The Loss Process of Messages in an M/M/1/K Queue

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Abstract

Our purpose in this paper is to obtain the distribution of the number of lost packets within a sequence of n consecutive packet arrivals into a finite buffer M/M/1 queue. We obtain explicit expressions for the multi-dimensional generating function of these probabilities based on a recursive scheme recently introduced by Cidon et al. [1]. We then analyze the loss probabilities of a whole message, and analyze the effect of adding redundant packets. We show that in both heavy traffic as well as in light traffic conditions, adding redundant packets results in decreasing the message loss probabilities.

1 Introduction

The loss probabilities of packet arrivals into queueing networks is an important performance measure needed for the design of telecommunication networks and in other applications. In many cases, when messages are divided into several packets, the loss of a packet may result in the loss of the whole message. Thus, in order to compute the loss probability of messages containing n packets, one has to compute the probability of no losses among n consecutive packets. By adding redundancy, one may often reconstruct lost packets, and hence hope to reduce the probability of losses of messages. We refer to [1] and the references therein for details on how redundant packets are constructed and used for error recovery. The quantities of interest are then the probability of no more losses than one (or of k, in general) packets within n consecutive ones. Cidon, Khamisy and Sidi [1] recently obtained recursive formulas that enabled them to compute these probabilities numerically in a M/M/1/K queue. Surprisingly, in all numerical calculations presented in [1], the probabilities of losing a message turned out always to be higher when a redundancy packet was added. Considered were messages of length varying from 7 to 71, and a queue of size 20. The complexity of the recursive calculations in [1] is $O(Mn^2)$, where n is the size of the message and M is the queue size. However, simple heuristic arguments hint that there should be values of the parameters for which redundancy does improve the loss probability of messages. This motivated a deeper analysis of the problem. Admitedly, the Poisson assumption for the arrivals of packets is not the most realistic, but the study of the exponential case reveals essential features of the system and is a first step for more general (Erlang, Cox) assumptions.

In this paper we obtain explicit expressions for the multi-dimensional generating function of the required probabilities. Our aims in doing so are (1) to reduce significantly the complexity and the memory requirements for the calculations of the probabilities, (2) to obtain asymptotic expressions for large n, and large ρ, (3) to identify conditions under which adding redundancy results in a decrease in the probability of losing messages. The asymptotic results enable us to approximate the loss probabilities for large n, or for large or low ρ, with complexity of $O(1)$. Explicit expressions that we obtain for the probability of j losses out of n consecutive packets require $O(nj^2)$ calculations.

The structure of the paper is as follows. In Section 2 we describe the model and present the main results: the expressions for the multi-dimensional generating function of the loss probabilities of messages. The proofs are presented in Section 3. The asymptotic analysis is performed in Section 4. It is used in Section 5 to show that for both heavy traffic as well as light traffic regime, adding redundant packets results in decreasing the message loss probabilities. In Section 6 we present numerical results to illustrate the advantage of adding redundancy.

2 The model and the main results

We consider an M/M/1 queue with a buffer of size M. Service is exponential with parameter μ and interarrival times are exponential with parameter λ. A loss occurs whenever an arriving packet finds the queue full. Define $\rho = \lambda/\mu$. We summarize the recursive method of [1] for computing $P(j,n)$, which are the probabilities of j losses in a block of n packets. We shall use the same notation as in [1]. The steady state probabilities of having i packets in the system at an arbitrary epoch are given by

$$\Pi(i) = \frac{\rho^i}{\sum_{l=0}^{M} \rho^l}. \qquad (1)$$

Define $P_i^a(j,n)$, $i = 0,1,...,M$, $n \geq 1$, $0 \leq j \leq n$, to be the probability of j losses in a block of n packets, given that there are i packets in the system just before the arrival epoch of the first packet in the block. We have

$$P(j,n) = \sum_{i=0}^{M} \Pi(i) P_i^a(j,n). \qquad (2)$$

Further define $Q_i(k)$ to be the probability that k packets out of i leave the system during an interarrival period. Then

$$Q_i(k) = \rho\alpha^{k+1},\ 0 \leq k \leq i-1 \ \text{ and } \ Q_i(i) = \alpha^i, \qquad (3)$$

where $\alpha := (1+\rho)^{-1}$. The recursion scheme is then the following: for $n = 1$,

$$\begin{array}{rcl} P_i^a(0,1) &=& P_M^a(1,1) = 1 \ \text{ for } \ 0 \leq i < M, \\ P_i^a(j,1) &=& 0 \quad \text{otherwise.} \end{array} \qquad (4)$$

For $n \geq 2$ we have the recursive equations

$$\begin{aligned} P_i^a(j,n) &= \sum_{k=0}^{i+1} Q_{i+1}(k) P_{i+1-k}^a(j,n-1),\ 0 \leq i < M \\ P_M^a(j,n) &= \sum_{k=0}^{M} Q_M(k) P_{M-k}^a(j-1,n-1). \end{aligned} \qquad (5)$$

Next we state the main results, whose proof is given in the next section. Define

$$q(y,z) \stackrel{\text{def}}{=} \sum_{j=0}^{\infty}\sum_{n=1}^{\infty} y^j z^{n-1} P(j,n).$$

Let $x_1(z)$ and $x_2(z)$ be the solutions in x of the equation $x^2 - (1+\rho)x + z\rho = 0$, i.e.

$$x_1(z), x_2(z) = \frac{1+\rho \pm \sqrt{(1+\rho)^2 - 4\rho z}}{2}.$$

We shall often write simply x_1 and x_2 below, meaning $x_1(z)$ and $x_2(z)$. Both these functions are analytic in the disk $\{|z| < (1+\rho)^2/4\rho\}$. Define, for all k, $\delta_k = x_1^k - x_2^k$, $\phi_k = z\delta_{k-1} - \delta_k$. Also, let $R_M = (\sum_{l=0}^{M} \rho^l)^{-1}$.

