A Taylor series expansion for H^∞ Control of Perturbed Markov jump Linear Systems

Rachid El Azouzi
Département de Math & Info
Faculté des sciences
B.P 1014 Rabat
Maroc
relazouz@sophia.inria.fr

Mohammed Abbad
Département de Math & Info
Faculté des sciences
B.P 1014 Rabat
Maroc
abbad@fsr.ac.ma

Eitan Altman
INRIA B.P 93
2004 Route des Lucioles
06902 Sophia Antipolis Cedex
France
altman@sophia.inria.fr

Abstract

In a recent paper, Pan and Başar [9] have studied the H^∞ control of large scale Jump Linear systems in which the transitions of the jump Markov chain can be separated into sets having strong and weak interactions. They obtained an approximating reduced-order aggregated problem which is the limit as the rate of transitions of the faster time scale (which is a multiple of some parameter $1/\epsilon$) goes to infinity. In this paper we further investigate the solution of that problem as a function of the parameter ϵ. We show that the related optimal feedback policy and the value admit a Taylor series in terms of ϵ, and we compute its coefficients.

1 Introduction

Several papers have studied in recent years the control of piecewise-deterministic systems in the presence of unknown (continuous) disturbances. In [8, 10] the case of jump linear systems was studied, and in [1] the case of nonlinear systems. In many applications, the formulation of the problem in terms of these models leads to Markov chains with high dimensional state spaces which makes the computation infeasible or extremely sensitive to small inaccuracies. This motivated Pan and Başar [9] to study systems in which the transitions of the jump Markov chain can be separated into sets having strong and weak interactions. They obtained an approximating reduced-order aggregated problem which is the limit as the rate of transitions of the faster time scale (which is a multiple of some parameter $1/\epsilon$) goes to infinity.

In this paper we further investigate the solution of that problem as a function of the parameter ϵ. We show that the related optimal feedback policy and the value admit a Taylor series in terms of ϵ, and we compute its coefficients. We use the methodology in [5, 6] which considered the problem without disturbances.

2 General Model

The class of jump linear system under consideration is described by:

$$\frac{dx}{dt} = A(\theta(t))x(t) + B(\theta(t))u(t) + D(\theta(t))w(t), \qquad x(0) = x_0, \tag{1}$$

where x is the p-dimensional system state, u is a r-dimensional control input; w is a q-dimensional disturbance and $\theta(t)$ is a finite state Markov chain defined on the state space S, of cardinality s, with the infinitesimal generator matrix

$$\Lambda = (\lambda_{ij}), \qquad i, j \in S.$$

and an initial distribution $\pi_0 := [\pi_{01}, ..., \pi_{0s}]$.
The control input u is generated by a control policy of the form

$$u(t) = \varphi(t, x_{[0,t]}, \theta_{[0,t]}), \qquad t \in [0, +\infty), \tag{2}$$

where $\varphi : [0, +\infty) \times \mathcal{H}_x \times \Omega \to \mathcal{H}_u$ is piecewise continuous in its first argument, and piecewise Lipschitz continuous in its second argument and measurable in θ, further satisfying the given causality condition. Let us denote the class of all admissible controllers by $\mathcal{M}$.

Associated with this system is the infinite-horizon quadratic performance index:

$$L = \int_0^{+\infty} (|x(t)|^2_{Q(\theta(t))} + |u(t)|^2)dt \qquad Q(.) > 0.$$

The system state x, inputs u and w, each belong to appropriate ($\mathcal{L}^2$) Hilbert spaces $\mathcal{H}_x$, $\mathcal{H}_u$, $\mathcal{H}_w$ respectively, defined on the time interval $[0, +\infty)$. The underlying probability space is the triple (Ω, F, P) and let E denote the expectation with respect to the underlying probability space.

