The distribution of delays of dispersed messages in an M/M/1 queue

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Abstract

We analyze the distribution of the delay of messages in an an infinite capacity M/M/1 queue. A message is composed of n packets, and the arrival of the packets to the queue is Poisson. Our calculations are based on recursive schemes introduced in [2]. We obtain explicit expressions for the Laplace-Stieltjes Transform (LST) of the delays, which enables to obtain exact expressions for the moments of the delay in a complexity smaller than the one obtained by using recursive schemes. We repeat the above calculations for the case that messages are dispersed, i.e. packets from several sources arrive to an M/M/1 queue and are served according to the FIFO discipline. Hence, several packets of other messages may arrive between consecutive packets of a given message.

1 Introduction

The distribution of delay of an arriving packet to a queueing network is well known under a variety of assumptions on the service and interarrival distribution. When a group of n packets is transmitted, one is naturally interested in the delay of the whole group, i.e. the time between the arrival of the first packet till the departure of the last packet of the group. We consider the natural situation where the n packets do not arrive simultaneously into the queue (batch arrival), but are instead dispersed in time, due possibly to nonnegligible transmission times in previous nodes. As indicated in [2], "*understanding the message delay behavior is important for the proper design of timeout mechanisms for data applications, such as end-to-end protocols for reliable transmission in ATM networks where the retransmitted quantity is the message and not the individual cells. Another example is the time-out for message retransmissions in data link protocols such as go-back-N protocol, in which the whole window (or message) is retransmitted*".

We obtain in this paper explicit expressions for the LST of the message delay. Our starting point is a set of functional recursions that were introduced in [2]. The advantage of obtaining explicit expressions rather than restrict to recursions is that it allows computations in lower complexity, and allows to obtain asymptotic expansions. Indeed, we obtain the variance of the delay in a complexity of $O(n)$, compared to a complexity of $O(n^2)$ that is obtained using the functional recursions in [2]. Asymptotic expressions based on the LST will be reported in [1].

In Section 2 we derive the PGF of the message delay for the case of a single session, where packets that arrive to a queue are all from the same source. We further obtain the second moment of the the message delay. We then analyze in Section 3 the case of a multiple session, where packets arrive from several sources so that several packets of messages from other sources may arrive between consecutive packets of a given message.

2 Single Session Systems

2.1 The Model

We consider an M/M/1 queue with an infinite buffer. Service is FIFO, exponentially distributed with parameter μ. Interarrival times of packets are exponential with parameter λ. Define $\rho = \lambda/\mu$. We shall assume throughout that $\rho < 1$, so that the system operates in its stationary regime. n consecutive arrivals form a message. Following [2], we consider the instant of an arrival of an arbitrary message to the system in steady state, i.e. the instant of the arrival of the first packet of the message, which we call the header. Let d_n be the random variable denoting the time delay between the arrival epoch of the header and the departure instant of the last packet of the message (called the tail). Denote by $\mathcal{D}_n(s)$ the Laplace-Stieltjes Transform (LST) of d_n.

Using the notation of [2] we define $d^a_{i,m}$, $m \geq 1$, $i \geq 0$ to be the random variable denoting the time delay between an arbitrary epoch, to the departure epoch of the last packet of the next m packets that leave the system (transmitted), given that i packets are present in the system at that arbitrary epoch. Denote $\mathcal{D}^a_{i,m}(s)$ the Laplace-Stieltjes Transform (LST) of $d^a_{i,m}$. We have [2] for $n \geq 1$,

$$\begin{aligned}\mathcal{D}_n(s) &= \frac{(\mu-\lambda)(\mu\lambda)^{n-1}}{(s+\mu-\lambda)(s+\mu)^{2n-2}} \\ &\quad + \frac{\mu-\lambda}{s+\mu}\sum_{j=0}^{n-2}\left(\frac{\lambda}{s+\mu}\right)^j \mathcal{D}^a_{j,n-1}(s)\end{aligned} \tag{1}$$

and $\mathcal{D}_1(s) = (\mu-\lambda)/(s+\mu-\lambda)$ (see eq. (2), (3) in [2]). Consequently, in order to obtain $\mathcal{D}_n(s)$, it suffices to determine $\mathcal{D}^a_{i,n-1}(s)$ for $0 \leq i \leq n-2$. We introduce the recursive method of [2] for computing the latter.

$$\mathcal{D}^a_{0,1}(s) = \frac{\lambda\mu}{(s+\lambda)(s+\mu)} \tag{2}$$

$$\mathcal{D}^a_{0,m}(s) = \frac{\lambda}{s+\lambda}\mathcal{D}^a_{1,m}(s) \tag{3}$$

$$\begin{aligned}\mathcal{D}^a_{i,m}(s) &= \frac{\lambda+\mu}{s+\lambda+\mu}\left(\frac{\lambda}{\lambda+\mu}\mathcal{D}^a_{i+1,m}(s)\right. \\ &\quad \left.+\frac{\mu}{\lambda+\mu}\mathcal{D}^a_{i-1,m-1}(s)\right),\ 1 \leq i \leq m-1\end{aligned} \tag{4}$$

$$\mathcal{D}^a_{m,m}(s) = \left(\frac{\mu}{s+\mu}\right)^m \tag{5}$$

where $2 \leq m \leq n-1$.

