, \mathbf{a}), S_{p_2}(f, x, \mathbf{a})\}, \end{aligned} \tag{14}$$

for all $y = (x, \mathbf{a}) \in \mathbf{Y}$, $\mathbf{a} = (a_1, a_2, \ldots, a_N)$.

It is usually difficult to directly determine the optimal policy from the DP equation $J_\beta = T_\beta J_\beta$ (see (1)). An alternative approach is to use the well known *value iteration* algorithm (see Ross [8]).

A key result of this approach is the following (see Bertsekas [1, Sec. 5.4, Prop. 12]):

Proposition 3.1 *Assume that $C'(y, a) \geq 0$ for all $y \in \mathbf{Y}$ and $a \in \mathbf{A}$. Let f_0 be the zero function on $\mathcal{K}$. Then,*

$$\lim_{n \to \infty} T_\beta^n f_0(y) = J_\beta(y), \quad y \in \mathbf{Y}. \tag{15}$$

Proposition (3.1) will be used in the proof of Theorem 3.1.

3.3 The Value Iteration Algorithm Approach

The following notation will be used throughout this section.

We shall say that $g : \mathbb{N} \times \mathbf{A} \to \mathbb{R}$ satisfies assumption

R1 if $g(x, p_1) - g(x, p_2)$ is monotone increasing in x (i.e., g is supermodular); equivalently,

$$g(x, p_1) - g(x-1, p_1) \geq g(x, p_2) - g(x-1, p_2), \quad x \geq 1;$$

R2 if $g(x+1, p_2) - g(x, p_1)$ is monotone increasing in x; equivalently,

$$g(x, p_1) - g(x-1, p_1) \leq g(x+1, p_2) - g(x, p_2), \quad x \geq 1.$$

A mapping $h : \mathbb{N} \to \mathbb{R}$ will be said to satisfy assumption

R3 if $h(x)$ is integer-convex in x; equivalently,

$$h(x+1) - h(x) \geq h(x) - h(x-1), \quad x \geq 1;$$

R4 if $h(1) \geq h(0)$.

The proof of the main result (see Theorem 3.1) relies upon the following technical lemmas:

Lemma 3.1 *Let $h : \mathbb{N} \to \mathbb{R}$ be a nondecreasing function. Then, for all $x \geq 1$,*

$$\begin{aligned} &\bar{p}_2 b\, h(x) + (p_2 b + \bar{p}_2\bar{b})\, h(x+1) + p_2\bar{b}\, h(x+2) \\ &\quad \geq \bar{p}_1 b\, h(x-1) + (p_1 b + \bar{p}_1\bar{b})\, h(x) + p_1\bar{b}\, h(x+1). \end{aligned}$$

Proof. For $i = 1, 2$, $x \geq 1$, define

$$F_i(x) := \bar{p}_i b\, h(x-1) + (p_i b + \bar{p}_i\bar{b})\, h(x) + p_i\bar{b}\, h(x+1).$$

We must show that $F_2(x+1) \geq F_1(x)$ for $x \geq 1$. We have

$$\begin{aligned} F_2(x+1) - F_1(x) &= p_2\bar{b}\,[h(x+2) - h(x+1)] \\ &\quad + (1 - (p_1\bar{b} + \bar{p}_2 b))\,[h(x+1) - h(x)] \\ &\quad + \bar{p}_1 b\,[h(x) - h(x-1)]. \end{aligned}$$

The proof is then concluded by using the increasingness of h together with the fact that $p_1\bar{b} + \bar{p}_2 b \leq 1$. ∎

Lemma 3.2 *Let $f \in \mathcal{K}$ be such that $f(\bullet, \mathbf{v}, \bullet)$ satisfies* **R1**, **R2** *for some $\mathbf{v} \in A^{N-1}$. Then, $f(\bullet, \mathbf{v}, a)$ satisfies* **R3** *for all $a \in \mathbf{A}$.*

Proof. Let $f \in \mathcal{K}$ and $\mathbf{v} \in \mathbf{A}^{N-1}$ be such that $f(\bullet, \mathbf{v}, \bullet)$ satisfies **R1** and **R2**. Define $g(x,a) := f(x, \mathbf{v}, a)$ for $a \in \{p_1, p_2\}$. We have, for $x \geq 1$,

$$\begin{aligned} & g(x+1,p_1) - g(x,p_1) - [g(x,p_1) - g(x-1,p_1)] \\ & \quad \geq \quad g(x+1,p_1) - g(x,p_1) - [g(x+1,p_2) - g(x,p_2)], \\ & \quad \geq \quad 0, \end{aligned}$$

where the first (resp. second) inequality follows since g satisfies **R2** (resp. **R1**). On the other hand,

$$\begin{aligned} & g(x+1,p_2) - g(x,p_2) - [g(x,p_2) - g(x-1,p_2)] \\ & \quad \geq \quad g(x,p_1) - g(x-1,p_1) - [g(x,p_2) - g(x-1,p_2)], \\ & \quad \geq \quad 0, \end{aligned}$$

where again the first (resp. second) inequality follows since g satisfies **R2** (resp. **R1**). This establishes the proof. ∎

Lemma 3.3 *Let $f \in \mathcal{K}$ be such that $f(\bullet, \mathbf{a})$ satisfies* **R3**, **R4** *for all $\mathbf{a} \in \mathbf{A}^N$. Then, $S_A(f,1,\mathbf{a}) \geq S_A(f,0,\mathbf{a})$ for all $A \in \mathbf{A}$, $\mathbf{a} \in \mathbf{A}^N$.*

