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On loss probabilities in presence of redundant packets and several traffic sources

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Abstract

We study the effect of adding redundancy to an input stream on the losses that occur due to buffer overflow. We consider several sessions that generate traffic into a finite capacity queue. Using multi-dimensional probability generating functions, we derive analytical formulas for the loss probabilities and provide asymptotic analysis (for large n and small or large ρ). Our analysis allows us to investigate when does adding redundancy decrease the loss probabilities. In many cases, redundancy is shown to degrade the performance, as the gain in adding redundancy is not sufficient to compensate the additional losses due to the increased overhead. We show, however, that it is possible to decrease loss probabilities if a sufficiently large amount of redundancy is added. Indeed, we show that for an arbitrary stationary ergodic input process, if $\rho < 1$ then redundancy can reduce loss probabilities to an arbitrarily small value.

Keywords: Forward error correction; Loss probabilities; Multi-dimensional generating functions; M/M/1/K queue; Stationary ergodic arrival processes

1. Introduction

An important trend in telecommunications is to integrate different type of traffic in a single network. The various traffic types typically have different requirements on quality of services, and in particular, on loss probabilities. Rapid progress in the development of fiber optics allows to achieve a bit error rate of 10^{-14}; information loss is then essentially due to congested nodes and buffer overflow.

Often, a group of consecutive packets are grouped into a frame, and loss of one packet results in the loss of the whole frame. This is the case in ATM where a transport layer protocol (known as AAL) is responsible for this grouping, see e.g. Chapter 5 in [11]. In order to reduce the loss probabilities,

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one may add redundant packets into the frame, so that lost packets can be often reconstructed. Indeed, there exist erasure recovery codes that, by including additional k redundant packets in a frame, enable to reconstruct up to k losses (see [5,10,13,4] and references therein). We note, however, that by adding redundant packets, the workload increases and thus the loss probability of a packet increases.

Adding redundant packets to a frame is quite frequent in networks, especially in the ATM adaptation layer (AAL), see e.g. [3]. It also plays an important role in several applications on the Internet (see e.g. [2,12]). If the number of redundant packets j that is to be added to a set of n packets is one, the simplest way to do it is by letting the kth bit of the redundant packet be the modulo 2 sum of the kth bit of all n packets. For the case of $j \geq 2$ there are several known methods, see e.g. [4], or the Reed Solomon code [6]. The procedure of adding redundancy is known as Forward Error Correction (FEC). (This method is in contrast with feedback error correction methods based on retransmissions, which may require long delays due to the retransmission.)

We analyze the tradeoff between the effects of increase of workload and the recovery of lost packets, and calculate the probability of no more losses than k packets within n consecutive ones in the presence of k redundant packets. The computations are based on recursive formulas obtained by Cidon, Khamisy and Sidi [5]. We consider the possibility of multiplexing between several sources so that the packets of a given source to which redundancy is added may be separated in the queue by packets from other sources. This type of models (with more general arrival processes) was studied also in Kawahara et al. [10] who obtain a procedure for the numerical solution. By restricting in this paper to Poisson arrivals, we are able to obtain exact formulas for the loss probabilities.

In [13], the authors have used an approximation based on an assumption of independence between consecutive losses, and shown that redundancy results in decrease of loss probabilities by 10% to 100%. Exact numerical methods based on recursions [5] led to an opposite conclusion, i.e. that redundancy causes increase in loss probabilities. One of the advantages of our analytical approach, together with the asymptotic approximations which we present, is that they enable to study both qualitative and quantitative behavior of the effect of redundancy in a systematic way. As was already shown in [1,9] for the case of a single source, we show that for both light traffic as well as heavy traffic conditions, redundancy decreases loss probabilities.

In this paper we identify a fundamental property of losses with redundancy. We show that for any value ρ smaller than one of the traffic load of sessions to which we wish to add redundancy, adding redundancy in an appropriate way results in arbitrarily small loss probabilities. This property is shown to hold for any stationary ergodic arrival sequence. For the special case of Poisson arrivals we actually compute the rate of redundancy that has to be added.

The paper is structured as follows: in Section 2, we describe the model and we set the main results: probability generating function (etc.), the proofs are given in Section 3. The asymptotic analysis is presented in Section 4. In Section 5, we show that in light traffic, adding redundancy decreases the loss probabilities. Numerical examples which illustrate this improvement are given in Section 6. In Section 7 we show that frame losses can be almost completely eliminated, and we compute the required rate of redundancy. In Section 8 we extend some of these results to general arrival and service time distributions. We conclude with some remarks and future work in Section 9.

2. The model and the main results

We consider an M/M/1 queue with a finite buffer of size K served according to the FIFO (first in first served) discipline. We assume that packets arrive to the queue from S independent sources, i.e. the inter-arrival times and the transmission times of packets from each source are mutually independent. The arrival process from source s, $s = 1, 2, \ldots, S$, is assumed to be Poisson with rate λ_s. The overall arrival process to the system is then Poisson with rate $\lambda \triangleq \sum_{s=1}^{S} \lambda_s$. Define $p_s \triangleq \lambda_s/\lambda$ and $p_{\bar{s}} \triangleq 1 - p_s$, $\rho = \lambda/\mu$, $\rho_s = \lambda_s/\mu = p_s\rho$. We summarize the recursive scheme introduced in [5] for computing $P^s(j, n)$, $s = 1, 2, \ldots, S$ which are the probabilities of j losses among n consecutive ones originating from source s. For the system with Poisson arrivals with rate λ and exponential transmission rate μ, in steady state, the probability of finding i packets in the system at an arbitrary epoch is given by $\Pi(i) = \rho^i/(\sum_{l=0}^{K} \rho^l)$. Define $Q_i(k)$ to be the probability that k packets out of i leave the system during an inter-arrival epoch. We have

$$\begin{aligned} Q_i(k) &= \rho\alpha^{k+1}, \quad 0 \le k \le i-1, \\ Q_i(i) &= \alpha^i, \qquad \text{where } \alpha := (1+\rho)^{-1}. \end{aligned} \tag{1}$$

Denote by $P_i^{s,a}(j, n)$ resp. $P_i^{\bar{s},a}(j, n)$, $i = 0, 1, \ldots, K$, $s = 1, 2, \ldots, S$, $n \ge 1$, $0 \le j \le n$, the probabilities of j losses in a block of n packets coming from source s, given that there are i packets in the system just before the arrival of the first packet in the block, and just before the arrival of a packet from any other source (denoted by $\bar{s}$), respectively. Since the first packet in the block is arbitrary, we have

$$P^s(j, n) = \sum_{i=0}^{K} \Pi(i) P_i^{s,a}(j, n). \tag{2}$$

The probability that an arbitrary arrival is from source s is equal to λ_s/λ. The recursive scheme is

$$P_i^{s,a}(j, 1) = \begin{cases} 1, & j = 0, \\ 0, & j \ge 1, \end{cases} \qquad i = 0, 1, \ldots, K-1 \tag{3}$$

$$P_K^{s,a}(j, 1) = \begin{cases} 1, & j = 1, \\ 0, & j = 0,\ j \ge 2. \end{cases} \tag{4}$$

For $n \ge 2$, we have for $0 \le i \le K-1$ and for $i = K$, respectively:

$$\begin{aligned} P_i^{s,a}(j, n) &= \sum_{k=0}^{i+1} Q_{i+1}(k) \left[p_s P_{i+1-k}^{s,a}(j, n-1) + p_{\bar{s}} P_{i+1-k}^{\bar{s},a}(j, n-1) \right], \\ P_K^{s,a}(j, n) &= \sum_{k=0}^{K} Q_K(k) \left[p_s P_{K-k}^{s,a}(j-1, n-1) + p_{\bar{s}} P_{K-k}^{\bar{s},a}(j-1, n-1) \right], \end{aligned} \tag{5}$$

where $P_i^{\bar{s},a}(j,n)$ for $n \geq 1$ is given by

$$P_i^{\bar{s},a}(j,n) = \sum_{k=0}^{i+1} Q_{i+1}(k)\left[p_s P_{i+1-k}^{s,a}(j,n) + p_{\bar{s}} P_{i+1-k}^{\bar{s},a}(j,n)\right], \quad 0 \leq i \leq K-1,$$
$$P_K^{\bar{s},a}(j,n) = P_{K-1}^{\bar{s},a}(j,n). \tag{6}$$

The complexity of these recursions is $\mathrm{O}(K^2nj)$ in arithmetic operations and $\mathrm{O}(K^2)$ in memory space. Next, we state the main results, whose detailed proofs are given in next section. Define:

$$q_s(y,z) \triangleq \sum_{j=0}^{\infty}\sum_{n=1}^{\infty} y^j z^{n-1} P^s(j,n).$$

Let $x_1(z)$ and $x_2(z)$ be the solutions in x of $x^2 - (1+\rho)x + \rho(p_{\bar{s}} + p_s z) = 0$:

$$x_1(z) = \left(1+\rho+\sqrt{(1+\rho)^2 - 4\rho(p_{\bar{s}} + p_s z)}\right)/2,$$
$$x_2(z) = \left(1+\rho-\sqrt{(1+\rho)^2 - 4\rho(p_{\bar{s}} + p_s z)}\right)/2.$$

We shall often write simply x_1 and x_2 for $x_1(z)$ and $x_2(z)$. Both these functions are analytic in the disk $\{|z| < \left((1+\rho)^2 - 4\rho_{\bar{s}}\right)/4\rho_s\}$. Define, for all k, $\delta_k = x_1^k - x_2^k$, $\phi_k = (p_{\bar{s}} + p_s z)\delta_{k-1} - \delta_k$. Let $R_K = \left(\sum_{l=0}^{K} \rho^l\right)^{-1}$.

Proposition 1. *The probability generating function q_s is given by*

$$q_s(y,z) = \frac{R_K}{(1-z)}\left(\frac{\rho^{K-1}(1-z)[\delta_{K+1}]^2}{z\phi_K}\left[\frac{1}{z\delta_K - \delta_{K+1} - \rho z\phi_K y}\right] + R_K^{-1} + \frac{\rho^{K-1}\delta_{K+1}}{z\phi_K}\right). \tag{7}$$

Once the probability generating function is obtained by Proposition 1, one can obtain the required probabilities by inverting q_s. We focus in the sequel on $P_\rho^s(>j,n)$, which is the probability of losing more than j packets out of n consecutive ones coming from source s. We investigate in particular the case of $j = 0, 1$, in order to be able to decide when does including one redundant packet in each frame results in a decrease of the loss probability.

We shall use the notation $[z^k]f(z)$ to denote the coefficient of z^k in the Taylor expansion of the function $f(z)$, i.e. if $f(z) = \sum_k f_k z^k$ then $[z^k]f(z) = f_k$.

Corollary 2. *The probability of losing more than j packets out of n consecutive packets that arrive from source s is*

$$P_\rho^s(>j,n) = R_K\rho^{K+j}\left[z^{n-1-j}\right]\frac{1}{z-1}\left[\frac{\delta_{K+1}}{z\delta_K - \delta_{K+1}}\right]\left(\frac{\phi_K}{z\delta_K - \delta_{K+1}}\right)^j. \tag{8}$$

In the following, we obtain a simple recursion on n, for computing the probabilities $P_\rho^s(>j,n)$. Thus, we avoid a recursion on j and a resolution of a set of linear equations of size K for all j and all n required in [5].

Define β, γ and θ as

$$\gamma = 1+\rho, \quad \beta = \sqrt{(1+\rho)^2 - 4\rho_{\bar{s}}}, \quad \theta = \frac{4\rho_s}{(1+\rho)^2 - 4\rho_{\bar{s}}}. \tag{9}$$

Let $G = K/2$ for K even and $(K+1)/2$ otherwise, and set

$$a_n = (-\theta)^n \sum_{k=n}^{G} \binom{K+1}{2k+1} \binom{k}{n} \beta \left(\frac{\beta}{\gamma}\right)^{2k},$$

$$b_n = (-\theta)^{n-1} \sum_{k=n}^{G} \binom{K+1}{2k+1} \binom{k}{n} \left(\frac{4\gamma(2k+1)n}{\beta(K+1)(K+1-2k)} + \theta\beta\right) \left(\frac{\beta}{\gamma}\right)^{2k}$$

(we use the convention that $\sum_{k=n}^{G} = 0$ if $n > G$).

Corollary 3. *For $n \geq 1$ we have*

$$P_\rho^s(>0, n) = R_K \rho^K + \frac{1}{a_0} \sum_{k=1}^{n-1} \left(b_{n-k} P_\rho^s(>0, k) + R_K \rho^K a_k\right). \tag{10}$$

For $j < n$, $n \geq 1$, $P_\rho^s(>j, n)$ is given by the expression

$$\frac{(-1)^j}{A_{j+1,0}} \left[\sum_{k=j}^{n-2} H_{j+1,n-1-k} P_\rho^s(>j, k+1) + R_K \rho^K \sum_{k=0}^{n-j-1} R_{j,k} \right], \tag{11}$$

where

$$H_{k,n} = \sum_{r=0}^{k} \sum_{m=r}^{n} \binom{k}{r} (-1)^{k-r} A_{k-r,n-m} B_{r,m-r},$$

$$R_{k,n} = \sum_{r=0}^{k} \sum_{m=0}^{n} \binom{k}{r} (-1)^{k-r} A_{k-r+1,n-m} B_{r,m},$$

with

$$A_{k,n} = [z^n]\,(\delta_{K+1})^k = \sum_{m=0}^{n} \sum_{r=0}^{k} \binom{k}{r} (-1)^{k-r} a'_{r(K+1),m} b'_{(k-r)(K+1),n-m},$$

$$B_{k,n} = [z^n]\,(\delta_K)^k = \sum_{m=0}^{n} \sum_{r=0}^{k} \binom{k}{r} (-1)^{k-r} a'_{rK,m} b'_{(k-r)K,n-m},$$

where

$$a'_{k,n} = \theta^n \left(\frac{\gamma}{2}\right)^k \sum_{r=0}^{k} \binom{k}{r} \left(\frac{\beta}{\gamma}\right)^r \frac{\Gamma(n-r/2)}{n!\,\Gamma(-r/2)},$$

$$b'_{k,n} = \theta^n \left(\frac{\gamma}{2}\right)^k \sum_{r=0}^{k} \binom{k}{r} \left(\frac{-\beta}{\gamma}\right)^r \frac{\Gamma(n-r/2)}{n!\,\Gamma(-r/2)}.$$

For computing the probabilities $P^s_\rho(>0, n)$, we first compute the terms a_k and b_k, $k = 0, \ldots, n$ which requires $\mathrm{O}(Kn)$ arithmetic operations, then we compute the sum, which is a simple recursion on n, with complexity of $\mathrm{O}(n^2)$ arithmetic operations and $\mathrm{O}(Kn)$ in memory space if we consider that all the values $\binom{k}{r}$, $0 \le r \le n$, $r \le k \le K/2$ remain in memory (and need not be computed). In the case $j > 0$, we proceed in the same manner; we first compute the terms $a'_{k,m}$, $b'_{k,m}$, $k = K+1, \ldots, j(K+1)$, $m = 0, \ldots, n$ which requires $\mathrm{O}(Knj^2)$ arithmetic operations, after this, we compute the terms $A_{k,m}$, $B_{k,m}$, $k = 1, \ldots, j$, $m = 0, \ldots, n$ with complexity of $\mathrm{O}(j^2n^2)$, after what, we compute the terms $H_{j+1,m}$ and $R_{j,m}$, $m = 0, \ldots, n$ which requires $\mathrm{O}(jn^2)$. Finally, the probabilities are computed from Eq. (11) with complexity of $\mathrm{O}(n^2)$ arithmetic operations. Thus, the complexity of this procedure is $\mathrm{O}(j^2n^2 + Knj^2)$ in arithmetic operations and $\mathrm{O}(K^2j^2)$ in memory space if we consider, again, that all values $\binom{k}{r}$, $0 \le r \le k$, $K+1 \le k \le j(K+1)$ are stored beforehand in the memory and need not be computed.