Proposition 2.1 *The probability generating function q is given by*

$$\begin{aligned} q(y,z) &= \frac{R_M}{1-z}\left(\frac{1-\rho^{M+1}}{1-\rho} + \frac{\rho^{M+1}\delta_{M+1}}{z\phi_M}\right. \\ &+ \left.\frac{\rho^{M+2}(1-z)[\delta_{M+1}]^2}{z\phi_M}\left[\frac{1}{\phi_{M+1} - z\phi_M y}\right]\right). \end{aligned} \qquad (6)$$

Once the probability generating function is obtained by Proposition 2.1, one can obtain the required probabilities $P(j,n)$ by inverting q. We focus in the sequel on $P_\rho(> j,n)$, which is defined as the probability of losing more than j packets out of n. We investigate in particular the cases of $j = 0,1$, in order to be able to decide when does adding a redundancy packet to each message result in a decrease of the loss probability.

We shall use the notation $[z^k]f(z)$ to denote the coefficient of z^k in the Taylor expansion of the function $f(z)$. Let c_j be coefficients $[u^j]\sqrt{1-u}$, i.e. $\sqrt{1-u} = \sum_{j=0}^{\infty} c_j u^j$. They are given by:

$$c_j = -\frac{2(2j-2)!}{4^j(j-1)!j!} \ \text{ if } j \geq 1, \quad c_0 = 0. \qquad (7)$$

Corollary 2.1 *The probability of losing more than j packets out of n consecutive packets is given by*

$$\begin{aligned} &\rho^{-M} R_M^{-1} P_\rho(> j,n) \qquad (8) \\ &\quad = \rho^j \left[z^{n-1-j}\right] \frac{\delta_{M+1}}{\phi_{M+1}} \frac{1}{z-1}\left(\frac{\phi_M}{\phi_{M+1}}\right)^j. \end{aligned}$$

In the next corollary we obtain a simpler expression for $P_\rho(> j,n)$; the term $P_\rho(> j,n)/(\rho^M R_M)$, surprisingly, does not depend on M when n is less than $M+j+1$. The following corollary provides formulas for actually computing their value.

Corollary 2.2 *Let $j+1 \leq n \leq M+j+1$. Then*

$$\begin{aligned} &\rho^{-M} R_M^{-1} P_\rho(> j,n) \\ &\quad = -\rho^{j+1}\left[z^{n-j-1}\right] \frac{1}{x_1^j(1-x_1)(1-z)}. \end{aligned} \qquad (9)$$

Corollary 2.3 *For $1 \leq n \leq M+1$,*

$$P_\rho(> 0,n) = \frac{\rho^M R_M}{2}\left(2n + \sum_{m=1}^{n-1} c_m \frac{(n-m)(4\rho)^m}{(1+\rho)^{2m-1}}\right). \qquad (10)$$

For $j+1 \leq n \leq M+j+1$,

$$\rho^{-j} P_\rho(> j,n) = -\rho^M R_M \sum_{m=0}^{n-1}\sum_{k=0}^{j-1} u_m^k + P_\rho(> 0,n), \qquad (11)$$

where:

$$u_m^k \stackrel{\text{def}}{=} [z^m]\frac{x_2^k}{\rho^k} = \left(\frac{1+\rho}{2\rho}\right)^k \left(\frac{4\rho}{(1+\rho)^2}\right)^m \sum_{r=0}^{k}\binom{k}{r}(-1)^r\binom{m-r/2-1}{m} . \tag{12}$$

In particular, for $2 \le n \le M+2$,

$$P_\rho(>0,n) = \rho^M R_M + \frac{1}{\rho}P_\rho(>1,n) , \tag{13}$$

which implies, for $2 \le n \le M+2$:

$$P_\rho(>1,n) = \frac{\rho^{M+1}R_M}{2}\big(2(n-1)+\sum_{j=1}^{n-1}c_j\frac{(n-j)(4\rho)^j}{(1+\rho)^{2j-1}}\big). \tag{14}$$

As can be seen from Corollary 2.2, the number of calculations required in order to obtain $P_\rho(>j,n)$ for $j+1 \le n \le M+j+1$ is of order $O(nj^2)$. This is a substantial improvement over the complexity of $O(Mn^2)$ needed in [1].

3 Proofs of the main results

To simplify the notation, let $\beta = 4\rho/(1+\rho)^2$. If $\rho \ne 1$, then $\beta < 1$.