We introduce the disturbance attenuation problem where the initial state of the system is completely unknown. The initial condition x_0 and the input w are generated by a strategy $\delta := (\delta_0, \delta_1)$, according to

$$\begin{aligned} x_0 &= \delta_0(\theta(0)) \\ w(t) &= \delta_1(t, x_{[0,t]}, \theta_{[0,t]}) \end{aligned} \tag{3}$$

where $\delta_0 : S \to \mathbb{R}^p$, and $\delta_1 : [0, +\infty) \times \mathcal{H}_x \times \Omega \to \mathcal{H}_w$ is piecewise continuous in its first argument, and piecewise Lipschitz continuous in its second argument and measurable in θ, further satisfying the given causality condition. Let us denote the class of all admissible policies δ by $\mathcal{D}$. The H^∞ optimal control problem is to find the optimal performance level γ^* due to the following $\mathcal{L}_2$ type gain:

$$\inf_{\varphi \in \mathcal{M}} \sup_{\delta \in \mathcal{D}} \frac{\sqrt{E[L(\varphi, \delta)]}}{(E\{\int_0^{+\infty} |w(t)|^2 dt + |x_0|^2_{Q_0(\theta(0))}\})^{\frac{1}{2}}} = \gamma^*, \quad Q_0(.) > 0.$$

(we added (φ, δ) to the notation L to show its dependence on the policy of both players). Moreover, we wish to find a policy f^* that guarantees this level to player 1, i.e. achieving

$$\sup_{\delta \in \mathcal{D}} \frac{\sqrt{E[L(f^*, \delta)]}}{\left(E\{\int_0^{+\infty} |w(t)|^2 dt + |x_0|^2_{Q_0(\theta(0))}\}\right)^{\frac{1}{2}}} = \gamma^*. \tag{4}$$

This H^∞ optimal control problem is known to be closely related to a class of zero-sum differential games for the jump linear system (1) (see [8, 10]), with the following γ-parametrized cost function:

$$J_\gamma(\varphi, \delta) \quad = \quad E\Big\{ \int_0^\infty (|x(t)|^2_{Q(\theta(t))} + |u(t)|^2 - \gamma^2 |w(t)|^2) dt - \gamma^2 |x_0|^2_{Q_0(\theta(0))} \Big\} \tag{5}$$

where player 1 (the minimiser) chooses a strategy $\varphi \in \mathcal{M}$ so as to minimise the expected cost function (5), and the second player chooses a strategy $\delta \in \mathcal{D}$ so as to maximize the same expected cost function:

$$\hat{J}_\gamma = \inf_{\varphi \in \mathcal{M}} \sup_{\delta \in \mathcal{D}} J_\gamma(\varphi, \delta). \tag{6}$$

It should be noted that the upper value of the game, $\hat{J}_\gamma$, is bounded below by 0, because player 2 can choose x_0 and $w(t)$ to be zero. Because of the linear-quadratic nature of the problem, the upper value of the game will be infinite if $\hat{J}_\gamma > 0$. As in standard H^∞-control, we will study a parametrization of the solution to this problem in term of γ. The threshold γ^* is the smallest value of $\gamma > 0$ such that the above game admits a finite upper value (which is equal to zero) and a saddle point.

If there exists a policy $f^* \in \mathcal{M}$ that guarantees the disturbance attenuation γ^* in (4) then it also guarantees (to player one) the value of the game (6), i.e.

$$\sup_{\delta \in \mathcal{D}} J_\gamma(f^*, \varphi) = \inf_{\varphi \in \mathcal{M}} \sup_{\delta \in \mathcal{D}} J_\gamma(\varphi, \delta).$$

Instead of obtaining f^* defined above, we will in fact solve a parameterized class of controllers, $\{f^\gamma, \gamma > \gamma^*\}$, where f^γ is obtained from $\sup_{\delta \in \mathcal{D}} J_\gamma(f^\gamma, \delta) = \inf_{\varphi \in \mathcal{M}} \sup_{\delta \in \mathcal{D}} J_\gamma(\varphi, w)$. The controller f^γ will clearly have the property that it ensures a performance level γ^2 for the index adopted for

$$\sup_{\delta \in \mathcal{D}} \frac{\sqrt{E[L(f^\gamma, \delta)]}}{(E\{\int_0^{+\infty} |w(t)|^2 dt + |x_0|^2_{Q_0(\theta(0))}\})^{\frac{1}{2}}} = \gamma.$$

We now make the following basic assumptions for the problem formulated above.