Denote

$$D(x,y,s) = \sum_{m=1}^{\infty}\sum_{i=0}^{m}\mathcal{D}^a_{i,m}(s)x^i y^{m-1}.$$

It is easily seen that $D(x,y,s)$ is analytic in the region $\Re(s) \geq 0, |y| < 1, |x| < 1$. This is because $\mathcal{D}^a_{i,m}(s)$ is the Laplace-Stieltjes Transform of a distribution function, and as such, it is not greater than one in modulus for $\Re(s) \geq 0$. In particular,

$$\begin{aligned}D(x,y,0) &= \frac{1}{1-x}\left[\frac{1}{1-y} - x^2\frac{1}{1-xy}\right] \\ &= \frac{1+x-xy}{(1-y)(1-xy)}.\end{aligned}$$

It can be seen from (1) that in order to determine $\mathcal{D}_n(s)$, or its PGF (Probability Generating Function) $V(y,s) := \sum_{n=2}^{\infty}\mathcal{D}_n(s)y^{n-2}$, it suffices to determine the sum $\delta_{n-1}(s) := \sum_{j=0}^{n-2}(\lambda/(s+\mu))^j\mathcal{D}^a_{j,n-1}(s)$, rather than to determine each $\mathcal{D}^a_{j,n}(s)$. In order to do that we define

$$\begin{aligned}\Delta(y,s) &= \sum_{m=1}^{\infty}\left[\sum_{i=0}^{m-1}\left(\frac{\lambda}{\mu+s}\right)^i\mathcal{D}^a_{i,m}(s)\right]y^{m-1} \\ &= \sum_{m=1}^{\infty}\delta_m(s)y^{m-1}.\end{aligned}$$

We note that

$$\begin{aligned}&\frac{\mu-\lambda}{s+\mu-\lambda}\sum_{n=2}^{\infty}\frac{(\mu\lambda)^{n-1}}{(s+\mu)^{2n-2}}y^{n-2} \\ &= \frac{\mu-\lambda}{s+\mu-\lambda}\sum_{k=0}^{\infty}\frac{(\mu\lambda)^{k+1}}{(s+\mu)^{2(k+1)}}y^k \\ &= \frac{\mu-\lambda}{s+\mu-\lambda}\left(\frac{\mu\lambda}{(s+\mu)^2}\right)\frac{1}{1-\frac{\mu\lambda}{(s+\mu)^2}y}.\end{aligned}$$

It then follows from (1) that

$$\begin{aligned}&V(y,s) \\ &= \frac{\mu-\lambda}{s+\mu-\lambda}\times\frac{\mu\lambda}{(s+\mu)^2}\left(1-\frac{\mu\lambda}{(s+\mu)^2}y\right)^{-1} \\ &\quad +\frac{\mu-\lambda}{s+\mu}\sum_{n=2}^{\infty}\sum_{j=0}^{n-2}\left(\frac{\lambda}{s+\mu}\right)^j\mathcal{D}^a_{j,n-1}(s)y^{n-2} \\ &= \frac{\mu-\lambda}{s+\mu-\lambda}\times\frac{\mu\lambda}{(s+\mu)^2}\left(1-\frac{\mu\lambda}{(s+\mu)^2}y\right)^{-1} + \\ &\quad \frac{\mu-\lambda}{s+\mu}\sum_{m=1}^{\infty}\sum_{j=0}^{m-1}\left(\frac{\lambda}{s+\mu}\right)^j\mathcal{D}^a_{j,m}(s)y^{m-1} \\ &= \frac{\mu-\lambda}{s+\mu-\lambda}\times\frac{\mu\lambda}{(s+\mu)^2-\lambda\mu y} \\ &\quad +\frac{\mu-\lambda}{s+\mu}\Delta(y,s).\end{aligned} \tag{6}$$