Proof. Let $f \in \mathcal{K}$ be such that $f(\bullet, \mathbf{a})$ satisfies **R3**, **R4** for all $\mathbf{a} \in \mathbf{A}^N$. ¿From the definition of $S_A(f,x,\mathbf{a})$ (see (12)-(13)) it is seen that for $A \in \mathbf{A}$, $\mathbf{a} = (a_1, a_2, \ldots, a_N) \in \mathbf{A}^N$,

$$\begin{aligned} & S_A(f,1,\mathbf{a}) - S_A(f,0,\mathbf{a}) \\ & \quad = \quad a_1 \bar{b} \, [f(2, D(\mathbf{a}), A) - f(1, D(\mathbf{a}), A) \\ & \qquad\quad -(f(1, D(\mathbf{a}), A) - f(0, D(\mathbf{a}), A))] \\ & \qquad\quad +(1 - \bar{a}_1 b) \, [f(1, D(\mathbf{a}), A) - f(0, D(\mathbf{a}), A)], \end{aligned} \tag{16}$$

which is nonnegative since $f(\bullet, \mathbf{a})$ satisfies **R3** and **R4**, which concludes the proof. ∎

Lemma 3.4 *Let $f \in \mathcal{K}$ be such that $f(\bullet, \mathbf{v}, \bullet)$ satisfies* **R1** *for all $\mathbf{v} \in \mathbf{A}^{N-1}$. Then, $S_{(\bullet)}(f, \bullet, \mathbf{a})$ satisfies* **R1** *for all $\mathbf{a} \in \mathbf{A}^N$.*

Proof. Let $f \in \mathcal{K}$ be such that $f(\bullet, \mathbf{v}, \bullet)$ satisfies **R1** for all $\mathbf{v} \in \mathbf{A}^{N-1}$. Fix $\mathbf{a} = (a_1, a_2, \ldots, a_N) \in \mathbf{A}^N$. Since $f(\bullet, D(\mathbf{a}), \bullet)$ satisfies **R1**, it is seen that for $x \geq 2$

$$\begin{aligned} & S_{p_1}(f,x,\mathbf{a}) - S_{p_1}(f,x-1,\mathbf{a}) \\ & \quad = \quad \bar{a}_1 b \, [f(x-1, D(\mathbf{a}), p_1) - f(x-2, D(\mathbf{a}), p_1)] \\ & \qquad\quad + (a_1 b + \bar{a}_1 \bar{b})[f(x, D(\mathbf{a}), p_1) - f(x-1, D(\mathbf{a}), p_1)] \\ & \qquad\quad + a_1 \bar{b} \, [f(x+1, D(\mathbf{a}), p_1) - f(x, D(\mathbf{a}), p_1)], \\ & \quad \geq \quad \bar{a}_1 b \, [f(x-1, D(\mathbf{a}), p_2) - f(x-2, D(\mathbf{a}), p_2)] \\ & \qquad\quad + (a_1 b + \bar{a}_1 \bar{b})[f(x, D(\mathbf{a}), p_2) - f(x-1, D(\mathbf{a}), p_2)] \\ & \qquad\quad + a_1 \bar{b} \, [f(x+1, D(\mathbf{a}), p_2) - f(x, D(\mathbf{a}), p_2)], \\ & \quad = \quad S_{p_2}(f,x,\mathbf{a}) - S_{p_2}(f,x-1,\mathbf{a}). \end{aligned}$$

On the other hand, using (16) we see that

$$\begin{aligned} & S_{p_1}(f,1,\mathbf{a}) - S_{p_1}(f,0,\mathbf{a}) \\ & \quad = \quad a_1 \bar{b} \, [f(2, D(\mathbf{a}), p_1) - f(1, D(\mathbf{a}), p_1)] \\ & \qquad\quad +(1 - \bar{a}_1 b - a_1 \bar{b}) \, [f(1, D(\mathbf{a}), p_1) - f(0, D(\mathbf{a}), p_1)], \\ & \quad \geq \quad a_1 \bar{b} \, [f(2, D(\mathbf{a}), p_2) - f(1, D(\mathbf{a}), p_2)] \\ & \qquad\quad +(1 - \bar{a}_1 b - a_1 \bar{b}) \, [f(1, D(\mathbf{a}), p_2) - f(0, D(\mathbf{a}), p_2)], \\ & \quad = \quad S_{p_2}(f,1,\mathbf{a}) - S_{p_2}(f,0,\mathbf{a}), \end{aligned}$$

by using the assumption that $f(\bullet, \mathbf{v}, \bullet)$ satisfies **R1** and the fact that $\bar{a}_1 b - a_1 \bar{b} \leq 1$. ∎

Lemma 3.5 *Let $f \in \mathcal{K}$, $\mathbf{v} \in \mathbf{A}^{N-1}$ and $A \in \mathbf{A}$ be such that (i) $f(\bullet, \mathbf{v}, \bullet)$ satisfies* **R1**, **R2** *and (ii) $f(\bullet, \mathbf{v}, A)$ satisfies* **R4**. *Then, $S_A(f, \bullet, \bullet, \mathbf{v})$ satisfies* **R1** *and* **R2**.

Proof. Let $f \in \mathcal{F}$, $\mathbf{v} \in \mathbf{A}^{N-1}$, $A \in \mathbf{A}$ be such that conditions (i) and (ii) in the lemma hold. We first establish that $S_A(f, \bullet, \bullet, \mathbf{v})$ satisfies **R1**. For $x \geq 1$, we have

$$\begin{aligned} & S_A(f,x,p_1,\mathbf{v}) - S_A(f,x,p_2,\mathbf{v}) \\ & \quad = \quad (\bar{p}_1 - \bar{p}_2) b f(x-1, \mathbf{v}, A) \\ & \qquad\quad + ((p_1 - p_2) b + (\bar{p}_1 - \bar{p}_2) \bar{b}) f(x, \mathbf{v}, A) \\ & \qquad\quad + (p_1 - p_2) \bar{b} f((x+1, \mathbf{v}, A)), \\ & \quad = \quad (p_1 - p_2) \left\{ \bar{b} \, [f(x+1, \mathbf{v}, A) - f(x, \mathbf{v}, A)] \right. \\ & \qquad\quad \left. + \, b \, [f(x, \mathbf{v}, A) - f(x-1, \mathbf{v}, A)] \right\}, \end{aligned}$$