Proof of Proposition 2.1: Define:

$$\pi_{j,n}(x) := \sum_{i=0}^{M} x^i P_i^a(j,n), \qquad n \ge 1, j \ge 0.$$

Making use of the recurrence (5) and of (3) yields, after some straightforward algebra, to:

$$\begin{aligned}\pi_{j,n}(x) &= \frac{\rho\alpha^2}{1-\alpha x}\left(\frac{1}{\alpha x}\pi_{j,n-1}(x) - (\alpha x)^M\pi_{j,n-1}(\alpha^{-1})\right) \\ &- \left(\alpha\rho\left(\frac{1}{\alpha x}-(\alpha x)^M\right)+1-(\alpha x)^M\right)\frac{\alpha\pi_{j,n-1}(0)}{1-\alpha x} \\ &+ \alpha\rho(\alpha x)^M\pi_{j-1,n-1}\left(\alpha^{-1}\right)+\alpha(\alpha x)^M\pi_{j-1,n-1}(0),\end{aligned} \tag{15}$$

where we use the fact that $\pi_{j,n}(0) = P_0^a(j,n)$. We now define, with some abuse of notation, the generating function of $P_i^a(j,n)$

$$\pi(x,y,z) = \sum_{j=0}^{\infty}\sum_{n=1}^{\infty} y^j z^{n-1}\pi_{j,n}(x).$$

When y and $|z| < 1$ are fixed, this is a polynomial in x and therefore an analytic function. In order to use (15) which holds only for $n \ge 2$ and $j \ge 1$, we note that

$$\sum_{j=1}^{\infty}\sum_{n=2}^{\infty} y^j z^{n-1}\pi_{j,n}(x) = \pi(x,y,z) - \pi(x,0,z) - \pi(x,y,0) + \pi(x,0,0).$$

It follows from (4) that

$$\pi(x,0,0) = \frac{1-x^M}{1-x} \quad\text{and}\quad \pi(x,y,0) = \frac{1-x^M}{1-x} + yx^M.$$

In these expressions as well as in the rest of the paper, we shall understand that for $x=1$ and all M, $(1-x^M)/(1-x) = M$. From (15) we obtain the following:

$$\begin{aligned}&[\pi(x,y,z)-\pi(x,0,z)]\left[(1-\alpha x)\alpha x - z\rho\alpha^2\right] \\ &= (1-\alpha x)\alpha y x^{M+1} \\ &\quad +z\rho\alpha(\alpha+x)^{M+1}\left(y(1-\alpha x)-\alpha\right) \\ &\qquad \left[\pi(\alpha^{-1},y,z)+\rho^{-1}\pi(0,y,z)\right] \\ &\quad +z\alpha^2(x-\rho)\pi(0,y,z) \\ &\quad +z\rho\alpha^2(\alpha x)^{M+1}\left[\pi(\alpha^{-1},0,z)+\rho^{-1}\pi(0,0,z)\right] \\ &\quad -z\alpha^2(x-\rho)\pi(0,0,z).\end{aligned} \tag{16}$$

We now make the following observation. In order to establish the proof of Proposition 2.1 it follows from (2) that it suffices to obtain $\pi(x,y,z)$ at $x=\rho$, since

$$q(y,z) = R_M\ \pi(\rho,y,z) . \tag{17}$$

By using the relation $\alpha + \alpha\rho = 1$ we get from (16):

$$\begin{aligned}[\pi(\rho,y,z)-\pi(\rho,0,z)](1-z) &= y\rho^M \\ &+ z(\alpha\rho)^{M+1}(y-1)\left[\pi(\alpha^{-1},y,z)+\rho^{-1}\pi(0,y,z)\right] \\ &+ z(\alpha\rho)^{M+1}\left[\pi(\alpha^{-1},0,z)+\rho^{-1}\pi(0,0,z)\right].\end{aligned} \tag{18}$$

Our next objective is to compute the functions in the square brackets, as well as $\pi(\rho,0,z)$. This is done as above by first expressing $\pi_{0,n}(x)$ (similarly to the calculations in (15)), and then by taking the generating function with respect to n. This leads to:

$$\begin{aligned}&\left[(1-\alpha x)\alpha x - z\rho\alpha^2\right]\pi(x,0,z) \\ &= \frac{1-x^M}{1-x}(1-\alpha x)\alpha x \\ &\quad -z\rho\alpha^2(\alpha x)^{M+1}\left[\pi(\alpha^{-1},0,z)+\rho^{-1}\pi(0,0,z)\right] \\ &\quad +z\alpha^2\pi(0,0,z)(x-\rho)\end{aligned} \tag{19}$$

from which we obtain

$$\begin{aligned}(1-z)\pi(\rho,0,z) &= \frac{1-\rho^M}{1-\rho} \\ &- z(\alpha\rho)^{M+1}\left[\pi(\alpha^{-1},0,z)+\rho^{-1}\pi(0,0,z)\right].\end{aligned} \tag{20}$$

We now make the following observation. For each $i = 1, 2$, when $x = x_i(z)$, the term that multiplies $\pi(x, 0, z)$ in the left hand side of equation (19) vanishes. Since $\pi(x, 0, z)$ is analytic in x, the left hand side of (19) vanishes at $x = x_i(z)$. Thus, when substituting x_i for x in (19), we obtain for each z two equations with two unknowns: $\pi(\alpha^{-1}, 0, z) + \rho^{-1}\pi(0, 0, z)$ and $\pi(0, 0, z)$. Solving these equations yields

$$\begin{aligned}&\pi(\alpha^{-1}, 0, z) + \rho^{-1}\pi(0, 0, z)\\ &= \frac{\alpha^{-(M+1)}\left(x_1^M - x_2^M\right)}{x_2^{M+1}(x_1 - \rho) - x_1^{M+1}(x_2 - \rho)}. \end{aligned} \quad (21)$$