2.2 Main results

Define for $y \neq 0$, x_1 and x_2 as the two roots of

$$-\mu x^2 y + x(s+\lambda+\mu) - \lambda = 0, \tag{7}$$

In particular, x_1 is given by

$$x_1 = \frac{s+\lambda+\mu-\sqrt{(s+\lambda+\mu)^2-4\lambda\mu y}}{2\mu y}. \tag{8}$$

Proposition 2.1 *For $|y| < 1, |x| < 1, \Re(s) \geq 0$, $D(x,y,s)$, $\Delta(y,s)$ and $V(y,s)$ are given by:*

$$D(x,y,s) = \frac{1}{-\mu x^2 y + x(s+\lambda+\mu) - \lambda}\times \tag{9}$$

$$\left[-\frac{x\mu-\lambda}{1+s/(\mu x_1)-y}+\frac{x^2\mu(s+\lambda+\mu-\mu xy)}{\mu+s-\mu xy}\right]$$

$$\begin{aligned}\Delta(y,s) &= -\frac{s(s+\mu)}{\lambda[(s+\mu)-\mu y]}\times\frac{1}{1+s/(\mu x_1)-y} \\ &\quad -\frac{\mu(s+\mu)^2}{[\mu y-(s+\mu)][(\mu+s)^2-\mu\lambda y]} \\ &= \frac{s(s+\mu)}{s+\mu-\mu y}\times\frac{-1}{\lambda(1-y)+syx_2} \\ &\quad -\frac{\mu(s+\mu)^2}{[\mu y-(s+\mu)][(\mu+s)^2-\mu\lambda y]}, \qquad (10)\end{aligned}$$

$$\Delta(y,0)=\frac{1}{(1-y)(1-\rho y)}, \qquad (11)$$

$$\begin{aligned}V(y,s) &= \frac{\mu(\mu-\lambda)}{(s+\mu(1-y))(\mu-\lambda+s)} \qquad (12) \\ &\quad +\frac{\mu}{\lambda}\frac{\lambda(1-y)+syx_1}{s+\mu(1-y)}\left(\frac{\lambda}{s+\lambda(1-y)}\right. \\ &\quad \left.-\frac{\mu}{s+\mu(y-1)}\right). \qquad (13)\end{aligned}$$

For $y=0$ we have

$$D(x,0,s)=\frac{\lambda\mu}{(s+\lambda)(s+\mu)}+\frac{\mu x}{s+\mu},$$

$$\Delta(0,s)=\frac{\lambda\mu}{(s+\lambda)(s+\mu)},$$

$$\begin{aligned}V(0,s) &= \frac{(\mu-\lambda)\mu\lambda}{(s+\mu)^2}\left[\frac{1}{s+\mu-\lambda}+\frac{1}{s+\lambda}\right] \\ &= \frac{(\mu-\lambda)\lambda\mu(2s+\mu)}{(s+\mu)^2(s+\mu-\lambda)(s+\lambda)}.\end{aligned}$$

Proof: For $y=0$, the result follows immediately from (3) and (4). We thus consider $y\neq 0$. We decompose $D(x,y,s)$ into three parts:

$$\begin{aligned}&D(x,y,s) \qquad (14) \\ &= \sum_{m=2}^{\infty}\sum_{i=1}^{m-1}\mathcal{D}^a_{i,m}(s)x^iy^{m-1} \\ &\quad +\sum_{m=1}^{\infty}\mathcal{D}^a_{0,m}(s)y^{m-1}+\sum_{m=1}^{\infty}\mathcal{D}^a_{m,m}(s)x^my^{m-1}.\end{aligned}$$

Using (4), the first part can be written as

$$\begin{aligned}&\sum_{m=2}^{\infty}\sum_{i=1}^{m-1}\mathcal{D}^a_{i,m}(s)x^iy^{m-1} \\ &= \frac{1}{\lambda+\mu+s}\sum_{m=2}^{\infty}\sum_{i=1}^{m-1}\big(\lambda\mathcal{D}^a_{i+1,m}(s) \\ &\quad +\mu\mathcal{D}^a_{i-1,m-1}(s)\big)x^iy^{m-1} \\ &= \frac{\lambda}{x(\lambda+\mu+s)}\sum_{m=2}^{\infty}\sum_{i=2}^{m}\mathcal{D}^a_{i,m}(s)x^iy^{m-1} \\ &\quad +\frac{\mu xy}{\lambda+\mu+s}\sum_{m=1}^{\infty}\sum_{i=0}^{m-1}\mathcal{D}^a_{i,m}(s)x^iy^{m-1} \\ &= \frac{\lambda}{x(\lambda+\mu+s)}\left[D(x,y,s)-\sum_{m=1}^{\infty}\mathcal{D}^a_{1,m}(s)xy^{m-1}\right. \\ &\quad \left.-\sum_{m=1}^{\infty}\mathcal{D}^a_{0,m}(s)y^{m-1}\right]+ \qquad (15) \\ &\quad \frac{\mu xy}{\lambda+\mu+s}\left[D(x,y,s)-\sum_{m=1}^{\infty}\mathcal{D}^a_{m,m}(s)x^my^{m-1}\right].\end{aligned}$$