and the latter is increasing in x since $p_1 \geq p_2$ and since by Lemma 3.2. $f(\bullet, \mathbf{a})$ satisfies **R3** for all $\mathbf{a} \in \mathbf{A}^N$. It remains to show that the monotonicity property also holds at the boundary. We have

$$\begin{aligned}
& S_A(f,1,p_1,\mathbf{v}) - S_A(f,1,p_2,\mathbf{v}) \\
& \quad -[S_A(f,0,p_1,\mathbf{v}) - S_A(f,0,p_2,\mathbf{v})] \\
&= (p_1-p_2)\,[b\,(f(1,\mathbf{v},A) - f(0,\mathbf{v},A)) \\
& \quad +\bar{b}\,(f(2,\mathbf{v},A) - f(1,\mathbf{v},A) \\
& \quad -(f(1,\mathbf{v},A) - f(0,\mathbf{v},A))],
\end{aligned}$$

which is nonnegative since $f(\bullet,\mathbf{v},A)$ satisfies **R4** and since (see Lemma 3.2) $f(\bullet,\mathbf{a})$ satisfies **R3** for all $\mathbf{a} \in \mathbf{A}^N$. This completes the proof that $S_A(f,\bullet,\bullet,\mathbf{v})$ satisfies **R1**.

Let us now show that $S_A(f,\bullet,\bullet,\mathbf{v})$ satisfies **R2**. For $x \geq 2$, it is easily seen that

$$\begin{aligned}
& S_A(f,x,p_1,\mathbf{v}) - S_A(f,x-1,p_1,\mathbf{v}) \\
& \quad -[S_A(f,x+1,p_2,\mathbf{v}) - S_A(f,x,p_2,\mathbf{v})] \\
&= \bar{p}_1 b\,[f(x-1,\mathbf{v},A) - f(x-2,\mathbf{v},A)] \\
& \quad +(p_1 b + \bar{p}_1\bar{b})\,[f(x,\mathbf{v},A) - f(x-1,\mathbf{v},A)] \\
& \quad +p_1\bar{b}\,[f(x+1,\mathbf{v},A) - f(x,\mathbf{v},A)] \\
& \quad -\bar{p}_2 b\,[f(x,\mathbf{v},A) - f(x-1,\mathbf{v},A)] \\
& \quad -(p_2 b + \bar{p}_2\bar{b})\,[f(x+1,\mathbf{v},A) - f(x,\mathbf{v},A)] \\
& \quad -p_2\bar{b}\,[f(x+2,\mathbf{v},A) - f(x+1,\mathbf{v},A)], \\
&= -p_2\bar{b}\,[f(x+2,\mathbf{v},A) - f(x+1,\mathbf{v},A) \\
& \quad -\,(f(x+1,\mathbf{v},A) - f(x,\mathbf{v},A))] \\
& \quad -(1-p_1\bar{b}-\bar{p}_2 b)\,[f(x+1,\mathbf{v},A) - f(x,\mathbf{v},A) \\
& \quad -(f(x,\mathbf{v},A) - f(x-1,\mathbf{v},A))] \\
& \quad -\bar{p}_1 b\,[f(x,\mathbf{v},A) - f(x-1,\mathbf{v},A) \\
& \quad -\,(f(x-1,\mathbf{v},A) - f(x-2,\mathbf{v},A))], \\
&\leq 0,
\end{aligned}$$

where the above inequality again follows from the fact that $f(\bullet,\mathbf{a})$ satisfies **R3** for all $\mathbf{a} \in \mathbf{A}^N$. It remains to examine the case when $x = 1$. We have

$$\begin{aligned}
& S_A(f,1,p_1,\mathbf{v}) - S_A(f,0,p_1,\mathbf{v}) \\
& \quad -[S_A(f,2,p_2,\mathbf{v}) - S_A(f,1,p_2,\mathbf{v})] \\
&= -p_2\bar{b}\,[f(3,\mathbf{v},A) - f(2,\mathbf{v},A) \\
& \quad -\,(f(2,\mathbf{v},A) - f(1,\mathbf{v},A))] \\
& \quad -(1-p_1\bar{b}-\bar{p}_2 b)\,[f(2,\mathbf{v},A) - f(1,\mathbf{v},A) \\
& \quad -(f(1,\mathbf{v},A) - f(0,\mathbf{v},A))] \\
& \quad -\bar{p}_1 b\,[f(1,\mathbf{v},A) - f(0,\mathbf{v},A)],
\end{aligned}$$

which is nonnegative since $f(\bullet,\mathbf{v},A)$ satisfies both conditions **R3** by Lemma 3.2 and **R4**. Hence, $S_A(f,\bullet,\bullet,\mathbf{v})$ satisfies **R2**, which concludes the proof. ∎

Lemma 3.6 *Assume that $f(\bullet,\mathbf{v},\bullet)$ satisfies* **R1** *for all $\mathbf{v} \in \mathbf{A}^{N-1}$. Then, for every $\mathbf{a} \in \mathbf{A}^N$, (i) there exists a (possibly infinite) number $l(\mathbf{a}) \in \mathbb{N}$ such that*

$$\begin{aligned}
S_{p_1}(f,x,\mathbf{a}) - S_{p_2}(f,x,\mathbf{a}) &< 0, \quad \textit{for } 0 \leq x < l(\mathbf{a}); \\
S_{p_1}(f,x,\mathbf{a}) - S_{p_2}(f,x,\mathbf{a}) &\geq 0, \quad \textit{for } x \geq l(\mathbf{a}).
\end{aligned}$$

Moreover, if $\mathbf{a}^1 = (p_1,\mathbf{v})$ and $\mathbf{a}^2 = (p_2,\mathbf{v})$ for $\mathbf{v} \in \mathbf{A}^{N-1}$, then (ii) $l(\mathbf{a}^1) \leq l(\mathbf{a}^2)$ and (iii) $l(\mathbf{a}^2) \leq l(\mathbf{a}^1)+1$.