The same arguments applies to (16) and allows to calculate the function $\pi(\alpha^{-1}, y, z) + \rho^{-1}\pi(0, y, z)$ of the right-hand side of (18). We obtain:

$$\begin{aligned}\pi(\alpha^{-1}, y, z) + \rho^{-1}\pi(0, y, z) = \alpha^{-(M+1)} \quad (22)\\ \frac{x_1^M[(x_2 - \rho)y - 1] - x_2^M[(x_1 - \rho)y - 1]}{x_1^{M+1}(x_2 - \rho)(1 - yx_2) - x_2^{M+1}(x_1 - \rho)(1 - yx_1)}.\end{aligned}$$

Substituting (20), (21) and (22) into (18) yields finally (6) after a few lines of calculations, and the introduction of the notation δ_m and ϕ_m. ■

Proof of Corollary 2.1: From (6),

$$\begin{aligned}P_\rho(> j, n) &= R_M \left[z^{n-1}\right] \sum_{k=j+1}^{\infty} \left[y^k\right] \pi(\rho, y, z)\\ &= R_M\rho^M \left[z^{n-1}\right] \frac{(\delta_{M+1})^2}{(\phi_{M+1})^2} \sum_{k=j+1}^{\infty} \left(\frac{\rho z\phi_M}{\phi_{M+1}}\right)^{k-1}\\ &= R_M\rho^M \left[z^{n-1}\right] \frac{(\delta_{M+1})^2}{(\phi_{M+1})^2} \left(\frac{\rho z\phi_M}{\phi_{M+1}}\right)^j \frac{1}{1 - \dfrac{\rho z\phi_M}{\phi_{M+1}}}.\end{aligned}$$

Equation (8) is then obtained by noting that

$$1 - \frac{\rho z\phi_M}{\phi_{M+1}} = -\frac{\delta_{M+1}}{\phi_{M+1}}(1 - z)$$

since $\phi_{M+1} - \rho z\phi_M = \rho\phi_M + (z - 1)\delta_M - \rho z\phi_M = (1 - z)(\rho\phi_M - \delta_M) = -(1 - z)\delta_{M+1}$. ■

Proof of Corollary 2.2: We first note that $x_1(0) = 1+\rho$, $x_2(0) = 0$; moreover, $x_2(z) = O(z)$ in the vicinity of $z = 0$, and hence $x_2^{M+1}(z) = O(z^{M+1})$ there, and $\left[z^{n-1}\right] x_2^{M+1}(z) = 0$ for $n - 1 \leq M + 1$. Consequently, for $k < M + 1$

$$\left[z^k\right] \delta_{M+1}(z) = \left[z^k\right] x_1^{M+1}(z),$$

$$\left[z^k\right] \phi_{M+1}(z) = \rho^{-1} \left[z^k\right] x_1^{M+1}(z)(1 - x_1(z)).$$

(Recall that $x_i - \rho = 1 - x_j$, $i \neq j$). For $0 \leq k - j < M + 1$ we get

$$\begin{aligned}&\left[z^k\right] \frac{\delta_{M+1}}{\phi_{M+1}} \frac{1}{1 - z} \left(\frac{\rho z\phi_M}{\phi_{M+1}}\right)^j\\ &= \left[z^{k-j}\right] \rho \frac{x_1^{M+1}}{x_1^{M+1}(1 - x_1)} \frac{\rho^j}{1 - z} \left(\frac{x_1^M(1 - x_1)}{x_1^{M+1}(1 - x_1)}\right)^j\\ &= \left[z^{k-j}\right] \frac{\rho^{j+1}}{x_1^j(1 - x_1)} \frac{1}{1 - z},\end{aligned}$$

from which (9) follows.

Proof of Corollary 2.3: Letting $j = 0$ in (9),

$$\begin{aligned}&\rho^{-M} R_M^{-1} P_\rho(> 0, n)\\ &= -\rho \left[z^{n-1}\right] \frac{1}{1 - z} \frac{2}{1 - (1 + \rho) - \sqrt{(1 + \rho)^2 - 4\rho z}}\\ &= \left[z^{n-1}\right] \frac{1}{2(1 - z)^2} \left[1 - \rho + \sqrt{(1 + \rho)^2 - 4\rho z}\right] \quad (23)\\ &= \frac{1}{2} \left[z^{n-1}\right]\left[2 + (1 + \rho) \sum_{j=1}^{\infty} c_j(\beta z)^j\right]\left[\sum_{j=0}^{\infty}(j + 1)z^j\right],\end{aligned}$$

from which (10) follows.

To obtain equation (11) from (9), we first note that:

$$\begin{aligned}\frac{\rho^j z^j}{x_1^j(1 - x_1)} &= \frac{x_2^j}{1 - x_1}\\ &= \frac{x_2^{j-1}(1 + \rho - x_1)}{1 - x_1} = \frac{x_2^{j-1}}{1 - x_1} + x_2^{j-1},\end{aligned}$$

and therefore:

$$\begin{aligned}&\rho^{-j} P_\rho(> j, n)\\ &= \rho^M R_M[z^{n-1}] \frac{x_2^{j-1}}{\rho^{j-1}} \frac{1}{1 - z} + \rho^{-j+1} P_\rho(> j - 1, n).\end{aligned}$$

Equation (9) then follows because $[z^n]f(z)/(1 - z) = \sum_0^n [z^m]f(z)$. Formula (12) is obtained from the expression:

$$\begin{aligned}x_2^j &= \left(\frac{1 + \rho}{2}\right)^j \left(1 - \sqrt{1 - \beta z}\right)^j\\ &= \left(\frac{1 + \rho}{2}\right)^j \sum_{r=0}^{j} \binom{j}{r} (-1)^r (1 - \beta z)^{r/2}\\ &= \left(\frac{1 + \rho}{2}\right)^j \sum_{r=0}^{j} \binom{j}{r} (-1)^r \sum_{s=0}^{\infty} \binom{s - r/2 - 1}{s} \beta^s z^s.\end{aligned}$$

The last equality follows from e.g. Eq. (2.1) in [2] p. 219.