In order to evaluate (15) and (14) we note that

$$\begin{aligned}&\sum_{m=1}^{\infty}\mathcal{D}^a_{m,m}(s)x^my^{m-1} \qquad (16) \\ &= \frac{1}{y}\sum_{m=1}^{\infty}\left(\frac{\mu xy}{s+\mu}\right)^m \\ &= \frac{1}{1-\mu xy/(s+\mu)}\left(\frac{\mu x}{s+\mu}\right)=\frac{\mu x}{\mu+s-\mu xy}\end{aligned}$$

(this follows from (5));

$$\sum_{m=1}^{\infty}\mathcal{D}^a_{0,m}(s)y^{m-1}=D(0,y,s) \qquad (17)$$

(which is a function that will be determined); and finally, it follows from (3) that

$$\sum_{m=1}^{\infty}\mathcal{D}^a_{1,m}(s)xy^{m-1}=\frac{s+\lambda}{\lambda}xD(0,y,s). \qquad (18)$$

By combining (14)-(18) we obtain

$$\begin{aligned}&D(x,y,s)-D(0,y,s)-\frac{\mu x}{\mu+s-\mu xy}= \\ &\quad \frac{\lambda}{x(s+\lambda+\mu)}\big[D(x,y,s)- \\ &\quad \big(\frac{s+\lambda}{\lambda}x+1\big)D(0,y,s)\big] \\ &\quad +\frac{xy\mu}{s+\lambda+\mu}\left[D(x,y,s)-\frac{\mu x}{\mu+s-\mu xy}\right].\end{aligned}$$

This yields by multiplying by $x(s+\lambda+\mu)$,

$$\begin{aligned}&D(x,y,s)\left[-x^2y\mu+x(s+\lambda+\mu)-\lambda\right] \qquad (19) \\ &= (x\mu-\lambda)D(0,y,s)+\frac{x^2\mu(s+\lambda+\mu-\mu xy)}{\mu+s-\mu xy}.\end{aligned}$$

Next we show that $|D(x_1,y,s)|<\infty$ for all s,y such that $\Re(s)\ge 0, |y|<1$. It is easy to check that $|D(x,y,s)|\le D(|x|,|y|,\Re(s))$. It thus suffices to check that for all real valued y and s, such that $0<y<1$, $s\ge 0$, $D(|x_1|,y,s)<\infty$. Let $\alpha := 4\lambda\mu y/(s+\lambda+\mu)^2$. It follows from the stability condition ($\lambda<\mu$) that $0\le\alpha<1$ if s and y satisfy the above conditions. Hence x_1 is real, and it can be checked that $1-\sqrt{1-\alpha}<\alpha$ for $0<\alpha<1$. Using again the stability condition, this implies that

$$0\le x_1\le \frac{s+\lambda+\mu}{2\mu y}\alpha<1, \qquad 0\le\alpha<1.$$

Consequently, by the definition of x_1, the left hand side of (19) vanishes at $x=x_1$. We note that for $j,i=1,2,\ i\ne j$, (7) implies that $\mu y x_i x_j=\lambda$ and $\mu y x_i = s+\lambda+\mu-\mu y x_j$. All this yields

$$\begin{aligned}
-D(0,y,s) &= \frac{x_1^2\mu(s+\lambda+\mu-\mu x_1 y)}{(\mu+s-\mu x_1 y)(x_1\mu-\lambda)}\\
&= \frac{x_1^2\mu x_2\mu y}{(\mu+s-\mu x_1 y)(x_1\mu-\lambda)}\\
&= \frac{\lambda\mu x_1}{(\mu+s-\mu x_1 y)(x_1\mu-\lambda)}\\
&= \frac{\mu x_1}{\mu x_1(y-1)-s}=\frac{1}{y-1-s/\mu x_1}\\
&= -\frac{\lambda}{\lambda(1-y)+syx_2}.
\end{aligned}$$

Combining this with (19) yields the expression (9).