Remark 4. All the results in [1], who considered a single source, can be obtained as special case of our results by substituting 1 for p_s.

3. Proof of the main results

Proof of Proposition 1. The following derivation is a generalization of the one given in [1] for the case of no exogenous flow, i.e. $p_{\bar{s}} = 0$. Define

$$\pi^s_{j,n}(x) \triangleq \sum_{i=0}^{K} x^i P^{s,a}_i(j,n),$$

$$\pi^{\bar{s}}_{j,n}(x) \triangleq \sum_{i=0}^{K} x^i P^{\bar{s},a}_i(j,n),$$

$$\Phi^{s\bar{s}}_{j,n}(x) \triangleq \sum_{i=0}^{K} x^i \left[p_s P^{s,a}_i(j,n) + p_{\bar{s}} P^{\bar{s},a}_i(j,n)\right], \qquad n \ge 1,\ j \ge 0.$$

It follows from Eq. (5) that

$$\pi^s_{j,n}(x) = \sum_{i=0}^{K-1} x^i \sum_{k=0}^{i+1} Q_{i+1}(k)\left[p_s P^{s,a}_{i+1-k}(j,n-1) + p_{\bar{s}} P^{\bar{s},a}_{i+1-k}(j,n-1)\right]$$
$$+ x^K \sum_{k=0}^{K} Q_K(k)\left[p_s P^{s,a}_{K-k}(j-1,n-1) + p_{\bar{s}} P^{\bar{s},a}_{K-k}(j-1,n-1)\right].$$

Next, we substitute Eq. (1) as well as the definition of $\Phi^{s\bar{s}}_{j,n}(x)$ into the last equation. Using the fact that $\Phi^{s\bar{s}}_{j,n}(0) = p_{s P_0^{s,a}}(j,n) + p_{\bar{s}} P_0^{\bar{s},a}(j,n)$, we obtain for $n \geq 2$, $j \geq 1$:

$$\begin{aligned}
\pi^s_{j,n}(x) = {} & \sum_{i=0}^{K-1} x^i \left(\sum_{k=0}^{i} \rho\alpha^{k+1} \left[p_s P^{s,a}_{i+1-k}(j,n-1) + p_{\bar{s}} P^{\bar{s},a}_{i+1-k}(j,n-1) \right] \right. \\
& \left. \qquad + \alpha^{i+1} \left[p_{s P_0^{s,a}}(j,n-1) + p_{\bar{s}} P_0^{\bar{s},a}(j,n-1) \right] \right) \\
& + x^K \left(\sum_{k=0}^{K-1} \rho\alpha^{k+1} \left[p_{s P^{s,a}_{K-k}}(j-1,n-1) + p_{\bar{s}} P^{\bar{s},a}_{K-k}(j-1,n-1) \right] \right. \\
& \left. \qquad + \alpha^K \left[p_{s P_0^{s,a}}(j-1,n-1) + p_{\bar{s}} P_0^{\bar{s},a}(j-1,n-1) \right] \right) \\
= {} & \frac{\rho\alpha^2}{1-\alpha x} \left(\frac{1}{\alpha x} \Phi^{s\bar{s}}_{j,n-1}(x) - (\alpha x)^K \Phi^{s\bar{s}}_{j,n-1}(\alpha^{-1}) \right) - \frac{\rho\alpha^2}{1-\alpha x} \left(\frac{1}{\alpha x} - (\alpha x)^K \right) \Phi^{s\bar{s}}_{j,n-1}(0) \\
& + \alpha \frac{1-(\alpha x)^K}{1-\alpha x} \Phi^{s\bar{s}}_{j,n-1}(0) + \alpha\rho(\alpha x)^K \Phi^{s\bar{s}}_{j-1,n-1}(\alpha^{-1}) + \alpha(\alpha x)^K \Phi^{s\bar{s}}_{j-1,n-1}(0). \qquad (12)
\end{aligned}$$

Proceeding similarly as above, we obtain from Eq. (6) for $n \geq 1$, $j \geq 0$:

$$\begin{aligned}
\pi^{\bar{s}}_{j,n}(x) = {} & \frac{\rho\alpha^2}{1-\alpha x} \left(\frac{1}{\alpha x} \Phi^{s\bar{s}}_{j,n}(x) - (\alpha x)^K \Phi^{s\bar{s}}_{j,n}(\alpha^{-1}) \right) - \frac{\rho\alpha^2}{1-\alpha x} \left(\frac{1}{\alpha x} - (\alpha x)^K \right) \Phi^{s\bar{s}}_{j,n}(0) \\
& + \alpha \frac{1-(\alpha x)^K}{1-\alpha x} \Phi^{s\bar{s}}_{j,n}(0) + \alpha\rho(\alpha x)^K \Phi^{s\bar{s}}_{j,n}(\alpha^{-1}) + \alpha(\alpha x)^K \Phi^{s\bar{s}}_{j,n}(0). \qquad (13)
\end{aligned}$$

Define, with some abuse of notation, the generating functions of $P_i^{s,a}(j,n)$ resp. $P_i^{\bar{s},a}(j,n)$:

$$\pi^s(x,y,z) \triangleq \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} y^j z^{n-1} \pi^s_{j,n}(x), \quad \text{resp.} \quad \pi^{\bar{s}}(x,y,z) \triangleq \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} y^j z^{n-1} \pi^{\bar{s}}_{j,n}(x).$$

Define also, with some abuse of notation, the generating function of $p_s P_i^{s,a}(j,n) + p_{\bar{s}} P_i^{\bar{s},a}(j,n)$:

$$\Phi^{s\bar{s}}(x,y,z) \triangleq \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} y^j z^{n-1} \Phi^{s\bar{s}}_{j,n}(x) = p_s \pi^s(x,y,z) + p_{\bar{s}} \pi^{\bar{s}}(x,y,z). \qquad (14)$$

When we fix y and $|z| < 1$, the three generating functions are polynomials in x, and therefore analytic functions. In order to use Eq. (12), which holds only for $n \geq 2$ and $j \geq 1$, we note that

$$\begin{aligned}
\sum_{j=1}^{\infty} \sum_{n=2}^{\infty} y^j z^{n-1} \pi^s_{j,n}(x) &= \pi^s(x,y,z) - \sum_{n=1}^{\infty} z^{n-1} \pi^s_{0,n}(x) - \sum_{j=0}^{\infty} y^j \pi^s_{j,1}(x) + \pi^s_{0,1}(x) \\
&= \pi^s(x,y,z) - \pi^s(x,0,z) - \pi^s(x,y,0) + \pi^s(x,0,0).
\end{aligned}$$

From Eqs. (3) and (4) we get

$$\pi^s(x,0,0) = \frac{1-x^K}{1-x}, \qquad \pi^s(x,y,0) = \frac{1-x^K}{1-x} + yx^K. \tag{15}$$

In Eq. (15), as well as in the rest of the paper, we consider that for $x = 1$ and for all K, $(1-x^K)/(1-x) = K$ (in particular, $R_K = 1/(K+1)$ for $\rho = 1$). From Eq. (12), after substituting Eq. (14), we obtain

$$\begin{aligned}
\pi^s(x,y,z) - \pi^s(x,0,z) &= yx^K + \frac{\rho\alpha^2}{1-\alpha x}\frac{1}{\alpha x} z\left[\Phi^{s\bar{s}}(x,y,z) - \Phi^{s\bar{s}}(x,0,z)\right] \\
&\quad - \frac{\rho\alpha^2}{1-\alpha x}(\alpha x)^K z\left[\Phi^{s\bar{s}}(\alpha^{-1},y,z) - \Phi^{s\bar{s}}(\alpha^{-1},0,z)\right] \\
&\quad - \frac{\rho\alpha^2}{1-\alpha x}\left(\frac{1}{\alpha x} - (\alpha x)^K\right) z\left[\Phi^{s\bar{s}}(0,y,z) - \Phi^{s\bar{s}}(0,0,z)\right] \\
&\quad + \alpha\frac{1-(\alpha x)^K}{1-\alpha x} z\left[\Phi^{s\bar{s}}(0,y,z) - \Phi^{s\bar{s}}(0,0,z)\right] \\
&\quad + \alpha\rho(\alpha x)^K zy\left[\Phi^{s\bar{s}}(\alpha^{-1},y,z) + \Phi^{s\bar{s}}(0,y,z)\right] \\
&= yx^K + \frac{\rho\alpha^2}{1-\alpha x}\frac{1}{\alpha x} z\left[\Phi^{s\bar{s}}(x,y,z) - \Phi^{s\bar{s}}(x,0,z)\right] \\
&\quad + \rho\alpha(\alpha x)^K\left(y - \frac{\alpha}{1-\alpha x}\right) z\left[\Phi^{s\bar{s}}(\alpha^{-1},y,z) + \frac{1}{\rho}\Phi^{s\bar{s}}(\alpha^{-1},0,z)\right] \\
&\quad + \frac{\rho\alpha^2(\alpha x)^K}{1-\alpha x} z\left[\Phi^{s\bar{s}}(\alpha^{-1},0,z) + \frac{1}{\rho}\Phi^{s\bar{s}}(0,0,z)\right] \\
&\quad + \left(\frac{-\rho\alpha^2}{1-\alpha x}\frac{1}{\alpha x} + \frac{\alpha}{1-\alpha x}\right) z\left[\Phi^{s\bar{s}}(0,y,z) - \Phi^{s\bar{s}}(0,0,z)\right].
\end{aligned} \tag{16}$$

Similarly, from Eq. (13) after substituting Eq. (14), we have

$$\begin{aligned}
\pi^{\bar{s}}(x,y,z) &= \rho\alpha(\alpha x)^K\left(1 - \frac{\alpha}{1-\alpha x}\right)\left[\Phi^{s\bar{s}}(\alpha^{-1},y,z) + \frac{1}{\rho}\Phi^{s\bar{s}}(\alpha^{-1},0,z)\right] \\
&\quad + \frac{\rho\alpha^2}{1-\alpha x}\frac{1}{\alpha x}\Phi^{s\bar{s}}(x,y,z) + \frac{\alpha^2(x-\rho)}{(1-\alpha x)\alpha x}\Phi^{s\bar{s}}(0,y,z).
\end{aligned} \tag{17}$$

By using the relation $\alpha + \rho\alpha = 1$, we get from Eq. (17) and Eq. (14)

$$\pi^{\bar{s}}(\rho,y,z) = \Phi^{s\bar{s}}(\rho,y,z) = \pi^s(\rho,y,z). \tag{18}$$

This means that the distributions of the number of the customers in the queues taken at the arrival times of the packets from source s are the same when taken at the arrival times of the other packets (packets coming from other sources $\bar{s}$). (This is due to the Pasta property.)

We note that in order to establish the proof of Proposition 1, it follows from Eq. (2) that it suffices to obtain $\pi^s(x,y,z)$ at $x = \rho$, since

$$q_s(y,z) = R_K\pi^s(\rho,y,z). \tag{19}$$

From Eqs. (16) and (18) we have

$$\begin{aligned}\left[\pi^s(\rho,y,z)-\pi^s(\rho,0,z)\right](1-z) &= y\rho^K + z(\rho\alpha)^{K+1}\left[\Phi^{s\bar{s}}(\alpha^{-1},0,z)-\Phi^{s\bar{s}}(0,0,z)\right]\\ &\quad + z(y-1)(\rho\alpha)^{K+1}\left[\Phi^{s\bar{s}}(\alpha^{-1},y,z)-\Phi^{s\bar{s}}(0,y,z)\right]. \qquad (20)\end{aligned}$$

In order to compute the function $\pi^s(\rho,y,z)$ it suffices to compute the functions in the square brackets as well as $\pi^s(\rho,0,z)$. To do that, we first compute $\pi^s_{0,n}$ by proceeding in the same manner as in Eq. (12). Since $P_K^{s,a}(0,n)=0$ we have

$$\begin{aligned}\pi^s_{0,n} &= \frac{\rho\alpha^2}{1-\alpha x}\frac{1}{\alpha x}\Phi^{s\bar{s}}_{0,n-1}(x) - \frac{\rho\alpha^2}{1-\alpha x}(\alpha x)^K\Phi^{s\bar{s}}_{0,n-1}(\alpha^{-1})\\ &\quad + \alpha\frac{1-(\alpha x)^K}{1-\alpha x}\Phi^{s\bar{s}}_{0,n-1}(0) - \frac{\rho\alpha^2}{1-\alpha x}\left(\frac{1}{\alpha x}-(\alpha x)^K\right)\Phi^{s\bar{s}}_{0,n-1}(0).\end{aligned}$$

By taking the generating function of both sides of the above equation and substituting Eq. (15), we can write

$$\begin{aligned}(1-\alpha x)\alpha x\pi^s(x,0,z) &= \frac{1-x^K}{1-x}(1-\alpha x)\alpha x + \rho\alpha^2 z\Phi^{s\bar{s}}(x,0,z) - \rho\alpha^2(\alpha x)^{K+1}z\\ &\quad\times\left[\Phi^{s\bar{s}}(\alpha^{-1},0,z)+\frac{1}{\rho}\Phi^{s\bar{s}}(0,0,z)\right] + \alpha^2(x-\rho)z\Phi^{s\bar{s}}(0,0,z), \qquad (21)\end{aligned}$$

from which we get, for $x=\rho$, and after substituting Eq. (18)

$$(1-z)\pi^s(\rho,0,z) = R^{-1}_{K-1} - (\rho\alpha)^{K+1}z\left[\Phi^{s\bar{s}}(\alpha^{-1},0,z)+\frac{1}{\rho}\Phi^{s\bar{s}}(0,0,z)\right]. \qquad (22)$$

