Queueing Analysis of Early Message Discard Policy

Parijat Dube[1] Eitan Altman[1,2]

***Abstract*— We consider in this paper packets which arrive according to a Poisson process into a finite queue. A group of consecutive packets forms a frame (or a message) and one then considers not only the quality of service of a single packet but also that of the whole message. In order to improve required quality of service, either on the frame loss probabilities or on the delay, discarding mechanisms have to be used. We analyze in this paper the performance of the Early Message Discard (EMD) policy at the buffer, which consists of (1) rejecting an entire message if upon the arrival of the first packet of the message, the buffer occupancy exceeds a threshold K, and (2) if a packet is lost, then all subsequent arrivals that belong to the same message are discarded.**

***Index Terms*— EMD policy, packet model, queue-length distribution, goodput.**

I. Introduction

Quite often quality of service have to be studied with respect to not only a single packet, but to a whole message or a frame. For example, in ATM a transport layer protocol (AAL) is responsible for grouping packets into a frame, and a lost packet implies the corruption of the whole frame. Selective Message Discarding (and EMD in particular, on which we focus here) have been proposed to achieve the twin goals of increased goodput and reduced network congestion by discarding the packets which do not belong to (or have potentials of not belonging to) *good* messages (a message is good if it is entirely received at the destination). Rejecting entire messages could also serve to guarantee an acceptable average delay bound for accepted messages. The goal of this paper is to present *explicit expressions* for the queue-length distribution and the goodput (defined as in [10] as the ratio between total packets comprising good messages exiting the network node and the total arriving packets at the input). Our starting point is the Markovian model proposed in [10]: a Poisson process of packet arrivals, geometrically distributed frame size, and exponentially distributed service times of packets. In [10], recursive procedures have been proposed for the computation of the performance measures, but explicit expressions have not been obtained. Our analytical results on closed form expressions for performance metrics (in particular the queue-length distribution and the goodput) may be quite useful in dimensioning the buffer size that should be used for a given goodput, in the study of the sensitivity of the goodput to different parameters for e.g., the message length, the buffer size, the load and most importantly in finding an estimate of the optimal discarding threshold etc.

In a previous work [5], we analyzed the Partial Message Discard (PMD) policy in which only if some packet of a message is lost, subsequent packets are rejected (but entire messages are not discarded, in contrast with EMD). As the packet level analysis turns to be quite complex and involved, we studied in [3], [4], [5], [6] some fluid approximations. Some other references on numerical studies of PMD and EMD policies are [8], [7].

[1]INRIA, B.P. 93, 06902, Sophia Antipolis Cedex, France. E-mail: {Parijat.Dube,Eitan.Altman}@sophia.inria.fr.
[2]CESIMO, Universidad de Los Andes, Facultad de Ingeneria, Merida, Venezuela

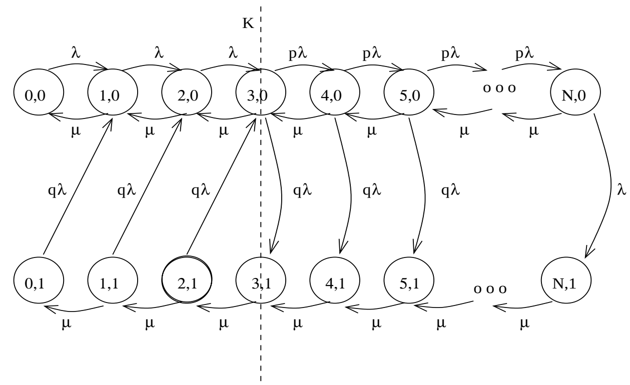

Fig. 1. Transition structure under the EMD policy

In Section II we describe our queueing model and present our main results on the z-transform of the queue-length distribution and then the explicit expressions for the steady-state probabilities. In Section III we present an approach for obtaining the explicit expression for the goodput ratio using algebraic techniques.