To obtain the expression for (10) we note that

$$\begin{aligned}
\Delta(y,s) &= D(x_3,y,s)-\sum_{m=1}^{\infty}\mathcal{D}^a_{m,m}(s)x_3^m y^{m-1}\\
&= D(x_3,y,s)+\frac{\lambda\mu}{\lambda\mu y-(\mu+s)^2}, \qquad (20)
\end{aligned}$$

where $x_3=\lambda/(s+\mu)$. We shall evaluate first $D(x_3,y,s)$ by using (19). We note that $-y\mu x_3^2+(s+\lambda+\mu)x_3-\lambda = x_3(\lambda-\mu x_3 y)$ and that $x_3\mu-\lambda=-sx_3$. It thus follows that

$$\begin{aligned}
D(x_3,y,s) &= \frac{-sx_3}{x_3(\lambda-\mu x_3 y)}D(0,y,s)\\
&\quad+\frac{x_3^2\mu(s+\lambda+\mu-\mu x_3 y)}{x_3(\lambda-\mu x_3 y)(\mu+s-\mu x_3 y)}\\
&= -\frac{s(s+\mu)}{\lambda(s+\mu)-\mu\lambda y}D(0,y,s)\\
&\quad-\frac{\mu(s+\mu)^2}{[\mu y-(s+\mu)][(\mu+s)^2-\mu\lambda y]}\\
&\quad-\frac{\lambda\mu}{\mu\lambda y-(s+\mu)^2}.
\end{aligned}$$

Combining this with (20) yields (10). (10) and (6) imply

$$\begin{aligned}
V(y,s) &= (\mu-\lambda)\left[\frac{1}{s+\mu-\lambda}\times\frac{\mu\lambda}{(s+\mu)^2-\mu\lambda y}\right.\\
&\quad-\frac{s}{s+\mu-\mu y}\times\frac{1}{\lambda(1-y)+syx_2}\\
&\quad\left.-\frac{\mu(s+\mu)}{[\mu y-(s+\mu)][(\mu+s)^2-\mu\lambda y]}\right]\\
&= \frac{\mu-\lambda}{s+\mu(1-y)}\left(\frac{\mu}{\mu-\lambda+s}\right.\\
&\quad\left.-\frac{s}{\lambda(1-y)+syx_2}\right), \qquad (21)
\end{aligned}$$

for $y\ne 0$. Now,

$$\begin{aligned}
&\frac{1}{\lambda(1-y)+syx_2}\\
&= \frac{\lambda(1-y)+syx_1}{(\lambda(1-y)+syx_2)(\lambda(1-y)+syx_1)}\\
&= \frac{\mu}{\lambda}\left(\frac{\lambda(1-y)+syx_1}{\lambda\mu(1-y)^2+\lambda(1-y)s(s+\lambda+\mu)+s^2y}\right)\\
&= \frac{\mu}{\lambda}\left(\frac{\lambda(1-y)+syx_1}{(\lambda y-\lambda-s)(\mu y-\mu-s)}\right)\\
&= \frac{\mu(\lambda(1-y)+syx_1)}{\lambda s(\mu-\lambda)}\left(\frac{\lambda}{\lambda(y-1)-s}\right.\\
&\quad\left.-\frac{\mu}{\mu(y-1)-s}\right).
\end{aligned}$$

Combining this with (21) yields (12). ∎

2.3 Inverting the PGF, and calculating moments

We shall use the notation $[z^k]f(z)$ to denote the coefficient of z^k in the Taylor expansion of the function $f(z)$. Define

$$c_j=\begin{cases}\dfrac{2(2j-2)!}{4^j(j-1)!j!} & \text{if } j\ge 1\\[2ex] 0 & \text{if } j=0.\end{cases}$$

The c_j are the coefficients $[u^j]\,1-\sqrt{1-u}$, i.e. $1-\sqrt{1-u}=\sum_{j=0}^{\infty}c_j u^j$. For the sake of brevity, we omit in this section the standard but cumbersome calculations. These will appear in [1]. One obtains from (12):

Proposition 2.2

$$V(y,s) = \sum_{n=0}^{\infty}y^n\left\{\frac{\mu-\lambda}{\mu-\lambda+s}\left(\frac{\mu}{\mu+s}\right)^{n+1}+\sum_{i=0}^{n}d_i e_{n-i}\right\}$$

where $d_0 = \lambda$,

$$d_1 = -\frac{\lambda(\lambda+\mu)}{s+\lambda+\mu},$$

$$d_n = c_n \frac{(4\lambda\mu)^n}{(s+\lambda+\mu)^{2n-1}} \frac{s}{2\mu},\ n \geq 2;$$

and

$$e_n = -\left(\frac{\mu}{\mu+s}\right)^{n+1} \frac{\mu(n+1)}{\lambda(\mu+s)} + \frac{1}{s(1-\rho)}\left(\frac{\mu}{\mu+s}\right)^{n+1} - \frac{1}{s(1-\rho)}\left(\frac{\lambda}{\lambda+s}\right)^{n+1}.$$