From Eqs. (14), (16) and (17) we have

$$\begin{aligned}&\left((1-\alpha x)\,\alpha x - \rho\alpha^2\,(p_{\bar{s}}+p_s z)\right)\left[\Phi^{s\bar{s}}(x,y,z)-\Phi^{s\bar{s}}(x,0,z)\right]\\ &\quad = (p_{\bar{s}}+p_s z)\left(\alpha^2(x-\rho)\right)\left[\Phi^{s\bar{s}}(0,y,z)-\Phi^{s\bar{s}}(0,0,z)\right]\\ &\qquad + \rho\alpha(\alpha x)^{K+1}\left[p_{\bar{s}}\alpha\,(\rho-x)+(y\,(1-\alpha x)-\alpha)\,p_s z\right]\left[\Phi^{s\bar{s}}(\alpha^{-1},y,z)+\frac{1}{\rho}\Phi^{s\bar{s}}(0,y,z)\right]\\ &\qquad + p_s(1-\alpha x)\alpha y x^{K+1} + \rho\alpha^2(\alpha x)^{K+1}\,(p_{\bar{s}}(x-\rho)+p_s z)\left[\Phi^{s\bar{s}}(\alpha^{-1},0,z)+\frac{1}{\rho}\Phi^{s\bar{s}}(0,0,z)\right]. \qquad (23)\end{aligned}$$

Also, from Eqs. (14) and (17) for $y=0$ and Eq. (21) we obtain

$$\begin{aligned}&\left((1-\alpha x)\,\alpha x - \rho\alpha^2\,(p_{\bar{s}}+p_s z)\right)\Phi^{s\bar{s}}(x,0,z)\\ &\quad = \frac{1-x^K}{1-x}(1-\alpha x)\alpha x p_s + \alpha^2(x-\rho)\,(p_{\bar{s}}+p_s z)\,\Phi^{s\bar{s}}(0,0,z)\\ &\qquad + \rho\alpha^2(\alpha x)^{K+1}\,(p_{\bar{s}}(x-\rho)+p_s z)\left[\Phi^{s\bar{s}}(\alpha^{-1},0,z)+\frac{1}{\rho}\Phi^{s\bar{s}}(0,0,z)\right]. \qquad (24)\end{aligned}$$

We now apply the 'kernel method' for solving the functional equations Eqs. (23) and (24). For each $i = 1, 2$, when $x = x_i(z)$, the term that multiplies $\Phi^{s\bar{s}}(x, 0, z)$ in the left-hand side of Eq. (24) (the kernel) vanishes. Since $\Phi^{s\bar{s}}(x, 0, z)$ is polynomial in x and therefore analytic in x, the left-hand side of Eq. (24) vanishes at $x = x_i(z)$. Thus by substituting x_i for x into Eq. (24), we obtain for each z two equations with two unknowns: $\Phi^{s\bar{s}}(0, 0, z)$ and $[\Phi^{s\bar{s}}(\alpha^{-1}, 0, z) + \frac{1}{\rho}\Phi^{s\bar{s}}(0, 0, z)]$. Solving these equations yields

$$\Phi^{s\bar{s}}(\alpha^{-1}, 0, z) + \frac{1}{\rho}\Phi^{s\bar{s}}(0, 0, z) = p_s \frac{\alpha^{-(K+1)}(x_1^K - x_2^K)}{x_2^K(x_2 - \rho_{\bar{s}})(x_1 - \rho) - x_1^K(x_1 - \rho_{\bar{s}})(x_2 - \rho)}, \tag{25}$$

$$\Phi^{s\bar{s}}(0, 0, z) = \frac{\rho_s}{(x_1 - \rho)(x_2 - \rho)}\left[-1 + x_1^K + \frac{\rho_s x_1^K(x_1 - \rho_{\bar{s}})(x_2 - \rho)(x_1^K - x_2^K)}{x_2^K(x_2 - \rho_{\bar{s}})(x_1 - \rho) - x_1^K(x_1 - \rho_{\bar{s}})(x_2 - \rho)}\right]. \tag{26}$$

We use again the same argument as above, for each $i = 1, 2$, when $x = x_i(z)$ the term that multiplies $\Phi^{s\bar{s}}(x, y, z) - \Phi^{s\bar{s}}(x, 0, z)$ in the left-hand side of Eq. (23), vanishes. Since $\Phi^{s\bar{s}}(x, y, z)$ and $\Phi^{s\bar{s}}(x, 0, z)$ are both analytic in x, after substituting Eqs. (25) and (26) into Eq. (23), for $x = x_i(z)$, we obtain two equations with two unknowns: $[\Phi^{s\bar{s}}(\alpha^{-1}, y, z) + \frac{1}{\rho}\Phi^{s\bar{s}}(0, y, z)]$ and $\Phi^{s\bar{s}}(0, y, z)$. Solving these equations yields

$$\begin{aligned}\Phi^{s\bar{s}}(\alpha^{-1}, y, z) &+ \frac{1}{\rho}\Phi^{s\bar{s}}(0, y, z)\\ &= \frac{p_s\alpha^{-(K+1)}\left(x_1^K(y(x_2 - \rho) - 1) - x_2^K(y(x_1 - \rho) - 1)\right)}{x_1^K(x_2 - \rho)\left[x_1(1 - yx_2) - \rho_{\bar{s}}(1 - y)\right] - x_2^K(x_1 - \rho)\left[x_2(1 - yx_2) - \rho_{\bar{s}}(1 - y)\right]}.\end{aligned} \tag{27}$$

Finally, by substituting Eqs. (22), (25) and (27) into Eq. (20), we obtain

$$\begin{aligned}(1 - z)\pi^s(\rho, y, z) &= y\rho^K + R_{K-1}^{-1} + z(y - 1)(\rho\alpha)^{K+1}\left[\Phi^{s\bar{s}}(\alpha^{-1}, y, z) + \frac{1}{\rho}\Phi^{s\bar{s}}(0, y, z)\right]\\ &= y\rho^K + R_{K-1}^{-1}\\ &\quad + \frac{p_s z(y - 1)\rho^{K+1}\left(x_1^K(y(x_2 - \rho) - 1) - x_2^K(y(x_1 - \rho) - 1)\right)}{x_1^K(x_2 - \rho)\left[x_1(1 - yx_2) - \rho_{\bar{s}}(1 - y)\right] - x_2^K(x_1 - \rho)\left[x_2(1 - yx_2) - \rho_{\bar{s}}(1 - y)\right]}.\end{aligned} \tag{28}$$

In the derivation of the above, we used the following relations: $x_1x_2 = p_{\bar{s}} + p_s z$, $x_1 + x_2 = 1 + \rho$ and $\rho_s(1 - z) = (x_i - \rho)(x_i - 1)$, $i = 1, 2$. Moreover, $\rho\phi_K = \delta_K - \delta_{K+1}$ since

$$\begin{aligned}\delta_{K+1} &= x_1^{K+1} - x_2^{K+1} = x_1^{K-1}[\alpha^{-1}x_1 - \rho(p_{\bar{s}} + p_s z)] - x_2^{K-1}[\alpha^{-1}x_2 - \rho(p_{\bar{s}} + p_s z)]\\ &= \alpha^{-1}\delta_K - \rho(p_{\bar{s}} + p_s z)\delta_{K-1} = \delta_K - \rho\phi_K.\end{aligned}$$

The proposition, finally, follows from Eq. (19). □

Proof of Corollary 2.

$$\begin{aligned}
P_\rho^s(>j,n) &= R_K\left[z^{n-1}\right]\sum_{k=j+1}^{\infty}\left[y^k\right]\pi^s(\rho,y,z)\\
&= R_K\left[z^{n-1}\right]\frac{\rho^{K-1}}{z\phi_K}\frac{\left[\delta_{K+1}\right]^2}{\left[z\delta_K-\delta_{K+1}\right]}\sum_{k=j+1}^{\infty}\left(\frac{\rho z\phi_K}{z\delta_K-\delta_{K+1}}\right)^k\\
&= R_K\left[z^{n-1}\right]\rho^K\left[\frac{\delta_{K+1}}{z\delta_K-\delta_{K+1}}\right]^2\sum_{k=j+1}^{\infty}\left(\frac{\rho z\phi_K}{z\delta_K-\delta_{K+1}}\right)^{k-1}\\
&= R_K\left[z^{n-1}\right]\rho^K\left[\frac{\delta_{K+1}}{z\delta_K-\delta_{K+1}}\right]^2\left(\frac{\rho z\phi_K}{z\delta_K-\delta_{K+1}}\right)^j\frac{1}{1-\dfrac{\rho z\phi_K}{z\delta_K-\delta_{K+1}}}.
\end{aligned}$$

Eq. (8) is obtained by noting that

$$z\delta_K-\delta_{K+1}-\rho z\phi_K = z(\delta_K-\rho\phi_K)-\delta_{K+1} = z\delta_{K+1}-\delta_{K+1} = -(1-z)\delta_{K+1}. \qquad \square$$

Proof of Corollary 3. From Eq. (8), it follows that

$$(z\delta_K-\delta_{K+1})^{j+1}\left(\sum_{n=1}^{\infty}z^{n-1}P_\rho^s(>j,n)\right) = -R_K\rho^K\frac{1}{1-z}\delta_{K+1}\left(\rho z\phi_K\right)^j. \tag{29}$$

Particularly, for $j=0$, by computing the coefficient of $[z^{n-1}]$ in both sides of Eq. (29), given that $[z^{n-1}]f(z)/(1-z)=\sum_{k=0}^{n}[z^k]f(z)$, we get

$$\begin{aligned}
&\left[z^{n-1}\right](z\delta_K-\delta_{K+1})\left(\sum_{n=1}^{\infty}z^{n-1}P_\rho^s(>0,n)\right) = \sum_{k=0}^{n-1}\left[z^{n-1-k}\right](z\delta_K-\delta_{K+1})\,P_\rho^s(>0,k+1)\\
&\quad= \left[z^0\right](z\delta_K-\delta_{K+1})\,P_\rho^s(>0,n)+\sum_{k=1}^{n-1}\left[z^{n-k}\right](z\delta_K-\delta_{K+1})\,P_\rho^s(>0,k)\\
&\quad= -R_K\rho^K\left[z^0\right]\delta_{K+1}-R_K\rho^K\sum_{k=1}^{n-1}\left[z^k\right]\delta_{K+1}.
\end{aligned}$$

Eq. (10) follows by noting that a_n and b_n defined below Eq. (9) are given by

$$a_n = \left[z^n\right]\left(\frac{\gamma}{2}\right)^K\frac{1}{\sqrt{1-\theta z}}\delta_{K+1},$$

$$b_n = \left[z^n\right]\left(\frac{\gamma}{2}\right)^K\frac{1}{\sqrt{1-\theta z}}\left(z\delta_K-\delta_{K+1}\right),$$

with

$$\left[z^0\right]\left(\frac{\gamma}{2}\right)^K \frac{1}{\sqrt{1-\theta z}}\left(z\delta_K-\delta_{K+1}\right)=\left[z^0\right]\left(\frac{\gamma}{2}\right)^K \frac{-1}{\sqrt{1-\theta a z}}\delta_{K+1}=-a_0.$$

By proceeding similarly as above, for $j>0$, we have

$$\begin{aligned}\left[z^{n-1}\right]\left(z\delta_K-\delta_{K+1}\right)^{j+1}\left(\sum_{n=1}^{\infty} z^{n-1}P_\rho^s(>j,n)\right)&=\sum_{k=0}^{n-1}\left[z^{n-1-k}\right]\left(z\delta_K-\delta_{K+1}\right)^{j+1}P_\rho^s(>j,k+1)\\ &=(-1)^{j+1}\left[z^0\right]\delta_{K+1}^{j+1}P_\rho^s(>j,n)+\sum_{k=j}^{n-2}\left[z^{n-1-k}\right]\left(z\delta_K-\delta_{K+1}\right)^{j+1}P_\rho^s(>j,k+1)\\ &=-R_K\rho^K\sum_{k=0}^{n-j-1}\left[z^k\right]\delta_{K+1}\left(\delta_K-\delta_{K+1}\right)^j. \end{aligned}\tag{30}$$

Eq. (11) follows from Eq. (30) by setting

$$H_{j+1,n}\triangleq\left[z^n\right]\left(z\delta_K-\delta_{K+1}\right)^{j+1}$$

and

$$R_{j,n}\triangleq\left[z^n\right]\delta_{K+1}\left(\delta_K-\delta_{K+1}\right)^j,$$

and noting that $P_\rho^s(>j,n)=0$, for $j\geq n$. Moreover,

$$H_{j+1,n}=\sum_{k=0}^{j+1}\sum_{m=k}^{n}\binom{j+1}{k}(-1)^{j+1-k}\left[z^{m-k}\right]\delta_K^k\left[z^{n-m}\right]\delta_{K+1}^{j+1-k}.$$

Thus $H_{k,n}$ and $R_{k,n}$ are obtained as functions of $A_{k,n}$ and $B_{k,n}$ by using Newton's binomial, where

$$\begin{aligned}A_{k,n}=\left[z^n\right]\left(\delta_{K+1}\right)^k=\left[z^n\right]\left(x_1^{K+1}-x_2^{K+1}\right)^k&=\sum_{r=0}^{k}\binom{k}{r}(-1)^{k-r}\left[z^n\right]x_1^{r(K+1)}x_2^{(k-r)(K+1)}\\ &=\sum_{r=0}^{k}\sum_{m=0}^{n}\binom{k}{r}(-1)^{k-r}\left[z^m\right]x_1^{r(K+1)}\left[z^{n-m}\right]x_2^{(k-r)(K+1)},\end{aligned}$$

and $B_{k,n}=\left[z^n\right]\left(\delta_K\right)^k$, which is obtained in the same way. Finally, $a'_{k,n}$ and $b'_{k,n}$ in Corollary 3 are the coefficients $\left[z^n\right]x_1^k$ and $\left[z^n\right]x_2^k$ respectively. □

4. Asymptotic behavior

In order to reduce the complexity of calculations of $P^s_\rho(>j,n)$, we shall approximate it by an expression $\tilde{P}^s_\rho(>j,n)$ which we derive below. From Eq. (8) we have

$$\begin{aligned}
P^s_\rho(>j,n) &= -R_K\rho^K\left[z^{n-1}\right]\frac{1}{1-z}\left(\frac{x_1^{K+1}}{x_1^K(z-x_1)}\left(\frac{zx_1^K(1-x_1)}{x_1^K(z-x_1)}\right)^j + f_j(z)\right)\\
&= -R_K\rho^K\left[z^{n-1}\right]\frac{1}{1-z}\left(\left(\frac{\rho_{\bar{s}}}{x_1-\rho_{\bar{s}}}+\frac{1}{1-x_1}\right)\left(x_2-\frac{\rho_{\bar{s}}(x_1-\rho)}{x_1-\rho_{\bar{s}}}\right)^j + f_j(z)\right)\\
&\triangleq \tilde{P}^s_\rho(>j,n) - R_K\rho^K\left[z^{n-1}\right]\frac{1}{1-z}f_j(z).
\end{aligned}$$

We show in Proposition A.1 that the term $\left[z^{n-1}\right]f_j(z)$ can be neglected for large n and $n<K$ $(j=0,1)$ and hence $P^s_\rho(>j,n)\cong\tilde{P}^s_\rho(>j,n)$, which we compute next.

