Adversarial Control in a Delay Tolerant Network*

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Abstract. We consider a multi-criteria control problem that arises in a delay tolerant network with two adversarial controllers: the source and the jammer. The source's objective is to choose transmission probabilities so as to maximize the probability of successful delivery of some content to the destination within a deadline. These transmissions are subject to interference from a jammer who is a second, adversarial type controller, We solve three variants of this problem: (1) the static one, where the actions of both players, u and w, are constant in time; (2) the dynamic open loop problem in which all policies may be time varying, but independent of state, the number of already infected mobiles; and (3) the dynamic closed-loop feedback policies where actions may change in time and may be specified as functions of the current value of the state (in which case we look for feedback Nash equilibrium). We obtain some explicit expressions for the solution of the first game, and some structural results as well as explicit expressions for the others. An interesting outcome of the analysis is that the latter two games exhibit switching times for the two players, where they switch from pure to mixed strategies and *vice versa*. Some numerical examples included in the paper illustrate the nature of the solutions.

Keywords: Delay-tolerant networks, nonzero-sum game, switching strategies.

1 Introduction

We consider in this paper a delay tolerant network, i.e. a sparse network of mobile relay nodes, where connectivity is very low. There is some source that transmits a file to mobiles that are in the communication range. Each mobile is assumed to be in range with the source at some instants that form a Poisson process. A node that receives a copy of the file stores it so that it may transmit it to some potential destinations that may search for a copy of the file. We consider two controllers whose goals are not aligned: the source and the jammer. They both determine at

* The work of the second author is supported by an INRIA-UIUC research collaboration grant, as well as by an AFOSR Grant. The other authors were supported by the Indo-French Centre for the Promotion of Advanced Research (IFCPAR), project 4000-IT-1 and by the INRIA association program DAWN.

T. Alpcan, L. Buttyan, and J. Baras (Eds.): GameSec 2010, LNCS 6442, pp. 87–106, 2010.

each time the probability of transmission. Transmission at a time t is successful if and only if the source attempts transmission while the jammer is silent.

We consider three frameworks, which lead to three different games:

(1) The static one, where the actions of both players, that is the probabilities of transmission and of jamming, u and w, respectively, are considered to be constant in time;

(2) The dynamic open-loop problem. Here, all policies may be time varying, but dependent only on the initial state. In solving the open-loop problem, we first show that the game is equivalent (strategically) to a zero-sum differential game, and then seek the saddle-point solution of that game.

(3) The dynamic closed-loop framework, where actions that may change in time are allowed to depend on the current value of the state (the number of mobiles with a copy of the file). In this case the underlying game is a genuine nonzero-sum differential game, where the solution sought is the feedback Nash equilibrium.

This work is another step in our effort of developing a control methodology for delay tolerant networks, which we initiated with our paper [1]. In contradistinction with the simple threshold structure of [1], we obtain here a much richer set of possible structures for the equilibrium policies, exhibiting in some cases multiple switching times between pure and proper mixed strategies.

The use of game theory for jamming problems. Jamming problems are among the first capturing conflicts in networks that have been modeled and solved using tools and the conceptual framework of game theory. The first publications on these games go back almost thirty years with the pioneering work [6] The question of the capacity achievable in channels prone to jamming was one of the main concerns, and was thus naturally studied within the information theory community, as for example in [7,9]. For a recent survey on wireless games that includes jamming games, see [12]. Jamming of specific wireless local area networks were investigated in [11] who study the jamming of IEEE802.11 and [10] who study the jamming of slotted ALOHA. Our current paper falls in this category of papers by specializing to the context of DTNs.

The paper is organized as follows. The next section (Section 2) provides a precise formulation of the problem, which is followed by Sections 3 and 4 which discuss the static and dynamic cases, respectively. These are followed by Section 5 which includes a number of numerical examples, and the concluding remarks of Section 6 concludes the paper.

2 Model and Problem Formulation

2.1 Model

In the model adopted in this paper, there are n relay mobile nodes, a source, and a destination which is assumed to be static. The network serves as a channel that enables the information to reach the destination. Whenever a relay mobile meets the source, the source may forward a packet to it. We consider the two-hop routing scheme [4] in which a mobile that receives a copy of the packet from the source can only forward it if it meets the destination. It cannot copy it into the memory of another mobile. The details of the basic model are as follows:

The source meets each relay node according to a Poisson process with a parameter λ. Each relay node meets the destination according to another Poisson process, with parameter ν. The source attempts to maximize the probability that a packet arrives successfully at a given destination by time ρ. A second transmitter (jammer), however, tries to jam the transmission, and hence attempts to minimize this probability. The jammer is assumed to be located close to the source. Jamming relay nodes is a separate problem that will be considered later. Note that we consider only two hop routing. Therefore jamming at the relays means jamming when transmitting to the destination.

Let $X(t), u_t, w_t$ denote, respectively, the fraction of mobiles with the message, the source's control, and the jammer's control. Here u_t is the probability to transmit at time t if at that time the source meets a relay and w_t is the probability of jamming at time t. We assume that if jamming and transmission occur simultaneously, then the transmitted packet is lost.

Let $x_t = E[X(t)]$ be the expected value of $X(t)$. Then x_t is generated by

$$\dot{x}_t = u_t(1 - w_t)\lambda(n - x_t), \tag{1}$$

with known initial condition x_0 at $t = 0$, and this constitutes the system dynamics.

2.2 Performance Measure: Successful Delivery Probability

During the incremental time interval $[t, t + dt)$, the number of copies of the packet in the network is $X(t)dt$. Then the number of packets that the destination receives during this time interval is a Poisson random variable with parameter $\nu X(t)dt$. In particular, the probability of not receiving any copy of the packet during $[0, \rho]$, conditioned on $X(t)$, is given by

$$P(T > \rho | X(t), 0 \le t \le \rho) = \exp\left(-\int_0^\rho \nu X(t)dt\right)$$

where T is the random variable describing the instant when the packet first reaches the destination. Its expectation (over $X(t)$) gives the failure probability, i.e. the complementary of the probability of successful delivery.

Instead of minimizing $P(T > \rho)$, we will minimize a bound on that quantity:

$$P(T > \rho) = E\left[\exp\left(-\int_0^\rho \nu X(t)dt\right)\right] \quad \le \quad \exp\left(-E\int_0^\rho \nu X(t)dt\right) \tag{2}$$

where the inequality is obtained by applying Jensen's inequality to the concave function $\exp(-x)$. Minimizing the latter (and hence the upper bound on $P(T > \rho)$) is equivalent to maximizing the quantity

$$J(u, w) := \int_0^\rho \nu x_t dt\,, \tag{3}$$

which we will take as the utility function of the source.

