On the Workload Process in a Fluid Queue with Bursty Input and Selective Discarding

Parijat Dube[a], Eitan Altman[a]

[a]INRIA, 2004 Route des Lucioles, B.P.93, 06902, Sophia Antipolis Cedex, France
Parijat.Dube,Eitan.Altman@sophia.inria.fr

In this paper we analyse a finite buffer fluid queue with an on-off Markov Modulated Fluid Process (MMFP) as the arrival process. We call each on-period of the arrival process a message/burst. The queue incorporates a threshold message discarding policy. We work within the framework of Poisson Counter Driven Stochastic Differential equations and obtain the moment generating function and hence the probability density function of the stationary workload process. We then comment on the stability of this fluid queue.

1. Introduction

In this paper we analyse a finite buffer fluid queue having some selective message discarding policy. Fluid arrive at the queue in bursts whose sizes are exponentially distributed. The intervals between the end of a burst and the beginning of the next burst are independently exponentially distributed. The queue employs a threshold discarding policy in the sense that only those messages at whose commencement epoch [1], the workload (i.e., the amount of fluid in the buffer) is less then some preset threshold are accepted. The input to the queue can be modelled as an on-off Markov modulated source with constant fluid arrival rate in the on-state.

One particular motivation for our model is the performance analysis of selective message discarding policies, which have been proposed [3] and implemented in routers (e.g., in Cisco BP 8600 series) to prevent network congestion. This is particularly the case with the router supporting UBR (Unspecified bit rate) service class of ATM where message (i.e., a frame) discarding is employed to achieve the twin goals of reduced network congestion and increased goodput [6]. Message discarding is based on the idea that loss of a single packet results in the corruption of the entire message (to which it belongs) and hence it is advantageous to discard the entire remaining message. Two discarding mechanisms are frequently used: the *partial discarding*, in which packets that belongs to an already corrupted message is discarded, and the *early discarding*, in which an admission control is applied to reject an entire message if upon arrivals of its first packet, the queue exceeds some value K [6]. We have focused both on discrete as well as on fluid analysis of the first mechanism in [1]. Here we focus on the second mechanism and also present the combination of both.

[1]By commencement epoch of a message we mean the time instant when the modulating process goes from off to on state.

Also as a special case of our model we can obtain the probability density function of the workload process in an MMFP/D/1 queue with constrained workload. Modelling of PCM (pulse code modulation) coded voice sources as a two state Markov Modulated Poisson Process (MMPP) is a standard acceptable practice and also there have been works on modelling the aggregate arrival due to the superposition of ATM traffic as a two state MMPP (see [5],[7]). In such scenarios our model will be handy in fluid analysis of corresponding queues with message discarding.

In Section 2 we describe the selective discarding model and state the main results, which we prove in Section 3. The infinite buffer case along with its stability analysis is presented in Section 4. Section 5 presents the combination of the two discarding mechanisms. Finally we conclude in Section 6.

2. Model and Main Results

Our fluid source is a Markov modulated on-off source. The fluid arrival rate is h in on-state and 0 in off-state; the server has a constant capacity c. Let the buffer size be B and the threshold be K, $K < B$. Let θ denote the state of the Markov chain. The source sends fluid at rate h when in on ($\theta = 1$) state and at rate 0 when in off ($\theta = 0$ state). The off and on periods are exponentially distributed with rates λ_1 and λ_2 respectively. The sojourn time in the on-state corresponds to the length of a burst [2]. Thus the message lengths are exponentially distributed with parameter λ_2 and the silence-period (off-period) between messages is exponentially distributed with parameter λ_1. The discarding policy is such that if at the commencement epoch of a message the workload process $v(t)$ is less than K, the message is admitted, otherwise not. In [2] we analysed the same system with infinite buffer. We extend the analysis here to the finite buffer case and obtain expressions for the stationary distribution of the workload process. We assume that $c < h$ [3].

We write the dynamics of the system in terms of *Poisson Counter Driven Stochastic Differential Equations* [4]. Let N_1 and N_2 be Poisson counters with parameters λ_1 and λ_2. We define a new variable $x \in \{0, 1\}$ as the indicator of *virtual arrival-process* to the buffer. $x(t)$ captures the behaviour of the discarding policy. The dynamics of $x(t)$ and $v(t)$ is,

$$dx(t) = (-x(t) + 1)dN_1\mathbf{I}(v(t) < K) - x(t)dN_2 \quad (1)$$
$$dv(t) = -c\mathbf{I}(v(t) > 0)dt + hx(t)\mathbf{I}(v(t) < B)dt \quad (2)$$

Remark 1 *$\theta(t)$ is a two-state MMFP with rates λ_1 and λ_2. Since the policy does not accept those batches at whose commencement epoch the workload process is greater than or equal to K, hence even if at time t, θ shifts from off to on state but $v(t) \geq K$, the workload process will continue evolving as if $\theta = 0$. Thus, we define a virtual arrival process $x(t)$ which captures this effect. In fact x is the arrival process being admitted to the queue and is not Markovian. A sample of $\theta(t)$, $x(t)$ and $v(t)$ is provided in Fig. 1.*

We proceed to find a recursive formula for the moments $E[v^n]$ for all positive integers n. The Laplace-Stieltjes transform (LST), $V(s)(= E[e^{-sv}])$ of v will then be obtained

[2]In the subsequent discussion he terms *message* and *burst* are used interchangeably.