II. Packet Model

The packet model is the same as the one proposed in [10]. We first describe the model in brief. In terms of packet the network element is a $M/M/1/N$ queue with arrival rate λ and service rate μ and the load $\rho = \frac{\lambda}{\mu}$. A message length (in terms of packets) is considered to be geometrically distributed with parameter q. Under the EMD policy, a threshold level K (K is an integer, $0 \leq K \leq N$) is fixed. If a message starts to arrive when the buffer occupancy is at or above K packets, then all the packets of that message are discarded. Also, if a packet belonging to an accepted message is discarded due to buffer overflow then all the subsequent packets belonging to the same message are also discarded. To model the policy, two modes for working of the network element are defined: the *normal mode*, in which packets are admitted, and the *discarding mode*, in which arriving packets are discarded. The state transition diagram for EMD policy under this model is shown in Figure (1). Let $P_{i,j} (0 \leq i \leq N, j = 0, 1)$ be the steady-state probability of having i packets in the system and the system is in mode j ($j = 0$ for normal; $j = 1$ for discarding). We now define the transform functions $A_j(z) = \sum_{i=0}^{K} z^i P_{i,j}$, $B_j(z) = \sum_{i=K+1}^{N} z^i P_{i,j}$ and $Q_j(z) = A_j(z) + B_j(z)$ for $j = 0, 1$.

A. PGF and distribution of the number of packets in the queue

Proposition 1: The probability generating functions $A_j(z)$ and $B_j(z)$ for $j = 0, 1$ are given by,

$$A_0(z) = \Big[(q\rho + 1 - z^{-1})(P_{0,0}(1 - z^{-1}) - P_{K,0}\rho z^{K+1} + P_{K+1,0} z^K) + P_{K,1} q\rho z^K$$

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$$
\begin{aligned}
& \quad +P_{0,1}(1-z^{-1})q\rho z\big] D_1 \\
A_1(z) &= \frac{P_{0,1}(1-z^{-1})+P_{K,1}(q\rho+1)z^K}{(q\rho+1-z^{-1})} \\
B_0(z) &= \frac{P_{K,0}r\rho z^{K+1}-P_{N,0}r\rho z^{N+1}-P_{K+1,0}z^K}{(\rho+1-r\rho z-z^{-1})} \\
B_1(z) &= \big[P_{K,1}(z^{-1}+r\rho z-\rho-1)-P_{K+1,0}q\rho \\
& \quad +P_{N,0}\rho z^{N-K}((\rho+1-z^{-1})(1-q) \\
& \quad -r\rho z)+P_{K,0}\rho q(\rho+1-z^{-1})\big] z^K D_2
\end{aligned}
$$

where,

$$
\begin{aligned}
D_1 &= \left((\rho+1-\rho z-z^{-1})(q\rho+1-z^{-1})\right)^{-1} \\
D_2 &= (\rho+1-r\rho z-z^{-1})(1-z^{-1})^{-1} \\
P_{K,0} &= \frac{q\rho(1-\rho)}{D} \quad P_{K,1}=\rho\left(1-\frac{r\delta_{K-N}}{\delta_{K-N-1}}\right)P_{K,0} \\
P_{0,1} &= \frac{1}{q(1+q\rho)^{K-1}}\left(1-\frac{r\delta_{K-N}}{\delta_{K-N-1}}\right)P_{K,0} \\
P_{0,0} &= \frac{P_{K,0}\rho^{-K}}{1-\rho}\left(1-\frac{r\rho\delta_{K-N}}{\delta_{K-N-1}}\right)\left[\frac{(1-\rho)(1-q)}{(q\rho+1-\rho)}\right. \\
& \quad \left.\left(\rho^2+\frac{\rho^K}{q(1+q\rho)^{K-1}}\right)\right] \\
P_{N,0} &= \frac{(W_1W_2)^{K-N-1}(W_2-W_1)r\rho P_{K,0}}{\delta_{K-N-1}} \\
P_{K+1,0} &= \frac{r\rho\delta_{K-N}}{\delta_{K-N-1}}P_{K,0} \\
D &= q\rho\left(\frac{\rho^{-K}}{1-\rho}\left(1-\frac{r\rho\delta_{K-N}}{\delta_{K-N-1}}\right)\left(\frac{(1-\rho)(1-q)}{(q\rho+1-\rho)}\right.\right. \\
& \quad \left.\left.\left(\rho^2+\frac{\rho^K(1-\rho)}{q(1+q\rho)^{K-1}}\right)\right)+\frac{1}{q(1+q\rho)^{K-1}}\right. \\
& \quad \left.\left(1-\frac{r\delta_{K-N}}{\delta_{K-N-1}}\right)\right)-(1-\rho)(1-q)r\rho^3 \\
& \quad \left(\frac{(W_1W_2)^{K-N-1}(W_2-W_1)}{\delta_{K-N-1}}\right) \\
& \quad +\rho^2(1-\rho+K\rho q)\left(1-\frac{r\delta_{K-N}}{\delta_{K-N-1}}\right) \\
& \quad +(-1+\rho+Kq\rho^2)\frac{r\rho\delta_{K-N}}{\delta_{K-N-1}} \\
& \quad -\rho^2(1+q+Kq\rho)
\end{aligned}
$$