In order to calculate moments of the message delay, define

$$e_n^{(k)}(0) = \left.\frac{de_n(s)^k}{ds}\right|_{s=0}, \qquad d_n^{(k)}(0) = \left.\frac{dd_n(s)^k}{ds}\right|_{s=0}.$$

Calculations yield the following expressions which we shall use:

$$e_n^{(k)}(0) = (-1)^k k!\mu \left\{ \frac{1}{\mu-\lambda}\left(\frac{1}{\lambda^{k+1}} - \frac{1}{\mu^{k+1}}\right) - \frac{k+1}{\lambda\mu^{k+1}} \right\} \binom{k+n+1}{n}$$

$$e_n^{(k)}(0) - e_{n-1}^{(k)}(0) = (-1)^k k!\mu \left\{ \frac{1}{\mu-\lambda}\left(\frac{1}{\lambda^{k+1}} - \frac{1}{\mu^{k+1}}\right) - \frac{k+1}{\lambda\mu^{k+1}} \right\} \binom{k+n}{n}$$

$$d_n^{(k)}(0) = \frac{c_n k (4\lambda\mu)^n (-1)^{k-1}}{2\mu(\lambda+\mu)^{2n+k-2}} \binom{2n+k-2}{k-1}.$$

The first moment of the message delay is given in [2]. We now compute the second moment:

$$\begin{aligned} ED_{n+2}^2 &= [y^n] \left.\frac{\partial^2 V(y,s)}{\partial s^2}\right|_{s=0} \qquad (22) \\ &= \frac{2}{(\mu-\lambda)^2} + \frac{2(n+1)}{\mu(\mu-\lambda)} + \frac{(n+1)(n+2)}{\mu^2} \\ &\quad +2\sum_{i=0}^{n} d_i^{(1)}(0) e_{n-i}^{(1)}(0) \\ &\quad +\lambda(e_n^{(2)}(0) - e_{n-1}^{(2)}(0)), \end{aligned}$$

where

$$d_n^{(1)}(0) = \frac{c_n(4\lambda\mu)^n}{2\mu(\lambda+\mu)^{2n-1}},$$

$$e_n^{(1)}(0) = \frac{(\lambda-\mu)(n+1)(n+2)}{2\lambda^2\mu},$$

and

$$e_n^{(2)}(0) - e_{n-1}^{(2)}(0) = \frac{(n+2)(n+1)}{\mu^2\lambda^3}(\mu^2 + \lambda\mu - 2\lambda^2).$$

Evaluating the second moment of the message delay using (22) requires a number of elementary operations $(+, \times)$ of order n, which is a significant improvement of the complexity of $O(n^2)$ obtained in [2] using a direct recursive approach.

3 Multiple Session Systems

3.1 The Model

We assume now that packets are generated by S independent Poisson processes with arrival rates λ_g, $g = 1, ..., S$. The overall arrival rate is $\lambda = \sum_{g=1}^{S} \lambda_g$. The transmission time of a packet is exponential with rate μ. Here again we assume that $\rho < 1$ where $\rho = \lambda/\mu$, so that the system is ergodic. Fix some g, $g = 1, ..., S$, and consider an arbitrary message of size n arriving to the system from source g, in steady-state. Denote $\lambda_{\overline{g}} := \lambda - \lambda_g$. Following the notation in [2], we define d_n, $n \geq 1$, to be the RV of the message delay and denote by $\mathcal{D}_n(s)$ its LST. Let $d_{i,n}$, $i \geq 0, n \geq 1$, be a RV of the time delay from an arbitrary epoch to the departure epoch of the last packet of the next n packets that arrive to the system from source g, given that i packets are present in the system at that arbitrary epoch. Denote $\mathcal{D}_{i,n}(s)$ its LST and define

$$\mathcal{D}_{i,0}(s) = \left(\frac{\mu}{s+\mu}\right)^i \qquad i \geq 0.$$

Define

$$G_n(s,z) = \sum_{i=0}^{\infty} \mathcal{D}_{i,n}(s) z^i, \quad \Gamma(s,z,x) = \sum_{n=0}^{\infty} G_n(s,z) x^n,$$

$$W(s,x) = \sum_{n=1}^{\infty} \mathcal{D}_n(s) x^{n-1},$$

for $|z| < 1, |x| < 1$ and for all s such that $\Re(s) \geq 0$. Using the relations

$$\begin{aligned} \mathcal{D}_n(s) &= \frac{1-\rho}{\rho} \sum_{i=1}^{\infty} \rho^i \mathcal{D}_{i,n-1}(s) \\ &= \frac{1-\rho}{\rho}(G_{n-1}(s,\rho) - G_{n-1}(s,0)) \end{aligned}$$

(see eq. (16),(17) in [2]), we get

$$W(s,x) = \frac{1-\rho}{\rho}[\Gamma(s,\rho,x) - \Gamma(s,0,x)]. \tag{23}$$

Thus in order to obtain $W(s,x)$ we shall evaluate $\Gamma(s,z,x) - \Gamma(s,0,x)$.