[3]For the case $c \geq h$ the workload will always be 0 w.p. 1

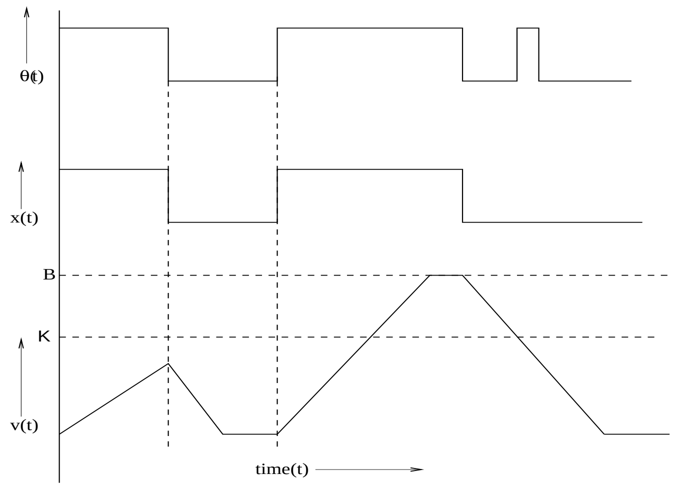

Figure 1. A realization of $\theta(t)$, $x(t)$ and $v(t)$.

from which we shall obtain the stationary probability density function, $\rho(v)$ of v. We first present our main results.

Proposition 1 *$V(s)$ is given by*

$$\begin{aligned} V(s) &= \frac{(p_2-p_1)g_6}{1+g_5 s}\left[\frac{(h-c)}{\lambda_2+(h-c)s}\left(e^{-Ks}-e^{-Bs}e^{\frac{-\lambda_2(B-K)}{h-c}}\right)\right. \\ &\left.+\frac{(h-c)}{\lambda_2}\left(e^{\frac{-\lambda_2(B-K)}{h-c}}-1\right)\right]+g_3\frac{(1-p_2)(e^{-Bs}-1)}{1+g_5 s}+1 \\ &+\frac{s}{1+g_5 s}\left(\frac{h^2 g_0(1-p_2)}{c(\lambda_1+\lambda_2)}-\frac{hg_5}{c}E[x]\right)-\frac{g_8(1-p_2)s(e^{-Bs}-1)}{1+g_5 s} \end{aligned} \tag{3}$$

where $p_1 = Prob(v<K)$, $p_2 = Prob(v<B)$, $E[x] = \frac{\lambda_1}{\lambda_1+\lambda_2}(1-(p_2-p_1)(1-c/h))$ *and*

$$g_0 = \left(1-\tfrac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}, \qquad g_3 = \left[1-\left(\frac{h\lambda_2}{c(\lambda_1+\lambda_2)}+1\right)g_0\right], \; g_5 = \frac{(h-c)}{(\lambda_1+\lambda_2)}g_0$$

$$g_6 = \left[1-\tfrac{\lambda_2}{\lambda_1+\lambda_2}g_0\right]\tfrac{1}{(h-c)E[Y_1]}, \quad g_8 = \frac{-hg_0}{(\lambda_1+\lambda_2)}, \; E[Y_1] = \frac{1}{\lambda_2}\left(1-e^{-\frac{B-K}{h-c}\lambda_2}\right) \tag{4}$$

Corollary 1 *The stationary probability density function $\rho(.)$ of $v(t)$ is given by*

$$\rho(m) = \begin{cases} \rho_1(m) & \text{for } 0 \le m \le K \\ \rho_2(m) & \text{for } K < m < B \\ 1-p_2 & \text{for } m = B \end{cases} \quad \textit{where,}$$

$$\begin{aligned} \rho_1(m) &= \frac{(p_2-p_1)(h-c)g_6}{g_5\lambda_2}\left(e^{\frac{-\lambda_2(B-K)}{h-c}}-1\right)e^{-g_5^{-1}m} \\ &+\delta(m)\left[1+\frac{h}{c}\left(\frac{hg_0(1-p_2)}{g_5(\lambda_1+\lambda_2)}-E[x]\right)+\frac{g_8}{g_5}(1-p_2)\right] \end{aligned}$$

$$+\left[g_5^{-1}\left(-\frac{g_8}{g_5}(1-p_2)-\frac{h}{c}\left(\frac{hg_0(1-p_2)}{g_5(\lambda_1+\lambda_2)}-E[x]\right)\right)-\frac{g_3(1-p_2)}{g_5}\right]e^{-g_5^{-1}m} \qquad (5)$$

$$\rho_2(m) = \rho_1(m)+\frac{(p_2-p_1)g_6}{\left(1-\frac{\lambda_2 g_5}{(h-c)}\right)}\left(e^{\frac{-\lambda_2(m-K)}{(h-c)}}-e^{-g_5^{-1}(m-K)}\right) \qquad (6)$$