For $j=0,1$, we have

$$\begin{aligned}
\tilde{P}^s_\rho(>0,n) = {}& \frac{R_K\rho^K}{2(1-\rho_{\bar{s}})}\left[z^{n-1}\right]\Bigg[\frac{1}{(1-z)^2}\left(1-\rho+\sqrt{(1+\rho)^2-4\rho_{\bar{s}}-4\rho_s z}\right)\\
&-\frac{\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\frac{1}{1-z}\left(1+\rho_s-\rho_{\bar{s}}-\sqrt{(1+\rho)^2-4\rho_{\bar{s}}-4\rho_s z}\right)\\
&+\frac{\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\frac{1}{\rho_{\bar{s}}-z}\left(1+\rho_s-\rho_{\bar{s}}-\sqrt{(1+\rho)^2-4\rho_{\bar{s}}-4\rho_s z}\right)\Bigg]\\
\triangleq {}& R_K\rho^K\left[z^{n-1}\right]\psi_0(z)
\end{aligned}\tag{31}$$

and

$$\begin{aligned}
\tilde{P}^s_\rho(>1,n) = {}& \frac{R_K\rho^K}{2(1-\rho_{\bar{s}})^2}\left[z^{n-1}\right]\Bigg[\frac{\rho_s}{(1-z)^2}\left(2z-1-\rho+\sqrt{(1+\rho)^2-4\rho_{\bar{s}}-4\rho_s z}\right)\\
&-\frac{\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\frac{1}{1-z}\left((1+\rho_s)^2+(\rho_{\bar{s}}-1)^2-1-2\rho_s z-(1+\rho_s-\rho_{\bar{s}})\sqrt{(1+\rho)^2-4\rho_{\bar{s}}-4\rho_s z}\right)\\
&+\frac{\rho_s\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\frac{1}{z-\rho_{\bar{s}}}\left(2z-1-\rho+\sqrt{(1+\rho)^2-4\rho_{\bar{s}}-4\rho_s z}\right)\\
&+\frac{\rho_{\bar{s}}^3}{1-\rho_{\bar{s}}}\frac{1}{z-\rho_{\bar{s}}}\left((1+\rho_s)^2+(\rho_{\bar{s}}-1)^2-1-2\rho_s z-(1+\rho_s-\rho_{\bar{s}})\sqrt{(1+\rho)^2-4\rho_{\bar{s}}-4\rho_s z}\right)\\
&+\frac{\rho_{\bar{s}}^3}{(\rho_{\bar{s}}-z)^2}\left((1+\rho_s)^2+(\rho_{\bar{s}}-1)^2-1-2\rho_s z-(1+\rho_s-\rho_{\bar{s}})\sqrt{(1+\rho)^2-4\rho_{\bar{s}}-4\rho_s z}\right)\Bigg]\\
\triangleq {}& \frac{R_K\rho^K}{2(1-\rho_{\bar{s}})^2}\left[z^{n-1}\right]\Psi_0(z).
\end{aligned}\tag{32}$$

Proposition 5. *For $n \geq 1$, for ρ fixed, and for $n < K$, $\tilde{P}^s_\rho(>0, n)$* is given by

$$\begin{cases} \dfrac{R_K\rho^K}{1-\rho_{\bar{s}}}\left[(1-\rho)n + \left(\dfrac{\rho_s}{1-\rho} - \dfrac{\rho_s\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\right) + \theta^n \mathrm{O}\left(n^{-3/2}\right)\right] & \text{if } \rho < 1, \\ \dfrac{1}{K+1}\dfrac{1}{p_s}\left[\left(2\sqrt{p_s} + \dfrac{p_{\bar{s}}}{\sqrt{p_s}n}\right)\dfrac{\sqrt{n}}{\sqrt{\pi}}\left(1+\mathrm{O}\left(\dfrac{1}{n}\right)\right) - p_{\bar{s}}\right] & \text{if } \rho = 1, \\ 1 - \left(\dfrac{4\rho_s^2}{\rho(\rho-1)^3} - \dfrac{\rho_s\rho_{\bar{s}}}{\rho(1-\rho_{\bar{s}})}\left(\dfrac{1}{\rho-1} - \dfrac{\rho-1}{(1+\rho_s-\rho_{\bar{s}})^2}\right)\right), & \\ \quad \times\, \theta^{n-1}\dfrac{\beta n^{-3/2}}{(1-\rho_{\bar{s}})\sqrt{\pi}}\,(1+\mathrm{o}\,(1)) & \text{if } \rho > 1 \end{cases} \tag{33}$$

and $\tilde{P}^s_\rho(>1, n)$ is given by

$$\begin{cases} \dfrac{R_K\rho^K\rho_s}{(1-\rho_{\bar{s}})^2}\left[(1-\rho)n + \dfrac{\rho_s+\rho-1}{1-\rho} - \dfrac{\rho_s\rho_{\bar{s}}}{1-\rho_{\bar{s}}} + \theta^n \mathrm{O}\left(n^{-3/2}\right)\right] & \text{if } \rho < 1, \\ \dfrac{1}{K+1}\dfrac{1}{p_s}\left[\left(2\sqrt{p_s} + \dfrac{2p_{\bar{s}}}{\sqrt{p_s}n}\right)\dfrac{\sqrt{n}}{\sqrt{\pi}}\left(1+\mathrm{O}\left(\dfrac{1}{n}\right)\right) - (1+p_{\bar{s}})\right] & \text{if } \rho = 1, \\ 1 - \dfrac{\rho_s}{1-\rho_{\bar{s}}}\left(\dfrac{4\rho_s^2}{\rho(\rho-1)^3} + \dfrac{\rho_{\bar{s}}(\rho-1)}{\rho(1-\rho_{\bar{s}})(1+\rho_s-\rho_{\bar{s}})}\left(\dfrac{(1+\rho_s-\rho_{\bar{s}})^2}{(\rho-1)^2}\right.\right. & \\ \quad \left.\left. -\, \dfrac{\rho_s}{(1+\rho_s-\rho_{\bar{s}})} + \rho_{\bar{s}}^2 + \dfrac{4\rho_s\rho_{\bar{s}}^2(1-\rho_{\bar{s}})}{(1+\rho_s-\rho_{\bar{s}})^2}\right)\right)\dfrac{\theta^{n-1}\beta n^{-3/2}}{(1-\rho_{\bar{s}})\sqrt{\pi}}\,(1+\mathrm{o}\,(1)) & \text{if } \rho > 1. \end{cases} \tag{34}$$

Proof. From Eq. (31), we get, for $\rho < 1$,

$$\begin{aligned} \psi_0(z) = \psi_1(z) &+ \frac{1-\rho}{(1-z)^2}\left(1 + \sqrt{1 - \frac{4\rho_s}{(1-\rho)^2}(z-1)}\right) \\ &- \frac{\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\frac{1}{1-z}\left(1+\rho_s-\rho_{\bar{s}}(1-\rho)\sqrt{1-\frac{4\rho_s}{(1-\rho)^2}(z-1)}\right) \\ &= \frac{1-\rho}{(1-z)^2}\left(2 + \sum_{j=1}^{\infty} c_j\left(\frac{4\rho_s}{(1-\rho)^2}(z-1)\right)^j\right) \\ &- \frac{\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\frac{1}{1-z}\left(2\rho_s - (1-\rho)\sum_{j=1}^{\infty} c_j\left(\frac{4\rho_s}{(1-\rho)^2}(z-1)\right)^j\right) \\ &+ \psi_1(z) = \frac{2(1-\rho)}{(1-z)^2} + \left(\frac{1}{1-\rho} - \frac{\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\right)\frac{2\rho_s}{1-z} + \psi_2(z) + \psi_1(z), \end{aligned}$$

where

$$\psi_2(z) = (1-\rho)\sum_{j=2}^{\infty} c_j \left(\frac{4\rho_s}{(1-\rho)^2}\right)^j (z-1)^{j-2} + \frac{\rho_{\bar{s}}(1-\rho)}{1-\rho_{\bar{s}}}\sum_{j=1}^{\infty} c_j \left(\frac{4\rho_s}{(1-\rho)^2}\right)^j (z-1)^{j-1}. \tag{35}$$

$\psi_2(z)$ is analytic at $z=1$ and has a singularity of type $\sqrt{\bullet}$ at $z = z_s \triangleq \left((1+\rho)^2 - 4\rho_{\bar{s}}\right)/4\rho_s$. Therefore, when z is close to z_s,

$$\psi_2(z) = \mathrm{O}\left(\sqrt{\frac{(1+\rho)^2 - 4\rho_{\bar{s}}}{4\rho_s} - z}\right).$$

It is easily checked that

$$\begin{aligned}\psi_1(z) &= \frac{\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\frac{1}{\rho_{\bar{s}} - z}\left(1 + \rho_s - \rho_{\bar{s}} - \sqrt{(1+\rho)^2 - 4\rho_{\bar{s}} - 4\rho_s z}\right) \\ &= \frac{\rho_{\bar{s}}(1+\rho_s-\rho_{\bar{s}})}{1-\rho_{\bar{s}}}\frac{1}{\rho_{\bar{s}} - z}\left(1 - \sqrt{1 - \frac{4\rho_s}{(1+\rho_s-\rho_{\bar{s}})^2}(z-\rho_{\bar{s}})}\right) \\ &= \frac{\rho_{\bar{s}}(1+\rho_s-\rho_{\bar{s}})}{1-\rho_{\bar{s}}}\sum_{j=1}^{\infty} c_j \left(\frac{4\rho_s}{(1+\rho_s-\rho_{\bar{s}})^2}\right)^j (z-\rho_{\bar{s}})^{j-1}. \end{aligned} \tag{36}$$

$\psi_1(z)$ is also analytic at $z = \rho_{\bar{s}}$ and for the same argument as above, when z is close to $\left((1+\rho)^2 - 4\rho_{\bar{s}}\right)/4\rho_s$,

$$\psi_1(z) = \mathrm{O}\left(\sqrt{\frac{(1+\rho)^2 - 4\rho_{\bar{s}}}{4\rho_s} - z}\right).$$

In addition to this singularity, ψ_0 is seen to have a pole of degree 2 at $z=1$. We get

$$\left[z^{n-1}\right]\psi_0(z) = 2(1-\rho)n + 2\rho_s\left(\frac{1}{1-\rho} - \frac{\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\right) + \left(\frac{4\rho_s}{(1+\rho)^2 - 4\rho_{\bar{s}}}\right)^n \mathrm{O}\left(n^{-3/2}\right).$$

This is obtained, by applying Theorem 1 of Flajolet and Odlyzko [8]. This theorem is applicable, since $\psi_0(z)$ is analytic in the whole complex plan except the segment along the real axis $z \in [((1+\rho)^2 - 4\rho_{\bar{s}})/4\rho_s, \infty[$.

For $\rho = 1$ we have

$$\psi_0(z) = \frac{2\sqrt{p_s}}{(1-z)^{3/2}} + \frac{2p_{\bar{s}}}{\sqrt{p_s}}\frac{1}{(1-z)^{1/2}} + \psi_1(z)$$

and

$$\psi_1(z) = 2p_{\bar{s}}\sum_{j=1}^{\infty} c_j \left(\frac{1}{p_s}\right)^j (z-p_{\bar{s}})^{j-1}$$

is analytic at $z = p_{\bar{s}}$ and has a singularity o type $\sqrt{\bullet}$ at $z = 1$ i.e when z is close to 1, $\psi_1(z) = \mathrm{O}\left(\sqrt{1-z}\right)$. We get

$$\left[z^{n-1}\right]\psi_0(z) = 2\sqrt{p_s}\frac{\Gamma(n+1/2)}{\Gamma(3/2)}\frac{1}{(n-1)!} + \frac{2p_{\bar{s}}}{\sqrt{p_s}}\frac{\Gamma(n-1/2)}{\Gamma(1/2)}\frac{1}{(n-1)!} - 2p_{\bar{s}} + \mathrm{O}\left(n^{-3/2}\right),$$

we obtain the corresponding equation in Proposition 5, by using Proposition 1 of [8] as well as the fact that $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \sqrt{\pi}/2$:

$$\frac{\Gamma(n+1/2)}{\Gamma(3/2)}\frac{1}{(n-1)!} = \frac{2}{\sqrt{\pi}}\sqrt{n}\left(1+\mathrm{O}\left(\frac{1}{n}\right)\right).$$

For $\rho > 1$ we note that

$$\psi_0(z) = \frac{2(1-\rho_{\bar{s}})\rho}{\rho-1}\left(\frac{1}{1-z}\right) - \psi(z) \tag{37}$$

where

$$\begin{aligned}\psi(z) = &-\frac{1-\rho}{(1-z)^2} + \frac{\rho_{\bar{s}}(1+\rho_s-\rho_{\bar{s}})}{1-\rho_{\bar{s}}}\frac{1}{1-z} - \frac{\rho_{\bar{s}}(1+\rho_s-\rho_{\bar{s}})}{1-\rho_{\bar{s}}}\frac{1}{\rho_{\bar{s}}-z}\\ &+\frac{\rho_{\bar{s}}\sqrt{(1+\rho)^2-4\rho_{\bar{s}}}}{1-\rho_{\bar{s}}}\frac{1}{\rho_{\bar{s}}-z}\sqrt{1-\frac{4\rho_s}{(1+\rho)^2-4\rho_{\bar{s}}}z} - \frac{2(1-\rho_{\bar{s}})\rho}{\rho-1}\frac{1}{1-z}\\ &-\frac{\sqrt{(1+\rho)^2-4\rho_{\bar{s}}}}{(1-z)^2}\sqrt{1-\frac{4\rho_s}{(1+\rho)^2-4\rho_{\bar{s}}}z} - \frac{\rho_{\bar{s}}\sqrt{(1+\rho)^2-4\rho_{\bar{s}}}}{1-\rho_{\bar{s}}}\frac{1}{1-z}\sqrt{1-\frac{4\rho_s}{(1+\rho)^2-4\rho_{\bar{s}}}z}.\end{aligned}$$

