Competitive Routing in Multicast Communications*

T. Boulogne[1,2], E. Altman[1,3]

Abstract— **We consider competitive routing in multicast networks from a non-cooperative game theoretical perspective. N users share a network, each has to send an amount of packets to a different set of addressees (each address must receive the same packets), to do this it has only to send one copy of a packet, the network making the duplications of the packets at appropriate nodes (depending on the chosen trees). The routing choice of a user is how to split its flow between different multicast trees. We present different criteria of optimization for this type of games. We treat two specific networks, establish the uniqueness of the Nash equilibrium in several networks, as well as the uniqueness of links' utilization at Nash equilibria for specific cost functions in networks with general topology.**

I. Introduction

In view of the deregulation of the telecommunication market, it has been recognized that optimal decision making concerning the network operation (such as routing) at the level of service providers cannot be modeled in the framework of centralized optimization. The natural framework to study this issue is non-cooperative game theory, and the optimality concept is the Nash equilibrium (see e.g. [11], [14]).

Within this framework, we consider in this paper the optimal routing problems, in which each service provider (that will be called "user") has to determine which paths to use and how to split the flow of its subscribers between these paths. This problem, known as "comptetitive routing" has received much attention in the framework of point-to-point communications, see e.g. [7], [8], [11]. Related competitive models have also been studied in the context of road traffic even earlier, see e.g. [5].

The unit entity that is routed is called a *packet*. There are infinitely many packets that we approximate by a continuous flow. A finite number of users share the network. Each one has to send an amount of packets (possibly of different sessions) from one or more sources to a set of (possibly source dependent) destinations. The routing decision of a user consists of how to split its flow between various multicast virtual paths which are represented as trees. We consider in this paper not only the point-to-multipoint situation but also the case of multipoint-to-multipoint. Yet even in the latter, we shall assume that the routing from each source is performed using virtual paths which are represented as trees.

Into a tree, a user has only to send one copy of each packet, and the network will duplicate the information at appropriate nodes: At each node of the tree the network will duplicate the packets so that a copy of the packet goes through each out-going links (which belong to the tree) of this node. This feature makes it impossible to use standard methods from games that arise in road traffic or in point-to-point communications (in which there is a single source and a single destination per each communications) that are based on flow conservation at each node (such as [11]).

Objective functions to be minimized in multicast competitive routing are also different from those that arise in unicast communications and in road traffic. We treat the case when the objective is to minimize a cost function that is obtained through the sum of link costs. In that case, our analysis can be used to relate the pricing of links (as a function of congestion) to the cost obtained at equilibrium. We also analyze two different types of cost related to delays. Surprisingly, in the case of multicast, the cost representing delay cannot be taken as a simple special case of the previous type of cost, as will be discussed.

The structure of the paper is as follows. After introducing the mathematical model, we define in Section II the optimality concept of Nash equilibrium, establish its existence in our networking routing game and present explicetly three different criteria of optimization. We then study two networks with specific topologies (Sections III and IV), in which we establish the uniqueness of the Nash equilibrium for the three criteria presented; in Section III, we also show the convergence to equilibrium of a best reply algorithm in the case where the network is shared by two users and give an example where there exist several distinct equilibria if the orthogonality of the strategy sets does not hold. In Section V, we present an extended version of the well known Wardrop equilibria [16], relate it to Nash equilibrium with specific cost functions. We then obtain uniqueness of the equilibrium for a general topology. Finally in Section VI we present a numerical example. We conclude with a section that discusses further extensions and some remaining open problems.

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[1] INRIA, 2004 route des Lucioles, 06902 Sophia Antipolis, France. Tel: +33-4-92-38-77-77, Fax: +33-4-92-38-77-65, E-mail: {thomas.boulogne,eitan.altman}@sophia.inria.fr

[2] Université Pierre et Marie Curie - Paris 6, 4 place Jussieu 75230 Paris, France.

[3] C.E.S.I.M.O., Universidad de Los Andes, Faculdad de Ingeneria, Merida, Venezuela.

II. Model

We consider a general network. We denote $\mathcal{N}$ the set of nodes, and $\mathcal{L} \subset \mathcal{N} \times \mathcal{N}$ the set of unidirectional links. The unit entity that is routed through the network is called a *packet*. Each packet j has a source and a set of addressees, we call this pair an origin-destinations pair (sD *pair*). We denote the origin, or the source by $s(j)$ and the set of destinations by $D(j)$. The network is shared by N users, we denote $\mathcal{I}$ the set of users. Each user $i \in \mathcal{I}$ has a set of sD pairs and for each sD pair a certain amount of packets to route from s to D, we call this amount the flow demand of user i for the pair sD and we denote it by ϕ^i_{sD}. A user with a single source (and several destinations) represents the, so called, session of point-to-multipoint. When a user has several sources and several destinations, the scenario is called a multipoint-to-multipoint communication.

A tree a from $s \in \mathcal{N}$ to $D \subset \mathcal{N}$ is a subnetwork, constituting by a set of nodes and a set of unidirectional links; an example of a tree belonging to the pair sD is

$$a = (su, uv, vd_1, vd_2, uw, wx, xd_3, xd_4, xd_5)$$

where $D = \{d_1, d_2, d_3, d_4, d_5\}$ and u, v, w and x are intermediate nodes, in this tree a duplication is made in u, in v and two in x, Figure 1.

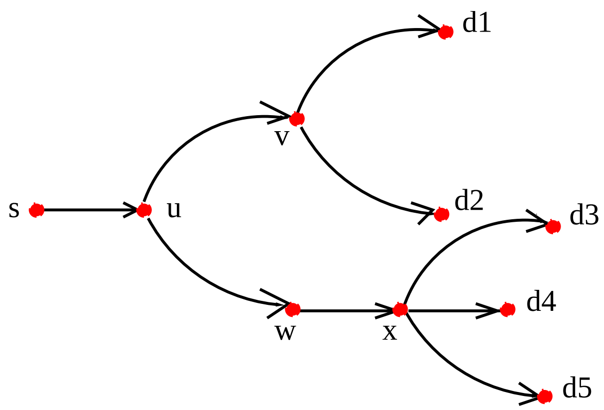

Fig. 1. Example of a tree

$\mathcal{A}$ is the set of all possible trees. Each node of a tree (except for the source) has only one in-going link belonging to this tree. For a user $i \in \mathcal{I}$ we denote by $\mathcal{E}^i$ its set of sD pairs, by $\mathcal{A}^i$ its set of possible trees and by $\mathcal{A}^i_{sD}$ its set of possible trees which go from s to D. Then we have $\mathcal{A} = \cup_{i\in\mathcal{I}}\mathcal{A}^i$ and $\mathcal{A}^i = \cup_{sD\in\mathcal{E}^i}\mathcal{A}^i_{sD}$.

Each user has to choose a (set of) tree(s) to route its packets. For $i \in \mathcal{I}$ and $a \in \mathcal{A}^i$, we denote by $\mathbf{x}^i_{(a)}$ the amount of flow sent into tree a by user i.

Denoting $x^i_{(a)l}$ the flow that user i sends into a tree a which goes through link l (this notation will only be used once), then by definition of our model we have

$$\forall i \in \mathcal{I},\ \forall a \in \mathcal{A}^i,\ \forall l \in a, \quad x^i_{(a)l} = \mathbf{x}^i_{(a)}$$

We introduce the incident indicator δ_{la},
$\delta_{la} = \begin{cases} 1 & \text{if } l \in a \\ 0 & \text{otherwise} \end{cases}.$

The flow on a link $l \in \mathcal{L}$ sent by user $i \in \mathcal{I}$ is $x^i_l = \sum_{a\in\mathcal{A}^i} \delta_{la}\mathbf{x}^i_{(a)}$ and the aggregate flow on link l is

$$x_l = \sum_{i\in\mathcal{I}}\sum_{a\in\mathcal{A}^i} \delta_{la}\mathbf{x}^i_{(a)} = \sum_{i\in\mathcal{I}} x^i_l$$

For every user $i \in \mathcal{I}$, its routing decision $\mathbf{x}^i \in R^{n^i}$ ($n^i = \#\mathcal{A}^i$) has to satisfy

$$\begin{aligned}\mathbf{x}^i \in \tilde{\mathcal{X}}^i \quad &= \quad \{\mathbf{x}^i \in R^{n^i} \mid \forall sD \in \mathcal{E}^i,\ \forall a \in \mathcal{A}^i_{sD}, \\ &\qquad \mathbf{x}^i_{(a)} \geq 0, \quad \sum_{a\in\mathcal{A}_{sD}} \mathbf{x}^i_{(a)} = \phi^i_{sD}\}\end{aligned}$$

We denote the set of all admissible flow configuration $\mathbf{x} = (\mathbf{x}^i)_{i\in\mathcal{I}}$ by $\mathcal{X}$, we call it the total strategy set. $\mathcal{X} \subset \tilde{\mathcal{X}} = \tilde{\mathcal{X}}^1 \times \tilde{\mathcal{X}}^2 \times \ldots \times \tilde{\mathcal{X}}^N$ is convex, compact and nonempty. $\tilde{\mathcal{X}}$ is called an orthogonal policy space.