where $\delta(.)$ is the Dirac delta function and p_1, p_2 solve

$$\begin{aligned} p_1 &= (p_2-p_1)\frac{g_6(h-c)}{\lambda_2}\left(e^{\frac{-\lambda_2(B-K)}{h-c}}-1\right)\left(1-e^{-g_5^{-1}K}\right)+(1-g_3(1-p_2)) \\ &\quad +\left[\left(\frac{g_8}{g_5}+g_3\right)(1-p_2)+\frac{h}{c}\left(\frac{hg_0(1-p_2)}{g_5(\lambda_1+\lambda_2)}-E[x]\right)\right]e^{-g_5^{-1}K} \qquad (7) \\ p_2-p_1 &= \left((p_2-p_1)\frac{g_6(h-c)}{\lambda_2}\left(e^{\frac{-\lambda_2(B-K)}{h-c}}-1\right)+\left(\frac{g_8}{g_5}+g_3\right)(1-p_2)\right. \\ &\quad \left.+\frac{h}{c}\left(\frac{hg_0(1-p_2)}{g_5(\lambda_1+\lambda_2)}-E[x]\right)\right)\left(e^{-g_5^{-1}K}-e^{-g_5^{-1}B}\right) \\ &\quad +\frac{(p_1-p_2)g_6(h-c)}{\lambda_2}\frac{\left(e^{\frac{-\lambda_2(B-K)}{h-c}}-e^{-g_5^{-1}(B-K)}\right)}{\left(1-\frac{g_5\lambda_2}{h-c}\right)} \qquad (8) \end{aligned}$$

3. Model Analysis and Proofs of the Main Results

Observe that the fluid level v never remains steady at K (visiting it only at isolated points in time) and hence we take the density function of v to be continuous at K. Note that the density function need not be *differentiable* at K. However, the density function has a mass at $v = B$ due to the buffer size being finite. Observe that, by using the result from stochastic calculus we can write from Eqs. (1) and (2),

$$\begin{aligned} dv^2(t) &= -2cv(t)\mathbf{I}(v(t)>0)dt+2v(t)hx(t)\mathbf{I}(v(t)<B)dt \\ d(v(t)x(t)) &= v(t)dx(t)+x(t)dv(t) \\ &= v(t)\left((-x(t)+1)dN_1\mathbf{I}(v(t)<K)-x(t)dN_2\right)+x(t)\left(-cI(v(t)>0)dt\right. \\ &\quad \left.+hx\mathbf{I}(v(t)<B)dt\right) \end{aligned}$$

Note that $P(v = B) = 1 - p_2$. From the last two equations we get [4],

$$\begin{aligned} dE[v^2] &= -2cE[v]dt+2hE[vx|v<B]p_2dt \qquad (9) \\ dE[vx] &= \lambda_1 dtE[v(-x+1)\mathbf{I}(v<K)]-\lambda_2 dtE[vx]-cE[x]dt+hE[x^2|v<B]p_2dt \end{aligned}$$

Observe that, $E[x^2|v < B]p_2 = E[x^2] - E[x^2|v = B](1 - p_2) = E[x] - 1 + p_2$. This is because $v = B$ only when $x = 1$ and also $E[x^2] = E[x]$. Thus we get,

$$\begin{aligned} dE[vx] &= (h-c)E[x]dt-h(1-p_2)dt+E[v|v<K]p_1\lambda_1dt-E[vx|v<K]\lambda_1p_1dt \\ &\quad -E[vx|v<B]\lambda_2p_2dt-B\lambda_2(1-p_2)dt \qquad (10) \end{aligned}$$

[4]For notational convenience we shall not henceforth show t in parentheses

By total probability argument, we can write,

$$\begin{aligned} E[v|v<B]p_2 &= E[v|v<K]p_1 + E[v|K<v<B](p_2-p_1) \\ E[vx|v<B]p_2 &= E[vx|v<K]p_1 + E[vx|K<v<B](p_2-p_1) \end{aligned}$$

Thus we can write Eqs. (9) and (10) as,

$$\begin{aligned} dE[v^2] &= -2cE[v]dt + 2h\left(E[vx|v<K]p_1 + E[vx|K<v<B](p_2-p_1)\right)dt \\ dE[vx] &= (h-c)E[x]dt - h(1-p_2)dt + E[v|v<K]p_1\lambda_1 dt - E[vx|v<K]\lambda_1 p_1 dt \\ &\quad - \left(E[vx|v<K]p_1 + E[vx|K<v<B](p_2-p_1)\right)\lambda_2 dt - B\lambda_2(1-p_2)dt \end{aligned}$$

Thus if a steady state exists then

$$\begin{pmatrix} p_1\lambda_1 & 0 & -p_1(\lambda_1+\lambda_2) & -\lambda_2(p_2-p_1) \\ -cp_1 & -c(p_2-p_1) & hp_1 & h(p_2-p_1) \end{pmatrix} \begin{pmatrix} E(v|v<K) \\ E(v|K\le v<B) \\ E(vx|v<K) \\ E(vx|K\le v<B) \end{pmatrix} + \begin{pmatrix} (h-c)E[x]-(1-p_2)(h+B\lambda_2) \\ -cB(1-p_2) \end{pmatrix} = 0 \tag{11}$$

Thus we shall have two equations in six unknowns (the four conditional expectations and p_1 and p_2). However an important observation to be made here is the fact that $E[v|K<v<B]$ and $E[vx|K<v<B]$ can be calculated alternatively as shown below.