with $W_{1,2}=\frac{(\rho+1)\pm\left((\rho+1)^2-4r\rho\right)^{\frac{1}{2}}}{2r\rho}$ and $\delta_y=W_1^y-W_2^y$ for any y.
Proof: Refer to Appendix A.

Corollary 1: The transition probabilities are given by,
For $1\le i\le K$,

$$
\begin{aligned}
P_{i,0} &= P_{K,0}\left(\frac{1-\rho^{-(K+1-i)}}{1-\rho^{-1}}-\frac{\delta_{K+1-i}}{\delta}\right) \\
& \quad +\frac{P_{K+1,0}}{\rho}\left(\frac{\delta_{K-i}}{r\delta}-\frac{1-\rho^{-(K-i)}}{1-\rho^{-1}}\right)+\frac{P_{N,0}}{\delta}\delta_{N+1-i} \\
& \quad -\frac{P_{K,1}q}{(1+q\rho)}\left(\frac{1-\rho^{-(K+1-i)}}{(1-\rho^{-1})(\rho^{-1}-(1+q\rho)^{-1})}\right. \\
& \quad \left.+\frac{(1+q\rho)^{-(K+1-i)}}{(1-(1+q\rho)^{-1})(\rho^{-1}-(1+q\rho)^{-1})}\right), \\
P_{i,1} &= P_{K,1}((1+q\rho)^{-K+i}-1) \\
& \quad +P_{N,0}\frac{(\rho\delta_{N-i+1}-\delta_{N-i})}{\delta} \\
& \quad +P_{K,0}\left(\rho+\frac{\delta_{K-i+1}-r\rho\delta_{K-i+2}}{\delta}\right) \\
& \quad -P_{K+1,0}\left(1+\frac{\delta_{K-i}-r\rho\delta_{K+1-i}}{\delta}\right).
\end{aligned}
$$

And for $K+1\le i\le N$,

$$
P_{i,0}=\frac{P_{N,0}\delta_{N+1-i}}{\delta}, \quad P_{i,1}=\frac{P_{N,0}(\rho\delta_{N-i+1}-\delta_{N-i})}{\delta}
$$

where $\delta_1=\delta$.
Proof: Refer to Appendix B.
Having obtained the explicit expressions for the stationary distribution of the queue-length we next proceed to obtain the expression for the goodput ratio (as defined in Section I).

III. Goodput ratio

Let $\mathcal{W}$ be the random variable that represents the length (number of packets) of an arriving message. Let $\mathcal{V}$ be the random variable representing the success of a message, $\mathcal{V}=1$ for a good message, and $\mathcal{V}=0$ for a message which has one or more dropped packets. Then $\mathcal{G}$ can be expressed as (see [10])

$$
q\sum_{n=1}^{\infty}nq(1-q)^{n-1}\sum_{i=0}^{N}P(\mathcal{V}=1|\mathcal{W}=n,\mathcal{Q}=i)P(\mathcal{Q}=i)
$$