Assumption 1

The initial probability distribution of the form process $\pi_{0i} > 0$ for al $i \in S$.

Assumption 2

The pair $(A(\theta(t)), B(\theta(t)))$ is stochastically stabilizable and the pair $(A(i), Q(i))$ are observable for each $i \in S$.

3 The perturbed solution process.

In many applications, because of the various sources uncertainties, the Markov chain involved is of inevitably large dimensional. This size brings about many numerical difficulties in the solution. Moreover, these systems may be sensitive to small perturbations of the parameter value. Often, when the Markov chain is large, as here, it is natural to think of the large number of states to be grouped into different collections of states, based on whether the interaction between any two states is weak of strong. Presence of such a phenomenon may be expressed mathematically by taking the probability transition rate matrix $\Gamma_a^\epsilon = (\lambda_{iaj}^\epsilon)$ in an appropriate singularly perturbed form, as already discussed in [3] and [4]:

$$\lambda_{ij}^\epsilon = \lambda_{ij} + \frac{1}{\epsilon}\mu_{ij}, \tag{7}$$

where $(\lambda_{ij})_{s\times s}$ and $(\mu_{ij})_{s\times s}$ are probability transition rate matrices corresponding to, respectively weak and strong interactions within the form process. The scalar ϵ is a small positive number.

For ϵ fixed, and any $\gamma > \gamma^*(\epsilon)$, the associated zero-sum differential game has a zero upper value [8]. A control policy that guarantees this upper value, which is then a control policy that guarantees an H^∞ performance level of γ, is given by (see Theorem 3.2 and Remark 3.2 in [8]):

$$\varphi_\gamma^*(t, x(t), \theta(t)) = -B^T(\theta(t))P_\epsilon^\gamma(\theta(t))x(t), \tag{8}$$

where $P_\epsilon^\gamma(i)$, $i \in S$ are the unique minimal positive-definite matrices solutions to the following set of coupled generalized algebraic Riccati equations:

$$\begin{aligned} A^T(i)P_\epsilon^\gamma(i) \quad &+ \quad P_\epsilon^\gamma(i)A(i) + Q(i) - P_\epsilon^\gamma(i)\Big(B(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i)\Big)P_\epsilon^\gamma(i) \\ &+ \quad \sum_{j\in S}\lambda_{ij}^\epsilon P_\epsilon^\gamma(j) = 0, \quad i \in S. \end{aligned} \tag{9}$$

Furthermore, this solution satisfies the condition :

$$\gamma^2 Q_0(i) - P_\epsilon^\gamma(i) > 0, \quad i \in S. \tag{10}$$

Remark 1

Note that on Theorem 3.2 in [8], the spectral radius condition (10) is not with a strict inequality. We need the strict inequality which we is shown to hold in the Appendix.

The problem addressed in this paper is to find for any $\gamma > \gamma^*(\epsilon)$ a solution of (9) which satisfies (10) for any ϵ small enough.

Theorem 3.1

The solution of equation (9), has the following form:

$$P_\epsilon^\gamma(i) = \sum_{n=0}^{+\infty} P_n^\gamma(i)\epsilon^{\frac{n}{M}}, \quad i \in S, \tag{11}$$

where $\Big(P_n^\gamma(i)\Big)_n$ *are* $p\times p$ *symmetric matrices for each* $i \in S$ *and* M *is positive integer.*

Proof

From Puiseux's Theorem [11], the solution of equation (9) is Puiseux's series which means that, there exists a positive integer M such that:

$$P_\epsilon^\gamma(i) = \sum_{n=-K}^{+\infty} P_n^\gamma(i)\epsilon^{\frac{n}{M}}, \quad i \in S,$$

where K is a nonnegative integer, and from Theorem 1 in [9], we have that

$$P_\epsilon^\gamma(i) = \bar{P}^\gamma(i) + O(\epsilon),$$

then we conclude that the terms of negative power of ϵ are equal to zero. This completes the proof of theorem 3.1. ∎