Usually $\mathcal{X} = \tilde{\mathcal{X}}$, but we may add some extra constraint as link capacities and then $\mathcal{X} \subset \tilde{\mathcal{X}}$. [1]

A. Definition of Nash equilibrium

The cost function of a user $i \in \mathcal{I}$ is denoted J^i, $J^i : R^n \to [0, \infty]$ ($n = \sum_{i\in\mathcal{I}} n^i$).

The aim of each user $i \in \mathcal{I}$ is to minimize its cost function (according to the strategy set), that is find a $\mathbf{x}^i$ such that

$$\mathbf{x}^i \in \min_{\mathbf{y}^i\in\tilde{\mathcal{X}}^i} \{J^i(\mathbf{x}^1, \ldots, \mathbf{y}^i, \ldots, \mathbf{x}^N) | (\mathbf{x}^1, \ldots, \mathbf{y}^i, \ldots, \mathbf{x}^N) \in \mathcal{X}\}$$

Let $(\mathbf{x}^{-i}, \mathbf{y}^i)$ be the flow configuration where class j $(j \neq i)$ uses strategy $\mathbf{x}^j$ and class i uses strategy $\mathbf{y}^i$.

Definition: $\mathbf{x} \in \mathcal{X}$ is a *Nash equilibrium* if and only if

$$\forall i \in \mathcal{I},\ \forall y^i \text{ s.t. } (\mathbf{x}^{-i}, \mathbf{y}^i) \in \mathcal{X}, \quad J^i(\mathbf{x}) \leq J^i(\mathbf{x}^{-i}, y^i)$$

B. Cost Functions

We denote the cost function of a link $l \in \mathcal{L}$ for a user $i \in \mathcal{I}$ by f^i_l, $f^i_l : [0, \infty) \to [0, \infty]$ depends only of the

[1] Note that constraints such as link capacities may always be represented in the orthogonal policy space as well by introducing infinity for costs of policies that do not satisfy the constraints. But when doing so, we may loose the continuity of the cost due to jumps to infinity that may occur. As we shall see, such constraints may result in nonuniqueness of the equilibrium. However, there are some specific cost functions, such as that obtained from the expected delay in an M/M/1 queue, that include capacity constraints (by imposing infinite delays when capacity is attained) but have the property that the cost remains continuous everywhere. For such costs, we may often obtain the uniqueness of equilibria, see e.g. [11].

flow which goes through link l.

In the kind of model presented in this paper, we may have different criteria of optimization. Then we have different types of cost functions.

The link's cost functions may be of two types:
(A) Cost functions similar to toles in road traffic
(B) Cost functions as delay

In case *(A)* each user wants to minimize its total cost (that is the sum of the links' cost functions over all the links which it uses). In the unicast case (one destination per source) or even in the case of multiple sources and single destination, there is no essential difference between "cost" and "delay": the cost can be taken as the delay.

This is not the case in multicast problems, and delay has to be treated differently. To see that, consider a single user in a simple network consisting of nodes (s, x, d_1, d_2); s is the source, d_1 and d_2 are two destinations. The links are sx, xd_1, xd_2 and there is a single tree that contains these three links. Assume that the (load dependent) cost of using each link is 1 unit. The total cost is then 3 since there are three links. But if the cost corresponds to the average delay then it is 2 units, since the delay between the source and each destination is 2 units. If we consider the total delay then it is 4 units (2 units between the source and each destination).

In case *(B)* we consider two different criteria of optimization, either each of the users wishes to minimize its total delay *(B.1)* (as in the example above) or it wishes to minimize its maximum delay over paths *(B.2)* (a path being a sequence of unidirectional links between a source and a destinations, any path belongs to a tree). The latter type of criterion has been advocated and used in the pòint-to-point framework for ad-hoc networks, see [4].

For any tree a and all nodes $u \in a$, we denote by $a_{|u}$ the subtree of of a which begins at node u and by τ^i_{ua} the number of destinations of the subtree $a_{|u}$, where a belongs to $\mathcal{A}^i$, $\tau^i_{ua} = \#\{d \in D(a) \mid d \in D(a_{|u})\}$.

The different users' cost functions are

(A) $J^i(\mathbf{x}) = \sum_{a \in \mathcal{A}^i} \mathbf{x}^i_{(a)} \sum_{l \in a} f^i_l(x_l)$

(B.1) $J^i(\mathbf{x}) = \sum_{a \in \mathcal{A}^i} \mathbf{x}^i_{(a)} \sum_{uv \in a} \tau^i_{va} f^i_{uv}(x_{uv})$

(B.2) $J^i(\mathbf{x}) = \max_{a \in \mathcal{A}^i, p \in a} \sum_{l \in p} f^i_l(x_l)$

C. *Existence of Nash Equilibrium*

When the strategy set is convex, compact and nonempty, and when for each user $i \in \mathcal{I}$, for all $\mathbf{x} \in \mathcal{X}$, the cost function $J^i(\mathbf{x})$ is continuous in $\mathbf{x}$ and convex in $\mathbf{x}^i$ for each fixed value of $\mathbf{x}^{-i}$, then a Nash equilibrium exists (see for example [14, Thm 1]).

Note that in order to obtain the existence of a Nash equilibrium, the differentiability of the J^i's is not required. Then to satisfy conditions of existence of equilibria, the f^i_l's have to be continuous and increasing (resp. convex) for cases *(A)* and *(B.1)* (resp. *(B.2)*), so that the J^i's be continuous in $\mathbf{x}$ and convex in $\mathbf{x}^i$ for each fixed value of $\mathbf{x}^{-i}$.

In the rest of paper we shall furthermore impose frequently the following assumption to obtain uniqueness of equilibrium in the cases studied.

Assumption ($\mathcal{G}$): $f^i_l : [0, \infty) \to [0, \infty]$ is continuously differentiable (wherever finite), strictly increasing and convex.

D. *Characterization of the equilibrium*

For cases *(A)* and *(B.1)*, for all $a \in \mathcal{A}$, we denote by $K^i_{(a)}(\mathbf{x})$ the derivative of $J^i(\mathbf{x})$ with respect to $\mathbf{x}^i_{(a)}$, *i.e.*, $K^i_{(a)}(\mathbf{x}) := \frac{d}{d\mathbf{X}^i_{(a)}} J^i(\mathbf{x})$, then

$$K^i_{(a)}(\mathbf{x}) = \sum_{uv \in a} c^i_{uv} \left(x^i_{uv} \nabla f^i_{uv}(x_{uv}) + f^i_{uv}(x_{uv}) \right)$$

where $c^i_{uv} = \begin{cases} 1 & \text{if } J^i \in \text{(A)} \\ \tau^i_{va} & \text{if } J^i \in \text{(B.1)} \end{cases}$ and ∇ denotes the derivative with respect to the argument of the function.

In case *(B.2)*, remark that for any $i \in \mathcal{I}$, J^i is convex in $\mathbf{x}^i$, continuous in $\mathbf{x}$ but no more differentiable.