3.1. Calculation of $E[v|K<v<B]$ and $E[vx|K<v<B]$

When $v \ge K$ and the state of the modulating process changes from 0 to 1 for the first time (and succeeding times) then the incoming fluid is not accepted. This is done until $v<K$. Also even if the current on-period started when $v<K$ (and hence accepted) but v reaches the level B before the on-period ends, then the excess fluid corresponding to this on-period(which arrives when $v=B$) is rejected. Thus we have a probability mass at $v=B$. A realization of the queue length evolution process under this policy is shown in Figure 2. The queue length can only cross the threshold of K at any time t, if $x(t)=x(t^-)=1$. Since the sojourn time in a state is exponentially distributed, the excess time the Markov chain spends in state 1 after time t is again exponentially distributed given that the chain was in state 1 at time t. The fluid level will rise steadily with a rate $(h-c)$ from t until the Poisson process N_2 causes x to change from 1 to 0 or until $v=B$ (whichever occurs earlier). Then either v will stay at B (if the excess time the Markov chain spends in state 1 after time t is greater than $\frac{B-K}{h-c}$) or v will start decreasing at a steady rate of c until the buffer level is K. During this period (when $v>K$), even if a new message arrives (with fluid arriving at rate h), the fluid is not accepted (the fact highlighted by the presence of an indicator function $\mathbf{I}(v<K)$ in Eq. (1)) and even if the current on-period started when $v<K$ (and hence accepted) but v reaches the level B before the on-period ends, then the excess fluid corresponding to this on-period(which arrives when $v=B$) is rejected(the fact highlighted by the presence of an indicator function $\mathbf{I}(v<B)$ in Eq. (2)). Let T_i, $i=1,2,\ldots$ be the random variable denoting the time spent by v after crossing K at the ith cross and during which the condition

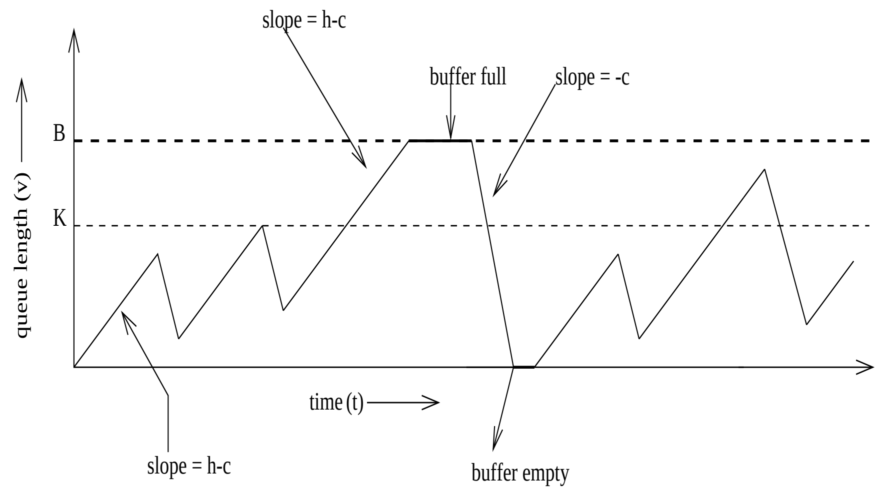

Figure 2. A realization of the queue length evolution process

$K < v < B$ is true. Thus,

$$T_i = Y_i + \frac{Y_i(h-c)}{c} = Y_i \frac{h}{c} \text{ where } Y_i = \min\left(\tfrac{B-K}{h-c}, X_i\right). \tag{12}$$

and X_i is the excess sojourn time in state 1. Notice that the condition $v(t) > K$ implies that at the time of crossing the level K, say at time $t_1 \leq t$, $x(t_1) = 1$. Thus, the sequence $\{T_i\}$ is independent and identically distributed (since $X_i \sim \exp \lambda_2$)

$$E[Y_i] = E\left[X_i \mathbf{I}\left(X_i < \frac{B-K}{h-c}\right) + \left(\frac{B-K}{h-c}\right)\mathbf{I}\left(X_i > \frac{B-K}{h-c}\right)\right]$$

which gives Eq. (4). Observe that we can write,

$$\begin{aligned} E[v|K<v<B] &= E[v|K<v<B, \frac{dv}{dt} = (h-c)]P(\frac{dv}{dt} = (h-c)|K<v<B) \\ &+ E[v|K<v<B, \frac{dv}{dt} = -c]P(\frac{dv}{dt} = -c|K<v<B) \end{aligned} \tag{13}$$

But observe that, from Eq. (12) we have $Y_i = \frac{c}{h}T_i$. Thus we have, $P(\frac{dv}{dt} = (h-c)|K < v < B) = \frac{c}{h}$ and $P(\frac{dv}{dt} = -c|K < v < B) = 1 - P(\frac{dv}{dt} = (h-c)|K < v < B) = 1 - \frac{c}{h}$. Thus from Eq. (13) we can write,

$$E[v|K<v<B] = E[v|K<v<B, \frac{dv}{dt} = (h-c)]\frac{c}{h} + E[v|K<v<B, \frac{dv}{dt} = -c](1 - \frac{c}{h})$$

Now, observe that when the condition $(\frac{dv}{dt} = h-c, K < v < B)$ is true, then $v(t) := v_1(t) = (h-c)t + K$. Thus we have, $E[v|K < v < B, \frac{dv}{dt} = (h-c)] = \frac{E[\int_0^{Y_1} v_1(t)dt]}{E[Y_1]}$. By *Renewal Reward Theorem*[9] we can show that,

$$E[v|K<v<B, \frac{dv}{dt} = (h-c)] = K + (h-c)\frac{E[Y_1^2]}{2E[Y_1]} = E[v|K<v<B, \frac{dv}{dt} = -c] \tag{14}$$

We proceed to evaluate $E[vx|K<v<B]$. By similar arguments, it is given by

$$E[vx|K<v<B,\frac{dv}{dt}=(h-c)]\frac{c}{h}+E[vx|K<v<B,\frac{dv}{dt}=-c](1-\frac{c}{h})$$