We consider the mean field limit (when we have large number of nodes), in which the randomness in the number of mobiles that have a copy of the nodes as a function of time disappears (we obtain a deterministic time varying limit). In this regime, the difference between the objective function (the delivery failure probability) and the bound (2) vanishes. Indeed, the bound was obtained by exchanging the order of expectation and exponent (using Jensen's inequality), but in the mean-field regime, Jensen's inequality is obtained with equality since the randomness vanishes.

2.3 Related Game Theory Concepts and Some Properties

Saddle-point, maximin and minimax policies: Let $J(u, w)$ be the utility function of the source, as introduced earlier by (3). We assume that the jammer wishes to minimize this quantity and the source wishes to maximize it.

Let Π_c be a set of policies for the controller (both source and relay mobiles) and let Π_j be a set of policies for the jammer. (We will introduce later specific classes of policies.)

We say that $u^* \in \Pi_c$ and $w^* \in \Pi_j$ are saddle-point policies for the game (J, Π_c, Π_j) if for every $u \in \Pi_c$ and $w \in \Pi_j$ we have[1]

$$J(u, w^*) \le J(u^*, w^*) \le J(u^*, w)$$

$J(u^*, w^*)$ is then called the value of the game.

In a general zero-sum game saddle-point need not exist. In that case, we are interested in the upper and lower values ($\overline{V}$ and $\underline{V}$) which are always well defined:

$$\overline{V} = \inf_{w \in \Pi_j} \sup_{u \in \Pi_c} J(u, w), \quad \underline{V} = \sup_{u \in \Pi_c} \inf_{w \in \Pi_j} J(u, w),$$

w^* is optimal for the minimax problem if $\overline{V} = \sup_{u \in \Pi_c} J(u, w^*)$. Given such a w^*, the controller u^* is a best response policy if $\overline{V} = J(u^*, w^*)$. Likewise, u^* is optimal for the maximin problem if $\underline{V} = \inf_{w \in \Pi_j} J(u^*, w)$, and given such a u^*, w^* is a best response policy if $\underline{V} = J(u^*, w^*)$.

A policy is said to be *open loop* if it does not depend on the state of the system. It is said to be *Markov* (or a *feedback* policy) if it takes at time t an action that is allowed to depend not only on t but also on the state at time t. A *pure policy* is one for which the actions at all times are deterministic. For example, a pure policy u for the source is a mixed strategy that takes as values only 0 or 1, with a possibility of switching between the two values, depending on t and possibly also the state.[2]

[1] By some abuse of notation, we will be using u and w both as policies as well as the realized values of these policies under the adopted information structures which also characterize the sets of policies for the two players (controller and jammer).

[2] Note that this definition is somewhat unconventional, and is made to capture the realization that the 'actions' of the players here are actually probabilities, and hence if these probabilities take the extreme values, 0 or 1, and if this is true for all t, then we call the underlying policies *pure*.

A multiple-criteria game: We next introduce a multiple-criteria problem (game) as follows. The source wishes to maximize with respect to u the function $L^u(x_0, u, w)$, where

$$L^u(x_0, u, w) = J(x, u, w) - \mu \int_0^\rho u_t dt \,,$$

and the jammer wishes to minimize with respect to w the function

$$L^w(x_0, u, w) = J(x, \rho, u, w) + \theta \int_0^\rho w_t dt \,,$$

where we have included x_0 in the set of arguments of J (defined earlier by (3)) to emphasize the dependence on the initial state. The pair (u^*, w^*) is a Nash equilibrium for this multiple-criteria problem (nonzero-sum game) if u^* maximizes $L^u(x, \rho, u, w^*)$ over $u \in \Pi_c$ and w^* minimizes $L^w(x, \rho, u^*, w)$ over $w \in \Pi_j$.

Note that in the multi-criteria game, there is antagonism between the two players (related to success probability), but yet it is not a zero-sum game because each player has in addition a second term in his objective function, its own energy cost. However, we can show that this nonzero-sum game is strategically equivalent to a zero-sum game [3], as long as the underlying information structure is open loop; hence every open-loop Nash equilibrium of the multi-criteria game is a saddle-point equilibrium for that particular zero-sum game and *vice versa*.

A strategically equivalent zero-sum game: Let the information structure be open loop for both players, and introduce the objective function

$$L(x, u, w) := J(x, u, w) - \mu \int_0^\rho u_t dt + \theta \int_0^\rho w_t dt \,,$$

which is obtained by adding $\theta \int_0^\rho w_t dt$ to L^u or equivalently by subtracting $\mu \int_0^\rho u_t dt$ from L^w. Let G_{zs} be the zero-sum game in which the source maximizes $L(x, \rho, u, w)$ and the jammer minimizes it. Note that the addition and subtraction of these additional terms have not changed the Nash equilibrium of the multi-criteria game, because the first term does not depend on the control of the source and the second term does not depend on the control of the jammer, that is[3]

$$\max_u L(x, u, w) = \max_u L^u(x, u, w) + \theta \int_0^\rho w_t dt = [\max_u L^u(x, u, w)] + \theta \int_0^\rho w_t dt,$$

$$\min_w L(x, u, w) = \min_w L^w(x, u, w) - \mu \int_0^\rho u_t dt = [\min_w L^w(x, u, w)] - \mu \int_0^\rho u_t dt \,,$$

where the first one holds for all open-loop w and the second one for all open-loop u. Then clearly if (u^*, w^*) is an open-loop Nash equilibrium for $(L^u, -L^w)$

[3] This argument is not valid if the control policies depend on the state, that is if they are for example feedback policies.

where both players are maximizers, it is also an open-loop Nash equilibrium for $(L, -L)$, and hence an open-loop saddle-point of L (that is game G_{zs}). Likewise, any open-loop saddle-point solution of the zero-sum game G_{zs} is also an open-loop Nash equilibrium of $(L, -L)$, and hence of $(L^u, -L^w)$.

2.4 The Constrained Problem: Energy Constraints

We introduce the constrained game as finding the saddle-point of $J(x, \rho, u, w)$ subject to the following constraints on the source and the jammer controls

$$\int_0^\rho u_t dt \leq D_s, \quad \text{and} \quad \int_0^\rho w_t dt \leq D_j, \quad \text{respectively.}$$

This constrained problem turns out to be related to the open-loop zero-sum game in the following sense: (u^*, w^*) is a saddle-point if and only if u^* is optimal against w^* and *vice versa.* By the Karush-Kuhn-Tucker (KKT) conditions, there exists $\mu \geq 0$ such that u^* is optimal against w^* if u^* achieves the maximum of $L^\mu(x, u, w^*)$, where

$$L^\mu(x, u, w) = J(x,\ u, w) - \mu(\int_0^\rho u_t dt - D_s).$$

Similarly, there exists $\theta \geq 0$ such that w^* is optimal against u^* if w^* achieves the minimum of $L^\theta(x, u^*, w)$, where

$$L^\theta(x, u, w) = J(x, u, w) + \theta(\int_0^\rho w_t dt - D_j)$$

Hence (u^*, w^*) is an equilibrium in the constrained problem if it is a saddle-point in the zero-sum game

$$L(x, u, w) = J(x, u, w) - \mu \int_0^\rho u_t dt + \theta \int_0^\rho w_t dt \quad \text{for some } \mu \text{ and } \theta.$$

As indicated earlier, we will take the success delivery probability as a performance measure, so that

$$J(x, u, w) = \int_0^\rho \nu x_t dt\,,$$

where x_t is generated by (1), with initial state x_0.