For cases *(A)* and *(B.1)*, if $\mathcal{X} = \tilde{\mathcal{X}}$, the equilibrium condition is equivalent to

$$\exists \alpha = \alpha(\mathbf{x}),\ \alpha^T = (\alpha^i_{sD})_{i \in \mathcal{I},\ sD \in \mathcal{E}^i}$$

such that for all $i \in \mathcal{I}$, $sD \in \mathcal{E}^i$ and $a \in \mathcal{A}^i_{sD}$

$$K^i_{(a)}(\mathbf{x}) - \alpha^i_{sD} \geq 0; \quad (K^i_{(a)}(\mathbf{x}) - \alpha^i_{sD}) \mathbf{x}^i_{(a)} = 0 \quad [2] \quad (1)$$

Let us denote by $\partial_a J^i(\mathbf{x})$ the subdifferential (which is the set of subgradients) of $J^i(\mathbf{x})$ according to $\mathbf{x}^i_{(a)}$. Then for case *(B.2)* ($\mathcal{X} = \tilde{\mathcal{X}}$), $\mathbf{x}$ is a Nash equilibrium if and only if it satisfies the condition

$$\begin{array}{ll} & \exists \quad \alpha = \alpha(\mathbf{x}),\ \alpha^T = (\alpha^i_{sD})_{i \in \mathcal{I},\ sD \in \mathcal{E}^i} \\ \text{and} & \exists \quad \beta = \beta(\mathbf{x}),\ \beta^T = (\beta^i_{(a)})_{i \in \mathcal{I},\ a \in \mathcal{A}^i} \end{array}$$

[2](1) are Kuhn-Tucker conditions where α^i_{sD} is the Lagrange multiplier associated to the constraint $\phi^i_{sD} = \sum_{a \in \mathcal{A}^i_{sD}} \mathbf{x}^i_{(a)}$.

such that for all $i \in \mathcal{I}$, $sD \in \mathcal{E}^i$ and $a \in \mathcal{A}^i_{sD}$

$$\beta^i_{(a)} \geq 0,\ \beta^i_{(a)} \mathbf{x}^i_{(a)-\lambda} = 0 \tag{2}$$

$$0 \in \{\partial_a J^i(\mathbf{x}) - \alpha^i_{sD} - \beta^i_{(a)}\} \tag{3}$$

Let $(\mathbf{x}, \mathbf{y}^i_{(a)})$ be the flow configuration $\mathbf{x}$ where the coordinate $\mathbf{x}^i_{(a)}$ is replaced by $\mathbf{y}^i_{(a)}$.

We denote by $J^i_{a-}(\mathbf{x})$ (resp. $J^i_{a+}(\mathbf{x})$) the left (resp. right) side derivative of $J^i(\mathbf{x})$ according to $\mathbf{x}^i_{(a)}$, that is

$$\lim_{\lambda \to 0+} \frac{J^i(\mathbf{x}, \mathbf{x}^i_{(a)}) - J^i(\mathbf{x})}{\lambda}$$
$$\left(\text{resp. } \lim_{\lambda \to 0+} \frac{J^i(\mathbf{x}, \mathbf{x}^i_{(a)} + \lambda) - J^i(\mathbf{x})}{\lambda}\right)$$

(a convex function has always one-side derivative).

Since the function J^i takes values in R, then its subdifferential at the point $\mathbf{x}$ with respect to $\mathbf{x}^i_{(a)}$, is the set of vectors

$$\{\mathbf{x}^* \mid J^i_-(\mathbf{x}) \leq \mathbf{x}^* \leq J^i_+(\mathbf{x})\}$$

(see for example Rockafellar [13, Ch 23,24]).

Remark 1: Case *(B.2)*. For a user $i \in \mathcal{I}$, the decision of others $(\mathbf{x}^{-i})$ being fixed, for any tree $a \in \mathcal{A}^i$ and for any path $p \in a$, the cost of p $(\sum_{l \in p} f^i_l(x_l))$ is continuous and increasing in $\mathbf{x}^i_{(a)}$. If $\mathcal{X} = \tilde{\mathcal{X}}$ and $\#\mathcal{E}^i = 1$, then the strategy of user i which minimize $J^i(., \mathbf{x}^{-i})$ will be a $\mathbf{x}^i$ such that
$\forall a, b \in \mathcal{A}^i$, s.t. $\mathbf{x}^i_{(a)} > 0$,

$$\max_{p \in a} \sum_{l \in p} f^i_l(x_l) \leq \max_{q \in b} \sum_{l \in q} f^i_l(x_l) \tag{4}$$

(if $\mathcal{X} \neq \tilde{\mathcal{X}}$, this may not hold anymore, since we can have a link saturated, for example). For this $\mathbf{x}$, typically $J^i(\mathbf{x})$ will not be differentiable.

Remark 2: If we replace the cost *(B.2)* by
$J^i(\mathbf{x}) = \max_{a \in \mathcal{A}^i} \sum_{uv \in a} \tau^i_{va} f^i_{uv}(x_{uv})$, that is instead of minimizing its maximum delay over paths, a user will minimize its maximum delay over trees, then the equation (4) becomes $\forall a, b \in \mathcal{A}^i$, s.t. $\mathbf{x}^i_{(a)} > 0$,

$$\sum_{uv \in a} \tau^i_{va} f^i_{uv}(x_{uv}) \leq \sum_{uv \in b} \tau^i_{va} f^i_{uv}(x_{uv}) \tag{5}$$

The equation (5) looks as a natural extension of the characterization of the so called Wardrop [16] equilibrium to the context of trees in a multicast network. This is an equilibrium notion adapted from the context of road traffic in which a decision maker is a single infinitesimal packet (as opposed of our original definition in which the decision maker for a large number of packets is the so called "user"). More details and a result of uniqueness for theses cost functions are given in Section V.

E. Comments on Bidirectional Links

When we consider a bidirectional link uv, we assume that for every user $i \in \mathcal{I}$, $f^i_{uv} = f^i_{vu}$ and moreover the cost of this function depends only on the aggregative flow which goes trough this link, $x_{uv} = x_{uv\to} + x_{vu\to}$.

With this assumption, we know that a bidirectional link may be transformed into a network of unidirectional ones where some are of null cost, see [3, Appendix B]. Hence the existence of the Nash equilibrium also holds for networks with both unidirectional and bidirectional links (the conditions of existence allow links with constant cost).

III. A Three Nodes Network

Consider a network with three nodes u, v and d, a bidirectional link between u and v and two unidirectional ones, one between u and d and the other between v and d (Figure 2). N users share this network. $\mathcal{I} = \mathcal{I}_I \cup \mathcal{I}_{II}$, where every user $i \in \mathcal{I}_I$ has to ship an amount ϕ^i of packets from u to v and d, and every user $i \in \mathcal{I}_{II}$ has to ship an amount ϕ^i of packets from v to u and d. To do this each user has two possible trees
a user $i \in \mathcal{I}_I$ has the tree (uv, ud) where the duplication is made in u and the tree (uv, vd) where the duplication is made in v
a user $i \in \mathcal{I}_{II}$ has the tree (vu, vd) where the duplication is made in v and the tree (vu, ud) where the duplication is made in u.

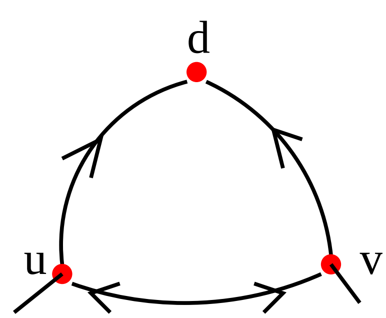

Fig. 2. A three nodes network

With the notations previously introduced, this gives
$\mathcal{I} = \mathcal{I}_I \cup \mathcal{I}_{II}$, for $i \in \mathcal{I}_I,\ s^i = u,\ D^i = \{v, d\}$,
$\mathcal{A}^i = \mathcal{A}^{\mathcal{I}_I} = \{(uv, ud), (uv, vd)\}$,
for $i \in \mathcal{I}_{II},\ s^i = v,\ D^i = \{u, d\}$,
$\mathcal{A}^i = \mathcal{A}^{\mathcal{I}_{II}} = \{(vu, vd), (vu, ud)\}$
for $i \in \mathcal{I},\ \mathcal{E}^i = \{s^i D^i\},\ \phi^i_{s^i D^i} = \phi^i$

Remark 3: No matter which strategy a user $i \in \mathcal{I}_I$ (resp. $i \in \mathcal{I}_{II}$) chooses, it has to send an amount of flow ϕ^i from u to v (resp. from v to u), hence the flow which goes through the link connecting u and v will be the constant $\phi = \sum_{i \in \mathcal{I}} \phi^i$.