$x=1$ when $\left(\frac{dv}{dt}=h-c,K<v<B\right)$ and $x=0$ when$\left(\frac{dv}{dt}=-c,K<v<B\right)$. Thus,

$$E[vx|K<v<B] \;=\; E[vx|K<v<B,\frac{dv}{dt}=(h-c)]\frac{c}{h}=\left(K+(h-c)\frac{E[Y_1^2]}{2E[Y_1]}\right)\frac{c}{h} \quad (15)$$

Having obtained $E[v|k<v<B]$ and $E[vx|K<v<B]$, we now return to the steady-state conditions (Eq. (11)). They imply

$$-cE[v]+h\left(E[vx|v<K]p_1+E[vx|K<v<B](p_2-p_1)\right) \;=\; 0$$

$$\begin{aligned}(h-c)E[x]-h(1-p_2)+E[v|v<K]p_1\lambda_1-E[vx|v<K](\lambda_1+\lambda_2)p_1&\\ -E[vx|K<v<B](p_2-p_1)\lambda_2-B\lambda_2(1-p_2)&=0\end{aligned}$$

In the next Lemma we obtain an expression for $E[x]$ in terms of p_1 and p_2.

Lemma 1 $E[x]$ *is given by*

$$E[x] \;=\; \frac{\lambda_1}{\lambda_1+\lambda_2}\left(1-(p_2-p_1)\left(1-\frac{c}{h}\right)\right) \quad (16)$$

Proof:
In steady state from Eq. (1), we get

$$\lambda_2E[x] \;=\; p_1\lambda_1-E[x|v<K]p_1\lambda_1 \quad (17)$$

Also,

$$\begin{aligned}E[x] &= E[x|v<K]p_1+E[x|K<v<B](p_2-p_1)+E[x|v=B](1-p_2)\\ &= E[x|v<K]p_1+\left(E[x|K<v<B,\frac{dv}{dt}=h-c]P(\frac{dv}{dt}=h-c|K<v<B)\right.\\ &\quad \left.+\,E[x|K<v<B,\frac{dv}{dt}=-c]P(\frac{dv}{dt}=-c|K<v<B)\right)(p_2-p_1)+(1-p_2)\\ &\quad \Rightarrow E[x|v<K]p_1=E[x]-\frac{c}{h}(p_2-p_1)-(1-p_2) \quad (18)\end{aligned}$$

The equivalence follows as $x=0$ for the case $\left(K<v<B,\frac{dv}{dt}=-c\right)$. From Eqs. (17) and (18) we get Eq. (16).

3.2. Proof of Proposition 1

From the steady state equations, after some calculations we get,

$$\begin{aligned}E[v|v<K] \;=\; &\left[\frac{h}{c}\left(\frac{E[vx|K<v<B]\lambda_1(p_2-p_1)+(h-c)E[x]-(B\lambda_2+h)(1-p_2)}{\lambda_1+\lambda_2}\right)\right.\\ &-\left.E[v|K<v<B](p_2-p_1)-B(1-p_2)\right]\left(p_1-\frac{hp_1\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\end{aligned}$$

Replacing the left hand side of the last equation by $p_1^{-1}(E[v]-E[v|K<v<B](p_2-p_1)-B(1-p_2))$ and using $E[vx|K<v<B]=E[v|K<v<B]\frac{c}{h}$ from Eq. (15), we get

$$\begin{aligned}E[v] &= \left[\frac{h(h-c)}{c(\lambda_1+\lambda_2)}\left(1-\frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\right]E[x]+E[v|K<v<B](p_2-p_1)\\ &\quad\left[1-\frac{\lambda_2}{\lambda_1+\lambda_2}\left(1-\frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\right]-(1-p_2)\left[B\left(1+\frac{h\lambda_2}{c(\lambda_1+\lambda_2)}\right)+\frac{h^2}{c(\lambda_1+\lambda_2)}\right]\\ &\quad\left(1-\frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}+B(1-p_2)\end{aligned}\tag{19}$$

Proceeding similarly, we will next write the steady state relation between $E[v^n]$ and $E[v^n|K<v<B]$ for any $n>1$. From Eqs. (1) and (2) we have

$$\begin{aligned}dv^{n+1} &= (n+1)v^n(-c\mathbf{I}(v>0)dt+hx\mathbf{I}(v<B)dt) \qquad (20)\\ dv^nx &= xnv^{n-1}dv+v^ndx\\ &= nv^{n-1}x(-c\mathbf{I}(v>0)dt+hx\mathbf{I}(v<B)dt)+v^n[(-x+1)dN_1\mathbf{I}(v<K)-xdN_2]\\ \Rightarrow dE[v^nx] &= -ncE[v^{n-1}x]dt+nhE[v^{n-1}x|v<B]p_2dt-E[v^nx|v<K]p_1\lambda_1dt\\ &\quad +E[v^n|v<K]p_1\lambda_1dt-E[v^nx|v<K]p_1\lambda_2dt\\ &\quad -E[v^nx|K<v<B](p_2-p_1)\lambda_2dt-B^n(1-p_2)\lambda_2dt\end{aligned}$$

Thus for the existence of steady state, the following should vanish.