Proof. Let $f \in \mathcal{K}$ be such that $f(\bullet,\mathbf{v},\bullet)$ satisfies **R1** for all $\mathbf{v} \in \mathbf{A}^{N-1}$. Fix $\mathbf{v} \in \mathbf{A}^{N-1}$ and let $\mathbf{a}^i = (p_i,\mathbf{v})$, $i = 1,2$.

The property (i) is a direct consequence of the fact that $S_{(\bullet)}(f,\bullet,\mathbf{a})$ satisfies **R1** for all $\mathbf{a} \in \mathbf{A}^N$ (see Lemma 3.4), where

$$l(\mathbf{a}) := \inf\{x \geq 0 : S_{p_1}(f,x,\mathbf{a}) \geq S_{p_2}(f,x,\mathbf{a})\}, \;\; \mathbf{a} \in \mathbf{A}^N.$$

Let us now turn to the proof of (ii). For $x \geq 1$ we have

$$\begin{aligned}
& S_{p_1}(f,x,p_1,\mathbf{v}) - S_{p_2}(f,x,p_1,\mathbf{v}) \\
& \quad -[S_{p_1}(f,x,p_2,\mathbf{v}) - S_{p_2}(f,x,p_2,\mathbf{v})] \\
&= \bar{p}_1 b\,[f(x-1,\mathbf{v},p_1) - f(x-1,\mathbf{v},p_2)] \\
& \quad +(p_1 b + \bar{p}_1\bar{b})[f(x,\mathbf{v},p_1) - f(x,\mathbf{v},p_2)] \\
& \quad +\,p_1\bar{b}\,[f(x+1,\mathbf{v},p_1) - f(x+1,\mathbf{v},p_2)] \\
& \quad -\bar{p}_2 b\,[f(x-1,\mathbf{v},p_1) - f(x-1,\mathbf{v},p_2)] \\
& \quad -\,(p_2 b + \bar{p}_2\bar{b})[f(x,\mathbf{v},p_1) - f(x,\mathbf{v},p_2)] \\
& \quad -\,p_2\bar{b}\,[f(x+1,\mathbf{v},p_1) - f(x+1,\mathbf{v},p_2)], \\
&= (p_1-p_2)\,\big\{\bar{b}\,[f(x+1,\mathbf{v},p_1) - f(x+1,\mathbf{v},p_2) \\
& \quad -(f(x,\mathbf{v},p_1) - f(x,\mathbf{v},p_2))] \\
& \quad +b\,[f(x,\mathbf{v},p_1) - f(x,\mathbf{v},p_2) - (f(x-1,\mathbf{v},p_1) \\
& \quad -f(x-1,\mathbf{v},p_2))]\big\}, \\
&\geq 0 \quad , \qquad\qquad (17)
\end{aligned}$$

since $p_1 \geq p_2$ and since $f(\bullet,\mathbf{v},\bullet)$ satisfies **R1**.

For $x = 0$ we have

$$\begin{aligned} & S_{p_1}(f,0,p_1,\mathbf{v}) - S_{p_2}(f,0,p_1,\mathbf{v}) \\ & \qquad -[S_{p_1}(f,0,p_2,\mathbf{v}) - S_{p_2}(f,0,p_2,\mathbf{v})] \\ & \quad = \quad (p_1 - p_2)\,\bar{b}\,[f(1,\mathbf{v},p_1) - f(1,\mathbf{v},p_2) \\ & \qquad -(f(0,\mathbf{v},p_1) - f(0,\mathbf{v},p_2))], \\ & \quad \geq \quad 0, \end{aligned} \tag{18}$$

since $f(\bullet,\mathbf{v},\bullet)$ satisfies **R1**. Therefore, it is seen from (17) and (18) that $S_{p_1}(f,x,p_1,\mathbf{v}) - S_{p_2}(f,x,p_1,\mathbf{v}) \geq S_{p_1}(f,x,p_2,\mathbf{v}) - S_{p_2}(f,x,p_2,\mathbf{v})$ for all $x \geq 0$, which proves (ii).

It remains to establish (iii). For $x \geq 1$, we have

$$\begin{aligned} & S_{p_1}(f,x,p_1,\mathbf{v}) - S_{p_2}(f,x,p_1,\mathbf{v}) \\ & \qquad -[S_{p_1}(f,x+1,p_2,\mathbf{v}) - S_{p_2}(f,x+1,p_2,\mathbf{v})] \\ & \quad = \quad \bar{p}_1 b\,[f(x-1,\mathbf{v},p_1) - f(x-1,\mathbf{v},p_2] \\ & \qquad +(p_1 b + \bar{p}_1\bar{b})[f(x,\mathbf{v},p_1) - f(x,\mathbf{v},p_2)] \\ & \qquad + p_1\bar{b}\,[f(x+1,\mathbf{v},p_1) - f(x+1,\mathbf{v},p_2)] \\ & \qquad -\bar{p}_2 b\,[f(x,\mathbf{v},p_1) - f(x,\mathbf{v},p_2)] \\ & \qquad -(p_2 b + \bar{p}_2\bar{b})[f(x+1,\mathbf{v},p_1) - f(x+1,\mathbf{v},p_2)] \\ & \qquad - p_2\bar{b}\,[f(x+2,\mathbf{v},p_1) - f(x+2,\mathbf{v},p_2)] \\ & \quad \leq \quad 0. \end{aligned} \tag{19}$$