A. Uniqueness of Nash Equilibrium

Lemma III.1: Assume $\mathcal{G}$ and $\mathcal{X} = \tilde{\mathcal{X}}$. Then in cases *(A)* and *(B.1)*, the Nash equilibrium is unique, and in case *(B.2)* for any two Nash equilibria, $\bar{\mathbf{x}}$, $\tilde{\mathbf{x}}$, we have $\forall i \in \mathcal{I}, \quad J^i(\tilde{\mathbf{x}}) = J^i(\bar{\mathbf{x}})$, and moreover the link's utilization is unique (at equilibrium).

Proof:

(A) From the *Remark 3* it follows that our networking game is equivalent to a classical routing game in a unicast network with two parallel links between a source s and a destination d, where a user $i \in \mathcal{I}_I$ (resp. $i \in \mathcal{I}_{II}$) has to ship a amount of flow ϕ^i from s to d with the links' cost functions $F_1^i(x) = f_{ud}^i(x)$, $F_2^i(x) = f_{vd}^i(x)$.

In such a network, under the assumption $\mathcal{G}$ the equilibrium $\mathbf{x}$ is known to be unique (see [11, Thm 2.1]).

Then at equilibrium, the cost for a user $i \in \mathcal{I}_I$ is:

$$\begin{aligned} J^i(\mathbf{x}) &= \mathbf{x}^i_{(uv,ud)}(f_{uv}^i(x_{uv}) + f_{ud}^i(x_{ud})) \\ &\quad + \mathbf{x}^i_{(uv,vd)}(f_{uv}^i(x_{uv}) + f_{vd}^i(x_{vd})) \\ &= \phi^i f_{uv}^i(\phi) + \mathbf{x}^i_{(uv,ud)} f_{ud}^i(x_{ud}) \\ &\quad + \mathbf{x}^i_{(uv,vd)} f_{vd}^i(x_{vd}) \end{aligned}$$

and for a user $i \in \mathcal{I}_{II}$

$$J^i(\mathbf{x}) = \phi^i f_{vu}^i(\phi) + \mathbf{x}^i_{(vu,vd)} f_{vd}^i(x_{vd}) + \mathbf{x}^i_{(vu,ud)} f_{ud}^i(x_{ud})$$

(B.1) In this case, the game is also equivalent to a classical routing game in a network of two parallel links, but due to the specifity of the total delay, we are faced to a change in the links' cost functions of the network of parallel links, they have to be for $i \in \mathcal{I}_I$ $F_1^i(x) = f_{ud}^i(x) + f_{uv}^i(\phi)$, $F_2^i(x) = f_{vd}^i(x) + 2f_{uv}^i(\phi)$ and for $i \in \mathcal{I}_{II}$ $F_1^i(x) = f_{ud}^i(x) + 2f_{vu}^i(\phi)$, $F_2^i(x) = f_{vd}^i(x) + f_{vu}^i(\phi)$.

Therefore, in this case again, the equilibrium is unique, and at equilibrium $\mathbf{x}$ the total delay for user i is:

if $i \in \mathcal{I}_I$

$$\begin{aligned} J^i(\mathbf{x}) &= \mathbf{x}^i_{(uv,ud)}(f_{uv}^i(\phi) + f_{ud}^i(x_{ud})) \\ &\quad + \mathbf{x}^i_{(uv,vd)}(2f_{uv}^i(\phi) + f_{vd}^i(x_{vd})) \end{aligned}$$

if $i \in \mathcal{I}_{II}$

$$\begin{aligned} J^i(\mathbf{x}) &= \mathbf{x}^i_{(vu,vd)}(f_{vu}^i(\phi) + f_{vd}^i(x_{vd})) \\ &\quad + \mathbf{x}^i_{(vu,ud)}(2f_{vu}^i(\phi) + f_{ud}^i(x_{ud})) \end{aligned}$$

(B.2) The cost function of a user $i \in \mathcal{I}$ is

$$J^i(\mathbf{x}) = max_{a \in \mathcal{A}^i, p \in a} \sum_{l \in p} f_l^i(x_l)$$

According to *Remark 1* at equilibrium $\mathbf{x}$ for $i \in \mathcal{I}_I$ we have if $\mathbf{x}^i_{(uv,ud)} > 0$

$$\max\{f_{uv}^i(\phi), f_{ud}^i(x_{ud})\} \le f_{uv}^i(\phi) + f_{vd}^i(x_{vd}) \quad (6)$$

and if $\mathbf{x}^i_{(uv,vd)} > 0$

$$f_{uv}^i(\phi) + f_{vd}^i(x_{vd}) \le \max\{f_{uv}^i(\phi), f_{ud}^i(x_{ud})\} \quad (7)$$

Firstly we prove that for any two equilibria, $\bar{\mathbf{x}}$, $\tilde{\mathbf{x}}$, we have for all $i \in \mathcal{I}$, $J^i(\bar{\mathbf{x}}) = J^i(\tilde{\mathbf{x}})$. Suppose that there exists $i \in \mathcal{I}_I$ such that $J^i(\bar{\mathbf{x}}) > J^i(\tilde{\mathbf{x}})$.

If $\tilde{\mathbf{x}}^i_{(a)} > 0$ for all $a \in \mathcal{A}^i$, we obtain that

$$f_{uv}^i(\phi) + f_{vd}^i(\tilde{x}_{vd}) < f_{uv}^i(\phi) + f_{vd}^i(\bar{x}_{vd})$$

and

$$max\{f_{uv}^i(\phi), f_{ud}^i(\tilde{x}_{ud})\} < max\{f_{uv}^i(\phi), f_{ud}^i(\bar{x}_{ud})\}$$

from which it follows that $\bar{x}_{vd} > \tilde{x}_{vd}$ and $\bar{x}_{ud} > \tilde{x}_{ud}$. This is impossible since $\bar{x}_{vd} + \bar{x}_{ud} = \tilde{x}_{vd} + \tilde{x}_{ud}$.

If there exists $a \in \mathcal{A}^i$ such that $\tilde{\mathbf{x}}^i_{(a)} = 0$, assume that $a = (uv, vd)$ (the other case is similar), then we obtain that

$$max\{f_{uv}^i(\phi), f_{ud}^i(\tilde{x}_{ud})\} < max\{f_{uv}^i(\phi), f_{ud}^i(\bar{x}_{ud})\}$$

Then $\bar{x}_{ud} > \tilde{x}_{ud}$, $f_{ud}^i(\bar{x}_{ud}) > f_{uv}^i(\phi)$ and there exists $j \in \mathcal{I}$ such that $\bar{x}_{ud}^j > \tilde{x}_{ud}^j$, which implies that $\tilde{x}_{vd} > \bar{x}_{vd}$ and $\tilde{x}_{vd}^j > \bar{x}_{vd}^j$.

If $j \in \mathcal{I}_{II}$, therefore $\tilde{\mathbf{x}}^j_{(vu,vd)} > \bar{\mathbf{x}}^j_{(vu,vd)}$ (we have also $\bar{\mathbf{x}}^j_{(vu,ud)} > \tilde{\mathbf{x}}^j_{(vu,ud)}$). Finally we obtain that

$$J^j(\tilde{\mathbf{x}}) \ge \max\{f_{vu}^j(\phi), f_{vd}^j(\bar{x}_{vd})\} \ge J^j(\bar{\mathbf{x}}) \quad (8)$$

and

$$J^j(\bar{\mathbf{x}}) > f_{vu}^j(\phi) + f_{ud}^j(\tilde{x}_{ud}) \ge J^j(\tilde{\mathbf{x}}) \quad (9)$$

A contradiction

If $j \in \mathcal{I}_I$, we obtain a similar result replacing $\mathbf{x}^j_{(vu,ud)}$ by $\mathbf{x}^j_{(uv,ud)}$ and $\mathbf{x}^j_{(vu,vd)}$ by $\mathbf{x}^j_{(uv,vd)}$.

Therefore for all $i \in \mathcal{I}_I$ $J^i(\bar{\mathbf{x}}) \le J^i(\tilde{\mathbf{x}})$. Interchanging $\tilde{\mathbf{x}}$ and $\bar{\mathbf{x}}$ we obtain that $J^i(\bar{\mathbf{x}}) = J^i(\tilde{\mathbf{x}})$, a similar result holds for users $i \in \mathcal{I}_{II}$.