Relation (13) is an easy consequence of (11) given that $u_m^0 = 1$ if $M = 0$ and 0 otherwise. Equations (10) and (13) imply (14).

■

4 Asymptotic behavior

In this section we investigate the asymptotic behavior of the expressions for $P_\rho(>j,n)$, $j=0,1$, that were obtained in Corollary 2.2. We are first interrested in the behavior of the loss probabilities as n becomes large, for large buffer sizes. As stated in Corollary 2.2, the loss probabilities are given as a product of some function of M by another function of n alone. Our approach is to obtain approximations for the function of n, and then the resulting expressions for the loss probabilities will hold for all M larger than n. Afterwards, we obtain approximations for the loss probabilities as ρ becomes large. The latter will allow us to conclude in the next section that for any size of message, redundancy improves the performance for ρ large enough.

Proposition 4.1 *Under the conditions of Corollary 2.2 we have*

$$P_\rho(>0,n) = \begin{cases} \dfrac{\rho^M(1-\rho)}{1-\rho^{M+1}}\left[(1-\rho)n+\dfrac{\rho}{1-\rho}+\ \beta^n O\big(n^{-3/2}\big)\right] & \text{if } \rho<1 \\ \dfrac{1}{M+1}\sqrt{n}\dfrac{2}{\sqrt{\pi}}\left(1+O\left(n^{-1}\right)\right) & \text{if } \rho=1 \\ 1-\dfrac{4\rho(1+\rho)}{\sqrt{\pi}(\rho-1)^3}\beta^{n-1}n^{-3/2}(1+o(1)) & \text{if } \rho>1 \end{cases}$$

$$P_\rho(>1,n) = \begin{cases} \dfrac{\rho^{M+1}(1-\rho)}{1-\rho^{M+1}}\left[(1-\rho)n+\dfrac{2\rho-1}{1-\rho}+\ \beta^n O\big(n^{-3/2}\big)\right] & \text{if } \rho<1 \\ \dfrac{1}{M+1}\left(\sqrt{n}\dfrac{2}{\sqrt{\pi}}\left(1+O\left(n^{-1}\right)-1\right)\right) & \text{if } \rho=1 \\ 1-\dfrac{4\rho^2(1+\rho)}{\sqrt{\pi}(\rho-1)^3}\beta^{n-1}n^{-3/2}(1+o(1)) & \text{if } \rho>1 \end{cases}$$

Proof: Define

$$\psi_0(z)=\frac{1}{2(1-z)^2}\left[1-\rho+\sqrt{(1+\rho)^2-4\rho z}\right] \quad (24)$$

Equation (23) relates $P_\rho(>0,n)$ to $\left[z^{n-1}\right]\psi_0(z)$. On the other hand, it is known from [2] that the asymptotic behavior of the coefficients $[z^n]f(z)$ are related to the nature and location of the singularities of $f(z)$. It is clear that $\psi_0(z)$ may have singularities at $z=1$ and $z=1/\beta$. For $\rho<1$, we have:

$$\psi_0(z)=\frac{1-\rho}{2(1-z)^2}\left(1+\sqrt{1-\frac{4\rho(z-1)}{(1-\rho)^2}}\right) \quad (25)$$

and a Taylor expansion of ψ_0 at point $z=1$ reveals that:

$$\psi_0(z)=\frac{1-\rho}{(1-z)^2}+\frac{\rho}{1-\rho}\left(\frac{1}{1-z}\right)+\psi_2(z), \quad (26)$$

where

$$\psi_2(z)=\frac{1-\rho}{2}\sum_{j=2}^{\infty}c_j\frac{(4\rho)^j(z-1)^{j-2}}{(1-\rho)^{2j}} \quad (27)$$

is analytic around $z=1$, and has a singularity of type $\sqrt{\bullet}$ at $z=1/\beta$. In other words, $\psi_2(z)=O(\sqrt{1/\beta-z})$ when $z\to1/\beta$. In addition to this singularity, ψ_0 is seen to have a pole at $z=1$. We get

$$\left[z^{n-1}\right]\psi_0(z)=(1-\rho)n+\frac{\rho}{1-\rho}+\beta^nO\left(n^{-3/2}\right)$$

by applying theorem 1 of Flajolet and Odlyzko [2]. This theorem is applicable since ψ_2 is analytic in the domain $\mathcal{D}$ which includes the whole complex plane, except for the segment along the real axis $z\in[1/\beta,\infty)$.

For $\rho=1$ we have

$$\psi_0(z)=(1-z)^{-3/2}\ ,$$

So that

$$\left[z^{n-1}\right]\psi_0(z)=\frac{\Gamma(n+1/2)}{\Gamma(3/2)(n-1)!}=\frac{2\sqrt{n}}{\sqrt{\pi}}\left(1+O(n^{-1})\right),$$

where we used Proposition 1 of [2], as well as the fact that $\Gamma(3/2)=\sqrt{\pi}/2$.