$$\begin{pmatrix} p_1\lambda_1 & 0 & -(\lambda_1+\lambda_2)p_1 & -\lambda_2(p_2-p_1)\\ -cp_1 & -c(p_2-p_1) & hp_1 & h(p_2-p_1)\end{pmatrix}\begin{pmatrix}E(v^n|v<K)\\ E(v^n|K<v<B)\\ E(v^nx|v<K)\\ E(v^nx|K<v<B)\end{pmatrix}$$
$$+\begin{pmatrix} n(h-c)E[v^{n-1}x]-(1-p_2)(nhB^{n-1}+B^n\lambda_2)\\ -cB^n(1-p_2)\end{pmatrix}$$

Equating the last expression to 0, we get in steady state, after some calculations

$$\begin{aligned}&E[v^n|v<K]=\\ &\left[\frac{h}{c}\left(\frac{E[v^nx|K<v<B]\lambda_1(p_2-p_1)+n(h-c)E[v^{n-1}x]-(1-p_2)B^n(nhB^{-1}+\lambda_2)}{\lambda_1+\lambda_2}\right)\right.\\ &\left.-E[v^n|K<v<B](p_2-p_1)-B^n(1-p_2)\right]\left(p_1-\frac{hp_1\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\end{aligned}\tag{21}$$

Similar to Eq. (15) we can show that for any $n\geq 1$, $E[v^n|K<v<B]=\frac{h}{c}E[v^nx|K<v<B]$. Thus from Eq. (21) we get

$$\begin{aligned}E[v^n|v<K] &= \left[\frac{nh(h-c)}{c(\lambda_1+\lambda_2)}E[v^{n-1}x]-E[v^n|K<v<B](p_2-p_1)\frac{\lambda_2}{\lambda_1+\lambda_2}\right.\\ &\quad\left.-B^n(1-p_2)\left(\frac{h(nhB^{-1}+\lambda_2)}{c(\lambda_1+\lambda_2)}+1\right)\right]\left(p_1-\frac{hp_1\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\end{aligned}$$

Again replacing the left hand side of the last equation by $p_1^{-1}(E[v^n] - E[v^n|K < v < B](p_2 - p_1) - B^n(1-p_2))$ we get the following expression for $E[v^n]$:

$$g_1 nE[v^{n-1}x] + g_2(p_2 - p_1)E[v^n|K < v < B] + g_3(1-p_2)B^n + g_4 n(1-p_2)B^{n-1} \tag{22}$$

with, $g_0 = \left(1 - \frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}$, $g_1 = \frac{h(h-c)}{c(\lambda_1+\lambda_2)}g_0$, $g_2 = \left[1 - \frac{\lambda_2}{\lambda_1+\lambda_2}g_0\right]$, $g_3 = \left[1 - \left(\frac{h\lambda_2}{c(\lambda_1+\lambda_2)} + 1\right)g_0\right]$ and $g_4 = \frac{-h^2}{c(\lambda_1+\lambda_2)}g_0$. We can find the expressions for $E[v^n|K < v < B]$ and $E[v^n x|K < v < B]$ for any $n > 1$ by arguments similar to that for the case of $n = 1$ in Section 3.1.

$$\text{Thus } E[v^n|K < v < B] \;=\; E\left[\frac{(K + Y_1(h-c))^{n+1} - K^{n+1}}{(n+1)(h-c)}\right](E[Y_1])^{-1} \tag{23}$$

Also with $E[v^n x] = E[v^n x\mathbf{I}(v < B)] + B^n(1-p_2)$, from the steady state behaviour of Eq. (20), we can write (for $n \geq 1$)

$$E[v^n x] \;=\; \frac{c}{h}E[v^n] + B^n(1-p_2) \tag{24}$$

From Eqs. (23), (24) and (22) we get, for $n \geq 2$

$$\begin{aligned} E[v^n] \;=\; & g_5 nE[v^{n-1}] + g_1(1-p_2)nB^{n-1} + g_6\frac{(p_2-p_1)}{(n+1)}E\left[(K + Y_1(h-c))^{n+1} - K^{n+1}\right] \\ & + g_3(1-p_2)B^n + g_4(1-p_2)nB^{n-1} \end{aligned} \tag{25}$$

And for $n = 1$ from Eqs. (14) and (19) we have

$$E[v^n] \;=\; g_1 E[x] + g_6\frac{(p_2-p_1)}{(n+1)}E\left[(K + Y_1(h-c))^{n+1} - K^{n+1}\right] + g_7(1-p_2)B^n \tag{26}$$

where $g_5 = g_1\frac{c}{h}$ and $g_6 = \frac{g_2}{E[Y_1](h-c)}$ ($E[Y_1]$ being obtained earlier in Eq. (4)) and $g_7 = \left[1 - \left(\frac{h^2B^{-1}+h\lambda_2}{c(\lambda_1+\lambda_2)} + 1\right)g_0\right]$. Thus we have a recursive relation between $E[v^n]$ and $E[v^{n-1}]$. With these we will proceed to find the LST of the stationary workload. Further multiplying both the sides of Eq. (25) by $\frac{(-s)^n}{n!}$ and summing from $n = 2$ to ∞, we can write (with $g_8 = g_1 + g_4$)

$$\begin{aligned} \sum_{n\geq 2}\frac{(-s)^n}{n!}E[v^n] \;=\; & g_5(-s)\sum_{n\geq 2}E[v^{n-1}]\frac{(-s)^{n-1}}{(n-1)!} + g_8(1-p_2)(-s)\sum_{n\geq 2}\frac{(-Bs)^{n-1}}{(n-1)!} \\ & + g_6(p_2-p_1)E\left[\frac{1}{(-s)}\sum_{n\geq 2}[(K + Y_1(h-c))^{n+1} - K^{n+1}]\frac{(-s)^{n+1}}{(n+1)!}\right] \\ & + g_3(1-p_2)\sum_{n\geq 2}B^n\frac{(-s)^n}{n!} \end{aligned} \tag{27}$$