Denote the conditional probabilities $S_{n,i}\stackrel{\triangle}{=}P(\mathcal{V}=1|\mathcal{W}=n,\mathcal{Q}=i)$. In [10], recursions for evaluating these probabilities and hence $\mathcal{G}$ were given. We will present here an explicit expression for $\mathcal{G}$. To do this we will use the multidimensional generating function for probabilities $S_{n,i}$ which was obtained in a different context in [1] and in [9]. We define the two-dimensional generating function of $S_{n,i}$ for $1\le n\le\infty$ and $0\le i\le N$, as $\bar{S}(x,y)$, i.e., $\bar{S}(x,y)=\sum_{n=1}^{\infty}\sum_{i=0}^{N}S_{n,i}x^{n-1}y^i$, We will next reproduce the Proposition we developed in [5] for the case of *Partial Message Discard(PMD)* policy. A PMD is an EMD with $K>N$.

Proposition 2: The probability generating function $\bar{S}(x,y)$ can be expressed as $\bar{S}(x,y)=\sum_{i=0}^{N}c_i(x)y^i$ where for $0\le i\le N-1$, $c_i(x)$ is equal to

$$
\begin{aligned}
& 1+K_3\left(A_1-A_2y_1^{N-(i+1)}-A_3y_2^{N-(i+1)}\right) \\
& +K_4\left(B_1y_1^{N-i}+B_2y_2^{N-i}\right)^{1}
\end{aligned}
$$

and $c_N(x)=0$ with,

$$
\begin{aligned}
y_{1,2} &= \frac{1+\rho\pm\sqrt{(1+\rho)^2-4\rho x}}{2} \\
K_3 &= -x\rho \\
K_4 &= \frac{x\rho(y_1^N-y_2^N)}{y_2^{N+1}(y_1-\rho)-y_1^{N+1}(y_2-\rho)} \\
A_1 &= 1/((1-y_1)(1-y_2)) \\
A_2 &= 1/((1-y_1)(y_1-y_2)) \\
A_3 &= 1/((1-y_2)(y_2-y_1)) \\
B_1=-B_2 &= (y_1-y_2)^{-1}
\end{aligned}
$$

For the EMD policy, the conditional probabilities $S_{n,i}$ are same as that for the PMD policy for $i < K$. If the head of the message arrives when the system occupancy is at or above the threshold the message as a whole is rejected. Thus for the EMD policy we define the transition probability generating function as $\bar{S}(x,y) = \sum_{i=0}^{K-1} c_i(x)y^i$, where $c_i(x)$ is given by Proposition (2). And the expression for *Goodput ratio* can be expressed (like in Proposition (3) in [5], with N in the summation replaced by $K-1$) as,

$$\begin{aligned}
\mathcal{G} &= q^2 \sum_{i=0}^{K-1} \left(\frac{d(xc_i(x))}{dx} \right)_{x=(1-q)} P(Q=i) \\
&= q^2 \left[(1-q) \left(\frac{d}{dx} \left(\sum_{i=0}^{K-1} c_i(x) P(Q=i) \right) \right)_{x=(1-q)} \right. \\
&\quad \left. + \sum_{i=0}^{K-1} c_i(1-q)P(Q=i) \right] \qquad (1)
\end{aligned}$$

where stationary probabilities $P(Q=i) = P_{i,0} + P_{i,1}$ are known from Corollary 1.

A. Exact expression for $\mathcal{G}$

In this section we aim to derive an exact expression for $\mathcal{G}$ for EMD policy. From (1) we find that we need an expression for $\sum_{i=0}^{K-1} c_i(x)P(Q=i)$. Observe that $\sum_{i=0}^{K-1} c_i(x)P(Q=i) = c_0(x)P(Q=0) + \sum_{i=1}^{K-1} c_i(x)P(Q=i)$. We have $P(Q=0) = P_{0,0} + P_{0,1}$ with $P_{0,0}$ and $P_{0,1}$ given by Corollary 1. We next find a general expression for $P(Q=i)$ for $i = 1$ to $K-1$. From Corollary 1, by adding $P_{i,0}$ and $P_{i,1}$ for $i = 1$ to $K-1$ we can express $P(Q=i)$ as,