Finally, for $\rho>1$, we obtain in a way similar to (25)

$$\psi_0(z)=\frac{\rho}{\rho-1}\left(\frac{1}{1-z}\right)-\psi_2(z) \quad (28)$$

where ψ_2 is given in (27). It is easily checked that as z tends to $1/\beta$, or equivalently, as $z-1$ tends to $1/\gamma$ (where $\gamma=4\rho/(\rho-1)^2$), we have

$$\begin{aligned}\psi_2 \ &= \ \frac{4\rho^2}{(\rho-1)^3}-\frac{8\rho^2(1+\rho)}{(\rho-1)^4}\sqrt{1-\gamma z} \\ &\quad +o\left(\sqrt{1-\gamma z}\right)\ .\end{aligned}$$

It follows from [2] (p. 219 and Corollary 1) that

$$\begin{aligned}\left[z^{n-1}\right]\psi_2(z) \ &= \ -\frac{8\rho^2(1+\rho)}{(\rho-1)^4}\left(\frac{4\rho}{(1-\rho)^2}\right)^{n-1} \\ &\quad \frac{(n-1)^{3/2}}{\Gamma(-1/2)}\left(1+O\left(n^{-1}\right)\right). \quad (29)\end{aligned}$$

Equations (23), (28) and (29) then imply

$$\begin{aligned}
&P_\rho(>0,n)\\
&= \rho^M \frac{\rho-1}{\rho^{M+1}-1}\left[z^{n-1}\right]\psi_0(z)\\
&= \frac{1}{1-\rho^{-(M+1)}}\left[z^{n-1}\right]\left(\frac{1}{1-z}+\frac{1-\rho}{\rho}\psi_2(z)\right)\\
&= 1-\frac{4\rho(1+\rho)}{\sqrt{\pi}(\rho-1)^3}\left(\frac{4\rho}{(1-\rho)^2}\right)^{n-1} n^{-3/2}(1+o(1)).
\end{aligned}$$

The asymptotics of $P_\rho(>1,n)$ are obtained from that of $P_\rho(>0,n)$ using relation (13).

■

Next we examine the asymptotics of the loss probabilities for large ρ.

Proposition 4.2 *For $1 \le n \le M+1$ and large ρ (i.e. $\rho >> 1$),*

$$P_\rho(>0,n) = 1 + c_{n+1}\frac{4^{n+1}}{2\rho^n} + \frac{1}{\rho^{M+1}} + O\left(\frac{1}{\rho^{n+1}}\right), \tag{30}$$

where c_n are given in (7). For $2 \le n \le M+2$ and large ρ (i.e. $\rho >> 1$),

$$P_\rho(>1,n) = 1+c_{n+1}\frac{4^{n+1}}{2\rho^{n-1}}+\frac{1}{\rho^{M+1}}+O\left(\frac{1}{\rho^n}\right). \tag{31}$$

Proof: It follows from (23) and (28) that

$$\begin{aligned}
&(1-\rho^{-(M+1)})P_\rho(>0,n)\\
&= \left[z^{n-1}\right]\left(\frac{1}{1-z}+\frac{1}{2\rho}\sum_{j=2}^{\infty} c_j \frac{(4\rho)^j(z-1)^{j-2}}{(1-\rho)^{2j-2}}\right)\\
&= 1+\frac{1}{2\rho}\sum_{j=n+1}^{\infty} c_j\frac{(4\rho)^j}{(1-\rho)^{2j-2}}\binom{j-1}{n-1}(-1)^{j-n-1},
\end{aligned}$$

from which (30) follows since $M+1 \ge n$.

The proof of (31) is obtained from (30) by applying Corollary 2.3.

■

Finally, we study the asymptotic behavior of the loss probabilities in light traffic.

Proposition 4.3 *For any n, M, $0 \le j \le n$, we have for small ρ*

$$P_\rho(>j,n) = \rho^{M+j}(n-j+O(\rho)).$$

Proof: The proof is similar to that of Corollary 2.2. We have $x_1(z) = 1+O(\rho)$, $x_2(z) = O(\rho)$ and hence $\delta_M = 1+O(\rho)$. This implies that

$$\left(\frac{\phi_M}{\phi_{M+1}}\right)^M = 1+O(\rho)\ , \qquad \frac{\delta_{M+1}}{\phi_{M+1}} = \frac{1}{z-1}+O(\rho).$$

Combining this with Corollary 2.1, we get:

$$\begin{aligned}
P_\rho(>j,n) &= \rho^{M+j}\left[z^{n-1}\right]\frac{z^j}{(z-1)^2}+O(\rho)\\
&= \rho^{M+j}(n-j+O(\rho)).
\end{aligned}$$

■

5 When is it better to add redundancy

In this section we compare the loss probabilities of messages of size n, with and without an additional redundancy. We assume that the rate at which *messages* arrive is the same for the two cases and is given by x. In the case of no redundancy, the rate at which *packets* arrive is $\rho = nx$. A message is lost if and only if at least one packet is lost out of n consecutive packets. In the case of redundancy, the rate at which packets arrive is $\rho = (n+1)x$. A message is lost if and only if more than one packet is lost out of $n+1$ consecutive ones. There is therefore a tradeoff between an increased loss probability for individual packets due to the increased load, and the decreased vulnerability of messages, which can now afford to lose one packet.

We are thus interested in the difference $\Delta = P_{nx}(>0,n) - P_{(n+1)x}(>1,n+1)$. If $\Delta > 0$ then the redundancy decreases the loss probabilities of messages.

Proposition 5.1 *(i) For any $1 \le n < M$ adding redundancy results in a decrease of the loss probabilities for all x large enough (heavy traffic regime) .*
(ii) For any n and M, adding redundancy results in a decrease of the loss probabilities for all x small enough (light traffic regime).