3.2 Main results

Define z_1 and z_2 as the two roots of

$$-\mu z^2 + z(s+\lambda+\mu) - \lambda_{\overline{g}} - \lambda_g x = 0, \tag{24}$$

In particular, z_1 is given by

$$z_1 = \frac{s+\lambda+\mu - \sqrt{(s+\lambda+\mu)^2 - 4\mu(\lambda_{\overline{g}} + \lambda_g x)}}{2\mu}. \tag{25}$$

Proposition 3.1 *For $|x| < 1, \Re(s) \geq 0$, $W(s,x)$ is given by:*

$$W(s,x) = \frac{1-\rho}{\rho s + \lambda_g(1-x)}\left[-\frac{s\lambda_g\mu z_1 x}{(\lambda_{\overline{g}} + \lambda_g x)s + \lambda_g \mu z_1(1-x)} + \frac{\lambda s + \lambda_g\mu}{s+\mu-\lambda}\right]. \tag{26}$$

Proof: The following recursion holds ([2] eq. (19)):

$$\begin{aligned} G_n(s,z)[-\mu z^2 + (s+\lambda+\mu)z - \lambda_{\overline{g}}] &= \lambda_g[G_{n-1}(s,z) - G_{n-1}(s,0)] \\ &\quad +(\mu z - \lambda_{\overline{g}})G_n(s,0), \end{aligned}$$

from which we get

$$\begin{aligned} &\Gamma(s,z,x)[-\mu z^2 + (s+\lambda+\mu)z - \lambda_{\overline{g}}] \\ &= \sum_{n=1}^{\infty} G_n(s,z)x^n[-\mu z^2 + (s+\lambda+\mu)z - \lambda_{\overline{g}}] \\ &\quad +G_0(s,z)[-\mu z^2 + (s+\lambda+\mu)z - \lambda_{\overline{g}}] \\ &= \lambda_g x \sum_{n=0}^{\infty}[G_n(s,z) - G_n(s,0)]x^n \\ &\quad +(\mu z - \lambda_{\overline{g}})\sum_{n=1}^{\infty} G_n(s,0)x^n \\ &\quad +G_0(s,z)[-\mu z^2 + (s+\lambda+\mu)z - \lambda_{\overline{g}}] \\ &= \lambda_g x[\Gamma(s,z,x) - \Gamma(s,0,x)] \\ &\quad +(\mu z - \lambda_{\overline{g}})[\Gamma(s,0,x) - \Gamma(s,0,0)] \\ &\quad +G_0(s,z)[-\mu z^2 + (s+\lambda+\mu)z - \lambda_{\overline{g}}]. \end{aligned}$$

By using

$$\Gamma(s,z,0) = G_0(s,z) = \frac{s+\mu}{s+\mu-\mu z}$$

we obtain

$$\begin{aligned} &\Gamma(s,z,x)[-\mu z^2 + (s+\lambda+\mu)z - \lambda_{\overline{g}} - \lambda_g x] \\ &= \Gamma(s,0,x)[\mu z - \lambda_{\overline{g}} - \lambda_g x] + \lambda_{\overline{g}} - \mu z \\ &\quad + \frac{s+\mu}{s+\mu-\mu z}[-\mu z^2 + (s+\lambda+\mu)z - \lambda_{\overline{g}}]. \end{aligned} \tag{27}$$

To calculate $\Gamma(s,0,x)$, we substitute z_1 (defined in (25)) into (27). We first show that $|\Gamma(s,z_1,x)| < \infty$ for all s,x such that $\Re(s) \geq 0, |x| < 1$. It is easy to check that $|\Gamma(s,z,x)| \leq \Gamma(\Re(s),|z|,|x|)$. It thus suffices to check that for all real valued x and s, such that $0 < x < 1$, $s \geq 0$, $\Gamma(s,|z_1|,x) < \infty$. Let $\alpha := 4\mu(\lambda_{\overline{g}} + \lambda_g x)/(s+\lambda+\mu)^2$, and note that

$$z_1 = \frac{s+\lambda+\mu}{2\mu}[1 - \sqrt{1-\alpha}].$$

It follows from the stability condition ($\lambda < \mu$) that $0 \leq \alpha < 1$. Hence z_1 is real, and it can be checked that $1 - \sqrt{1-\alpha} < \alpha$ for $0 < \alpha < 1$. Using again the stability condition, this implies that

$$0 \leq z_1 \leq \frac{s+\lambda+\mu}{2\mu}\alpha \leq \frac{2\lambda}{s+\lambda+\mu} < 1, \quad 0 \leq \alpha < 1.$$