$$\begin{aligned}
& P_{K,0} \left[\frac{\rho - \rho^{-(K+1-i)}}{1-\rho^{-1}} - \frac{r\rho}{\delta}\delta_{K-i+2} \right] \\
& + P_{N,0} \left[\frac{\delta_{N+1-i}(1+\rho) - \delta_{N-i}}{\delta} \right] \\
& - P_{K,1} \left[\frac{\rho}{\rho-1} - \frac{\rho^{-(K-i)}q}{(1-\rho^{-1})(1+q\rho-\rho)} \right. \\
& \left. + \frac{(1+q\rho)^{-(K-i)}}{(1+q\rho-\rho)}\rho(1-q) \right] \\
& + P_{K+1,0} \left[\frac{\delta_{K-i}}{\delta} \left(\frac{1}{r\rho} - 1 \right) + \frac{r\rho}{\delta}\delta_{K+1-i} \right. \\
& \left. + \frac{\rho - \rho^{K-i}}{1-\rho} \right] \qquad (2)
\end{aligned}$$

We will now find an expression for $\sum_{i=1}^{K-1} c_i P(Q=i)$ [2]. From (2) and Proposition 2 we write after some algebra (see [2] for details), $\sum_{i=1}^{K-1} c_i P(Q=i)$ as

$$\begin{aligned}
& \sum_{i=1}^{K-1} c_i \left[\frac{P_{K,0}\rho - P_{K,1} - P_{K+1,0}}{1-\rho^{-1}} \right] - \frac{\rho^K P_{K+1,0}}{1-\rho} \sum_{i=1}^{K-1} c_i \rho^{-i} \\
& - \sum_{i=1}^{K-1} c_i \rho^i \left[\frac{P_{K,1} q \rho^{-K}}{(1-\rho^{-1})(1+q\rho-\rho)} - \frac{P_{K,0}\rho^{-(K+1)}}{1-\rho^{-1}} \right] \\
& - \sum_{i=1}^{K-1} c_i (1+q\rho)^i \frac{P_{K,1}\rho(1-q)(1+q\rho)^{-K}}{1+q\rho-\rho} - \sum_{i=1}^{K-1} c_i y_1^{-i} \\
& \left[\frac{P_{K,0} r\rho y_1^{K+2} - P_{N,0} y_1^N (y_1(1+\rho)-1)}{\delta} - \frac{P_{K+1,0} y_1^K}{\delta} \right. \\
& \left. \left(\frac{1}{r\rho} - 1 + y_1 r\rho \right) \right] + \sum_{i=1}^{K-1} c_i y_2^{-i} \left[\frac{P_{K,0} r\rho y_2^{K+2}}{\delta} \right. \\
& - \frac{P_{N,0} y_2^N (y_2(1+\rho)-1)}{\delta} \qquad (3) \\
& \left. - \frac{P_{K+1,0} y_2^K}{\delta} \left(\frac{1}{r\rho} - 1 + y_2 r\rho \right) \right]
\end{aligned}$$

Observe that the last expression contains terms of the form $\sum_{i=1}^{K-1} c_i a^i$ (with $a = 1, y_1^{-1}, y_2^{-1}, \rho, (1+q\rho)$). We now obtain these terms from the expression for c_i from Proposition 2. For any a, $\sum_{i=1}^{K-1} c_i a^i$ is equal to

$$\begin{aligned}
& (K-1)(1+K_3A_1)\frac{a(1-a^{K-1})}{1-a} \\
& + (K_4B_1y_1^N - K_3A_2y_1^{N-1}) \frac{\frac{a}{y_1}\left(1 - \left(\frac{a}{y_1}\right)^{K-1}\right)}{1-\frac{a}{y_1}} \\
& + (K_4B_2y_2^N - K_3A_3y_2^{N-1}) \frac{\frac{a}{y_2}\left(1 - \left(\frac{a}{y_2}\right)^{K-1}\right)}{1-\frac{a}{y_2}}
\end{aligned}$$