The above inequality is obtained by applying Lemma 3.1 with $h(x) := f(x,\mathbf{v},p_1) - f(x,\mathbf{v},p_2)$ and by noting that h is nondecreasing since $f(\bullet,\mathbf{v},\bullet)$ satisfies **R1**. On the other hand,

$$\begin{aligned} & S_{p_1}(f,0,p_1,\mathbf{v}) - S_{p_2}(f,0,p_1,\mathbf{v}) \\ & \qquad -[S_{p_1}(f,1,p_2,\mathbf{v}) - S_{p_2}(f,1,p_2,\mathbf{v})] \\ & \quad = \quad -p_2\bar{b}\,[f(2,\mathbf{v},p_1) - f(2,\mathbf{v},p_2) \\ & \qquad -(f(1,\mathbf{v},p_1) - f(1,\mathbf{v},p_2))] \\ & \qquad -(1 - p_1\bar{b} - \bar{p}_2 b)\,[f(1,\mathbf{v},p_1) - f(1,\mathbf{v},p_2) \\ & \qquad -(f(0,\mathbf{v},p_1) - f(0,\mathbf{v},p_2))], \\ & \quad \leq \quad 0, \end{aligned} \tag{20}$$

since $f(\bullet,\mathbf{v},\bullet)$ satisfies **R1**. Combining (19) and (20) yields $l(\mathbf{a}^2) \leq l(\mathbf{a}^1) + 1$, which completes the proof. ∎

Lemma 3.7 *Let $f \in \mathcal{K}$ be such that $f(\bullet,\mathbf{v},\bullet)$ satisfies* **R1**, **R2** *for all $\mathbf{v} \in \mathbf{A}^{N-1}$ and $f(\bullet,\mathbf{a})$ satisfies* **R4** *for all $\mathbf{a} \in \mathbf{A}^N$. Then, $T_\beta f(\bullet,\bullet,\mathbf{v})$ satisfies* **R1**, **R2** *for all $\mathbf{v} \in \mathbf{A}^{N-1}$ and $T_\beta f(\bullet,\mathbf{a})$ satisfies* **R4** *for all $\mathbf{a} \in \mathbf{A}^N$.*

Proof. Let $f \in \mathcal{K}$ be such that $f(\bullet,\mathbf{v},\bullet)$ satisfies **R1**, **R2** for all $\mathbf{v} \in \mathbf{A}^{N-1}$ and $f(\bullet,\mathbf{a})$ satisfies **R4** for all $\mathbf{a} \in \mathbf{A}^N$. Fix $\mathbf{v} \in \mathbf{A}^{N-1}$.

We first establish that $T_\beta f(\bullet,\bullet,\mathbf{v})$ satisfies **R1**. To do so, we must show that $G(x)$ defined as

$$\begin{aligned} G(x) &:= \beta^{-1}\,(T_\beta f(x,p_1,\mathbf{v}) - T_\beta f(x,p_2,\mathbf{v}) \\ & \qquad -[T_\beta f(x+1,p_1,\mathbf{v}) - T_\beta f(x+1,p_2,\mathbf{v})])\,, \\ &= \min\{S_{p_1}(f,x,p_1,\mathbf{v}), S_{p_2}(f,x,p_1,\mathbf{v})\} \\ & \quad - \min\{S_{p_1}(f,x,p_2,\mathbf{v}), S_{p_2}(f,x,p_2,\mathbf{v})\} \\ & \quad - [\min\{S_{p_1}(f,x+1,p_1,\mathbf{v}), S_{p_2}(f,x+1,p_1,\mathbf{v}) \\ & \quad - \min\{S_{p_1}(f,x+1,p_2,\mathbf{v}), S_{p_2}(f,x+1,p_2,\mathbf{v})\}]\,, \end{aligned} \tag{21}$$

is nonpositive for all $x \geq 0$.

Recall the definition of $\mathbf{a}^i$ and $l(\mathbf{a}^i)$ for $i = 1,2$ (see Lemma 3.6). Also recall that $l(\mathbf{a}^1) \leq l(\mathbf{a}^2) \leq l(\mathbf{a}^1) + 1$.

We shall distinguish the following cases:

(1) $x \geq l(\mathbf{a}^2)$;

(2) $\max(0, l(\mathbf{a}^1) - 1) \leq x < l(\mathbf{a}^2)$;

(3) $x \leq \max(0, l(\mathbf{a}^1) - 2)$.

Case (1): $x \geq l(\mathbf{a}^2)$.

We have, cf. (21),

$$\begin{aligned} G(x) &= \min\{S_{p_1}(f,x,p_1,\mathbf{v}), S_{p_2}(f,x,p_1,\mathbf{v})\} \\ & \quad -S_{p_2}(f,x,p_2,\mathbf{v}) \\ & \quad -[S_{p_2}(f,x+1,p_1,\mathbf{v}) - S_{p_2}(f,x+1,p_2,\mathbf{v})]\}, \\ &\leq \quad S_{p_2}(f,x,p_1,\mathbf{v}) - S_{p_2}(f,x,p_2,\mathbf{v}) \\ & \quad -[S_{p_2}(f,x+1,p_1,\mathbf{v}) - S_{p_2}(f,x+1,p_2,\mathbf{v})]\,, \end{aligned}$$

which is nonpositive since $S_{p_2}(f,\bullet,\bullet,\mathbf{v})$ satisfies **R1** from Lemma 3.5.

Case (2): $\max(0, l(\mathbf{a}^1) - 1) \leq x < l(\mathbf{a}^2)$.

We have for all $x \in \mathbb{N}$, cf. (21),

$$G(x) = \min\{S_{p_1}(f,x,p_1,\mathbf{v}), S_{p_2}(f,x,p_1,\mathbf{v})\} - S_{p_1}(f,x,p_2,\mathbf{v}) - [S_{p_2}(f,x+1,p_1,\mathbf{v}) - \min\{S_{p_1}(f,x+1,p_2,\mathbf{v}), S_{p_2}(f,x+1,p_2,\mathbf{v})\}]. \quad (22)$$

For $x \geq 1$, we have from (13), (22),

$$\begin{aligned} G(x) &\leq S_{p_1}(f,x,p_1,\mathbf{v}) - S_{p_1}(f,x,p_2,\mathbf{v}) \\ &\quad - [S_{p_2}(f,x+1,p_1,\mathbf{v}) - S_{p_2}(f,x+1,p_2,\mathbf{v})], \\ &= (p_1-p_2)\{b[f(x,\mathbf{v},p_2) - f(x-1,\mathbf{v},p_1) \\ &\quad -(f(x+1,\mathbf{v},p_2) - f(x,\mathbf{v},p_1))] \\ &\quad + \bar{b}[f(x+1,\mathbf{v},p_2) - f(x,\mathbf{v},p_1) \\ &\quad - (f(x+2,\mathbf{v},p_2) - f(x+1,\mathbf{v},p_1))]\}, \end{aligned}$$

which is nonpositive since $f(\bullet,\mathbf{v},\bullet)$ satisfies **R2** and since $p_1 \geq p_2$.