Proof: (i) Using Proposition 4.2 and substituting (7) we get

$$\begin{aligned}
\Delta &\cong \frac{1}{2x^n}\left[c_{n+1}\frac{4^{n+1}}{n^n} - c_{n+2}\frac{4^{n+2}}{(n+1)^n}\right]\\
&= \frac{1}{\rho^n}\frac{2n!}{(n+1)!n!}\left[-\frac{1}{n^n}+\frac{2(2n+1)}{n+2}\frac{1}{(n+1)^n}\right].
\end{aligned}$$

Hence for all large enough x, $\Delta > 0$ iff

$$\frac{2(2n+1)}{n+2} > \left(\frac{n+1}{n}\right)^n.$$

By taking the log and differentiating twice the latter is seen to hold for all $n \ge 1$.

(ii) Using Proposition 4.3, we get

$$P_{nx}(>0,n) = (nx)^M(n+O(nx))$$

and

$$P_{(n+1)x}(>1, n+1) = ((n+1)x)^{M+1}(n + O((n+1)x)).$$

It thus follows that

$$\lim_{x\to 0} \frac{P_{(n+1)x}(>1, n+1)}{x P_{nx}(>0, n)} = \left(\frac{n+1}{n}\right)^M (n+1). \quad (32)$$

Hence, for all x small enough, $\Delta > 0$. ∎

Remark 5.1 Consider any fixed integers $0 \le k_1 < k_2$ and any n and M. Assume that we want to compare whether it is better to add k_1 or k_2 redundancy packets. Again, we consider a fixed rate x of arrival of messages, whose length is initially n. If k_j redundancy packets are added, then loss occur if and only if more than k_j packets are lost out of $n + k_j$ consecutive packets. The above proof for the light traffic case generalizes in a straightforward way to this generalization, and shows that if ρ is sufficiently small then the loss probability with k_2 redundant packets is smaller than for k_1 redundant packets.

6 Numerical examples

Although Proposition 5.1 shows that redundancy is required in light and in heavy traffic, a natural question is whether these cases are of any practical interest. Indeed, for ρ much larger than 1, the loss probabilities are already very close to one, so that even if adding redundancy results in improving the loss probabilities, this improvement may be quite negligible. Moreover, it does not seem reasonable that systems would operate with loss probabilities close to one. The light traffic region might also be of limited interest in cases that only for extremely small ρ, redundancy is of benefit; in that case, again one may argue that systems are seldom designed to operate under such conditions.

In the following numerical examples we show that, indeed, adding redundancy is profitable not only in extreme conditions, but also in practical cases of interest. Our approach below is to fix ρ and then obtain a family of n and M such that the behavior of the system will be the one characterized by the light traffic region, and hence redundancy will lead to better performance. We shall restrict to a family of n and M related by $n \cong \gamma M$, where γ is a constant to be determined, and we shall consider $n >> 1$. In fact, the asymptotic approximations turn out to be quite accurate even for moderate values of n and M. In Figures 1, we display the values of $P_\rho(>0, n)$ and of two approximations:

$$A_0(\rho) = \rho^M \frac{(1-\rho)^2}{1-\rho^{M+1}} n$$

$$B_0(\rho) = \rho^M \frac{(1-\rho)}{1-\rho^{M+1}} \left((1-\rho)n + \frac{\rho}{1-\rho}\right),$$

in the case $M = 7$, $n \le 7$ and $\rho = 0.3$ (see below). In Figure 2, we make the same comparison for $P_\rho(>1, n)$ and

$$A_1(\rho) = \rho^{M+1} \frac{(1-\rho)^2}{1-\rho^{M+1}} n$$

$$B_1(\rho) = \rho^{M+1} \frac{(1-\rho)}{1-\rho^{M+1}} \left((1-\rho)n + \frac{2\rho-1}{1-\rho}\right).$$

These approximations are obtained from Proposition 4.1 (case $\rho < 1$), by taking the first and the the two first terms, respectively, in the asymptotic expansion.

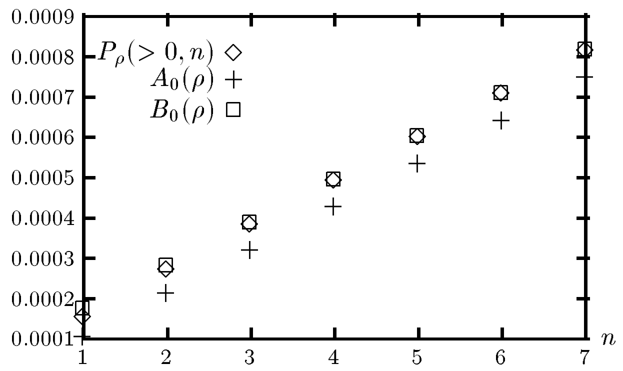

Figure 1: $P_\rho(>0, n)$ and its approximations

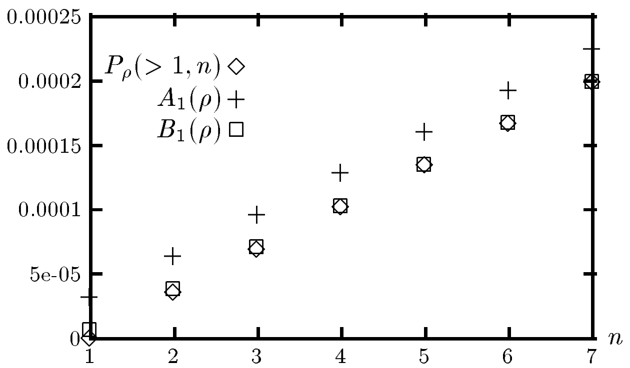

Figure 2: $P_\rho(>1, n)$ and its approximations

We use the following heuristic for obtaining γ. We wish to determine x^* for which $P_{(n+1)x^*}(>1, n+1) =$