Thus after some rearrangements we can express the expression for $\sum_{i=1}^{K-1} c_i P(Q=i)$ from (3) as [3]

$$\begin{aligned}
& (K-1)(1+K_3A_1)[K_5 - K_6 - K_9 - F_1 \\
& + F_2] + (K_4B_1y_1^N - K_3A_2y_1^{N-1})[F_3 - F_{11} \\
& - F_4 - F_5 + F_6] + (K_4B_2y_2^N - K_3A_3y_2^{N-1}) \\
& [F_7 - F_{12} - F_8 - F_9 + F_{10}]
\end{aligned}$$

where,

$$\begin{aligned}
K_5 &= \frac{\rho(1-\rho^{K-1})}{1-\rho}K_7, \quad & K_6 &= \tfrac{(1+q\rho)(1-(1+q\rho)^{K-1})}{1-(1+q\rho)}K_8 \\
F_1 &= E_1(y_1)E_2(y_1), & F_2 &= E_1(y_2)E_2(y_2) \\
F_3 &= E_1(y_1/\rho)K_7, & F_4 &= E_1(y_1/(1+q\rho))K_8 \\
F_5 &= E_1(y_1^2)E_2(y_1), & F_6 &= E_1(y_1y_2)E_2(y_2) \\
F_7 &= E_1(y_2/\rho)K_7, & F_8 &= E_1(y_2/(1+q\rho))K_8 \\
F_9 &= E_1(y_1y_2)E_2(y_1), & F_{10} &= E_1(y_2^2)E_2(y_2) \\
F_{11} &= E_1(\rho y_1), & F_{12} &= E_1(\rho y_2)
\end{aligned}$$

with, $E_1(y) = \frac{y^{-1}(1-y^{-K+1})}{1-y^{-1}})$ and $E_2(y)$ is equal to

$$\begin{aligned}
& \frac{P_{K,0} r\rho y^{K+2}}{\delta} - \frac{P_{N,0}y^N(y(1+\rho)-1)}{\delta} \\
& + \frac{P_{K+1,0}y^K}{\delta}\left(\frac{1}{r\rho} - 1 + yr\rho\right)
\end{aligned}$$

[2] We shall not explicitly show x in parentheses for function c_i

[3] It should be noted that $A_1, A_2, A_3, B_1, B_2, K_3$ and $F_i, i = 1, 2, \ldots, 12$. are all functions of x.

and,

$$K_7 = \left[\frac{P_{K,1}\rho^{-K}}{(1-\rho^{-1})(1+q\rho-\rho)} - \frac{P_{K,0}\rho^{-(K+1)}}{1-\rho^{-1}}\right]$$
$$K_8 = \frac{P_{K,1}\rho(1-q)(1+q\rho)^{-K}}{1+q\rho-\rho}$$
$$K_9 = \frac{\rho(1-\rho^{K-1})P_{K+1,0}}{(1-\rho)^2}$$

Having obtained an expression for $\sum_{i=1}^{K-1} c_i P(Q=i)$, one can directly obtain the expression for $\mathcal{G}$ from (1) (however, we also need derivative of $\sum_{i=1}^{K-1} c_i P(Q=i)$ with respect to x which is easy to obtain).

IV. Conclusion

We provided explicit expressions for the stationary distribution of the queue-length and the goodput for the EMD policy. An interesting extension will be to study the asymptotic behavior of EMD policy, either from the generating functions or from the explicit expressions and to obtain simpler approximations valid for asymptotic regimes (large buffer, heavy traffic etc).