On the other hand, we see from (12), (13), (22) that

$$\begin{aligned} G(0) &\leq S_{p_1}(f,0,p_1,\mathbf{v}) - S_{p_1}(f,0,p_2,\mathbf{v}) \\ &\quad - [S_{p_2}(f,1,p_1,\mathbf{v}) - S_{p_2}(f,1,p_2,\mathbf{v})], \\ &= -(p_1-p_2)\{b[f(1,\mathbf{v},p_2) - f(0,\mathbf{v},p_2)] \\ &\quad + \bar{b}[f(2,\mathbf{v},p_2) - f(1,\mathbf{v},p_2) - (f(1,\mathbf{v},p_1) \\ &\quad - f(0,\mathbf{v},p_1))]\}, \end{aligned}$$

which is nonpositive since $f(\bullet,\mathbf{v},\bullet)$ satisfies **R2** and since $f(\bullet,\mathbf{a}')$ satisfies **R4** for all $\mathbf{a}' \in \mathbf{A}^N$.

Case (3): $x \leq \max(0, l(\mathbf{a}^1) - 2)$.

We obtain from (21)

$$\begin{aligned} G(x) &= S_{p_1}(f,x,p_1,\mathbf{v}) - S_{p_1}(f,x,p_2,\mathbf{v}) \\ &\quad - [S_{p_1}(f,x+1,p_1,\mathbf{v}) - S_{p_1}(f,x+1,p_2,\mathbf{v})], \end{aligned}$$

which is nonpositive for all $x \in \mathbb{N}$ since $S_{p_1}(f,\bullet,\bullet,\mathbf{v})$ satisfies **R1** by Lemma 3.5. This concludes the proof that $G(x) \leq 0$ for all $x \in \mathbb{N}$. Hence, $T_\beta f(\bullet,\bullet,\mathbf{v})$ satisfies **R1**.

Let us now show that $T_\beta f(\bullet,\mathbf{a})$ satisfies **R4** We have, cf. (14),

$$\begin{aligned} &\beta^{-1}[T_\beta f(1,\mathbf{a}) - T_\beta f(0,\mathbf{a})] \\ &= \beta^{-1}(c(1) - c(0)) + \min\{S_{p_1}(f,1,\mathbf{a}), S_{p_2}(f,1,\mathbf{a})\} \\ &\quad - \min\{S_{p_1}(f,0,\mathbf{a}), S_{p_2}(f,0,\mathbf{a})\}, \quad \mathbf{a} \in \mathbf{A}^N. \quad (23) \end{aligned}$$

Since $S_A(f,1,\mathbf{a}) \geq S_A(f,0,\mathbf{a})$ by Lemma 3.3 and since $c(1) \geq c(0)$ by assumption, we readily deduce from (23) that $T_\beta f(\bullet,\mathbf{a})$ satisfies **R4**.

It remains to establish that $T_\beta f(\bullet,\bullet,\mathbf{v})$ satisfies **R2**. This amounts to showing that

$$\begin{aligned} H(x) &:= \beta^{-1}(T_\beta f(x+1,p_2,\mathbf{v}) - T_\beta f(x,p_2,\mathbf{v}) \\ &\quad -[T_\beta f(x,p_1,\mathbf{v}) - T_\beta f(x-1,p_1,\mathbf{v})]), \end{aligned}$$

is nonnegative for all $x \geq 1$.

We have for $x \geq 1$,

$$\begin{aligned} H(x) &= \beta^{-1}[c(x+1) - c(x) - (c(x) - c(x-1))] \\ &\quad + \min\{S_{p_1}(f,x+1,p_2,\mathbf{v}), S_{p_2}(f,x+1,p_2,\mathbf{v})\} \\ &\quad - \min\{S_{p_1}(f,x,p_2,\mathbf{v}), S_{p_2}(f,x,p_2,\mathbf{v})\} \\ &\quad - [\min\{S_{p_1}(f,x,p_1,\mathbf{v}), S_{p_2}(f,x,p_1,\mathbf{v})\} \\ &\quad - \min\{S_{p_1}(f,x-1,p_1,\mathbf{v}), S_{p_2}(f,x-1,p_1,\mathbf{v})\}], \\ &\geq \min\{S_{p_1}(f,x+1,p_2,\mathbf{v}), S_{p_2}(f,x+1,p_2,\mathbf{v})\} \\ &\quad - \min\{S_{p_1}(f,x,p_2,\mathbf{v}), S_{p_2}(f,x,p_2,\mathbf{v})\} \\ &\quad - [\min\{S_{p_1}(f,x,p_1,\mathbf{v}), S_{p_2}(f,x,p_1,\mathbf{v})\} \\ &\quad - \min\{S_{p_1}(f,x-1,p_1,\mathbf{v}), S_{p_2}(f,x-1,p_1,\mathbf{v})\}], \end{aligned} \quad (24)$$

where (24) follows from the convexity of the cost function c.

For $x \geq l(\mathbf{a}^1) + 1$, we have from (24)

$$\begin{aligned} H(x) &\geq S_{p_2}(f,x+1,p_2,\mathbf{v}) - S_{p_2}(f,x,p_2,\mathbf{v}) \\ &\quad - [S_{p_2}(f,x,p_1,\mathbf{v}) - S_{p_2}(f,x-1,p_1,\mathbf{v})], \\ &\geq 0. \end{aligned}$$

The first inequality holds because $l(\mathbf{a}^2) \leq l(\mathbf{a}^1)+1$. The last inequality holds because $S_{p_2}(f,\bullet,\bullet,\mathbf{v})$ satisfies **R2** (cf. Lemma 3.5).