$P_{nx^*}(> 0, n)$. The interval $[0, x^*]$ will then describe the light traffic region, for which we should use redundancy. As in (32), it follows from Proposition 4.1 that

$$x^* \cong \frac{1}{n+1}\left(\frac{n}{n+1}\right)^M \cong \frac{1}{\gamma M}\left(1 - \frac{1}{\alpha M}\right)^M \cong \frac{e^{-1/\gamma}}{\gamma M}$$

Since $\rho \cong x^* \gamma M$, we have

$$\gamma = -1/\log \rho. \tag{33}$$

Remark 6.1 The fact that we used the case $\rho < 1$ of Proposition 4.1 above restricts this approach to cases where γ will turn out not to be greater than one. This implies that the approach is valid for $\rho \leq 1/e \cong 0.31$.

Denote $S(\rho) = P_{nx}(> 0, n)$, and $A(\rho) = P_{(n+1)x}(> 1, n+1)$.

In all the examples below we apply the above heuristic to determine the upper limit of the light traffic region, and then compare it to exact calculations.

We first choose $\rho = 0.3$. We get from (33) $\gamma \cong 0.83$ and thus $n \cong 0.83M$. Hence

$$S(0.3) \cong 0.3^M \frac{(1-0.3)^2}{1-0.3^{M+1}} \gamma M. \tag{34}$$

Suppose that we wish to have $S(\rho) < 10^{-3}$. Then (34) yields $M \geq 7$. With $M = 7$ we obtain $n \cong \gamma M \cong 6$. Exact calculation yields that $S(\rho) = 7.1 \cdot 10^{-4}$ and $A(\rho) = 5.9 \cdot 10^{-4}$. Moreover, the sets of values of ρ for which redundancy improves the performance for these M and n are the intervals $[0, 0.384]$, and $[2.123, \infty)$. Hence the upper limit of the light traffic region is 0.384, about 30% higher than the the value 0.3 for which the heuristic method predicts that limit be obtained. The right end of the first interval can be read on figure 4, whereas the right end of the second interval is easier to see on figure 3.

Next we choose $\rho = 0.2$. Equation (33) yields $\gamma = 0.62$. As in the previous calculation (34), in order to get a loss probability inferior to 10^{-3} we need $M \geq 5$. With $M = 5$ we obtain $n \cong \gamma M \cong 3$. Exact calculation yields $S(\rho) = 6.7 \cdot 10^{-4}$ and $A(\rho) = 6.0 \cdot 10^{-4}$. The sets of values of ρ for which redundancy improves the performance for these M and n are the intervals $[0, 0.235]$, and $[3.429, \infty)$ (see Figure 3). If we choose instead $M = 10$ we obtain $n = 6$. Exact calculation then yields $S(\rho) = 3.9 \cdot 10^{-7}$ and $A(\rho) = 4.1 \cdot 10^{-7}$. The sets of values of ρ for which redundancy improves the performance for these M and n are the intervals $[0, 0.208]$, and $[2.26, \infty)$ (see again Figures 3 and 4).

7 Conclusion

We have applied analytical tools to solve a recurrence for the loss probabilities of messages in a $M/M/1/K$ queue. This has resulted in more efficient algorithms to compute the loss probabilities, and, using asymptotics, to a heuristic allowing to find regions of the parameter space for which adding a redundancy packet decreases the loss probability of messages.

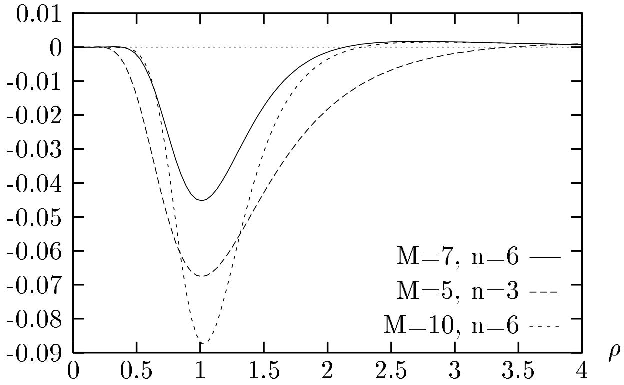

Figure 3: Difference $S(\rho) - A(\rho)$ for several M and n

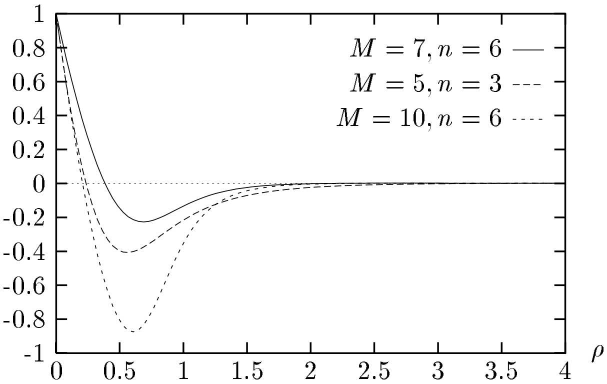

Figure 4: Difference $(S(\rho) - A(\rho))/S(\rho)$ for several M and n

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