Appendix A

We have the following set of equations from [10] (with $r = 1-q$).

$$\rho P_{0,0} = P_{1,0} \quad (4)$$
$$q\rho P_{0,1} = P_{1,1} \quad (5)$$
$$(\rho+1)P_{i,0} = \rho P_{i-1,0} + P_{i+1,0} + q\rho P_{i-1,1} \text{ for } 1 \le i \le K \quad (6)$$
$$(\rho+1)P_{i,0} = r\rho P_{i-1,0} + P_{i+1,0} \text{ for } K+1 \le i \le N-1 \quad (7)$$
$$(\rho+1)P_{N,0}r = \rho P_{N-1,0} \quad (8)$$
$$P_{N,1} = \rho P_{N,0} \quad (9)$$
$$(q\rho+1)P_{i,1} = P_{i+1,1} \; 1 \le i \le K-1 \quad (10)$$
$$P_{i,1} = P_{i+1,1} + q\rho P_{i,0} \quad K \le i \le N-1 \quad (11)$$

Taking *z-transforms* of (6), (7), (8) and (10) and from (4) (5) and (9) after some algebra we get $A_j(z)$ and $B_j(z)$ as in Proposition 1 in terms of $P_{0,0}, P_{0,1}, P_{K,0}, P_{K+1,0}, P_{K,1}$ and $P_{N,0}$. Next observe that all the transform functions $A_j(z), B_j(z)$, $j = 0,1$, are polynomial in z (N is finite) and hence analytic. Thus the numerator of the right hand side of the expressions for $A_j(z), B_j(z)$ vanishes at the zeros of the denominator of the right hand side. Thus, substituting the zeros of the denominator in the numerator and equating it to 0 in the expressions for $A_j(z), B_j(z)$ we get the following set of five independent equations:

$$q\rho(-P_{K,0}\rho + P_{K+1,0}) + P_{K,1}q\rho = 0$$
$$P_{K,1} + P_{K+1,0} = \rho P_{K,0} \quad (12)$$
$$P_{0,1}q\rho(1+q\rho)^{K-1} = P_{K,1} \quad (13)$$
$$(q\rho+1-\rho)\left(P_{0,0}(1-\rho) - P_{K,0}\rho^{-K} + P_{K+1,0}\rho^{-K}\right) + P_{K,1}q\rho^{-K+1} + P_{0,1}(1-\rho)q = 0 \quad (14)$$

$$P_{K,0}r\rho W_1^{K+1} - P_{N,0}r\rho W_1^{N+1} - P_{K+1,0}W_1^K = 0 \quad (15)$$
$$P_{K,0}r\rho W_2^{K+1} - P_{N,0}r\rho W_2^{N+1} - P_{K+1,0}W_2^K = 0 \quad (16)$$

with $W_{1,2}$ as defined in Proposition 1. We also have $\sum_{i=0}^{N}(P_{i,0} + P_{i,1}) = 1$ which is same as

$$A_0(1) + B_0(1) + A_1(1) + B_1(1) = 1 \quad (17)$$

Since, $A_0(z)$ and $B_1(z)$ have the form of $0/0$ at $z = 1$ we shall take, $A_0(1) = \lim_{z\to1} A_0(z)$ and $B_1(1) = \lim_{z\to1} B_1(z)$. Thus, we have the following equation from (17)

$$\begin{aligned} q\rho(1-\rho) = \; & q\rho(P_{0,0}+P_{0,1}) - \rho^2(1+q+Kq\rho)P_{K,0} \\ & +(-1+\rho+Kq\rho^2) - (1-\rho)(1-q)\rho^2 P_{N,0} \\ & P_{K+1,0} + \rho(1-\rho+K\rho q)P_{K,1} \end{aligned} \quad (18)$$

Thus, the six unknowns $P_{0,0}, P_{0,1}, P_{K,0}, P_{K+1,0}, P_{K,1}$ and $P_{N,0}$ can be obtained by solving the six equations, (12), (13), (14), (15), (16) and (18) in six unknowns.

Appendix B

We have,

$$\begin{aligned} Q_0(z) &= A_0(z) + B_0(z) \\ &= \frac{\left(P_{0,0}(1-z^{-1}) - P_{K,0}\rho z^{K+1} + P_{K+1,0}z^K\right)}{(\rho+1-\rho z - z^{-1})} \\ &\quad + \frac{\left(P_{K,1}q\rho z^K + P_{0,1}(1-z^{-1})q\rho z\right)}{(\rho+1-\rho z-z^{-1})(q\rho+1-z^{-1})} \\ &\quad + \frac{P_{K,0}r\rho z^{K+1} - P_{N,0}r\rho z^{N+1} - P_{K+1,0}z^K}{(\rho+1-r\rho z-z^{-1})} \end{aligned}$$