Consequently, and by the definition of z_1, the left hand side of (27) vanishes at $z = z_1$. This yields

$$\Gamma(s,0,x) = \frac{-\mu z_1 + \lambda_{\overline{g}} + \lambda_g x(s+\mu)/(s+\mu-\mu z_1)}{\lambda_{\overline{g}} + \lambda_g x - \mu z_1}. \tag{28}$$

We shall simplify (28) by noting that

$$\begin{aligned} &(-\mu z_1 + \lambda_{\overline{g}})(s+\mu-\mu z_1) + (s+\mu)\lambda_g x \\ &= \mu^2 z_1^2 - \mu z_1(s+\mu+\lambda_{\overline{g}}) + \lambda_{\overline{g}}(s+\mu) \\ &\quad +(s+\mu)\lambda_g x \\ &= \mu(\mu z_1^2 - z_1(s+\mu+\lambda) + \lambda_{\overline{g}} + \lambda_g x) \\ &\quad +(\lambda_{\overline{g}} + \lambda_g x)s + \lambda_g \mu z_1 \\ &= (\lambda_{\overline{g}} + \lambda_g x)s + \lambda_g\mu z_1 \end{aligned}$$

and similarly,

$$\begin{aligned} &(\lambda_{\overline{g}} + \lambda_g x - \mu z_1)(s+\mu-\mu z_1) \\ &= \mu z_1 \lambda_g(1-x) + s(\lambda_{\overline{g}} + \lambda_g x). \end{aligned}$$

Hence we obtain

$$\Gamma(s,0,x) = \frac{(\lambda_{\overline{g}} + \lambda_g x)s + \lambda_g\mu z_1}{(\lambda_{\overline{g}} + \lambda_g x)s + \lambda_g \mu z_1(1-x)}. \tag{29}$$

In order to evaluate $\Gamma(s,\rho,x)$, we use (27) and

$$-\mu\rho^2 + (s+\lambda+\mu)\rho - \lambda_{\overline{g}} = \rho s + \lambda_g,$$

$$\lambda_{\overline{g}} - \mu\rho = -\lambda_g,$$

$$\mu\rho - \lambda_{\overline{g}} - \lambda_g x = \lambda_g(1-x)$$

and obtain

$$\begin{aligned}
&\Gamma(s,\rho,x)[\rho s + \lambda_g(1-x)] \\
&\quad = \quad \Gamma(s,0,x)\lambda_g(1-x) - \lambda_g + \frac{s+\mu}{s+\mu-\lambda}(\rho s + \lambda_g).
\end{aligned}$$

Hence

$$\begin{aligned}
&[\Gamma(s,\rho,x) - \Gamma(s,0,x)][\rho s + \lambda_g(1-x)] \\
&\quad = \quad -\rho s\Gamma(s,0,x) + \frac{(s+\mu)\rho s + \lambda_g\lambda}{s+\mu-\lambda} \\
&\quad = \quad -\rho s \frac{(\lambda_{\overline{g}} + \lambda_g x)s + \lambda_g\mu z_1}{(\lambda_{\overline{g}} + \lambda_g x)s + \lambda_g\mu z_1(1-x)} \\
&\qquad\quad + \frac{(s+\mu)\rho s + \lambda_g\lambda}{s+\mu-\lambda} \\
&\quad = \quad -\rho s - \frac{s\lambda_g\lambda z_1 x}{(\lambda_{\overline{g}} + \lambda_g x)s + \lambda_g\mu z_1(1-x)} \\
&\qquad\quad + \frac{(s+\mu)\rho s + \lambda_g\lambda}{s+\mu-\lambda} \\
&\quad = \quad -\frac{s\lambda_g\lambda z_1 x}{(\lambda_{\overline{g}} + \lambda_g x)s + \lambda_g\mu z_1(1-x)} + \frac{\lambda^2 s + \lambda_g\lambda\mu}{\mu(s+\mu-\lambda)}.
\end{aligned}$$

Substituting this into (23) finally yields (26). ■

References

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