3 The Static Game

We first restrict the analysis to u and w that are constants in time, in which case we have the unique solution of (1), with initial state x_0, given by

$$x_t = n + (x_0 - n)\exp(-\lambda\kappa t) \tag{4}$$

where $\kappa := u(1-w)$. Then the objective function of the equivalent zero-sum game can be expressed as:

$$L(x_0,u,w) = \nu \int_0^{\rho} x_t dt - \rho(\mu u - \theta w) = -\nu(n-x_0)F(\kappa) + \nu n\rho - \rho(\mu u - \theta w)\,,$$

where

$$F(\kappa) := \frac{1-\exp(-\kappa\lambda\rho)}{\lambda\kappa}.$$

With F' denoting the first derivative of $F(\kappa)$ with respect to κ, and F'' its second derivative, we readily have, for $\kappa \in (0,1]$:

$$\begin{aligned} F'(\kappa) &= \frac{-1+(1+\kappa\lambda\rho)\exp(-\kappa\lambda\rho)}{\lambda\kappa^2} \\ F''(\kappa) &= \frac{-\kappa^3\lambda^2\rho^2\exp(-\kappa\lambda\rho)+2\kappa-2\kappa(1+\kappa\lambda\rho)\exp(-\kappa\lambda\rho)}{\lambda\kappa^4} \\ &= \frac{2-(2+2\kappa\lambda\rho+\kappa^2\lambda^2\rho^2)\exp(-\kappa\lambda\rho)}{\lambda\kappa^3} \\ &> \frac{2-2\exp(\kappa\lambda\rho)\exp(-\kappa\lambda\rho)}{\lambda\kappa^3} = 0 \end{aligned}$$

and for $\kappa = 0$,

$$F'(0) = -\frac{\lambda\rho^2}{2}, \quad F''(0) = \frac{\lambda^2\rho^3}{3}.$$

Hence $F(\kappa)$ is strictly convex in κ, on $[0,1]$, which implies that $L(x_0,u,w)$ is strictly convex in $\kappa = u(1-w)$ as long as $x_0 < n$. Since the additional terms in L that depend on u and w are linear, this readily implies that for each $x_0 < n$ $L(x_0,u,w)$ is strictly concave-convex in the pair (u,w) on $(0,1] \times [0,1)$, and concave-convex on the closed square $[0,1] \times [0,1]$. Hence, we have a concave-convex game defined on a closed and bounded subset of a finite-dimensional space, which is known to admit a saddle-point solution [3].This result is now captured in the following theorem, which also addresses the uniqueness and characterization:

Theorem 1. *Assume throughout that $x_0 < n$. Then:*

(i) The static zero-sum game has a saddle-point on $[0,1] \times [0,1]$, and it is unique.
(ii) If $\nu(n-x_0)\rho\lambda \le 2\mu$, $(u^* = 0, w^* = 0)$ is the unique saddle-point.
(iii) The game cannot have a saddle-point with $w = 1$.
(iv) If $\nu(n-x_0)\rho\lambda > 2\mu$, the unique saddle-point is in $(0,1] \times (0,1)$.

Proof:

i) As stated prior to the statement of the theorem, existence follows from a standard result in game theory. since we have a concave-convex game. Uniqueness will follow from the proofs of parts (ii) and (iv) below, carried out separately in two regions of the parameter space.

ii) Let $M(u) := -\nu(n - x_0)F(u) - \rho\mu u$, and note that $M'(u) = -\nu(n - x_0)F'(u) - \mu\rho$. Using the earlier expression for $F'(0)$, $M'(0) = \nu(n - x_0)\rho^2\lambda/2 - \mu\rho$ and thus $M'(0) \leq 0$ under the given condition. Further, since $F''(u) > 0$ for all u, $F'(u)$ is an increasing function of u and hence $M'(u)$ is decreasing for all u, which means that

$$M'(0) < 0 \text{ implies } M'(u) < 0 \text{ for all } u > 0.,$$

and hence that $M(u)$ attains its maximum uniquely at $u = 0$. This means that $u = 0$ is the unique best response to $w = 0$. Further, since $L(x_0, 0, w) = \nu x_0 \rho + \rho\theta w$, the unique minimizing response to $u = 0$ on $[0, 1]$ is $w = 0$. Hence, $(0, 0)$ is a saddle-point solution, and by the ordered interchangeability property of multiple saddle-points and the uniqueness of responses in this case, there can be no other saddle-point.

iii) This readily follows from the observation that the unique maximizing response to $w = 1$ is $u = 0$ while the unique minimizing response to $u = 0$ is $w = 0$. Hence $w = 1$ cannot be part of a saddle-point.

iv) From part (i), we already know that there exists a saddle-point under this condition. Suppose that the saddle-point is not unique, and let (u^*, w^*) and $(\tilde{u}, \tilde{w})$ be two such solutions. By ordered interchangeability of multiple saddle-points, $(\tilde{u}, w^*)$ and $(u^*, \tilde{w})$ are also saddle-point solutions. We know from part (iii) that $w^* \neq 1$, $\tilde{w} \neq 1$, and hence under each of them the objective function is strictly concave in u, which implies that the only way for both u^* and $\tilde{u}$ to be optimal responses to w^* (as well as $\tilde{w}$)) is if they are equal. Hence, u^* has to be unique. Now, if $u^* \neq 0$, then $L(x_0, u^*, w)$ is strictly convex in w, and hence the optimal response by the jammer is unique; hence $w^* = \tilde{w}$ if $u^* \neq 0$. This then leaves out only the case $u^* = 0$ not covered. We already know from the proof of part (ii) that the unique minimizing response to $u = 0$ on $[0, 1]$ is $w = 0$, and under the given condition $u = 0$ is not a maximizing response to $w = 0$ since $M'(0) > 0$. Hence, $u^* = 0$ is ruled out. What we then have is that the saddle-point solution (u^*, w^*) is unique, and necessarily $u^* \in (0, 1]$ and $w^* \in (0, 1)$. □