It remains to cover the case when $1 \leq x \leq l(\mathbf{a}^1)$.

Assume first that $l(\mathbf{a}^1) = 1$. We have for $x = 1$, cf. (24),

$$\begin{aligned} H(1) &\geq S_{p_2}(f,2,p_2,\mathbf{v}) \\ &\quad - \min\{S_{p_1}(f,1,p_2,\mathbf{v}), S_{p_2}(f,1,p_2,\mathbf{v})\} \\ &\quad - [\min\{S_{p_1}(f,1,p_1,\mathbf{v}), S_{p_2}(f,1,p_1,\mathbf{v})\} \end{aligned}$$

$$
\begin{aligned}
& -S_{p_1}(f,0,p_1,\mathbf{v})], \\
\geq \quad & S_{p_2}(f,2,p_2,\mathbf{v}) - S_{p_1}(f,1,p_2,\mathbf{v}) \\
& - [S_{p_2}(f,1,p_1,\mathbf{v}) - S_{p_1}(f,0,p_1,\mathbf{v})], \\
= \quad & p_2\bar{b}\,[f(3,\mathbf{v},p_2) - f(2,\mathbf{v},p_1) \\
& -(f(2,\mathbf{v},p_2) - f(1,\mathbf{v},p_1))] \\
& +(1-p_1\bar{b}-\bar{p}_2 b)\,[f(2,\mathbf{v},p_2) - f(1,\mathbf{v},p_1) \\
& -(f(1,\mathbf{v},p_2) - f(0,\mathbf{v},p_1))] \\
& +\bar{p}_1 b\,[f(1,\mathbf{v},p_2) - f(0,\mathbf{v},p_2)],
\end{aligned}
$$

which is nonnegative since $f(\bullet,\mathbf{v},\bullet)$ satisfies **R2** and since $f(\bullet,\mathbf{a})$ satisfies **R4**.

Assume now that $l(\mathbf{a}^1) \geq 2$. For $1 \leq x < l(\mathbf{a}^1)-1$, it is easily seen from (24) that

$$
\begin{aligned}
H(x) \geq & S_{p_1}(f,x+1,p_2,\mathbf{v}) - S_{p_1}(f,x,p_2,\mathbf{v}) \\
& - [S_{p_1}(f,x,p_1,\mathbf{v}) - S_{p_1}(f,x-1,p_1,\mathbf{v})],
\end{aligned}
$$

which is nonnegative since $S_{p_1}(f,\bullet,\bullet,\mathbf{v})$ satisfies **R2** (cf. Lemma 3.5).

For $x = l(\mathbf{a}^1)$, we have

$$
\begin{aligned}
H(x) \geq & S_{p_2}(f,x+1,p_2,\mathbf{v}) - S_{p_1}(f,x,p_2,\mathbf{v}) \\
& -[S_{p_2}(f,x,p_1,\mathbf{v}) - S_{p_1}(f,x-1,p_1,\mathbf{v})], \\
= \quad & \bar{p}_2 b\,[f(x,\mathbf{v},p_2) - f(x-1,\mathbf{v},p_1)] \\
& +(p_2 b+\bar{p}_2\bar{b})\,[f(x+1,\mathbf{v},p_2) - f(x,\mathbf{v},p_1)] \\
& +p_2\bar{b}\,[f(x+2,\mathbf{v},p_2) - f(x+1,\mathbf{v},p_1)] \\
& -\bar{p}_1 b\,[f(x-1,\mathbf{v},p_2) - f(x-2,\mathbf{v},p_1)] \\
& -(p_1 b+\bar{p}_1\bar{b})\,[f(x,\mathbf{v},p_2) - f(x-1,\mathbf{v},p_1)] \\
& -p_1\bar{b}\,[f(x+1,\mathbf{v},p_2) - f(x,\mathbf{v},p_1)],
\end{aligned}
$$

which is nonnegative by Lemma 3.1 with $h(x) := f(x+1,\mathbf{v},p_2) - f(x,\mathbf{v},p_1)$ (note that h is nondecreasing since $f(\bullet,\mathbf{v},\bullet)$ satisfies **R2**).

Assume now that $x = l(\mathbf{a}^1)-1$. When $l(\mathbf{a}^2) = l(\mathbf{a}^1)$ this case is seen to reduce to the case $x = l(\mathbf{a}^1)$. When $l(\mathbf{a}^2) = l(\mathbf{a}^1)+1$, we have

$$
\begin{aligned}
H(x) \geq & S_{p_1}(f,x+1,p_2,\mathbf{v}) \\
& - \min\{S_{p_1}(f,x,p_2,\mathbf{v}), S_{p_2}(f,x,p_2,\mathbf{v})\} \\
& - [\min\{S_{p_1}(f,x,p_1,\mathbf{v}), S_{p_2}(f,x,p_1,\mathbf{v})\} \\
& -S_{p_1}(f,x-1,p_1,\mathbf{v})], \\
\geq \quad & S_{p_1}(f,x+1,p_2,\mathbf{v}) - S_{p_1}(f,x,p_2,\mathbf{v}) \\
& - [S_{p_1}(f,x,p_1,\mathbf{v}) - S_{p_1}(f,x-1,p_1,\mathbf{v})],
\end{aligned}
$$

which is nonnegative since $S_{p_1}(f,\bullet,\bullet,\mathbf{v})$ satisfies **R2**, which concludes the proof. ∎

We are now ready to prove the main result of this article.

Theorem 3.1 *There exists an optimal threshold policy for the problem $\mathbf{Q}_\beta$. More precisely, for every $\mathbf{a} \in \mathbf{A}^N$ there exists a (possibly infinite) number $l(\mathbf{a}) \in \mathbb{N}$ such that if the system is in state $y = (x,\mathbf{a})$, $x \in \mathbb{N}$, then the optimal action is p_1 if $x < l(\mathbf{a})$, and p_2 otherwise. Further,*

$$l(\mathbf{a}^1) \leq l(\mathbf{a}^2) \leq l(\mathbf{a}^1)+1,$$

for all $\mathbf{a}^1 = (p_1,\mathbf{v})$, $\mathbf{a}^2 = (p_2,\mathbf{v})$, $\mathbf{v} \in \mathbf{A}^{N-1}$.