Adding to Eq. (27), $(-s)\times$ Eq. (26) we get

$$\begin{aligned} \sum_{n\geq 1}\frac{(-s)^n}{n!}E[v^n] \;=\; & g_5(-s)\sum_{n\geq 2}E[v^{n-1}]\frac{(-s)^{n-1}}{(n-1)!} + g_8(1-p_2)(-s)\sum_{n\geq 2}\frac{(-Bs)^{n-1}}{(n-1)!} \\ & + g_6(p_2-p_1)E\left[\frac{1}{(-s)}\sum_{n\geq 1}[(K + Y_1(h-c))^{n+1} - K^{n+1}]\frac{(-s)^{n+1}}{(n+1)!}\right] \\ & + g_3(1-p_2)\sum_{n\geq 1}B^n\frac{(-s)^n}{n!} - sg_1E[x] + \frac{h^2 g_0}{c(\lambda_1+\lambda_2)}s(1-p_2) \end{aligned} \tag{28}$$

Also observe that,

$$E\left[\frac{1}{-s}\sum_{n\geq 1}[(K+Y_1(h-c))^{n+1}-K^{n+1}]\frac{(-s)^{n+1}}{(n+1)!}\right]=E\left[\frac{e^{-Ks}}{s}\left(1-e^{-Y_1(h-c)s}\right)-Y_1(h-c)\right]$$

$$\begin{aligned}E\left[\frac{e^{-Ks}}{s}\left(1-e^{-Y_1(h-c)s}\right)-Y_1(h-c)\right] &= \frac{(h-c)}{\lambda_2+(h-c)s}\left(e^{-Ks}-e^{-Bs}e^{\frac{-\lambda_2(B-K)}{h-c}}\right)\\ &\quad +\frac{(h-c)}{\lambda_2}\left(e^{\frac{-\lambda_2(B-K)}{h-c}}-1\right)\end{aligned}$$

Furthermore, $V(s)=\sum_{n\geq 0}\frac{(-s)^n Ev^n}{n!}$, thus we get from Eq. (28)

$$\begin{aligned}V(s)-1 &= -g_5 s(V(s)-1)+g_6(p_2-p_1)\left[\frac{(h-c)}{\lambda_2+(h-c)s}\left(e^{-Ks}-e^{-Bs}e^{\frac{-\lambda_2(B-K)}{h-c}}\right)\right.\\ &\quad \left.+\frac{(h-c)}{\lambda_2}\left(e^{\frac{-\lambda_2(B-K)}{h-c}}-1\right)\right]+g_3(1-p_2)(e^{-Bs}-1)\\ &\quad +s\left(\frac{h^2 g_0(1-p_2)}{c(\lambda_1+\lambda_2)}-g_1E[x]\right)+g_8(1-p_2)(-s)(e^{-Bs}-1)\end{aligned}$$

which implies Eq. (3).

3.3. Proof of Corollary 1

Taking inverse LST of Eq. (3), we get, for $0\leq v\leq K$, Eq. (5). And for $K<v\leq B$ the inverse LST implies,

$$\rho(v)=\rho_2(v)=\rho_1(v)+\mathcal{L}^{-1}\left(\frac{(p_2-p_1)g_6(h-c)e^{-Ks}}{(1+g_5 s)(\lambda_2+(h-c)s)}\right) \tag{29}$$

where $\mathcal{L}^{-1}$ denotes the inverse LST [5]. Thus (with * denoting convolution operator)

$$\mathcal{L}^{-1}\left(\frac{1}{(s+g_5^{-1})(s+\frac{\lambda_2}{h-c})}\right)=e^{-g_5^{-1}v}*e^{\frac{-\lambda_2 v}{h-c}}=\frac{1}{\left(g_5^{-1}-\frac{\lambda_2}{(h-c)}\right)}\left(e^{\frac{-\lambda_2 v}{(h-c)}}-e^{-g_5^{-1}v}\right) \tag{30}$$

Eqs. (30) and (29) implies (6). Observe that $\int_0^K\rho_1(v)dv=p_1$ and $\int_K^B\rho_2(v)dv=p_2-p_1$. Thus we get from integrating Eqs. (5) and (6) with limits $[0,K]$ and $[K,B]$ respectively, we shall get two linear equations in two unknowns p_1 and p_2, Eqs. (7) and (8), which gives explicit closed form solutions for p_1 and p_2.