When z is close to $((1+\rho)^2-4\rho_{\bar{s}})/4\rho_s$, equivalently, as $(z-1)$ tends to $(1-\rho)^2/4\rho_s$, also, as $(z-\rho_{\bar{s}})$ tends to $(1+\rho_s-\rho_{\bar{s}})^2/4\rho_s$, we have

$$\begin{aligned}\psi(z) = &+\frac{\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\frac{4\rho_s}{(1+\rho_s-\rho_{\bar{s}})} - \frac{\rho_{\bar{s}}(1+\rho_s-\rho_{\bar{s}})}{1-\rho_{\bar{s}}}\frac{4\rho_s}{(1-\rho)^2} + \frac{2(1-\rho_{\bar{s}})\rho}{\rho-1}\frac{4\rho_s}{(1-\rho)^2}\\ &+\frac{(\rho-1)(4\rho_s)^2}{(1-\rho)^4} - \left(\frac{(4\rho_s)^2}{(1-\rho)^4} - \frac{4\rho_s\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\left(\frac{1}{(1-\rho)^2} - \frac{1}{(1+\rho_s-\rho_{\bar{s}})^2}\right)\right)\\ &\times\sqrt{(1+\rho)^2-4\rho_{\bar{s}}}\sqrt{1-\frac{4\rho_s}{(1+\rho)^2-4\rho_{\bar{s}}}z} + \mathrm{o}\left(\sqrt{1-\frac{4\rho_s}{(1+\rho)^2-4\rho_{\bar{s}}}z}\right).\end{aligned}$$

It follows from [2, p. 219 (2.2)] that

$$\begin{aligned}\left[z^{n-1}\right]\psi(z) = &-\left(\frac{(4\rho_s)^2}{(1-\rho)^4} - \frac{4\rho_s\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\left(\frac{1}{(1-\rho)^2} - \frac{1}{(1+\rho_s-\rho_{\bar{s}})^2}\right)\right)\left(\frac{4\rho_s}{(1+\rho)^2-4\rho_{\bar{s}}}\right)^{n-1}\\ &\times\frac{\sqrt{(1+\rho)^2-4\rho_{\bar{s}}}(n-1)^{-3/2}}{\Gamma(-1/2)}\left(1+\mathrm{O}\left(\frac{1}{n}\right)\right).\end{aligned} \tag{38}$$

Finally, from Eqs. (31), (37) and (38), we get

$$\begin{aligned}\tilde{P}^s_\rho(>0,n) &= \frac{R_K\rho^K}{2(1-\rho_{\bar{s}})}\left[z^{n-1}\right]\psi_0(z) = \frac{1}{1-\rho^{-(K+1)}}\left[z^{n-1}\right]\left(\frac{1}{1-z}+\frac{(1-\rho)}{2\rho(1-\rho_{\bar{s}})}\psi(z)\right)\\ &= -\left(\frac{4\rho_s^2}{\rho(\rho-1)^3}-\frac{\rho_s\rho_{\bar{s}}}{\rho(1-\rho_{\bar{s}})}\left(\frac{1}{\rho-1}-\frac{\rho-1}{(1+\rho_s-\rho_{\bar{s}})^2}\right)\right)\left(\frac{4\rho_s}{(1+\rho)^2-4\rho_{\bar{s}}}\right)^{n-1}\\ &\quad\times\frac{\sqrt{(1+\rho)^2-4\rho_{\bar{s}}}\,n^{-3/2}}{(1-\rho_{\bar{s}})\sqrt{\pi}}\left(1+\mathrm{o}\left(1\right)\right).\end{aligned}$$

To obtain $\tilde{P}^s_\rho(>1,n)$, we proceed in the similar way. We shall identify the singularities of $\Psi_0(z)$. From Eq. (32) we have

$$\begin{aligned}\Psi_0(z) &= \Psi_1(z)-\frac{2\rho_s}{1-z}-\frac{2\rho_{\bar{s}}\rho_s}{1-\rho_{\bar{s}}}+\frac{\rho_s(1-\rho)}{(1-z)^2}\left(1+\sqrt{1-\frac{4\rho_s}{(1-\rho)^2}(z-1)}\right)\\ &\quad-\frac{\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\frac{1}{1-z}\left((\rho_{\bar{s}}-1)^2+\rho_s^2-((\rho_{\bar{s}}-1)^2-\rho_s^2)\sqrt{1-\frac{4\rho_s}{(1-\rho)^2}(z-1)}\right)\\ &= -\frac{2\rho_s}{1-z}-\frac{2\rho_{\bar{s}}\rho_s}{1-\rho_{\bar{s}}}+\frac{\rho_s(1-\rho)}{(1-z)^2}\left(2+\sum_{j=1}^{\infty}c_j\left(\frac{4\rho_s}{(1-\rho)^2}(z-1)\right)^j\right)\\ &\quad-\frac{\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\frac{1}{1-z}\left(2\rho_s^2-((\rho_{\bar{s}}-1)^2-\rho_s^2)\sum_{j=1}^{\infty}c_j\left(\frac{4\rho_s}{(1-\rho)^2}(z-1)\right)^j\right)\\ &\quad+\Psi_1(z) = \frac{2\rho_s(1-\rho)}{(1-z)^2}+2\rho_s\left(\frac{\rho_s+\rho-1}{1-\rho}-\frac{\rho_s\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\right)\frac{1}{1-z}-\frac{2\rho_{\bar{s}}\rho_s}{1-\rho_{\bar{s}}}+\Psi_2(z)+\Psi_1(z),\end{aligned}$$

where

$$\begin{aligned}\Psi_2(z) &= \rho_s(1-\rho)\sum_{j=2}^{\infty}c_j\left(\frac{4\rho_s}{(1-\rho)^2}\right)^j(z-1)^{j-2}\\ &\quad+\frac{\rho_{\bar{s}}(1-\rho)(1+\rho_s-\rho_{\bar{s}})}{1-\rho_{\bar{s}}}\sum_{j=1}^{\infty}c_j\left(\frac{4\rho_s}{(1-\rho)^2}\right)^j(z-1)^{j-1}.\end{aligned}\tag{39}$$

$\Psi_2(z)$ is analytic at $z=1$ and has the same singularities as $\psi_2(z)$.

It is, also, easy to check that

$$\begin{aligned}\Psi_1(z) &= \frac{2\rho_{\bar{s}}\rho_s}{1-\rho_{\bar{s}}}\left(1-\rho_{\bar{s}}^2\right)+\left(2\rho_s-\rho_{\bar{s}}^2\left(1+\rho_s-\rho_{\bar{s}}\right)\right)\frac{\rho_{\bar{s}}(1+\rho_s-\rho_{\bar{s}})}{1-\rho_{\bar{s}}}\sum_{j=1}^{\infty}c_j\left(\frac{4\rho_s}{(1+\rho_s-\rho_{\bar{s}})^2}\right)^j\\ &\quad\times(z-\rho_{\bar{s}})^{j-1}+\rho_{\bar{s}}^3\left(1+\rho_s-\rho_{\bar{s}}\right)^2\sum_{j=2}^{\infty}c_j\left(\frac{4\rho_s}{(1+\rho_s-\rho_{\bar{s}})^2}\right)^j(z-\rho_{\bar{s}})^{j-2}.\end{aligned}\tag{40}$$

$\Psi_1(z)$ is analytic at $z = \rho_{\bar{s}}$ and has the same singularities as $\psi_1(z)$. In addition, it has poles at $z = 1$.

For $\rho = 1$

$$\Psi_0(z) = \frac{2p_s\sqrt{p_s}}{(1-z)^{3/2}} + \frac{2p_{\bar{s}}\sqrt{p_s}}{(1-z)^{1/2}} - \frac{2p_s(1+p_{\bar{s}})}{1-z} - 2p_{\bar{s}} + \Psi_1(z).$$

When z is close to 1 $\Psi_1(z) = \mathrm{O}\left(\sqrt{1-z}\right)$.

We obtain the corresponding equation in Eq. (34) by using the same properties as those used for getting Eq. (33). The expression for $\rho > 1$ is obtained in similar way as we obtained it for $\tilde{P}^s_\rho(>0, n)$, which establishes the proof. □

Next, we examine the asymptotics of the loss probabilities for small ρ. For large ρ the probabilities are close to 1, thus, this last case is of no interest since systems are not supposed to work with such loss probabilities.

Proposition 6. *For $n \geq 1, 0 \leq j < n$, we have for small ρ*

$$P^s_\rho(>j, n) = \rho^K \rho_s^j (n - j + \mathrm{O}(\rho_s)).$$

Proof. For ρ small enough, we have $P^s_\rho(>j, n) = \rho^k \rho_s^j (n - j + \mathrm{O}(\rho))$, as $\rho \to 0$. The function O(1) here depends on ρ and z and it is uniformly bounded in the disk $|z| \leq h$ ($h > 0$ is a small constant) as $\rho \to 0$.

This implies

$$\frac{\delta_{K+1}}{z\delta_K - \delta_{K+1}} = \frac{1}{z-1} + \mathrm{O}(\rho), \qquad \frac{\Phi_K}{z\delta_K - \delta_{K+1}} = p_s + \mathrm{O}(\rho).$$

By substituting this into Eq. (8), since $R_K \cong 1$, we can write

$$P^s_\rho(>j, n) = \rho^K \rho_s^j [z^{n-j-1}] \left(\frac{1}{(1-z)^2} + \mathrm{O}(\rho) \right) = \rho^K \rho_s^j (n - j + \mathrm{O}(\rho)).$$

(We used the fact that if the function $\mathrm{O}(1) = \mathrm{O}(z, \rho)$ is uniformly bounded in the small disk $|z| < h$ as $\rho \to 0$, all its coefficients $[z^n]\mathrm{O}(z, \rho)$ are also bounded as $\rho \to 0$.)

In particular $P^s_\rho(>0, n) = \rho^k(n + \mathrm{O}(\rho_s))$, $P^s_\rho(>1, n) = \rho^{K+1}\rho_s(n - 1 + \mathrm{O}(\rho))$. □

5. When is it better to add redundancy

In this section we compare the loss probabilities of a whole group of n consecutive packets, which we call a block, with and without j additional redundant packets. The group of packets that includes the original block plus the additional redundant packets (if these are added) is called a frame. We still assume that the process of arrivals of packets is Poisson. If the number of packets of a frame (containing $j + n$ packets) that reach the destination is at least n then all the packets that have not reached the destination can be reconstructed. If not, all the packets of the frame are considered to be lost. In this section we restrict ourselves to the case of $j = 0$ and $j = 1$.

Without loss of generality, we may rescale time so that the service rate is one: $\mu = 1$. We assume that the rate at which frames arrive is the same for the two cases and is given by $x = p_{\bar{s}}x + p_{sx} = x' + x''$. We distinguish two cases:

(1) Adding redundancy for all sources. Hence, the rate at which packets arrive is $\lambda = \rho = (n+1)x$.
(2) Adding redundancy only for source s; the workload is then $\lambda = \rho = nx + x''$ and $\rho_{\bar{s}}$ stays the same.

The frame is lost, in both last cases, if and only if more than one packet is lost out of $n+1$ consecutive ones coming from source s.

We are thus interested in the difference

$$\Delta = \begin{cases} P^s_{nx}(>0,n) - P^s_{(n+1)x}(>1,n+1) & \text{if all sources add redundancy,} \\ P^s_{nx}(>0,n) - P^s_{nx+x''}(>1,n+1) & \text{if only source } s \text{ adds redundancy.} \end{cases}$$

If $\Delta > 0$ then the redundancy decreases the loss probabilities of frames.

Proposition 7. *For any n and K, adding redundancy results in a decrease of the loss probabilities for all x small enough (light traffic regime).*

Proof. We consider case 1. Case 2 follows similarly. From Proposition 6 we have $P^s_{nx}(>0,n) = (nx)^K(n + \mathrm{O}(nx''))$ and

$$P^s_{(n+1)x}(>1,n+1) = ((n+1)x)^{K+1} p_s(n + \mathrm{O}(nx'')).$$

The proof now follows by noting that

$$\lim_{x\to 0} \frac{P^s_{(n+1)x}(>1,n+1)}{P^s_{nx}(>0,n)} = \lim_{x\to 0} \left(\frac{n+1}{n}\right)^K (n+1)p_s x = 0 < 1. \qquad \square \tag{41}$$

6. Numerical examples

We have shown that adding redundancy is profitable in light traffic. A natural question is how small should the traffic be in order for this conclusion to hold in practice.

Below, we fix ρ, p_s and obtain a set of n and K for which redundancy will lead to better performance and for which the loss probability of frames is of a given order (e.g.: 10^{-8}). We shall restrict to a family of n and K that are inter-related by $n \cong \eta K$, where η is a constant to be determined, and we shall consider $n \gg 1$. In fact, the approximations turn out to be quite accurate even for moderate values of n and K.

In Fig. 1, we display the values of $P^s_\rho(>0,n)$ and its approximations (from Proposition 5):

$$A_0(>0,n) \triangleq \frac{R_K\rho^K}{(1-\rho_{\bar{s}})}\left((1-\rho)n + \rho_s\left(\frac{1}{1-\rho} - \frac{\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\right)\right),$$

$$B_0(>0,n) \triangleq \frac{R_K\rho^K}{(1-\rho_{\bar{s}})}(1-\rho)n$$

in the case $K = 10$, $n \le 10$, $\rho = 0.4$ and $p_s = 0.6$.

In Fig. 2, we make the same comparison for $P^s_\rho(>1,n)$ and

$$A_1(>1,n) \triangleq \frac{R_K\rho^K\rho_s}{(1-\rho_{\bar{s}})^2}\left((1-\rho)n + \frac{\rho+\rho_s-1}{1-\rho} - \frac{\rho_s\rho_{\bar{s}}}{1-\rho_{\bar{s}}}\right),$$

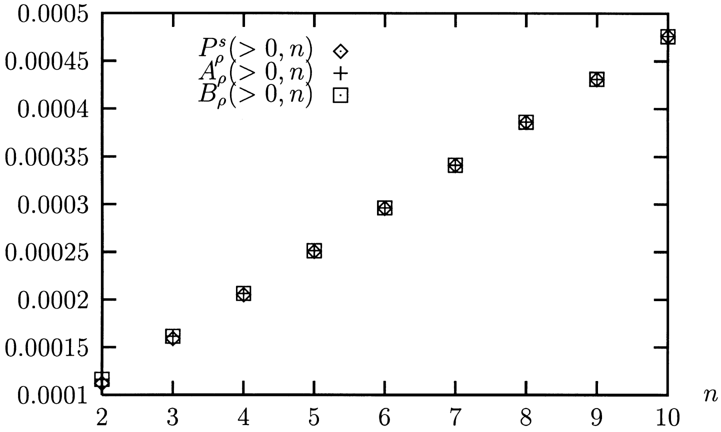

Fig. 1. $P^s_\rho(>0,n)$ and its approximations $\rho=0.4$, $p_s=0.6$ and $K=10$.