We now further elaborate on the case when the saddle-point is inside the square, which we know from part (iv) of the Theorem that it happens only when the condition $\nu(n - x_0)\rho\lambda > 2\mu$ holds. We also know that for an inner saddle-point solution, since the game kernel is strictly concave-convex, and jointly continuously differentiable, a necessary and sufficient condition is satisfaction of the stationarity conditions. Toward this end, let

$$K(\kappa) := -\nu(n - x_0)\frac{dF(\kappa)}{d\kappa} = -\nu(n - x_0) \times \frac{-1 + (1 + \kappa\lambda\rho)\exp(-\kappa\lambda\rho)}{\lambda\kappa^2}$$

Then, the inner saddle-point solution (u^*, w^*) uniquely solves

$$\frac{dL(x, u^*, w^*)}{du} = 0, \quad \frac{dL(x, u^*, w^*)}{dw} = 0\,,$$

which can be written as

$$K(\kappa)(1-w)-\rho\mu=0, \quad -K(\kappa)u+\rho\theta=0.$$

We thus conclude that $\theta(1-w^*)=\mu u^*$, which leads to

$$\kappa^*=u^*(1-w^*)=(u^*)^2\mu/\theta \quad \text{or} \quad u^*=\sqrt{\theta\kappa^*/\mu}$$

Finally, substituting this into the second stationarity condition above leads to a single equation for κ^* as below: $K(\kappa^*)\sqrt{\kappa^*}=\rho\sqrt{\theta\mu}$, which we know admits a unique solution in $(0,1)$. u^* and w^* are then obtained from

$$u^*=\sqrt{\theta\kappa^*/\mu} \quad \text{and} \quad w^*=1-(\mu/\theta)u^*\,.$$

4 The Dynamic Game

We now return to the original dynamic game, and discuss derivation of the equilibrium solution, first for the case of open-loop information and following that for the closed-loop feedback case.

4.1 Open-Loop Information

As discussed earlier, in the open-loop case, every Nash equilibrium of the original differential game is also saddle-point equilibrium of a related strategically equivalent zero-sum differential game. Following the standard derivation of open-loop saddle-point solution [3], we have the single Hamiltonian

$$H(u,w;x,p)=-\mu u+\theta w+\nu x+pu(1-w)\lambda(n-x), \tag{5}$$

which will be maximized over $u\in[0,1]$ and minimized over $w\in[0,1]$. Here p is the co-state variable, which satisfies the associated co-state equation:

$$\dot{p}=-\frac{\partial H}{\partial x}=pu(1-w)\lambda-\nu\,,\;\; p(\rho)=0\,, \tag{6}$$

which constitutes a two-point boundary value problem along with the original state equation

$$\dot{x}=u(1-w)\lambda(n-x)\,. \tag{7}$$

The source will be maximizing H, and the jammer will be minimizing the same, and if exists we seek a saddle-point solution (u^*,w^*) for the game, which necessarily will also be a saddle-point solution for the Hamiltonian for each t, that is

$$\max_{u\in[0,1]} H(u,w^*;x,p)=\min_{w\in[0,1]} H(u^*,w;x,p)=H(u^*,w^*;x,p)\,.$$

Now, maximizing $H(u,w;x,p)$ over $u \in [0,1]$ for each $w \in [0,1]$, and minimizing the same over $w \in [0,1]$ for each $u \in [0,1]$ we obtain the complete set of solutions:

$$\arg\max_{u\in[0,1]} H(u,w;x,p) = \begin{cases} 1 & \text{if} \quad p(1-w)\lambda(n-x) > \mu \\ 0 & \text{if} \quad p(1-w)\lambda(n-x) < \mu \\ [0,1] & \text{if} \quad p(1-w)\lambda(n-x) = \mu \end{cases} \tag{8}$$

$$\arg\min_{w\in[0,1]} H(u,w;x,p) = \begin{cases} 1 & \text{if} \quad pu\lambda(n-x) > \theta \\ 0 & \text{if} \quad pu\lambda(n-x) < \theta \\ [0,1] & \text{if} \quad pu\lambda(n-x) = \theta \end{cases} \tag{9}$$

Since $p(\rho) = 0$, the unique saddle-point of the Hamiltonian at the terminal time $t = \rho$ is clearly $u^* = w^* = 0$. And clearly, by continuity, the same holds in some left neighborhood of ρ. Integrating the co-state equation backwards from $t = \rho$ with $u = w = 0$, we obtain $p(t) = \nu(\rho - t)$. Note that $u^*(t) = w^*(t) = 0$ is a valid solution as long as

$$p(t)\lambda(n - x_t) < \mu\,, \tag{10}$$

and the first time (in retrograde time) this is violated will determine the switch time from $u^* = 0$ to some other action for the source. Further note that it is the inequality associated with the source and not the one associated with the jammer that will determine the switching time (in retrograde time) because the LHS of the inequality associated with the jammer, (9), is *zero* as long as $u = 0$. We denote this switching time by $\bar{t}_s$,

$$\bar{t}_s := \sup\{t \leq \rho : \nu(\rho - t)\lambda(n - x_t) < \mu\}$$

When $\theta > \mu$, there exists another threshold t_s such that during the interval $[t_s, \bar{t}_s]$,

$$\mu \leq p(t)\lambda(n - x_t) < \theta$$

and hence from (8) and (9) $u^* = 1$ while $w^* = 0$ during $[t_s, \bar{t}_s]$.

The above two switch times also depend on x_t and $p(t)$, which in turn are generated under the players' actions in the earlier stages of the game. Another observation worth pointing out is that it is not possible for $w^*(t) = 1$ for any t, because this would imply that $u^*(t) = 0$, which in turn implies that $w^*(t) = 0$, a contradiction.

All this reasoning leads to the following theorem which captures the saddle-point solution to the open-loop differential game.

Theorem 2. *(i) If $\theta > \mu$, there exist two switch times $t_s, \bar{t}_s$ with $t_s < \bar{t}_s \leq \rho$ and there exists a saddle-point solution given by*

$$u(t) = \begin{cases} \frac{\theta}{m(t)} & \textit{when} \;\; t < t_s \\ 1 & \textit{when} \;\; t_s < t < \bar{t}_s \\ 0 & \textit{when} \;\; t \geq \bar{t}_s \end{cases} \; \textit{and} \; w(t) = \begin{cases} 1 - \frac{\mu}{m(t)} & \textit{when} \;\; t < t_s \\ 0 & \textit{when} \;\; t_s < t < \bar{t}_s \\ 0 & \textit{when} \;\; t \geq \bar{t}_s. \end{cases}$$

(ii) When $\theta \le \mu$, there exists a single switch time t_s such that for $t > t_s$, the saddle-point solution dictates both players to play $u^*(t) = w^*(t) = 0$, and for $t < t_s$

$$u^*(t) = \frac{\theta}{m(t)}, \qquad w^*(t) = 1 - \frac{\mu}{m(t)} \text{ where}$$
$$m(t) := p(t)\lambda(n - \xi(t)),$$

with p and ξ solving the coupled set of mixed boundary differential equations:

$$\dot{\xi} = \frac{\theta\mu}{p^2\lambda(n-\xi)}, \ \ \xi(0) = x_0 \ ; \quad \dot{p} = \frac{\theta\mu}{p\lambda(n-\xi)^2} - \nu, \ \ p(t_s) = \nu(\rho - t_s)$$

and t_s is solved from $m(t_s) = \mu$.