The uniqueness of the links utilization at equilibrium is a trivial implication of $J^i(\bar{\mathbf{x}}) = J^i(\tilde{\mathbf{x}})$ for all $i \in \mathcal{I}$. $\square$

Remark 4: For the cases *(A)* and *(B.1)*, if the links' cost functions are the same for all users ($f_l^i = f_l$) and if all users has the same amount of packets to ship ($\phi^i = \phi$), it follows from [11, Lem 3.1] applied to the equivalent unicast networks defined in the proof of *Lemma III.1*, that $\mathbf{x}_{(a)}^i = x_{(a)}/\#\mathcal{I}_m,\ \forall m \in \{I, II\},\ i \in \mathcal{I}_m,\ a \in \mathcal{A}^{\mathcal{I}_m}$

B. $\mathcal{X} \neq \tilde{\mathcal{X}}$: An Example With Distinct Equilibria

Consider the previous network shared by two users, I and II, each user i has an flow amount of 1 to send from s^i to D^i and has the following cost functions:
$f_{uv}^i(x) = f_{uv}(x) = x$, $f_{ud}^i(x) = f_{ud}(x) = x$ and $f_{vd}^i(x) = f_{vd}(x) = 2x$.

Moreover the capacity of the link ud is limited: $x_{ud} \leq 1$. Then we have

$$\begin{aligned}\mathcal{X} &= \{\mathbf{x} \in R^n \mid \forall i \in \mathcal{I}, \forall a \in \mathcal{A}^i,\ \mathbf{x}_{(a)}^i \geq 0, \\ &\qquad \sum_{a \in \mathcal{A}^i} \mathbf{x}_{(a)}^i = \phi, \quad x_{ud} \leq 1\}\end{aligned}$$

We define the flow configuration $\tilde{\mathbf{x}}$ by $\tilde{\mathbf{x}}_{(uv,ud)} = 1$, $\tilde{\mathbf{x}}_{(uv,vd)} = 0$, $\tilde{\mathbf{x}}_{(vu,vd)} = 1$ and $\tilde{\mathbf{x}}_{(vu,ud)} = 0$ and the flow configuration $\bar{\mathbf{x}}$ by $\bar{\mathbf{x}}_{(uv,ud)} = 0$, $\bar{\mathbf{x}}_{(uv,vd)} = 1$, $\bar{\mathbf{x}}_{(vu,vd)} = 0$ and $\bar{\mathbf{x}}_{(vu,ud)} = 1$.

Obviously $\tilde{\mathbf{x}}$ and $\bar{\mathbf{x}}$ are both Nash equilibria, since in the first case user I has no interest to change its flow configuration, and user II has no other choice that send all its flow into the tree (vu, vd) and inversely for the second case (in fact every convex combination of these two flow configurations is a Nash equilibrium).

Then *Lemma III.1* is false if $\mathcal{X} \neq \tilde{\mathcal{X}}$.

C. Convergence to Nash Equilibrium

In this subsection we present a classical algorithm of updating flow configuration based on best reply strategy (see [11, Sec. 2.4] or [17, Sec. 2] for a detailed description of this algorithm). We show, under the restriction of twice-differentiability of the f_l^i's, that this algorithm converges to the unique equilibrium.

In the networking game presented previously, each user has only one decision which is the quantity of flow to send through tree (uv, vd), $\mathbf{x}_{(uv,vd)}$, since the other variable is $(\phi^I - \mathbf{x}_{(uv,vd)})$, similarly the decision of user II is $\mathbf{x}_{(vu,vd)}$. We denote y the strategy of user I and z this of user II.

Let the sequence $(y_n, z_n)_{n \geq 0}$ be defined as follows:
- step 0: (y_0, z_0) is a given initial flow configuration
- step 1: I updates its flow in order to minimize its cost function the strategy of II, z_0, being given, the resulting flow configuration is (y_1, z_1), where y_1 is a best reply to z_0 $(= z_1)$
-step 2: II updates its flow in order to minimize its cost function the strategy of I, y_1, being given, the resulting flow configuration is (y_2, z_2), where z_2 is a best reply to y_1 $(= z_2)$
...
- step $2n - 1$: I updates, the resulting flow configuration is (y_{2n-1}, z_{2n-1}), where y_{2n-1} is a best reply to z_{2n-2} $(= z_{2n-1})$
- step $2n$: II updates, the resulting flow configuration is (y_{2n}, z_{2n}), where z_{2n} is a best reply to y_{2n-1} $(= y_{2n})$.

Remark 5: The function which associates to a flow generated by a decision y (resp. z) the best reply set for user II (resp. I) is continuous, due to the continuity of the cost functions.

Assumption:
- $(\mathcal{C})\ \forall i \in \mathcal{I},\ l \in \mathcal{L},\ f_l^i$ is twice-differentiable wherever finite.

Lemma III.2: Assume $\mathcal{G}, \mathcal{C}$ and $\mathcal{X} = \tilde{\mathcal{X}}$. Given any initial flow configuration, the sequence generated by a succession of best reply strategies will converge to equilibrium in cases (A), (B.1) and (B.2).

Proof:
(A) It is sufficient to remark that the networking game is supermodular, *i.e.,*

$$\frac{d^2}{dydz} J^i(y, z) \geq 0 \quad \forall (y, z) \in [0, \phi^I] \times [0, \phi^{II}]$$

And the result follows from Yao [17, Thm 2.3].

The supermodularity of G_1 is trivial. Indeed we have

$$\begin{aligned}J^I(y, z) &= y(f_{uv}^I(\phi) + f_{vd}^I(y + z)) \\ &\qquad + (\phi^I - y)(f_{uv}^I(\phi) + f_{ud}^I(\phi - y - z))\end{aligned}$$

Then

$$\begin{aligned}&\frac{d^2}{dydz} J^I(y, z) = \nabla f_{vd}^I(y + z) + \nabla f_{ud}^I(\phi - y - z) \\ &\qquad + y\nabla^2 f_{vd}^I(y + z) + (\phi^I - y)\nabla^2 f_{ud}^I(\phi - y - z) \\ &\geq \ 0 \qquad \forall (y, z) \in [0, \phi^I] \times [0, \phi^{II}]\end{aligned}$$

where the inequality is due to assumptions $\mathcal{G}$ and $\mathcal{C}$ (we obtain the result for J^{II} by a similar way).

(B.1) similar to *(A)*.

(B.2) The proof is based on the fact that the sequence $(y_n, z_n)_{n \geq 2}$ is motone, more precisely we have $z_n \geq z_{n-1},\ y_n \geq y_{n+1}$ or $z_{n-1} \geq z_n,\ y_{n+1} \geq y_n$. *Remark 5* and the boundedness of the flows will imply the convergence to equilibrium of the sequence.

We will only only show

$$z_n \geq z_{n-1} \implies y_n \geq y_{n+1} \quad \forall n \geq 2 \text{ (n even)} \tag{10}$$

1) $\phi^I > y_n = y_{n-1} > 0$, due to *Remark 1*, we have

$$\begin{aligned} f^I_{uv}(\phi) &+ f^I_{vd}(y_{n-1} + z_{n-1}) \\ &= \max\{f^I_{uv}(\phi), f^I_{ud}(\phi - y_{n-1} - z_{n-1})\} \end{aligned} \tag{11}$$

Recall that $y_n = y_{n-1}$. By hypothesis, we obtain that

$$f^I_{uv}(\phi) + f^I_{vd}(y_n + z_n) \geq \max\{f^I_{uv}(\phi), f^I_{ud}(\phi - y_n - z_n)\}$$

If $y_{n+1} = 0$, (10) is checked. Suppose that $y_{n+1} = \phi^I$, then we have

$$\begin{aligned} f^I_{uv}(\phi) &+ f^I_{vd}(y_{n+1} + z_{n+1}) > f^I_{uv}(\phi) + f^I_{vd}(y_n + z_n) \\ &\geq \max\{f^I_{uv}(\phi), f^I_{ud}(\phi - y_n - z_n)\} \\ &\geq \max\{f^I_{uv}(\phi), f^I_{ud}(\phi - y_{n+1} - z_{n+1})\} \end{aligned}$$

but in order to $y_{n+1} = \phi^I$ be a best reply to z_n, it has to satisfy (cf.*Remark 1*)

$$\begin{aligned} f^I_{uv}(\phi) &+ f^I_{vd}(y_{n+1} + z_{n+1}) \\ &\leq \max\{f^I_{uv}(\phi), f^I_{ud}(\phi - y_{n+1} - z_{n+1})\} \end{aligned}$$

A contradiction.