Grouping the terms with the same constants of the type $P_{i,j}$ we express the expression for $Q_0(z)$ in the following format,

$$\begin{aligned} Q_0(z) = \; & \frac{-P_{0,0}}{(z-\rho^{-1})\rho} + P_{K,0}z^{K+2}\left(\frac{1}{(z-1)(z-\rho^{-1})}\right. \\ & \left.\frac{-1}{(z-W_1)(z-W_2)}\right) + \frac{P_{K+1,0}z^{K+1}}{\rho} \\ & \left(\frac{1}{r(z-W_1)(z-W_2)} - \frac{1}{(z-1)(z-\rho^{-1})}\right) \\ & + \frac{P_{N,0}z^{N+2}}{(z-W_1)(z-W_2)} - \left(\frac{P_{K,1}z^K}{(z-1)} + P_{0,1}\right) \\ & \frac{z^2}{(1+q\rho)(z-\rho^{-1})(z-(1+q\rho))} \end{aligned} \quad (19)$$

Applying partial fraction approach to the last equation and after some rearrangements (see [2] for details) we get $Q_0(z)$ equal to,

$$P_{K,0}\left[\frac{1}{(1-\rho^{-1})}\left(\frac{z^{K+2}-1}{z-1} - \frac{z^{K+2}-\rho^{-(K+2)}}{z-\rho^{-1}}\right) - \frac{1}{(W_1-W_2)}\left(\frac{z^{K+2}-W_1^{K+2}}{z-W_1} - \frac{z^{K+2}-W_2^{K+2}}{z-W_2}\right)\right]$$

$$
\begin{aligned}
&+\frac{P_{K+1,0}}{r\rho(W_1-W_2)}\left[\left(\frac{z^{K+1}-W_1^{K+1}}{z-W_1}-\frac{z^{K+1}-W_2^{K+1}}{z-W_2}\right)\right.\\
&\left.-\ \frac{r(W_1-W_2)}{(1-\rho^{-1})}\left(\frac{z^{K+1}-1}{z-1}-\frac{z^{K+1}-\rho^{-(K+1)}}{z-\rho^{-1}}\right)\right]\\
&+\frac{P_{N,0}}{(W_1-W_2)}\left(\frac{z^{N+2}-W_1^{N+2}}{z-W_1}-\frac{z^{N+2}-W_2^{N+2}}{z-W_2}\right)\\
&-\frac{P_{K,1}q}{(1+q\rho)}\left(\frac{z^{K+2}-1}{(1-\rho^{-1})(1-(1+q\rho)^{-1})(z-1)}\right.\\
&-\frac{(z^{K+2}-\rho^{-(K+2)})}{(1-\rho^{-1})(\rho^{-1}-(1+q\rho)^{-1})(z-\rho^{-1})}+\\
&\left.\frac{z^{K+2}-(1+q\rho)^{-(K+2)}}{(1-(1+q\rho)^{-1})(\rho^{-1}-(1+q\rho)^{-1})(z-(1+q\rho)^{-1})}\right)\\
&-\frac{P_{0,1}q}{(1+q\rho)(\rho^{-1}-(1+q\rho)^{-1})}\left(\frac{z^2-\rho^{-2}}{z-\rho^{-1}}\right.\\
&\left.-\frac{z^2-(1+q\rho)^{-2}}{z-(1+q\rho)^{-1}}\right)
\end{aligned}
$$

We can now find the inverse transform of $Q_0(z)$ and obtain the steady-state probabilities $P_{i,0}$ for $0 \le i \le K$. This can be easily done as the last equality contains terms of the form $\frac{z^m-a^m}{z-a}$. Thus, we by the inverse z-transform of last equation, we get the expression for $P_{i,0}$ as in Corollary 1. The expression for $P_{i,1}$ is obtained by finding the inverse z-transform of $Q_1(z)$ along similar lines. We are not providing here the details which can be found in [2].

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