Proof: Please see the Appendix, where also the computation of the two switch times, $t_s, \bar{t}_s$, are discussed. □

Remarks: The following are some observations on the saddle-point solution (equivalently Nash solution) obtained in Theorem 2:

- It is an open-loop Nash equilibrium, i.e., the policies obtained depend only upon the time t elapsed from the birth of the message and not on the state x, the number of already infected messages.
- When $\mu > \theta$, i.e., when the power constraint on the source is higher than that on the jammer, the jammer and source are active during the same period and switch off at the same time threshold (t_s of Theorem 2). In a way the jammer is dominating here as it has bigger power resources and hence keeps jamming whenever the source is active.
- When $\theta > \mu$, i.e., when the power constraint on the jammer is high, the jammer is forced to switch off even when the source is active (at time threshold t_s of Theorem 2). The source continues being active for a longer time, until time threshold $\bar{t}_s$. In fact after t_s, the policy is similar to situation without jammer ([1]): the source always infects the contacted mobiles till the threshold $\bar{t}_s$ after which it never infects any further mobiles.
- During the initial time interval, i.e., in the interval $[0, t_s]$ (when the policies are equalizing in nature), the source's probability of transmitting is high whenever the jammer's probability of jamming is low and *vice versa*.

4.2 Closed-Loop Feedback Information

Here we have to stay with the non-cooperative game framework, and seek for Nash equilibria (NE). Let V^u and V^w be the value functions for the two players, where again player u is maximizer and Player w is minimizer. Assuming that these value functions are continuously differentiable jointly in (x, t) (they can even be piecewise continuously differentiable solutions with possibly a finite number of discontinuities in the derivative), the associated HJB equations are ([3]):

$$\frac{\partial V^u}{\partial t} + \max_{u\in[0,1]} \left[\tfrac{\partial V^u}{\partial x} u(1-w^*)\lambda(n-x) + \nu x - \mu u\right] = 0 \quad (11)$$

$$\frac{\partial V^w}{\partial t} + \min_{w\in[0,1]} \left[\tfrac{\partial V^w}{\partial x} u^*(1-w)\lambda(n-x) + \nu x + \theta w\right] = 0 \quad (12)$$

with boundary conditions $V^u(\rho, x) \equiv V^w(\rho, x) \equiv 0$ where (u^*, w^*) is a NE. The corresponding feedback policies are:

$$u^*(x,t) = \arg\max_{u\in[0,1]} \left[\frac{\partial V^u}{\partial x} \lambda u(1-w^*)(n-x) - \mu u\right] \quad (13)$$

$$w^*(x,t) = \arg\min_{w\in[0,1]} \left[\frac{\partial V^w}{\partial x} \lambda u^*(1-w)(n-x) + \theta w\right] \quad (14)$$

Using these two dynamic programming equations, one can easily establish the following two lemmas.

Lemma 1. *Any feedback Nash equilibrium (NE) will feature a jammer policy taking values only in the semi-open interval* $[0, 1)$.

Proof: If w^* was 1, at some (t, x), then, from equation (13) the corresponding optimal controller would be $u^* = 0$. This in turn implies from equation (14) that $w^* = 0$, which is a contradiction. □

Lemma 2. *If* $\nu\lambda(n - x_0)\rho < \mu$ *then at NE,* $u^* = w^* \equiv 0$*, i.e., the optimal policies of both the jammer and the source are to never jam/transmit.*

Proof: From the pair of HJB equations (11) and (12), if it is possible to make the point-wise optimizers in both the Hamiltonians equal to zero, the solution of both PDEs would have been $V^u(x, t) = V^w(x, t) = \nu x(\rho - t)$ for all x, t. And this is exactly the case under the given hypothesis as for any $x \in [x_0, n]$, $t \in [0, \rho]$ and for any $w \in [0, 1]$,

$$\frac{\partial V^u}{\partial x} u(1-w)\lambda(n-x) = \nu(\rho-t)(1-w)\lambda(n-x) < \nu\rho\lambda(n-x_0) < \mu$$

and hence $u^* \equiv 0$, and thus from equation (14) $w^* \equiv 0$ □

The first lemma rules out the possibility of pure-strategy NE with nonzero jammer policy.[4] What this leaves as possibility is a NE which is 1) completely inner (or completely mixed NE, i.e., where both players' policies take values in the open interval $(0, 1)$) for some states and time and 2) with $w^* = 0$ for the rest of the states and time. Lemma 2 gives the condition under which the second

[4] Again, by *pure strategy* here we mean one that does not take the extreme values 0 or 1, for both players. A *mixed-strategy* NE in this context is one where at least one player's policy takes values in the open interval $(0, 1)$ for some time and state.

situation always (for all states and time) happens. We now consider the case in which this condition is negated, i.e., henceforth we assume that $\nu\lambda(n-x_0)\rho > \mu$. We show the existence of a switching time until which the first possibility occurs and beyond which the second scenario (that of $w^* = 0$) occurs.

Let us consider the first possibility. This would happen if the policies would actually be *equalizer rules*, with $u^* \in (0,1)$ making the expression to be minimized on the right-hand-side of (14) independent of w, and simultaneously $w^* \in (0,1)$ making the expression to be maximized on the right-hand-side of (13) independent of u. Such a (u^*, w^*) would be the solution of the fixed point equations:

$$\frac{\partial V^u}{\partial x}\lambda(1-w^*)(n-x(t)) = \mu \tag{15}$$

and

$$\frac{\partial V^w}{\partial x}\lambda u^*(n-x(t)) \;=\; \theta. \tag{16}$$

If there exist such solutions, then the HJB equations will be simplified to

$$\begin{aligned} \frac{\partial V^u}{\partial t} + \nu x = 0\;, \quad & \frac{\partial V^w}{\partial t} + \nu x + \theta 1_{\{u^*>0\}} = 0; \\ V^u(\rho, x) = 0 \quad = \quad & V^w(\rho, x) \quad \text{for all } x. \end{aligned} \tag{17}$$

The simplification in the second PDE is obtained using (16). For future reference we note that we would have arrived at these PDEs if both u and w were taken to be identically zero–a property we will have occasion to utilize shortly.