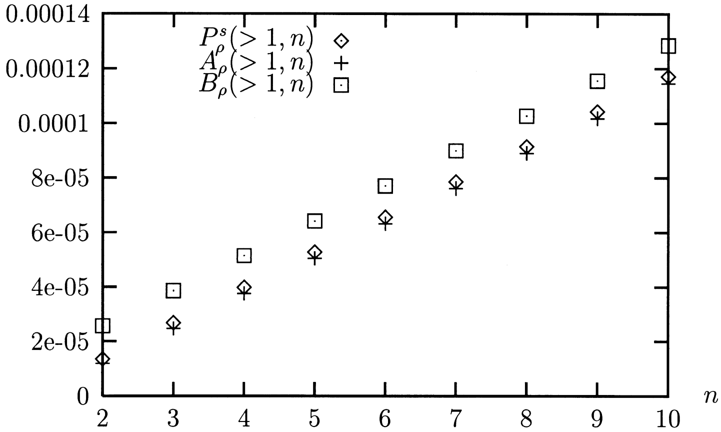

Fig. 2. $P^s_\rho(>1,n)$ and its approximations $\rho=0.4$, $p_s=0.6$ and $K=10$.

$$B_1(>1,n) \triangleq \frac{R_K\rho^K\rho_s}{(1-\rho_{\bar{s}})^2}(1-\rho)n.$$

These approximations are obtained from Proposition 5 ($\rho<1$), by taking the two first and the first term, respectively, in the asymptotic expansion (in n) of $\tilde{P}^s$.

6.1. Adding redundancy for all sources

We wish to determine x^* for which $P^s_{(n+1)x}(>1,n+1)-x^*P^s_{nx}(>0,n)=0$. We shall provide a heuristic approach to obtain the interval $[0,x^*]$ for which we should use redundancy, and confirm this by numerical examples. From Eq. (41), we have for large K

$$x^*=\frac{1}{p_s(n+1)}\left(\frac{n}{n+1}\right)^K \cong \frac{1}{p_s\eta K}\frac{n}{n+1}\left(1-\frac{1}{\eta K}\right)^K \cong \frac{n}{n+1}\frac{\exp(-1/\eta)}{p_s\eta K}.$$

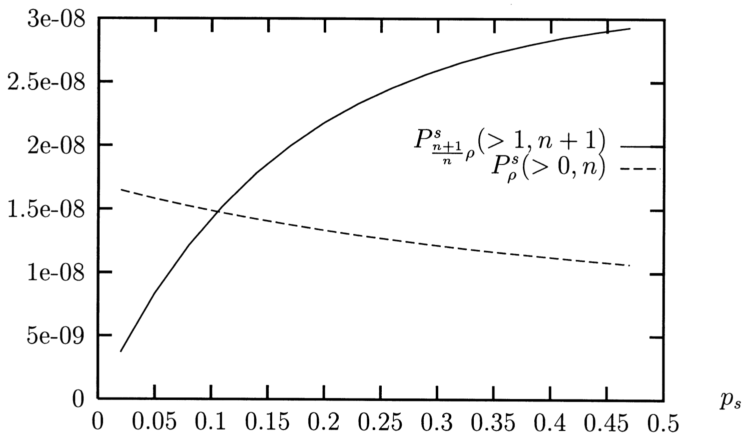

Fig. 3. $P^s_\rho(> j, n+j)$, $j = 0, 1$ as function of p_s for $\rho = 0.6$, $n = 19$ and $K = 39$.

Since $\rho_s^* = p_s x^* \eta K$, we have

$$\rho_s^* \cong \frac{n}{n+1}\exp(-1/\eta) \cong \exp(-1/\eta). \tag{42}$$

More generally, for $\rho_j = (n/(n+j))\rho^*$ adding j packets leads to better performance. This heuristic is quite optimistic and we obtain this experimental result: We fix $\rho = 0.6$ and for $p_s \in]0, p_s^*]$ where $p_s^* = \exp(-1/\eta)$, adding one redundant packet decreases the loss probabilities of frames and for the same values of p_s, for $\rho = (n/(n+j))0.6$, adding j packets leads to better performance than adding $0, \ldots, j-1$ packets.

Example 1. Let $\rho = 0.6$, $p_s = 0.1$, we wish to determine n and K for which the loss probability is of order 10^{-8} and redundancy leads to better performance. It follows from Eq. (42) that $\eta \geq 0.36$. From Proposition 5, we have

$$P^s_{0.6}(>0, n) \cong \frac{1-0.6}{1-0.6^{K+1}}\frac{0.6^K}{1-0.54}(1-0.6)0.36K \cong 10^{-8} \Rightarrow K \cong 39, \quad n = \eta K = 14.04.$$

The exact calculation for $A(\rho) \triangleq P^s_{nx}(>0, n)$ and $S(\rho) \triangleq P^s_{(n+1)x}(>1, n+1)$ yields that for $n = 14$, $A(0.6) = 1.10 \times 10^{-8}$ and $S(0.6) = 2.09 \times 10^{-8}$. For $n = 19$, $S(0.6) = 1.42 \times 10^{-8}$ and $A(0.6) = 1.48 \times 10^{-8}$. We have to take η greater than the value $\eta = 0.36$ obtained above.

If we choose $\eta = 0.5$ we obtain $K = 40$ and $n = 20$; $A(0.6) = 9.39 \times 10^{-9}$ and $S(0.6) = 8.59 \times 10^{-9}$. In Fig. 3, we display the probability with and without redundancy as a function of p_s for $\rho = 0.6$, $K = 39$ and $n = 19$. We note that for the sources whose proportion in the overall arrival stream does not exceed 10% ($p_s \leq 0.1$), redundancy decreases their loss probabilities, but not for the others. We note also, that the loss probability without redundancy decreases in p_s contrary to the loss probability with redundancy which increases in p_s.

In Fig. 4 we show the loss probabilities as function of the number of redundant packets for $\rho = (19/(19+4))0.6 \cong 0.5$, $p_s = 0.1$, $n = 19$, $K = 39$. We remark that adding four packets decreases the loss probabilities more than adding one up to three, but when we exceed four packets the loss probabilities begins to increase. It means that adding more redundant packets doesn't necessary result in

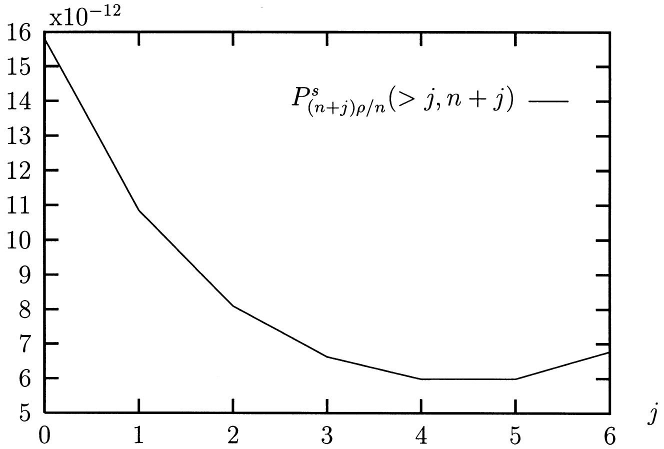

Fig. 4. $P^s_{(n+j)\rho/n}(> j, n+j)$ as function of j for $\rho = 0.5$, $p_s = 0.1$, $n = 19$ and $K = 39$.

a decrease of the loss probabilities. In fact, the adequate number of redundant packets which we should add in order to decrease, as much as possible, the loss probabilities strongly depends on the workload and the size of frames.

6.2. Adding redundancy only for source s

We are interested by the case when redundancy is added only for the source s. We assume that the rate at which frames arrive is the same with and without redundancy. When one redundant packet is added for source s, we have $\lambda = \rho = nx + p_s x$. We proceed similarly as above, we get for large K

$$\rho_s^* = \exp(-p_s/\eta). \tag{43}$$

Example 2. Let $\rho = 0.8$ and $p_s = 0.1$, we wish to determine n and K for which the loss probability is of order 10^{-9} and redundancy leads to better performance. From Eq. (43) we have $\eta = 0.04$ and from Proposition 5 we have

$$P^s_{0.8}(> 0, n) \cong \frac{1-0.8}{1-0.8^{K+1}} \frac{0.8^K}{1-0.72}(1-0.8)0.04K \cong 10^{-9} \Rightarrow K \cong 90, \quad n = \eta K \cong 4.$$

Exact calculation for $A(\rho) \triangleq P^s_{nx}(> 0, n)$ and $S'(\rho) \triangleq P^s_{nx+p_s x}(> 1, n+1)$ yields that for $n = 4$ and $K = 90$, $S(0.8) = 2.67 \times 10^{-9}$ and $A(\rho) = 1.28 \times 10^{-9}$ and for $n = 7$, $K = 90$ we have $S(0.8) = 1.86 \times 10^{-9}$ and $A(\rho) = 2.12 \times 10^{-9}$, thus for $n \geq 7$, $S(\rho) < A(\rho)$. We display in Fig. 5 the loss probabilities as function of the number of redundant packets for $n = 8$, $K = 90$, $\rho = 0.8$ and $p_s = 0.1$. We note that the larger size of frames leads to better performance in the presence of redundancy. For the same example, by taking $n = 14$, we reduce the loss probabilities of frames by an order of 10 when we add 4 redundant packets only for source s (Fig. 6). Note that this improvement has a negative effect on the other sources.

For the example above, if we consider only two sources s and $\bar{s}$, we have $p_{\bar{s}} = 0.9$ and $P^{\bar{s}}_{0.8} = 2.114 \times 10^{-9}$. When we add 4 redundant packets for s the workload becomes $\rho = 0.8 + \frac{4}{14}0.08 = 0.8228$, $p_s = \frac{18}{14}0.1 = 0.1285$ and $p_{\bar{s}} = 1 - 0.1285 = 0.8715$. For these values, we obtain $P^{\bar{s}}_{0.8228} = 2.305 \times 10^{-8}$.

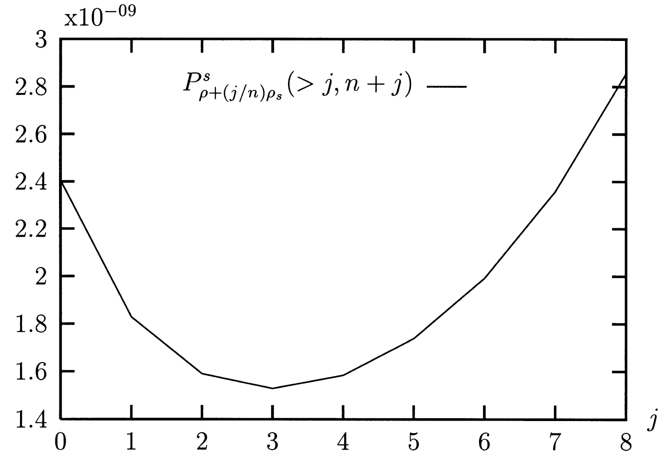

Fig. 5. $P^s_{\rho+(j/n)\rho_s}(> j, n+j)$ as function of j for $\rho = 0.8$, $p_s = 0.1$, $n = 8$ and $K = 90$.

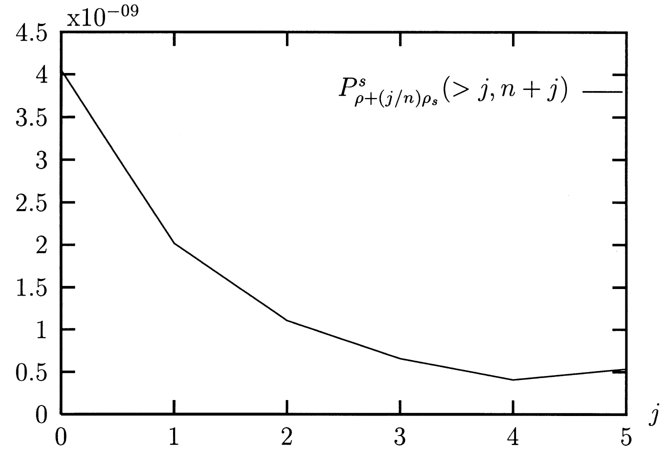

Fig. 6. $P^s_{(\rho+(j/n)\rho_s)}(> j, n+j)$ as function of j for $\rho = 0.8$, $p_s = 0.1$, $n = 14$ and $K = 90$.

Finally, we see that the redundancy in source s has an effect of increasing the loss probability for the other source (by the same order of magnitude as the decrease in the loss probabilities of s).

7. Eliminating frame losses

In this section we show that there exists a way of adding redundancy that yields arbitrarily small frame loss probabilities as long as $\rho < 1$. We compute the amount of redundancy that has to be added.

Suppose first, there is only one source of packets. We consider a redundancy rate of rate j/n i.e. we wish to add an amount of j redundant packets per group of n information packets.

However, instead of fixing n and studying the impact of the additional redundant packets, we fix here the rate j/n, and study the effect of using larger blocks. In other words, we are interested in the impact of grouping kn information packets together with kj redundant packets into a single frame, for large k. We shall show that for any $\rho < 1$, there exist rates j/n such that the frame loss probability is exponentially small in k. Hence, all the kn packets in a frame are reconstructed with a sufficiently large probability, which tends to 1 as $k \to \infty$. We further compute the required rates.

We further show that beginning from some initial large k, one can group consecutively k, $(k+1)$, $(k+2), \dots$ groups of n packets together with additional kj, $(k+1)j$, $(k+2)j, \dots$ redundant packets respectively. Then with a large probability *all* the transmitted frame will be reconstructed.

Fixing the rate of redundancy to j/n results in the rate of arrivals of packets of $\rho(n+j)/n$. Let us turn to the rigorous statements.

Lemma 8. (1) *Suppose that*

$$\frac{(\rho(n+j)/n)^K}{\sum_{l=0}^{K}(\rho(n+j)/n)^l} < \frac{j}{n+j}. \tag{44}$$

Then there exists a constant $h_0 = h_0(\rho, n, j) > 0$ *such that for all* $k > 0$

$$P\,(< kj, k(n+j)) \geq 1 - \exp(-h_0 k). \tag{45}$$

Moreover for some constant $h = h(\rho, n, j) > 0$ *and all* $k > 0$

$$P\left(\bigcap_{l=0}^{\infty}\{< (k+l)j, (k+l)(n+j)\}\right) \geq 1 - \exp(-hk). \tag{46}$$

(2) *Suppose now that the inequality inverse to* Eq. (44) *holds. Then there exists a constant* $h_1 = h_1(\rho, n, j) > 0$ *such that for all* $k > 0$.

$$P\,(< kj, k(n+j)) \leq \exp(-h_1 k). \tag{47}$$

Remark 9. The relation in Eq. (44) has the following interpretation. The left-hand side is the loss probability of an M/M/1 queue whose load, $\rho(n+j)/n$, corresponds to that obtained by adding j redundant packets for every group of n. The right-hand side is the maximal loss recovery rate that can be obtained due to the redundancy: as long as the rate of losses is less than $j/(n+j)$ we may expect for large enough blocks that all losses in the block can be recovered with high probability.