It rest us to consider the case where $\phi^I > y_{n+1} > 0$, in this case in order to y_{n+1} be a best reply to z_n, it is necessary that (11) holds also for $n+1$ (where $z_{n+1} = z_n$). Hence (10) is checked.

2) $y_n = \phi^I$, (10) is always true.

3) $y_n (= y_{n-1}) = 0$, we have

$$\begin{aligned} f^I_{uv}(\phi) &+ f^I_{vd}(y_{n+1} + z_{n+1}) \\ &\geq f^I_{uv}(\phi) + f^I_{vd}(y_{n-1} + z_{n-1}) \\ &\geq \max\{f^I_{uv}(\phi), f^I_{ud}(\phi - y_{n-1} - z_{n-1})\} \\ &\geq \max\{f^I_{uv}(\phi), f^I_{ud}(\phi - y_{n+1} - z_{n+1})\} \end{aligned} \tag{12}$$

where the second inequality comes from the fact that $y_{n-1} = 0$ is a best reply to z_{n-1}. If $y_{n+1} > 0$, then (12) holds with a strict inequality, and y_{n+1} is not a best reply to z_{n+1}, then (10) is checked.

Implication (10) holds also when $\leq$ is substituted to $\geq$, and when we interchange y and z for n odd. The result follows. □

IV. A Four Nodes Network

Consider a network with four nodes s, u, d_1 and d_2 and unidirectional links between them, N users share this network, each user $i \in \mathcal{I}$ has an amount of packets ϕ^i to ship from the source s to the destination d_1 and also from s to d_2. To do this each user has two possible trees either the tree $t_d = (sd_1, sd_2)$ (d for direct) where the duplication is made in s or the tree $t_s = (su, ud_1, ud_2)$ (s for split) where the duplication is made in u, we call this network N_1 (Figure 3).

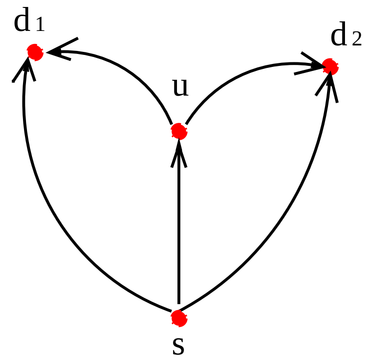

Fig. 3. A four nodes network

According to our mathematical notations we have $\mathcal{I} = \{1, \ldots, N\}$, $\forall i \in \mathcal{I}$, $s^i = s$, $D^i = \{d_1, d_2\}$, $\mathcal{E}^i = \{s^i D^i\}$, $\phi^i_{s^i D^i} = \phi^i$ and $\mathcal{A}^i = \mathcal{A} = \{t_d, t_s\}$.

A. Uniqueness of Nash Equilibrium

Lemma IV.1: Assume $\mathcal{G}$ and $\mathcal{X} = \tilde{\mathcal{X}}$. Then in cases *(A)* and *(B.1)*, the Nash equilibrium is unique, and in case *(B.2)* for any two Nash equilibria, $\bar{\mathbf{x}}$, $\tilde{\mathbf{x}}$, we have

$$\forall i \in \mathcal{I}, \quad J^i(\tilde{\mathbf{x}}) = J^i(\bar{\mathbf{x}})$$

and moreover the links' utilization is unique (at equilibrium).

Proof: We only present the proof for case *(A)*, the one for case *(B.1)* is similar, and those of *(B.2)* is identical to the one of *Lemma III.1* replacing (6) and (7) by if $\mathbf{x}^i_{(t_s)} > 0$

$$\begin{aligned} f^i_{su}(x_{su}) &+ \max\{f^i_{ud_1}(x_{ud_1}), f^i_{ud_2}(x_{ud_2})\} \\ &\leq \max\{f^i_{sd_1}(x_{sd_1}), f^i_{sd_2}(x_{sd_2})\} \end{aligned} \tag{13}$$

and if $\mathbf{x}^i_{(t_d)} > 0$

$$\begin{aligned} \max &\{ f^i_{sd_1}(x_{sd_1}), f^i_{sd_2}(x_{sd_2})\} \\ &\leq f^i_{su}(x_{su}) + \max\{f^i_{ud_1}(x_{ud_1}), f^i_{ud_2}(x_{ud_2})\} \end{aligned} \tag{14}$$

where $x_{sd_1} = x_{sd_2}$ and $x_{su} = x_{ud_1} = x_{ud_2}$.

Suppose that there exist two distinct equilibria $\tilde{\mathbf{x}}$ and $\bar{\mathbf{x}}$, then both of them have to satisfy the conditions: for $\mathbf{x} =$

$\tilde{\mathbf{x}},\ \mathbf{x} = \bar{\mathbf{x}}$

$$\exists \alpha = \alpha(\mathbf{x}),\ \alpha^T = (\alpha^i)_{i\in\mathcal{I}}$$

such that for all $i \in \mathcal{I}$ and $a \in \mathcal{A}$

$$K^i_{(a)}(\mathbf{x}) - \alpha^i \geq 0; \quad (K^i_{(a)}(\mathbf{x}) - \alpha^i)\mathbf{x}^i_{(a)} = 0 \qquad (15)$$

Firstly we prove that $\forall a \in \mathcal{A},\ \bar{x}_{(a)} = \tilde{x}_{(a)}$. Suppose that there exists a tree $a \in \mathcal{A}$ such that $\tilde{x}_{(a)} > \bar{x}_{(a)}$, therefore there exists a user $i \in \mathcal{I}$ such that $\tilde{\mathbf{x}}^i_{(a)} > \bar{\mathbf{x}}^i_{(a)}$, denoting by b the other tree we have that $\bar{x}_{(b)} > \tilde{x}_{(b)}$ and $\bar{\mathbf{x}}^i_{(b)} > \tilde{\mathbf{x}}^i_{(b)}$ (since $\mathbf{x}^i_{(a)} + \mathbf{x}^i_{(b)} = \phi^i,\ \forall i \in \mathcal{I}$). But by construction of our model we have that

$$\forall l \in a, \quad x_{(a)} = x_l \quad \text{and} \quad \mathbf{x}^i_{(a)} = x^i_l$$

(and similarly for tree b). Therefore from (15) it follows that

$$\begin{aligned} \tilde{\alpha}^i &= K^i_{(a)}(\tilde{\mathbf{x}}) = \sum_{l\in a}\left(\tilde{x}^i_l \nabla f^i_l(\tilde{x}_l) + f^i_l(\tilde{x}_l)\right) \qquad (16) \\ &> \sum_{l\in a}\left(\bar{x}^i_l \nabla f^i_l(\bar{x}_l) + f^i_l(\bar{x}_l)\right) \geq \bar{\alpha}^i \end{aligned}$$

due to the strict increase of f^i_l and the increase of ∇f^i_l.

But from the inequalities on tree b we obtain

$$\begin{aligned} \bar{\alpha}^i &= \sum_{l\in b}\left(\bar{x}^i_l \nabla f^i_l(\bar{x}_l) + f^i_l(\bar{x}_l)\right) \qquad (17) \\ &> \sum_{l\in b}\left(\tilde{x}^i_l \nabla f^i_l(\tilde{x}_l) + f^i_l(\tilde{x}_l)\right) \geq \tilde{\alpha}^i \end{aligned}$$

(17) contradicts (16), hence for $a = t_d,\ t_s$ we have that

$$\tilde{x}_{(a)} = \bar{x}_{(a)}$$

It rests us to show that $\forall i \in \mathcal{I}$, $a \in \mathcal{A}$, $\tilde{\mathbf{x}}^i_{(a)} = \bar{\mathbf{x}}^i_{(a)}$, which is trivial. Indeed suppose that there exists a user $i \in \mathcal{I}$ such that $\tilde{\mathbf{x}}^i_{(a)} > \bar{\mathbf{x}}^i_{(a)}$, then (16) and (17) are still valid and the conclusion follows. $\square$

B. *Equivalent Unicast Network*

The four nodes network presented in this section may be transformed in a equivalent parallel links unicast network. One more time we will only treat the case *(A)*, case *(B.1)* being similar.