One can easily solve and obtain the solution $V^u(t,x) = \nu x(\rho - t)$ and hence that $\partial V^u/\partial x = \nu(\rho - t)$. Hence the objective function in (13) is non-positive for all $t > t_c(x)$, where

$$t_c(x) := \frac{\rho\nu\lambda(n-x) - \mu}{\nu\lambda(n-x)} \quad \text{and hence} \quad u^*(x,t) = 0 \text{ for all } t \geq t_c(x).$$

This in turn yields from equation (14) that $w^*(x,t) = 0$ for all $t \geq t_c(x)$. Now the second PDE in (17) can be solved:

$$V^w(t,x) = \theta(t_c(x) - t)1_{\{t<t_c(x)\}} + \nu x(\rho - t).$$

Both PDEs can be brought to the above simplified form and hence the simplified solutions of the fixed point equations (15) and (16) can be obtained for all $t \leq t_c(x)$. By definition of t_c, whenever $t < t_c(x)$, the fixed point equation (15) can be satisfied with a $w^* \in (0,1)$ if we assume $\mu/\theta - \lambda(\rho - t_c(x_0)) > 1$ as then

$$\begin{aligned} \frac{\partial V^w(t,x)}{\partial x}\lambda(n-x) &= \left(\nu(\rho - t) - \frac{\mu\theta}{\nu\lambda(n-x)^2}\right)\lambda(n-x) \\ &> \mu - \frac{\mu\theta}{\nu(n-x)} = \theta\left(\frac{\mu}{\theta} - \lambda(\rho - t_c)\right) > \theta \end{aligned}$$

for all (t,x) with $t < t_c(x)$. Under this assumption, the fixed point equation (16) can also be satisfied with $0 < u^* < 1$. Thus we have

Theorem 3. *Under the assumption $\mu/\theta - \lambda(\rho - t_c(x_0)) > 1$, the closed-loop mixed strategy NE exists with the optimal state trajectory given as the solution of the following ODE:*

$$\dot{x} = f(x,t) \text{ with } f(x,t) := \frac{\mu\theta(n-x)}{(\rho-t)(\nu^2\lambda(\rho-t)(n-x)^2 - \mu\theta)} 1_{\{t \leq t_c(x)\}}.$$

and the optimal controls are given by,

$$u^*(t) = \frac{\theta 1_{\{t \leq t_c(x_t)\}}}{\lambda(n-x_t)\left(\nu(\rho-t) - \frac{\mu\theta}{\nu\lambda(n-x_t)^2}\right)}$$
$$w^*(t) = \left(1 - \frac{\mu}{\nu(\rho-t)\lambda(n-x_t)}\right) 1_{\{t \leq t_c(x_t)\}}. \qquad \square$$

Optimal controls for larger values of $\boldsymbol{\theta}$. Now we consider the cases that may not satisfy $\mu/\theta - \lambda(\rho - t_c(x_0)) > 1$. We may not find a $w^* \leq 1$ that satisfies the fixed point equation (16) for all $t \leq t_c$. Let us start with the extreme case: assume θ is very large ($\theta >> \mu$) such that the fixed point equation can not be satisfied for all (x,t) with $t \leq t_c(x)$. In this case, one can easily verify that the jammer's optimal strategy is to never jam (i.e., $w^* \equiv 0$) and the source's optimal policy is

$$\bar{u}(t,x) := 1_{\left\{\lambda(n-x)\frac{d\bar{V}^u}{dx} > \mu\right\}}$$

where $\bar{V}^u$ is the solution of the PDE

$$\frac{d\bar{V}^u}{dt} + \nu x + \left(\lambda(n-x)\frac{d\bar{V}^u}{dx} - \mu\right)\bar{u}(t,x); \quad \bar{V}^u(\rho,x) = 0 \tag{18}$$

and the optimal state trajectory x^* is the solution of $\dot{x}(t) = \lambda(n-x)\bar{u}(t,x)$. The corresponding Hamiltonian PDE for the jammer will be

$$\frac{d\bar{V}^w}{dt} + \nu x + \bar{u}(t,x)\lambda(n-x)\frac{d\bar{V}^w}{dx} = 0. \tag{19}$$

Thus, a sufficient condition for the optimal jammer policy to be *zero* is that

$$\frac{d\bar{V}^w}{dt}\lambda(n-x)\bar{u}(t,x) < \theta \text{ for all } (x,t). \tag{20}$$

This condition can only be verified on numerical examples.

Remark: The PDE solutions $\bar{V}^u$, $\bar{V}^w$ are both equal to $\nu x(\rho - t)$ for all (x,t) with $t > t_c(x)$ (the boundary condition is at left boundary $t = \rho$). Thus, $\bar{u}(t,x) = \bar{w}(t,x) = 0$ for all (x,t) with $t \leq t_c(x)$. $\square$

Continuing further, consider now the case when (20) is not true for some (x,t). Then there exists an $0 \leq \bar{t}_c(x) \leq t_c(x)$ such that,

$$\bar{t}_c(x) = \inf_{t < t_c(x)} \left\{\frac{d\bar{V}^w}{dx}\bar{u}(t,x)\lambda(n-x) > \theta\right\}. \tag{21}$$

Let $\tilde{V}^u$, $\tilde{V}^w$ represent the solutions of the PDEs,

$$\frac{d\tilde{V}^u}{dt} + \nu x + \left(\lambda(n-x)\frac{d\tilde{V}^u}{dx} - \mu\right)\bar{u}(t,x)1_{\{\bar{t}_c(x)\leq t<t_c(x)\}} = 0; \tilde{V}^u(\rho,x) = 0$$

$$\frac{d\tilde{V}^w}{dt} + \nu x + \bar{u}(t,x)\lambda(n-x)\frac{d\tilde{V}^w}{dx}1_{\{\bar{t}_c(x)\leq t<t_c(x)\}} + \theta 1_{\{t<\bar{t}_c(x)\}} = 0; \tilde{V}^w(\rho,x) = 0$$

Then the optimal controls will be given by,

$$u_t^* := \tilde{u}(t,x_t^*) \text{ with } \tilde{u}(t,x) := \bar{u}(t,x)1_{\{t\geq\bar{t}_c(x)\}} + 1_{\{t<\bar{t}_c(x)\}}\frac{\theta}{\frac{d\tilde{V}^w}{dx}\lambda(n-x)} \quad (22)$$

$$w_t^* := \tilde{w}(t,x_t^*) \text{ with } \tilde{w}(t,x) := 1_{\{t<\bar{t}_c(x)\}}\left(1 - \frac{\mu}{\frac{d\tilde{V}^u}{dx}\lambda(n-x)}\right) \quad (23)$$

where x_t^* is now the solution of the ODE,

$$\dot{x} = \lambda(n-x)\tilde{u}(t,x)(1-\tilde{w}(t,x)).$$

Remark. The PDE solutions ($\tilde{V}^u$, $\tilde{V}^w$) equal to ($\bar{V}^u$, $\bar{V}^w$) for all (x,t) with $t > \bar{t}_c(x)$. Further, the solution can be obtained numerically. □