Remark 10. We shall estimate h_0 appearing in Eq. (45) at the end of the section for $K = 1$.

Proof. Let us consider $k(n+j)$ consecutive arrivals of packets. Let η_i be a number of the packets in the buffer just before the ith arrival of a packet and let $\xi_i = 1\{\eta_i = K\}$. This means that $\xi_i = 1$ whenever the ith packet is lost and $\xi_i = 0$ otherwise. Then $\xi_1 + \xi_2 + \cdots + \xi_{k(n+j)}$ is the number of lost packets among $k(n+j)$ consecutive arriving packets and

$$\frac{\xi_1 + \xi_2 + \cdots + \xi_{k(n+j)}}{k(n+j)} \to \frac{(\rho(n+j)/n)^K}{\sum_{l=0}^{K}(\rho(n+j)/n)^l} \tag{48}$$

in probability as $k \to \infty$ with exponential rate (see below). In fact, the sequence η_i forms an irreducible aperiodic finite Markov chain on the state space $\{0, 1, \ldots, K\}$. The fraction in the left-hand side of Eq. (48) is the empirical measure of the time spent by this chain at the state K. It is well-known that it converges in probability to a stationary probability of the state K, which is in the right-hand side of Eq. (48). Assume that Eq. (44) holds. Then we can choose $\varepsilon > 0$ such that

$$\frac{(\rho(n+j)/n)^K}{\sum_{l=0}^{K}(\rho(n+j)/n)^l} + \varepsilon < \frac{j}{n+j}.$$

Then for some $h_0 > 0$ (which is a function of ε) and all $k > 0$

$$\begin{aligned} P\,(< kj, k(n+j)) &= P\left(\xi_1 + \xi_2 + \cdots + \xi_{k(n+j)} < kj\right) \\ &\geq P\left(\frac{\xi_1 + \xi_2 + \cdots + \xi_{k(n+j)}}{k(n+j)} < \frac{(\rho(n+j)/n)^K}{\sum_{l=0}^{K}(\rho(n+j)/n)^l} + \varepsilon\right) \\ &\geq 1 - \exp(-h_0 k). \end{aligned} \tag{49}$$

The last inequality follows from standard Large deviation arguments, see e.g. section 3.1 in [7]. Thus Eq. (45) is proved. The inequality Eq. (47) is derived by the same way from the convergence Eq. (48).

To get Eq. (46) we note that the convergence in Eq. (48) does not depend on an initial distribution, then for all $l > 0$

$$P\left(\{< (k+l)j, (k+l)(n+j)\} \Big| \bigcap_{i=0}^{l-1}\{< (k+i)j, (k+i)(n+j)\}\right) \geq 1 - \exp(-h_0(k+l)).$$

Then for some $h > 0$ and all $k > 0$

$$P\left(\bigcap_{l=0}^{\infty}\{< (k+l)j, (k+l)(n+j)\}\right) \geq \prod_{l=0}^{\infty}[1 - \exp(-h_0(k+l))] \geq 1 - \exp(-hk).$$

This establishes the proof. □

In the following lemma we show the way to find j for given $K > 0$, $\rho < 1$ and n such that the proposed strategy is valid, i.e. the inequality Eq. (44) is fulfilled.

Lemma 11. *Let us fix $K > 0$ and $\rho > 0$.*

If $\rho \geq 1$, the inequality Eq. (44) *does not hold for any pair of integers $j > 0$ and $n > 0$.*

If $\rho < 1$, then for all sufficiently large n, we can find an integer $j > 0$ such that Eq. (44) *holds. Moreover, the minimal j with this property is such that:*

$$j/n < (1-\rho)/\rho, \quad \text{if } K > \rho/(1-\rho); \qquad j/n > (1-\rho)/\rho, \quad \text{if } K < \rho/(1-\rho).$$

Proof. Denote by $\alpha := j/n$. Then, except when $\rho(1+\alpha) = 1$, Eq. (44) can be written as

$$\frac{(\rho(1+\alpha))^K\,(\rho(1+\alpha) - 1)}{(\rho(1+\alpha))^{K+1} - 1} < \frac{\alpha}{1+\alpha}. \tag{50}$$

This inequality holds for no $\alpha \geq 0$ under the assumption $\rho \geq 1$. If $\rho < 1$, then Eq. (50) implies

$$\begin{cases} \rho^K(1+\alpha)^{K+1} > \alpha/(1-\rho), \\ \rho(1+\alpha) > 1, \end{cases} \quad \text{and} \quad \begin{cases} \rho^K(1+\alpha)^{K+1} < \alpha/(1-\rho), \\ \rho(1+\alpha) < 1. \end{cases} \tag{51}$$

Let us introduce the function $f(\alpha) = \rho^K(1+\alpha)^{K+1} - \alpha/(1-\rho)$, which equals zero when $\alpha = 1/\rho - 1 = (1-\rho)/\rho$. The systems (51) mean that

$$f(\alpha) < 0 \quad \text{if } \alpha < (1-\rho)/\rho; \qquad f(\alpha) > 0 \quad \text{if } \alpha > (1-\rho)/\rho. \tag{52}$$

- Assume that $K > \rho/(1-\rho)$. This amounts to say that $f'((1-\rho)/\rho) > 0$. Then in the neighborhood of $\alpha = (1-\rho)/\rho$, we have Eq. (52). The elementary analysis of $f(\alpha)$ shows that there exists $\alpha_0 < (1-\rho)/\rho$, such that $f(\alpha_0) = 0$, $f(\alpha) > 0$ on $[0; \alpha_0)$, $f(\alpha) < 0$ on $(\alpha_0; (1-\rho)/\rho)$ and $f(\alpha) > 0$ on $((1-\rho)/\rho; \infty)$. To get Eq. (44), we take $\alpha \in (\alpha_0; \infty)$, or equivalently $j \in (n\alpha_0; \infty)$.
- Suppose now that $K < \rho/(1-\rho)$, i.e. in other words $f'((1-\rho)/\rho) < 0$. Then Eq. (52) does not hold in the neighborhood of $\alpha = (1-\rho)/\rho$ neither on $[0; (1-\rho)/\rho]$. However, there exists some $\alpha_0 > (1-\rho)/\rho$ such that $f(\alpha) > 0$ on $[\alpha_0; \infty)$. So, to obtain Eq. (44), we take the minimal integer j on $[n\alpha_0; \infty)$.

Note that for $\rho > 1$ the suggested strategy does not fit at all. □

To complete our investigation, we will also specify the estimation (45) from Lemma 8 in the case $K = 1$. The sequence η_i forms a Markov chain $\mathcal{L}$ on the state space $\{0, 1\}$ with the matrix of transition probabilities:

$$\begin{pmatrix} \dfrac{1}{1+\rho(n+j)/n} & \dfrac{\rho(n+j)/n}{1+\rho(n+j)/n} \\ \dfrac{1}{1+\rho(n+j)/n} & \dfrac{\rho(n+j)/n}{1+\rho(n+j)/n} \end{pmatrix}.$$

Let $\zeta_1, \zeta_2, \ldots$ be a sequence of i.i.d. random variables distributed as the time to return to the state 1 starting from it by the chain $\mathcal{L}$. Indeed, $E\zeta_1 = (1+\rho(n+j)/n)(\rho(n+j)/n)^{-1}$. Let also ζ_0 be a random variable distributed as the time to reach the state 1 at the stationary regime. Then, for all $\delta > 0$ such that $E\exp(\delta\zeta_i) < \infty$, $i = 0, 1$, and all n, j and k by Chernof inequality we have:

$$\begin{aligned} P(> kj, (n+j)k) &= P\left(\zeta_0 + \zeta_1 + \cdots + \zeta_{kj} \leq k(n+j)\right) \\ &\leq E\exp(\delta\zeta_0)\,(E\exp\delta(\zeta_1 - (n+j)/j))^{kj} \\ &= E\exp(\delta\zeta_0)\exp\left(\left[\log(E\exp\delta\zeta_1) - \delta(n+j)/j\right] jk\right). \end{aligned} \tag{53}$$

It is easy to verify that

$$E\exp(\delta\zeta_0) = \frac{\rho(n+j)/n}{1+\rho(n+j)/n - \exp\delta}, \qquad E\exp(\delta\zeta_1) = \frac{\rho(n+j)/n\exp\delta}{1+\rho(n+j)/n - \exp\delta}.$$

Let us assume that Eq. (44) holds, i.e.

$$\frac{j}{n+j} - \frac{\rho(n+j)/n}{1+\rho(n+j)/n} > 0. \tag{54}$$

Then the function $f(\delta) = -\log(E \exp \delta\zeta_1) + \delta(n+j)/j$ is increasing on $[0, \delta_0]$ and is decreasing on $[\delta_0, \ln(1+\rho(n+j)/n)]$, $(f(0)=0)$, where $\delta_0 = \ln(\rho + n/(n+j))$.

The estimation (53) with $\delta = \delta_0$ implies

$$P\,(> kj, (n+j)k) \le E \exp(\delta_0\zeta_0) \exp(-jf(\delta_0)k)$$

for all $k > 0$, where

$$E \exp(\delta_0\zeta_0) = \frac{\rho(n+j)/n}{1+\rho(n+j)/n}\frac{n+j}{j},$$

$$f(\delta_0) = -\ln\left(\rho\frac{n+j}{n}\right) + \frac{n+j}{j}\ln\left(\rho + \frac{n}{n+j}\right).$$

The constant $jf(\delta_0)$ is a Large deviation constant.

Let us now proceed with the case of many sources of packets. Suppose, we are interested in decreasing the losses of frames or of packets issued only from one source s. Hence, we add j redundant packets from n, originated from s, thus the total rate is $\rho_s(n+j)/n + \rho_{\bar{s}}$. Let as usual $\rho_s = \lambda_s/\mu$, $\rho_{\bar{s}} = \lambda_{\bar{s}}/\mu = (\lambda - \lambda_s)/\mu$. The strategy, that we use, is the same as in the previous case: kn, $(k+1)n, \dots$ packets from the source s are grouped together with the redundancy of kj, $(k+1)j, \dots$ respectively. In the case of one source, only for $\rho < 1$, there is a suitable j to render the strategy profitable. In the case of many sources, to find such a j, the restriction $\rho_s < 1$ remains necessary. However, the inequality $\rho = \rho_s + \rho_{\bar{s}} > 1$ is accepted.

Lemma 12. (1) *Suppose that*

$$\frac{(\rho_s(n+j)/n + \rho_{\bar{s}})^K}{\sum_{l=0}^{K}(\rho_s(n+j)/n + \rho_{\bar{s}})^l} < \frac{j}{n+j}. \tag{55}$$

Then there exists $h_0 > 0$ *such that for all* $k > 0$

$$P\,(< kj, k(n+j)) \ge 1 - \exp(-h_0 k). \tag{56}$$

Moreover for some $h > 0$ *and all* $k > 0$

$$P\left(\bigcap_{l=0}^{\infty}\{< kj, k(n+j)\}\right) \ge 1 - \exp(-hk). \tag{57}$$

(2) *Suppose that the inequality inverse to Eq.* (55) *holds. Then for some* $h_1 > 0$ *and all* k $P\,(< kj, k(n+j)) \le \exp(-h_1 k)$.

The proof is completely analogous to the proof of Lemma 8.

Lemma 13. *Assume that* $\rho_s > 0$, $\rho_{\bar{s}} > 0$, K *and* n *are fixed. There exists an integer* j *satisfying Eq.* (55) *if and only if* $\rho_s < 1$.

Proof. The proof is carried out as the proof of Lemma 11. Without going into details, we will point out the way to find the minimal integer j satisfying Eq. (55).

Denote by $\alpha := j/n$, then the inequality Eq. (55) is equivalent to the following:

$$\frac{(\rho_s(1+\alpha)+\rho_{\bar{s}})^K\,(\rho_s(1+\alpha)+\rho_{\bar{s}}-1)}{(\rho_s(1+\alpha)+\rho_{\bar{s}})^{K+1}-1} < \frac{\alpha}{1+\alpha}, \tag{58}$$

except when $\rho_s(1+\alpha)+\rho_{\bar{s}}=1$. It holds for no $\alpha \geq 0$ if $\rho_s \geq 1$.

- Assume that $\rho = \rho_s + \rho_{\bar{s}} > 1$, $\rho_s < 1$. Then, to get Eq. (58), we should take α satisfying the system

$$\begin{cases} (\rho_s(1+\alpha)+\rho_{\bar{s}})^K > \dfrac{\alpha}{(1-\rho_s)\alpha+1-\rho}, \\ \alpha > \dfrac{\rho_{\bar{s}}}{1-\rho_s} - 1. \end{cases}$$

  There exists minimal α_0 such that the system holds on $(\alpha_0; \infty)$, i.e. $j \in (n\alpha_0; \infty)$.
- Assume that $\rho = 1$. Then we have to take α satisfying the inequality $(1+\alpha\rho_s)^K > 1/(1-\rho_s)$.
- Assume now that $\rho < 1$. There are two cases.
  If $K > \rho_s/(1-\rho)$ then there is the minimal $\alpha_0 < (1-\rho_{\bar{s}})/\rho_s - 1 = (1-\rho)/\rho_s$ such that on the segment $(\alpha_0; (1-\rho)/\rho_s)$ the inequality

$$(\rho_s(1+\alpha)+\rho_{\bar{s}})^K < \frac{\alpha}{(1-\rho_s)\alpha+1-\rho}$$

  holds. We take $j \in (n\alpha_0; n(1-\rho)/\rho_s)$.
  If $K < \rho_s/(1-\rho)$ then there exists the minimal $\alpha_0 > (1-\rho)/\rho_s$ such that the inequality

$$(\rho_s(1+\alpha)+\rho_{\bar{s}})^K > \frac{\alpha}{(1-\rho_s)\alpha+1-\rho}$$

  holds on $(\alpha_0; \infty)$. We take $j \in (n\alpha_0; \infty)$. $\square$

8. General arrivals and service times

We relax in this section the probabilistic assumptions on the distributions of the arrival and service processes: we consider a stationary ergodic sequence $\{\sigma_n\}$, $n \in \mathbb{Z}$ (where $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$) of service times, and a stationary ergodic sequence $\{\tau_n\}$, $n \in \mathbb{Z}$, of interarrival times of packets.