Indeed consider the network N_2 shared by N users, with two nodes, a source s and a destination d, and two parallel links between them l_s and l_d, each user $i \in \mathcal{I}$ has to ship an amount of packets ϕ^i from s to d. The cost functions of the links are

$$F^i_{l_s}(x_{l_s}) = f^i_{su}(x_{l_s}) + f^i_{ud_1}(x_{l_s}) + f^i_{ud_2}(x_{l_s})$$

and

$$F^i_{l_d}(x_{l_d}) = f^i_{sd_1}(x_{l_d}) + f^i_{sd_2}(x_{l_d})$$

in order that the cost of the link l_s (resp. l_d) be the same as the cost of the tree t_s (resp. t_d) in the original model.

The total cost function for a user $i \in \mathcal{I}$ is

$$J^i(x) = x^i_{l_s} F^i_{l_s}(x_{l_s}) + x^i_{l_d} F^i_{l_d}(x_{l_d})$$

In such a network the Nash equilibrium is known to be unique (see Orda, Rom and Shimkin [11, Thm 2.1]) and moreover if the users are symmetric, that is the links' cost functions are the same for all users ($f^i_l = f_l$) and all users has the same amount of packets to ship from s to d ($\phi^i = \phi$) then this equilibrium is symmetric, that is $\mathbf{x}^i_l = \mathbf{x}^j_l$ for all $i,\ j \in \mathcal{I}$ and $l \in \mathcal{L}$ (then we have that $\mathbf{x}^i_l = \frac{x_l}{N}$) ([11, Lem 3.1]).

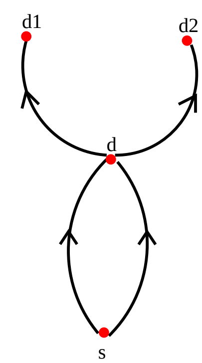

Fig. 4. Equivalent unicast network

Now we construct a new network, N_3, by adding to this network two extra nodes d_1 and d_2 and two links dd_1 and dd_2 of null cost (Figure 4). Assume that the packets which arrive in d are duplicated in order that one copy goes into dd_1 and the other one into dd_2. The results obtained for N_2 are still valid in this new network. Since this network (N_3) is equivalent to N_1, therefore the results (uniqueness of the Nash equilibrium and symmetry of this equilibrium for symmetrical users) are valid for the original network.

V. Extended Wardrop Equilibrium

Wardrop equilibrium is the concept of optimization in a network shared by an infinity of users where the decision of any user has a negligeable influence on the others' decisions (it is typically the case in road traffic).

We assume that the cost functions on the links are the same for any user and we group the users in classes according to their sD pair, therefore into a class $i \in \mathcal{I}$, any user $u \in i$ has the same sD pair and the same set of possible trees $\mathcal{A}^i$. We denote the cost (or delay) for a user $i \in \mathcal{I}$ of a tree $a \in \mathcal{A}^i$ when the strategy $\mathbf{x}$ is used by $F^i_{(a)}(\mathbf{x})$.

We define the extended Wardrop equilibrium through the condition
$\forall i \in \mathcal{I}, \forall a, b \in \mathcal{A}^i$,

$$\mathbf{x}^i_{(a)} > 0 \implies F^i_{(a)}(\mathbf{x}) \leq F^i_{(b)}(\mathbf{x}) \tag{18}$$

If $F^i_{(a)}(\mathbf{x}) = \sum_{uv \in a} \tau^i_{va} f^i_{uv}(x_{uv})$ we see that equation (18) is identical to equation (5) (Sec II-D, Rem 2). Hence The equilibrium characterizations for a Nash equilibrium where the cost functions are $J^i(\mathbf{x}) = \max_{a \in \mathcal{A}^i} \sum_{uv \in a} \tau^i_{va} f^i_{uv}(x_{uv})$ and for a Wardrop equilibrium are the same.

We know that in unicast networks links' utilization is unique at Wardrop equilibrium, this can be proved in two different ways (1) a variational inequalities approach, or (2) a transformation of the problem into another equivalent one in which we first transform the cost, and then consider a single entity that optimizes for everybody; the optimal routing is then equal to the equilibrium. For more details see e.g. [12].

We can still use the variational inequalities approach to show the uniquess of links' utilization at Wardrop equilibrium in multicast networks, yet we can't apply the second approach due to the presence of a factor depending of trees (τ^i_{va}) in the cost function. We give here the proof of the uniqueness based on the variational inequalities approach for general topology.

Assumption:

- For any $l \in \mathcal{L}$, f_l is stricly increasing

Notations:

Given the flow configuration $\mathbf{x} \in \mathcal{X}$, we denote the delay for user $i \in \mathcal{I}$ of a tree $a \in \mathcal{A}^i$ by $F^i_{(a)}(\mathbf{x})$, that is

$$F^i_{(a)}(\mathbf{x}) = \sum_{uv \in a} \tau^i_{va} f^i_{uv}(x_{uv})$$

and the vector of delay by

$$F(\mathbf{x}) = \left[F^i_{(a)}(\mathbf{x})\right]_{i \in \mathcal{I}, a \in \mathcal{A}^i}$$

Let Γ be the incident matrix (see appendix A) and A be the N-dimensional vector $A(\mathbf{x}) = \left[A^i(\mathbf{x})\right]_{i \in \mathcal{I}}$, where $A^i(\mathbf{x})$ denotes the minimal delay over trees for user i given the flow configuration $\mathbf{x}$, that is

$$A^i(\mathbf{x}) = \min_{a \in \mathcal{A}^i} F^i_{(a)}(\mathbf{x})$$

Lemma V.1: $\mathbf{x} \in \mathcal{X}$ is a Nash equilibrium if and only if $\mathbf{x}$ satisfies

$$F(\mathbf{x}) - \Gamma A(\mathbf{x}) \geq 0, \qquad (F(\mathbf{x}) - \Gamma A(\mathbf{x})) \cdot \mathbf{x} = 0 \tag{19}$$

$$\Gamma^T \mathbf{x} = \Phi, \qquad \mathbf{x} \geq 0 \tag{20}$$

Proof: We have just to note that the conditions (20) are equivalent to $\mathbf{x} \in \mathcal{X}$. □

Lemma V.2: $\mathbf{x} \in \mathcal{X}$ is a Nash equilibrium if and only if

$$F(\mathbf{x}) \cdot (\mathbf{y} - \mathbf{x}) \geq 0 \quad \forall \mathbf{y} \in \mathcal{X}. \tag{21}$$

Proof: Similar to the proof of [1, Lem. 3.2] ((21) holds if and only if $\mathbf{x}$ is solution of the linear program $\min_{\mathbf{y}} F(\mathbf{x}) \cdot \mathbf{y}$, s.t. $\Gamma^T \mathbf{y} = \Phi,\ \mathbf{y} \geq 0$). □

Lemma V.3: For arbitrary $\mathbf{x}$ and $\tilde{\mathbf{x}}$ ($\mathbf{x} \neq \tilde{\mathbf{x}}$), if $F(\mathbf{x})$ is finite or $F(\tilde{\mathbf{x}})$ is finite and if $\exists l \in \mathcal{L}$ such that $x_l \neq \tilde{x}_l$, then

$$(\mathbf{x} - \tilde{\mathbf{x}}) \cdot [F(\mathbf{x}) - F(\tilde{\mathbf{x}})] > 0 \tag{22}$$

Proof: Assume that $\exists l \in \mathcal{L}$ such that $x_l \neq \tilde{x}_l$. Then

$$\begin{aligned}
(\mathbf{x} &- \tilde{\mathbf{x}}) \cdot [F(\mathbf{x}) - F(\tilde{\mathbf{x}})] \\
&= \sum_{i \in \mathcal{I}} \sum_{a \in \mathcal{A}^i} \left(x^i_{(a)} - \tilde{x}^i_{(a)}\right) \left(F^i_{(a)}(\mathbf{x}) - F^i_{(a)}(\tilde{\mathbf{x}})\right) \\
&= \sum_{i \in \mathcal{I}} \sum_{a \in \mathcal{A}^i} \left(x^i_{(a)} - \tilde{x}^i_{(a)}\right) \\
&\qquad \times \left(\sum_{uv \in \mathcal{L}} \delta_{uva} \tau^i_{va} \left(f_{uv}(x_{uv}) - f_{uv}(\tilde{x}_{uv})\right)\right) \\
&= \sum_{i \in \mathcal{I}} \sum_{uv \in \mathcal{L}} \left[\left(x^i_{uv} - \tilde{x}^i_{uv}\right) \left(f_{uv}(x_{uv}) - f_{uv}(\tilde{x}_{uv})\right)\right] \\
&\qquad \times \left[\sum_{a \in \mathcal{A}^i} \tau^i_{va}\right] \\
&= \sum_{uv \in \mathcal{L}} \left[\left(x_{uv} - \tilde{x}_{uv}\right) \left(f_{uv}(x_{uv}) - f_{uv}(\tilde{x}_{uv})\right)\right] \\
&\qquad \times \left[\sum_{i \in \mathcal{I}} \sum_{a \in \mathcal{A}^i} \tau^i_{va}\right]
\end{aligned}$$