Remarks: The following are some observations on the closed-loop feedback NE policies:

– The solution is a genuine closed-loop feedback NE, i.e., the policies depend both upon the time t elapsed from the birth of the message and upon the state x, the number of already infected messages.
– The nature of these controls is exactly the same as that in the case of open loop controls.
  - When $\mu/\theta - \lambda(\rho - t_c(x_0)) > 1$ (the case of Theorem 3), i.e., when the power constraint on the source is higher than that on the jammer, the jammer and source are active during the same period and switch off at the same time threshold $(t_c(x),)$. The jammer is dominating even in closed-loop strategies, as it has bigger power resources and hence keeps jamming whenever the source is active. The switch off threshold t_c, unlike in the case of open loop strategies, also depends upon the number of infected mobiles, x.
  - When the power constraint on the jammer is high, the jammer is forced to switch off even when the source is active (at time threshold $\bar{t}_c(x)$ given by (21)). The source continues being active for a longer time, till time threshold $t_c(x)$. In fact after $\bar{t}_c(x)$, the policy is similar to situation without jammer ([1]): the source always infects the contacted mobiles till the threshold $\bar{t}_s$ after which it never infects any further mobiles.

5 Numerical Examples

We now compute the optimal policies obtained in the previous section for the closed-loop case for some numerical examples and verify the same using HJB equations. For example, to verify that (u^*, w^*) is a NE, we obtain a second set of PDEs by replacing the optimal value in the Hamiltonians of (11), (12) with the values evaluated at u^*, w^* respectively. We compare the solutions of this new set of PDEs with that of the HJB solutions.

The two sets of PDE solutions are compared in Figure 1. In Figure 1 the thick lines represent the solution of the simplified HJB equations (17) while thin dotted curves represent the corresponding ones of the PDEs with the optimal policies. We note that the two trajectories almost match, thereby reinforcing the existence of Mixed strategy NE. We also plot the optimal policies as a function of time in Figure 2. It is interesting to observe that the jammer jams with higher probability in the beginning while the probability, with which the source transmits, increases with time till it reaches the switching threshold. This behavior could be because, whenever the jammer jams with large probability, the source better attempt with smaller probability and use the resources at some other time point. Further both the jammer and the source do not transmit after the switching threshold. Note that this threshold is given by $\inf_t\{t > t_c(x_t^*)\}$ where x_t^* is the optimal state trajectory.

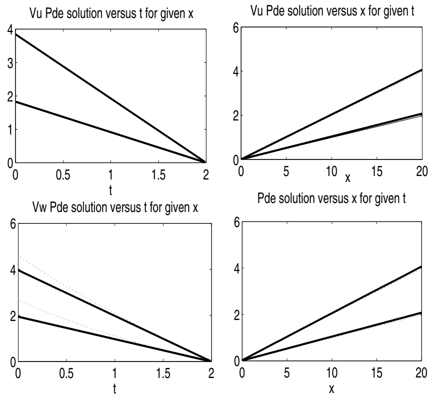

Fig. 1. HJB solutions versus PDE Solutions at the computed NE

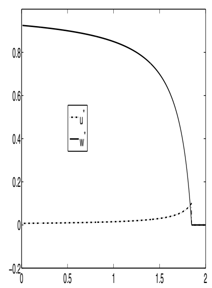

Fig. 2. Optimal Controls

We conclude this section with an example which considers two values of θ in Figures 3 4, 5 and 6. For $\theta = 9$, the condition of Theorem 2 is satisfied and hence the optimal control is given by Theorem 2. For $\theta = 200$, we compute the optimal policies using the procedure explained and verify the same by showing that the optimal policies satisfy the HJB equations given earlier in this section. We plot the optimal policies for both cases in Figure 4 while both optimal state trajectories are plotted in Figure 3. In this example, the switching period $t_c(x_0) \approx 1981$ is very close to $\rho = 2000$ and hence in both cases the source is active for almost all the time. We notice that with large θ, there exists another switching time $\bar{t}_c \approx 1698$ beyond which the source is completely active, while the jammer is completely inactive. For small θ $\bar{t}_c$ coincides with t_c. And before this switching time the optimal policies are always mixed in nature (*equalizer rules*) for all the cases. We finally verify that the optimal policies satisfy the HJB equations and the corresponding PDE solution is plotted in Figure 5 for large θ. Both the switching periods t_c, $\bar{t}_c$, when $\theta = 200$, are plotted as functions of x in Figure 6. We also plot the two optimal state trajectories in Figure 3. For larger values of θ the jammer is constrained more and hence the infected population size at any point in time is bigger with larger θ.