We consider a finite queue with capacity $K \geq 1$. Define $\rho \triangleq E\sigma_1/E\tau_1$.

8.1. Basic idea

We present in this subsection the general idea behind the elimination of losses. Assume that the process $\{\tau_n, \sigma_n\}$ is already the one observed after we included the redundancy of rate j: for each information packet we added j redundant ones. We denote by $\lambda(j+1)$ the input arrival rate of packets. Assume that this process feeds a finite FIFO queue, and that the joint process of arrivals and queue length is stationary ergodic.

We call a frame a sequence of $(j+1)k$ consecutive packets, where k is some parameter. We assume that all packets from the frame can be recovered if there are no more than jk losses within the frame.

Assume that the following reasonable property is satisfied: as $j \to \infty$, the output rate (the expected number of departures per time unit) from the queue approaches $\mu := 1/E\tau_1$. Fix $\varepsilon > 0$ such that $\rho + \varepsilon < 1$ and let j be such that the output rate from the queue is greater than $\mu(1-\varepsilon)$. Then the proportion p_j of lost packets satisfies

$$p_j = \frac{\text{input rate} - \text{output rate}}{\text{input rate}} < \frac{(j+1)\lambda - \mu(1-\varepsilon)}{(j+1)\lambda} = \frac{j}{j+1} + \frac{\rho+\varepsilon-1}{(j+1)\rho} < \frac{j}{j+1}.$$

Due to the stationarity and ergodicity assumptions, for any $\delta > 0$, the number of losses within a frame is less than $kj(p_j + \delta)$ with probability that approaches 1 as $k \to \infty$.

Thus with probability arbitrarily close to one, the actual number of losses per frame will be smaller than kj by choosing j and k sufficiently large, and all losses in the frame can be recovered.

8.2. Actual redundancy scheme

We restrict here to the case where the service times are i.i.d. and are independent of the interarrival times and $K = 1$.

We shall assume throughout that $\rho < 1$ before adding redundancy. Under this assumption, we show that by adding appropriately redundant packets, one can obtain loss probabilities as small as desired. We assume that the sequence of interarrival times $\{\tau_n\}$, $n \in Z$ of the original information packets (before we add redundancy) is stationary ergodic.

For some integer k that will be determined later, we call the group of packets number $nk+1, nk+2, \ldots, (n+1)k$ the nth block of information packets. We shall add jk redundant packets to the each block. The $(j+1)k$ packets which include the original block as well as the additional redundant packets are called a frame. We assume that the service times of the redundant packets added to a block have the same distribution as σ_1; σ_n will in fact denote the service time of the nth packet actually served, whether it is an information packet or a redundant one.

As long as the number of losses in a frame is less than or equal to jk, all the frame (and in particular, the original information packets) can be retrieved at the destination. Define

$$r = \frac{E\tau_1 + E\sigma_1}{2(j+1)}$$

and consider the following transport protocol:

(1) *Blocking phase*: Wait till a whole block of k information packets is generated at the source; as long as the whole block is not generated, we do not transmit any packet of the block.
(2) *Framing phase*: once all k packets have arrived, we compute the extra jk redundant packets.
(3) *Transmission phase*: Once the whole frame has been generated, all packets of the frame are put in a transmission buffer. Packets are transmitted from this buffer at a constant rate $1/r$, i.e. the time between transmission of two consecutive packets is r.

The above protocol requires buffering capability at the source of at least one frame. To make our protocol realistic we have to assume that

- The capacity of the transmission buffer at the source is finite.

This implies that losses may occur also due to buffer overflow at the source, and not only at the buffer inside the network. We shall show that the above protocol can render the total loss probabilities arbitrarily small.

Note that congestion at the source occurs typically at periods during which interarrival times are short. In order to minimize the buffer requirements at the source we shall thus assume that

• we deliberately drop the nth frame at the source if and only if $\sum_{i=1}^{k} \tau_{nk+i} < kr(j+1)$. In that case, the framing and transmission phase for frame n are not performed.

In case that the computation required in the framing phase takes a negligible amount of time, this assumption on dropping at the source implies that the total buffering required at the source is exactly of one frame and is thus minimal.

We shall assume that the above protocol has been used for at least one frame before packet 1 is transmitted, and that before it was used, the system was empty.

Theorem 14. *With $\rho < 1$, the above protocol results in frame loss probabilities that can be made arbitrarily small by an appropriate choice of k and j.*

Proof. Consider an arbitrary frame, say frame 1, and let $\hat{T}$ be the time at which the first packet of that frame arrives to the buffer inside the network. Define

$$\Omega_1 \triangleq \left(\sum_{i=1}^{k} \sigma_i + \eta > jkr\right), \quad \Omega_2 \triangleq \left(\sum_{i=1}^{k} \tau_i < (j+1)kr\right),$$

where η is the residual service time in the buffer inside the network at $\hat{T}$ (the packet that arrives at the network buffer might find there another packet from some previous frame that is still getting service; the remaining service time of that packet is called η, and it is considered to be zero if there is no such packet at time $\hat{T}$).

Let $\varepsilon > 0$ be an arbitrary small number. One can choose j and k such that $P(\Omega_1) < \varepsilon$ and $P(\Omega_2) < \varepsilon$. That this choice is possible follows since $\rho < 1$, since $P -$ a.s.

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \sigma_i = E\sigma_1 < jr, \quad \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} \tau_i = E\tau_1 > (j+1)r$$

and since for all $\delta > 0$ $P(\eta > k\delta) \to 0$ as $k \to \infty$. This fact needs some additional explanation (note that the distribution of η might depend on k and on the number of the frame that we consider). Let $\hat{S}$ be the time at which the last successful packet transmission occurred from the buffer inside the network before time $\hat{T}$. Let A_n denote the event that the number of packets that were blocked (and thus lost) in the buffer inside the network during the time interval $(\hat{S}, \hat{T})$ was exactly n. $n \geq 0$. (In the case $n = 0$ the last packet of the previous frame is served.) Then $P(\{\eta > k\delta\} \cap A_n) \leq P(\sigma_1 > nr + k\delta)$. Since $E\sigma_1 < \infty$,

$$P(\eta > k\delta) \leq \sum_{n=0}^{\infty} P(\sigma_1 > nr + k\delta) \to 0 \quad \text{as } k \to \infty. \tag{59}$$

On the event $\bar{\Omega}_2$ (the complement of Ω_2) the new frame is not dropped at the transmission buffer.

On $\bar{\Omega}_1$, the time T till the k first successful transmissions of packets occurs satisfies $T \leq \eta + \sum_{i=1}^{k} \sigma_i + kr \leq (j+1)kr$. Thus the number of packets successfully transmitted on the event $\bar{\Omega}_1 \cap \bar{\Omega}_2$ among the first frame is at least k, so that the probability of a successful transmission of the whole first frame is at least $1 - 2\varepsilon$. (Indeed, no later than $\hat{T} + \eta + r$, the first successful transmission in the frame begins, no later than $\hat{T} + \eta + \sigma_1 + 2r$, the 2nd successful transmission begins, etc.).

The same argument holds for any frame; since the bound in Eq. (58) is uniform for all frames, this establishes the proof. □

9. Discussion

In this paper, we have shown the effect of adding redundancy to losses of packets and of frames due to overflow in a finite queue. Explicit expressions for the loss probabilities of frames were obtained in the case of several traffic streams that are multiplexed at the input of a finite buffer. We have obtained schemes of adding redundancy that may almost eliminate loss probabilities for any given buffer size as long as the offered load of the traffic to which redundancy is added is lower than 1 (before adding the redundancy). The price to pay is long delays due to the need to consider redundancy of large blocks. The analysis of the required delay and the tradeoff between losses and delay are the issue of future work.

Appendix A

We return to the asymptotic behavior of $P^s_\rho(> j, n)$ and show that the terms $[z^{n-1}] f_0(z)$ and $[z^{n-1}] f_1(z)$ can be neglected as $n \to \infty$, $n < K$.

Proposition A.1. *We have*

$$P^s_\rho(> 0, n) = \tilde{P}^s_\rho(> 0, n)(1 + \mathrm{o}(1)), \quad \textit{as } n \to \infty,\ n < K \tag{A.1}$$

$$P^s_\rho(> 1, n) = \tilde{P}^s_\rho(> 1, n)(1 + \mathrm{o}(1)), \quad \textit{as } n \to \infty,\ n < K. \tag{A.2}$$

Moreover $\mathrm{o}(1)$ *is exponentially small in K (there exists* $0 < \beta < 1$: $|\mathrm{o}(1)| \leq \beta^K$).

Proof. Let us first prove Eq. (A.1). We have

$$\begin{aligned} P^s_\rho(> 0, n) &= -R_K \rho^K [z^{n-1}] \frac{1}{1-z} \frac{x_1}{z - x_1} \frac{1 - (x_2/x_1)^{K+1}}{1 - (x_2/x_1)^K (z - x_2)/(z - x_1)} \\ &= -R_K \rho^K [z^{n-1}] \frac{1}{1-z} \frac{x_1}{z - x_1} \left(1 + \frac{(x_2/x_1)^K \left((z - x_2)/(z - x_1) - x_2/x_1\right)}{1 - (x_2/x_1)^K (z - x_2)/(z - x_1)} \right). \end{aligned}$$

Let us denote for shortness

$$a_{n-1} = [z^{n-1}] \frac{1}{z-1} \frac{x_1}{z - x_1},$$

$$b_{n-1} = [z^{n-1}] \frac{(x_2/x_1)^K \left((z - x_2)/(z - x_1) - x_2/x_1\right)}{1 - (x_2/x_1)^K (z - x_2)/(z - x_1)},$$

$$c_{n-1} = [z^{n-1}]\frac{1}{1-z}\frac{x_1}{z-x_1}\frac{(x_2/x_1)^K\,((z-x_2)/(z-x_1)-x_2/x_1)}{1-(x_2/x_1)^K(z-x_2)/(z-x_1)}.$$

Then $\tilde{P}^s_\rho(>0,n) = -R_K\rho^K a_{n-1}$ and by Eq. (33) there are constants $C_1, C_2 > 0$ such that

$$C_1 < |a_n| < C_2 n < C_2 K. \tag{A.3}$$

Moreover, $P^s_\rho(>0,n) = -R_K\rho^K(a_{n-1}+c_{n-1})$, where $c_{n-1} = \sum_{k=0}^{n-1} a_{n-1-k}b_k$. Thus, it suffices to show that for some $0 < \beta < 1$

$$|c_n| \le |a_n|\beta^K. \tag{A.4}$$

By virtue of Eq. (A.3)

$$|c_{n-1}| \le C_2(K-1)\sum_{k=0}^{n-1}|b_k|. \tag{A.5}$$

Let us now estimate b_k. The function

$$g_0(z) = \frac{(x_2/x_1)^K\,((z-x_2)/(z-x_1)-x_2/x_1)}{1-(x_2/x_1)^K(z-x_2)/(z-x_1)} \quad = \frac{x_2^K z(x_1-x_2)}{x_1\left(x_1^K(z-x_1)-x^{K_2}(z-x_2)\right)}$$

is analytic in the disk $|z| < 1+\varepsilon$ for sufficiently small $\varepsilon > 0$. In fact

- $x_1 \ne 0$ for all z, $(\rho \ne 0)$;
- if $z - x_1 = 0$, then $x_2(z-x_2) \ne 0$;
- the branching point of $x_1(z)$ and $x_2(z)$ is outside the unit disk. The branches $x_1(z)$ and $x_2(z)$ have been chosen in such a way that $|x_2(0)|/|x_1(0)| < 1$. The equality $|x_2(z)|/|x_1(z)| = 1$ takes place only if $\mathrm{Re}\, z = 1 + (1-\rho)^2/4\rho\rho_{\bar{s}} > 1$. The function $|x_2(z)|/|x_1(z)|$ being continuous,

$$\inf_{|z|\le 1+\varepsilon}\left|\frac{x_2(z)}{x_1(z)}\right| = \gamma < 1.$$

Hence for sufficiently large K, $x_1^K(z-x_1) - x_2^K(z-x_2) \ne 0$ for all $z \in \{z\colon |z| \le 1+\varepsilon\}$. In addition for some $C_3 > 0$, $|g_0(z)| \le C_3\gamma^K$. Then for some $C_4 > 0$,

$$|b_k| = \left|\frac{1}{2\pi \mathrm{i}}\int_{|z|=1+\varepsilon}\frac{g_0(z)}{z^{k+1}}\,dz\right| \le C_4\gamma^K(1+\varepsilon)^{-k}.$$

Therefore, taking into account Eq. (A.5), we have for some $0 < \gamma_0 < 1$, $\gamma_0 > \gamma$,

$$|c_{n-1}| \le C_2(K-1)\gamma^K C_4\sum_{k=0}^{n-1}(1+\varepsilon)^{-k} \le \gamma_0^K.$$

This estimation together with Eq. (A.3) implies Eq. (A.4) and thus Eq. (A.1) is proved. Let us now turn to the case $j = 1$.

$$P_\rho^s(> 1, n) = -R_K \rho^K [z^{n-1}] \frac{1}{1-z} \frac{x_1}{z-x_1} \frac{z(1-x_1)}{z-x_1} \frac{1-(x_2/x_1)^{K+1}}{1-(x_2/x_1)^K (z-x_2)/(z-x_1)} \times \frac{1-(x_2/x_1)^K (1-x_2)/(1-x_1)}{1-(x_2/x_1)^K (z-x_2)/(z-x_1)}$$

$$= -R_K \rho^K [z^{n-1}] \frac{1}{1-z} \frac{x_1}{z-x_1} \frac{z(1-x_1)}{z-x_1} (1 + g_1(z) + g_2(z)),$$

where

$$g_1(z) = \frac{x_2^K \left(-x_1^K x_2 (z-x_1)^2 + 2x_1^{K+1} (z-x_2)(z-x_1) - x_1 x_2^K (z-x_2)^2\right)}{x_1 \left(x_1^K (z-x_1) - x_2^K (z-x_2)\right)^2},$$

$$g_2(z) = \frac{x_2^K (x_2^{K+1} - x_1^{K+1})(1-x_2)(z-x_1)^2}{x_1(1-x_1)\left(x_1^K (z-x_1) - x_2^K (z-x_2)\right)^2}.$$

The functions $g_1(z)$ and $g_2(z)$ are analytic in the disk $|z| < 1 + \varepsilon$ due to the same arguments as for $g_0(z)$. (The point $z = 1$, where $x_1(z) - 1 = 0$ can not be a pole of $g_2(z)$ because of $(z - x_1)^2$ in the numerator.) Further, the proof of Eq. (A.2) is carried out along the same lines as of Eq. (A.1), so the other details are skipped. $\square$

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