Proof. Define $\mathcal{G} \subset \mathcal{K}$ to be the set of all real-valued functions on $\mathbf{Y}$ such that $f(\bullet,\mathbf{v},\bullet)$ satisfies **R1**, **R2** for all $\mathbf{v} \in \mathbf{A}^{N-1}$ and $f(\bullet,\mathbf{a})$ satisfies **R4** for all $\mathbf{a} \in \mathbf{A}^N$. Since the zero function f_0 on $\mathcal{K}$ belongs to $\mathcal{G}$, it is seen from Lemma 3.7 that

$$T_\beta^n f_0 \in \mathcal{G}, \quad n \geq 1. \tag{25}$$

Assume first that $C'(y,a) \geq 0$ for all $y = (x,\mathbf{a}) \in \mathbf{Y}$, $a \in \mathbf{A}$. Then, Proposition 3.1 applies to give that

$$\lim_{n\to\infty} T_\beta^n f_0(y) = J_\beta(y), \quad y \in \mathbf{Y}. \tag{26}$$

Further, since $\mathcal{G}$ is closed under pointwise limits, we may deduce from (25) and (26) that $J_\beta \in \mathcal{G}$. So, in particular, J_β satisfies **R1**, which implies from Lemma 3.6 that for every $\mathbf{a} \in \mathbf{A}$ there a (possibly infinite) number $l(\mathbf{a}) \in \mathbb{N}$ such that

$$
\begin{aligned}
S_{p_1}(J_\beta,x,\mathbf{a}) - S_{p_2}(J_\beta,x,\mathbf{a}) &< 0, \quad \text{for } x < l(\mathbf{a}); \\
S_{p_1}(J_\beta,x,\mathbf{a}) - S_{p_2}(J_\beta,x,\mathbf{a}) &\geq 0, \quad \text{for } x \geq l(\mathbf{a}).
\end{aligned}
$$

Hence, it is seen from (11), (14) and Theorem 2.1 that the optimal action when the system is in state $y = (x,\mathbf{a}) \in \mathbf{Y}$ is p_1 if $x < l(\mathbf{a})$ and p_2 otherwise. The last part of the theorem also follows from Lemma 3.6.

The general case when $C'(y,a) \in \mathbb{R}$ readily deduces from the case when $C'(y,a) \geq 0$ by noting, from (9) and from the assumption that $c(x)$ is nondecreasing and convex, that there always exists a constant $0 \leq D < \infty$ such that $C'(y, a) + D \geq 0$ for all $y \in \mathbf{Y}$, $a \in \mathbf{A}$. This concludes the proof. ∎

Remark 3.1 If $N = 1$ then $l(p_1) \leq l(p_2) \leq l(p_1) + 1$.

4 Concluding Remarks

The existence of an optimal policy of a threshold type has been shown for a simple flow control problem in the case when the decision maker has only access to delayed state information (the so-called N-SDSI pattern). High-Speed Data Networks (HSDN's) provide a good instance where such information patterns are useful. Indeed, because of the very high throughputs that can be handled by HSDN's, the state of the network may have changed considerably by the time the messages on the state of the network reach the decision-maker. Therefore, only controls based on delayed information patterns may be used as far as closed-loop controls are concerned.

Extensions of our model to more general arrival processes (e.g., batch arrivals, Markov Arrival Processes, cf. Neuts [7], Lucantoni [5]) are however needed so as to capture the basic characteristics of the traffic in HSDN's (e.g., burstiness). This issue, as well as the long-run average version of the control problem considered in this article, will be addressed in a forthcoming paper.

Acknowledgements: The authors are grateful to A. Khamisy, N. Shimkin and to Prof. S. Stidham Jr. for useful discussions during the course of this work

References

[1] Bertsekas, D. P., *Dynamic Programming. Deterministic and Stochastic Models*. Prentice Hall, Inc. Englewoods Cliff, 1987.

[2] Grizzle, J. W. and S. I. Marcus, "A decentralized control strategy for multiaccess broadcast networks," *Large Scale Systems*, Vol. 3, pp. 75-88, 1982.

[3] Hernández-Lerma, O., *Adaptive Markov Control Processes*. Springer-Verlag, New York, 1989.

[4] Hsu, K. and I. M. Marcus, "Decentralized control of finite state Markov processes," *IEEE Trans. Aut. Contr.*, Vol. AC-27, No. 2, pp. 426-431, Apr. 1982.

[5] Lucantoni, D. M., "New results on the single server with a batch arrival Markovian arrival process," *Stochastic Models*, Vol. 6, pp. 1-46, 1991.

[6] Ma, D.-J. and A. M. Makowski, "Optimality results for a simple flow control problem," in *Proc. 26th Conf. on Decision and Control*, pp. 1852-1857, Los Angeles, CA, USA, Dec. 1987.

[7] Neuts, M. F., *Structured Stochastic Matrices of the M/G/1 Type and Their Applications*. Marcel Dekker Inc., New York, 1989.

[8] Ross, S. M., *Introduction to Stochastic Dynamic Programming*. Academic Press, New York, 1983.

[9] Schäl, M., "Conditions for optimality in dynamic programming and for the limit of the n-stage optimal policies to be optimal", *Z. Wahrscheinlichkeitsth.*, Vol. 32, pp. 179-196, 1975.

[10] Schoute, F. C. , "Decentralized control in packet switched satellite communication", *IEEE Trans. Aut. Contr.*, Vol. AC-23, No. 2, pp. 362-371, Apr. 1978.

[11] Stidham, S., "Optimal Control of Admission to a Queueing System," *IEEE Trans. Aut. Contr.*, Vol. AC-30, No. 8, pp. 705-713, Aug. 1985.