4. Infinite Buffer Case: the Workload Process and the Stability Analysis

Taking $B\to\infty$ in Corollary (1) we have

[5] For a random variable X with distribution $F(x)$ and LST $\mathcal{L}(s)(=\int_0^\infty e^{-sx}dF(x))$ we mean by $\mathcal{L}^{-1}$, the probability density of X, i.e., $\mathcal{L}^{-1}[\mathcal{L}(s)]=dF(x)/dx$

Corollary 2 ([2], Prop. 2) *The stationary probability density function* $\rho(.)$ *of* $v(t)$ *is*

$$\rho(m) = \begin{cases} \rho_1(m) & \text{for } 0 \le m < K \\ \frac{k_2(1-p)\frac{a\lambda_2}{(h-c)}}{\left(a-\frac{\lambda_2}{(h-c)}\right)}\left(e^{-\frac{\lambda_2}{(h-c)}(m-K)} - e^{-a(m-K)}\right) + \rho_1(m) & \text{for } m \ge K \end{cases}$$

$$\text{where, } \rho_1(m) = \delta(m)\left(1 - \frac{h}{c}E[x]\right) + \left(\frac{h}{ck_1}E[x] - \frac{k_2}{k_1}(1-p)\right)e^{-am},$$

$$k_1 = \left[\frac{(h-c)}{(\lambda_1+\lambda_2)}\left(1 - \frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\right] \quad k_2 = \left[1 - \frac{\lambda_2}{\lambda_1+\lambda_2}\left(1 - \frac{h\lambda_1}{c(\lambda_1+\lambda_2)}\right)^{-1}\right], \; a = k_1^{-1}$$

$$E[x] = \frac{\lambda_1}{\lambda_1+\lambda_2}\left(p(1-\frac{c}{h}) + \frac{c}{h}\right), \; p = \frac{1}{1+\frac{\alpha}{1-\frac{\alpha c}{h}}}, \; \alpha = \frac{\lambda_1 e^{-aK}h}{(\lambda_1+\lambda_2)c}\left(1 - k_2 + k_2 e^{-aK}\right)^{-1}$$

Next we establish the stability of the queue for the infinite buffer case. In particular we show that the workload process is a renewal process. To that end we consider the Markov chain V_n which is the workload process at the nth transition of $\theta(t)$ from state 0 to state 1. In other words, it is the workload as seen by the commencement of the nth potential arriving message. We shall show that V_n is a Harris recurrent Markov chain, and that the empty state is recurrent whenever $c > 0$. This will then imply that the original workload process is a renewal process. To that end, we recall the following sufficient condition for the Harris recurrence which follows from ([8], Theorem 14.0.1, p. 330).

Lemma 2 *Assume that* V_n *is* Ψ*-irreducible (for some* Ψ*) and aperiodic. Let there be a function* f*, some* $\epsilon > 0$*, and some small set* C *such that*

$$E[f(V_{n+1}) - f(V_n)|V_n = v] < -\epsilon + \mathbf{I}\{v \notin C\}. \tag{31}$$

Then the Markov chain V_n *is positive recurrent, it has a stationary probability* π*, and the* n*-step transition probabilities* P^n *converges to the* π *in total variation as* $n \to \infty$*.*

In the above Lemma, a set C is *small* if there exists some integer n, a constant $g > 0$ and a probability measure ϕ over the state space R such that

$$[P^n]_{xA} > g\phi(A) \text{ for all } x \in C \text{ and measurable set } A \subset R. \tag{32}$$

Theorem 1 *Assume that* $c > 0$*. Then the process* V_n *is Harris recurrent.*

Proof: Define $C = [0, K]$ and $f(v) = v$. Let T be an exponentially distributed random variable with parameter λ_2. Then for all $v \notin C$, $E[V_{n+1} - V_n | V_n = v] < -cE[\min(T, K/c)] =: -\epsilon$. Thus (31) holds. Next we check that C is indeed a small set. Let Z_i be an exponentially random variable with parameter λ_i, $i = 1, 2$. Viewing Z_1 as the length of the off-period and Z_2 as the length of the on-period, we have for any $v \in C$ and measurable $A \subset R$, $P^2]_{vA} \ge P(\frac{K}{h-c} < Z_2 < \frac{2K}{h-c}, Z_1 > \frac{3K}{c})P_{0A}$. Thus, (32) holds with $\phi(A) = P_{0A}$ and $g = P(Z)$ where $Z = \{\frac{K}{h-c} < Z_2 < \frac{2K}{h-c}, Z_1 > \frac{3K}{c}\}$. Note on Z, a message starts to arrive when $v < K$ and then no more messages are accepted until the system empties.

Observe that V_n is ϕ-irreducible ([8], pp. 70,87) since the probability of eventually reaching any measurable set $A \subset R$ from any state x is greater that $g\phi(A)$. Finally, the aperiodicity follows since the probability to go from state 0 to state 0 is strictly positive.

5. Combining partial and selective discarding

The difference between our previous model and the one in which we combine both discarding policies is that now when B is reached, immediately discarding begins. Hence the probability mass that we had at B disappears. Consider a sample path of $v(t)$ in our previous model. Let τ_i be the ith time $v(t)$ hits B and let σ_i be the ith time it leaves B. Define $S_i, i = 1, 2, ..$ to be the time interval $(\tau_i, \sigma_i]$. We now construct a new sample $\hat{v}(t)$ that is obtained by *eliminating* the periods S_i from $v(t)$ (by simply "cutting" them out). Then it is easy to see that $\hat{v}(t)$ has the same distribution as the process in the new model that combines both discarding mechanisms. We conclude that the stationary probability density function of the new model is given by $\rho(.)/p_2$ where ρ and p_2 (the probability of not being at B) are given in Corollary 1; We thus have also the probability distribution of the model with combined discarding mechanisms.

6. Conclusion

In this paper we have analysed a fluid queue fed by an on-off fluid process and incorporating selective (and partial) burst discarding policy. We obtain expressions for the LST of the stationary workload process and hence the stationary distribution of the workload process. We also discussed the stability of the fluid queue for the infinite buffer case.

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