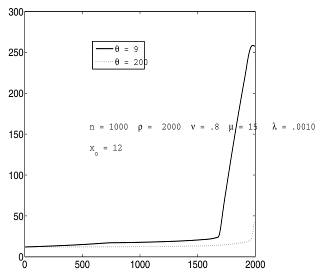

Fig. 3. Example 2: Optimal state trajectory

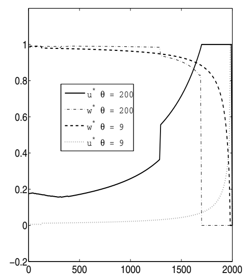

Fig. 4. Example 2: Optimal Controls

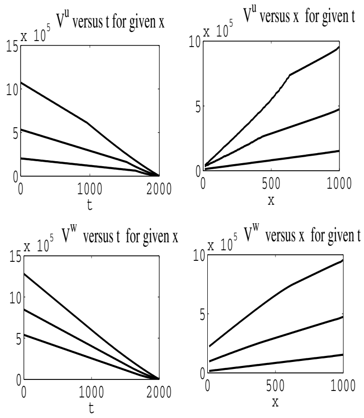

Fig. 5. PDE Solutions: When $\mu/\theta > 1 + \lambda(\rho - t_c(x_0))$

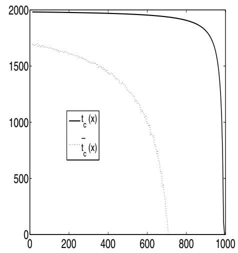

Fig. 6. With θ large, $t_c(x)$, $\bar{t}_c(x)$ versus x

6 Conclusions

We have considered a multi-criteria control problem that arises in a delay tolerant network with two adversarial controllers: the source and the jammer. The source's objective was to choose transmission probabilities so as to maximize the probability of successful delivery of some content to the destination within a given time interval, and the jammer's objective was to cause collisions. We considered two types of information structures; the closed loop and the open loop. In the closed loop structure we assume both the jammer and the source have the knowledge of the current number of mobiles with a copy of the message and in this case the game is a genuine nonzero-sum differential game. In the open loop structure they do not have such knowledge and the game becomes strategically equivalent to a zero-sum differential game. The structure of the policies are similar for both types of information structures. In both cases, the optimal policies have two or one switch time(s) depending upon the energy constraints of the source and the jammer. When the jammer has a tighter constraint on its energy resources than the source, the policies have two switch times. Before the first switch time, both the source and jammer policies are inner (i.e., the transmission probabilities are not one of the extreme cases, 0 or 1) and are given by equalizer policies. After the first switch time, the jammer switches off and the source continues transmitting at maximum probability and after the second switch time, both the source and jammer are off. When the source has a tighter constraint on its energy resources than the jammer, there exists only one switch time before which both use inner equalizer policies and after which both are switched off.

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Appendix: Proof of Theorem 2

Case 1: $\theta > \mu$: In the interval $[0, t_s]$, both u^* and w^* are simultaneously inner (i.e., have values in (0,1)). For this to happen, we need equalizer policies, i.e., u^* should make Hamiltonian (5) independent of w and w^* should make the same Hamiltonian independent of u simultaneously. Thus,

$$u(t) = \frac{\theta}{m(t)} \text{ and } w(t) = 1 - \frac{\mu}{m(t)} \text{ where } m(t) := p(t)\lambda(n - x(t)).$$

In the above, $p(t)$ and $x(t)$ are the co-state and state trajectories for the saddle-point that we are constructing and these are obtained in retrograde while constructing the saddle-point policy. The equalizer policies must be in open interval $(0, 1)$ and hence for all $t < t_s$,

$$m(t) > \max\{\theta, \mu\} = \theta.$$

Thus t_s is given by

$$t_s = \inf\{t : m(t) \le \theta\}$$

or by continuity it satisfies $m(t_s) = \theta$. Substituting the policies back in the state and co-state equations, the state and co-state trajectories in the interval $[0, t_s]$ are obtained by solving the ODE's:

$$\dot{x} = \frac{\mu\theta}{\lambda p^2(n - x)} \qquad \text{with } x(0) = x_0 \tag{24}$$

$$\dot{p} = \frac{\mu\theta}{p\lambda(n - x)^2} - \nu \qquad \text{with } p(t_s) = p_{t_s}. \tag{25}$$

where the expression for p_{t_s} will be given shortly.

In the interval $[t_s, \bar{t}_s]$, $u^* = 1$ and $w^* = 0$ and $\mu < m(t) \le \theta$ and hence the co-state trajectory can be solved for this interval as

$$p(t) = c(\bar{t}_s)e^t + \nu \text{ with } c(t) := e^{-t}\left(p_{\bar{t}_s} - \nu\right) \text{ for all } t \in [t_s, \bar{t}_s], \tag{26}$$

where the expression for $p_{\bar{t}_s}$ will also be given shortly. Now p_{t_s} is calculated in terms of $p_{\bar{t}_s}$ as:

$$p_{t_s} = c(\bar{t}_s)e^{t_s} + \nu = e^{t_s - \bar{t}_s}\left(p_{\bar{t}_s} - \nu\right) + \nu.$$

The state trajectory in this interval would be,

$$x(t) = n - (n - x(t_s))e^{-\lambda(t-t_s)} \text{ for all } t \in [t_s, \bar{t}_s]. \tag{27}$$

For all $t > \bar{t}_s$ $u^*(t) = 0 = w^*(t)$. Thus solving backwards,

$$p(t) = \nu(\rho - t) \text{ and } x(t) = x(\bar{t}_s) \text{ for all } t \in [\bar{t}_s, \rho].$$

Thus,

$$p_{\bar{t}_s} = \nu(\rho - \bar{t}_s) \text{ and hence } p_{t_s} = e^{t_s - \bar{t}_s}\left(\nu(\rho - \bar{t}_s) - \nu\right) + \nu. \tag{28}$$

Further,

$$m(t) = \lambda\nu(n - x(\bar{t}_s))(\rho - t) \text{ for all } t \in [\bar{t}_s, \rho]$$

and hence $m(t)$ is strictly decreasing for t beyond $\bar{t}_s$, i.e., in the interval $[\bar{t}_s, \rho]$. It is possible that there can be no $t \leq \rho$ for which $m(t) = \mu$ and in this case we define $\bar{t}_s = \rho$. In the other case we define $\bar{t}_s$ as the time which satisfies the equation $m(\bar{t}_s) = \mu$, i.e.,

$$\lambda\nu(n - x(\bar{t}_s))(\rho - \bar{t}_s) = \mu.$$

From (27),

$$x(\bar{t}_s) = n - (n - x(t_s))e^{-\lambda(\bar{t}_s - t_s)}$$

and hence

$$\lambda\nu(n - x(t_s))e^{-\lambda(\bar{t}_s - t_s)}(\rho - \bar{t}_s) = \mu. \tag{29}$$

From (26) and (27) for all $t \in [t_s, \bar{t}_s]$,

$$\begin{aligned} m(t) &= \lambda c(\bar{t}_s)e^{(1-\lambda)t}(n - (x_{t_s}))e^{\lambda t_s} + \lambda\nu(n - x(t_s))e^{-\lambda(t-t_s)} \\ &= \lambda\nu(n - x(t_s))\left(e^{-(\bar{t}_s - t)}(\rho - \bar{t}_s - 1) + 1\right)e^{-\lambda(t-t_s)} \end{aligned} \tag{30}$$

Further at t_s, $m(t_s) = \theta$ and hence we get the second equation in terms of t_s and $\bar{t}_s$:

$$m(t_s) = \lambda\nu(n - x(t_s))\left((\rho - \bar{t}_s - 1)e^{t_s - \bar{t}_s} + 1\right) = \theta \tag{31}$$

The thresholds t_s and $\bar{t}_s$ are obtained by solving (29) and (31), and by further using the solutions of the ODEs (24) and (25) with boundary condition (28).

Case 2: $\mu \geq \theta$: The solution can be obtained as in the previous case but now with $\bar{t}_s = t_s$. With $\mu \geq \theta$, at t_s $m(t) = \max\{\mu, \theta\} = \mu$ and hence it is not possible for u^* to be 1. Thus the solutions are obtained by solving the joint ODEs (24) and (25) where t_s is obtained from (29) after replacing $\bar{t}_s = t_s$. □