Since $\tau^i_{va} \geq 1$, the result follows from the strict increase of the f_l's. □

From the two previous lemmas it follows

Lemma V.4: For any two Wardrop equilibria $\mathbf{x}$ and $\tilde{\mathbf{x}}$, we have

$$x_l = \tilde{x}_l$$

Proof: See [1, Thm. 3.5] □

Remark 6: Obviously it follows from the previous lemmas, that in any multicast network if the cost function of any user $i \in \mathcal{I}$ is $J^i(\mathbf{x}) = \max_{a \in \mathcal{A}^i} \sum_{uv \in a} \tau^i_{va} f_{uv}(x_{uv})$

for any two Nash two equilibria $\mathbf{x}$ and $\tilde{\mathbf{x}}$, we have $\forall i \in \mathcal{I}, \quad J^i(\mathbf{x}) = J^i(\tilde{\mathbf{x}})$ and $\forall l \in \mathcal{L}, \quad x_l = \tilde{x}_l$

VI. Numerical Example

We consider the example of Section IV with a single source and two destinations. The costs of all links are taken to be linear. The cost of each direct link, $sd_1,\ sd_2$, is $2(x+1)$. The cost of the common link, su, as well as that of the individual links, $ud_1,\ ud_2$, are $1+x$ each.

(We have been told that linear costs for links are often used in networks for pricing purposes in France Telecom.) We consider the problem of minimizing the total cost. The global demand is ϕ and there are N symmetric users. Thus the demand per user is ϕ/N.

We look for a symmetric Nash equilibrium. We thus assume first that all players except for, say, player 1, send the same amount x over the tree that consists of direct links, t_d, and the rest over the tree with the common link, t_s. We compute the best response of player 1, who sends y over t_d. The cost for player 1 is

$$\begin{aligned} Z(y) &= 4y(1+(N-1)x+y) \\ &\quad +3(\phi-y)(1+(N-1)(\phi-x)+\phi-y) \end{aligned}$$

We now choose $\phi = 1$. The best response is given by

$$y = \frac{1}{14N}(2N - 7xN^2 + 7xN + 3)$$

The symmetric Nash equilibrium is then obtaining by equating $x = y$, which yields

$$x = y = \frac{1}{7}\left(\frac{2N+3}{N(N+1)}\right)$$

In Figure 5 we present this solution (the amount sent over t_d at equilibrium) as a function of the number of players. Figure 6 shows the ratio of the amount sent by a player over t_d and the amount sent over t_s as a function of N. We see that there is a limit of this ratio as N goes to infinity (which would correspond to the concept of Wardrop equilibrium in the context of multicast).

We finally depict in Figure 7 the sum of costs for all players as a function of N. This figure shows clearly that the Nash equilibrium becomes less and less efficient as the number of players grow. The lowest global cost is obtained as expected when there is a single player.

VII. Conclusion, discussion and further extensions

In this paper, we have studied competitive routing in multicast networks using game theoretic tools. We have introduced different criteria for optimization adapted to these networks and have established for these criteria the existence of equilibrium as well as their uniqueness for two specific networks. We further introduced an extension of the notion of Wardrop equilibrium and established its uniqueness in a general topology. The question of the uniqueness of Nash equilibrium in multicast networks with general topology is still open. In our model we assumed that the flow transmitted by a source node arrived fully to all the corresponding destinations.

An interesting extension of our model arises in the case of multirate transmission, which can be attained by hierarchically encoding real time signals. In this approach, a signal is encoded into a number of layers that can be incrementally combined to provide progressive refinements. Every layer is transmitted as a separate multicast group and receivers may choose to which groups they wish to subscribe (according to the capacity available or the congestion state). Internet protocols for adding and dropping layers. can be found in [9] and [10]. Multirate transmission has applications both in video [6], [15] as well as in audio [2].

To deal with such a model we introduce a new object μ_d^{sD}, where $sD \in \mathcal{E}^i$ for some $i \in \mathcal{I}$ and $d \in D$, it represents the proportion of the flow ϕ_{sD}^i sent by user i that the address $d \in D$ wants to receive. In such an extension we cannot apply the proofs of Section III, nevertheless the results of Section IV are still valid. The exact model as well as the detailed analysis of this extension to the network presented in Section IV can be found at http://www-sop.inria.fr/mistral/personnel/Thomas.Boulogne; it is entitled *Multicast Sessions with Multirate Transmission*.

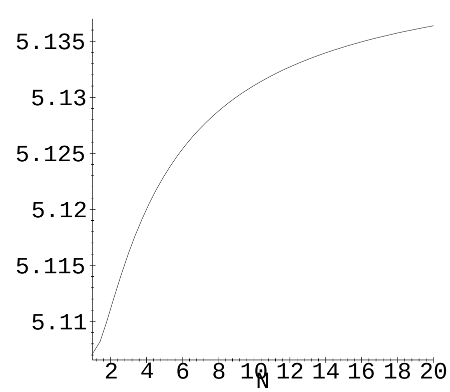

Fig. 7. Sum of the cost for all players as a function of N

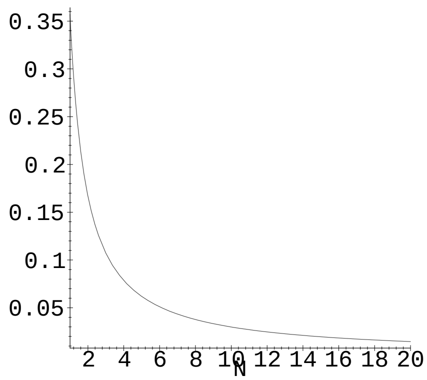

Fig. 5. Amount sent over t_d at equilibrium as a function of N

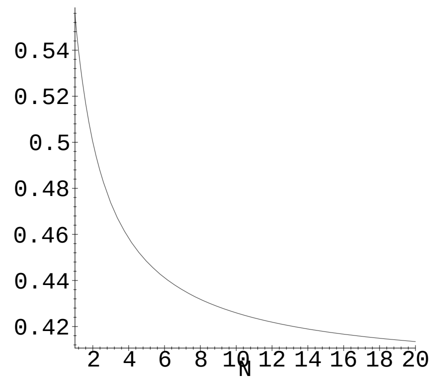

Fig. 6. Ratio of the amount sent by a player over t_d and the amount sent over t_s as a function of N

APPENDIX

A. Incident matrix

Let $a^i := \#\{a \in \mathcal{A}^i\}$, then the incident matrix Γ is

$$\Gamma = \begin{pmatrix} \gamma_1^{11} & \gamma_1^{21} & \dots & \gamma_1^{N1} \\ \gamma_2^{11} & \gamma_2^{21} & \dots & \gamma_2^{N1} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{a^1}^{11} & \gamma_{a^1}^{21} & \dots & \gamma_{a^1}^{N1} \\ \gamma_1^{12} & \gamma_1^{22} & \dots & \gamma_1^{N2} \\ \gamma_2^{12} & \gamma_2^{22} & \dots & \gamma_2^{N2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{a^N-1}^{1N} & \gamma_{a^N-1}^{2N} & \dots & \gamma_{a^N-1}^{NN} \\ \gamma_{a^N}^{1N} & \gamma_{a^N}^{2N} & \dots & \gamma_{a^N}^{NN} \end{pmatrix}$$

whose element (i, r) is γ_k^{ij}, where $i, j \in \mathcal{I}$, $k \in \mathcal{A}^i$, $r = k + \sum_{s=1}^{i-1} a^s$ and $\gamma_k^{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$

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