From Theorem 3.1 and (9), it follows that

$$\begin{aligned} & A^T(i)\sum_{n=0}^{+\infty} P_n^\gamma(i)\epsilon^{n/M} + \sum_{n=0}^{+\infty} P_n^\gamma(i)\epsilon^{n/M} A(i) + Q(i) \\ & -\sum_{n=0}^{+\infty}\sum_{k=0}^{n}\left(P_k^\gamma(i)\Big(B(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i)\Big)P_{n-k}^\gamma(i)\right)\epsilon^{n/M} \\ & +\sum_{n=0}^{+\infty}\sum_{j\in S}\lambda_{ij}P_n^\gamma(j)\epsilon^{n/M} + \sum_{n=-M}^{+\infty}\sum_{j\in S}\mu_{ij}P_{n+M}^\gamma(j)\epsilon^{n/M} = 0, \quad i \in S. \end{aligned}$$

Then we obtain the following set of equations:

If $-M \le n < 0$, then

$$\sum_{j\in S}\mu_{ij}P_{n+M}^\gamma(j) = 0 \qquad i \in S. \tag{12}$$

If $n = 0$, then

$$\begin{aligned} & A^T(i)P_0^\gamma(i) + P_0^\gamma(i)A(i) + Q(i) - P_0^\gamma(i)\Big(B(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i)\Big)P_0^\gamma(i) \\ & +\sum_{j\in S}\lambda_{ij}P_0^\gamma(j) + \sum_{j\in S}\mu_{ij}P_M^\gamma(j) = 0, \qquad i \in S. \end{aligned}$$

If $n > 0$, then

$$\begin{aligned} & A^T(i)P_n^\gamma(i) + P_n^\gamma(i)A(i) - \sum_{k=0}^{n} P_k^\gamma(i)\Big(B(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i)\Big)P_{n-k}^\gamma(i) \\ & +\sum_{j\in S}\lambda_{ij}P_n^\gamma(j) + \sum_{j\in S}\mu_{ij}P_{n+M}^\gamma(j) = 0, \quad i \in S. \end{aligned}$$

We consider the Markov chain associated with the transition probability rates $(\mu_{ij})_{i,j\in S}$. There exists a partition of S into a family of m recurrent classes and, a transient class T, $\bar{\xi}$ will denote the class of the recurrent state

$$S = (\cup_{n=1}^m \bar{\xi}_n) \cup T \quad with \quad \bar{\xi}_n \cap \bar{\xi}_{n'} = \emptyset \quad if \quad n \neq n'.$$

Hence

$$\mu_{ij} = 0 \quad if \quad i \in \bar{\xi}_n \quad and \quad j \in \bar{\xi}_{n'}, \quad n \neq n'.$$

To each class $\bar{\xi}$ is associated the invariant measure (row vector) $m_{\bar{\xi}}$ of the recurrent subchain defined on the class $\bar{\xi} \in \bar{S}$ where $\bar{S} = \{\bar{\xi}_1, \bar{\xi}_2, .., \bar{\xi}_m\}$. Let:

$$C = (C_{ij})_{1 \leq i,j \leq s} \quad where \quad C_{ij} = \mu_{ij}.$$

We shall denote by $q_{\bar{\xi}}(i)$ the probability to end in the class $\bar{\xi}$ starting from i.
The m functions (column vectors) $q_{\bar{\xi}}(.)$ are the solutions to $Cq_{\bar{\xi}} = 0$, and form a basis of the m-dimensional subspace Ker(C).

Remark 2 *If v is a solution to $\sum_{j \in S} \mu_{ij} v(j) = 0$, then*

$$\begin{aligned} v(i) : &= v(\bar{\xi}), \qquad \forall i \in \bar{\xi} \\ v(i) &= \sum_{\bar{\xi} \in \bar{S}} q_{\bar{\xi}}(i) v(\bar{\xi}), \qquad i \in T, \end{aligned}$$

where $(v(\bar{\xi}))_{\bar{\xi}}$ are some real numbers.

We start to solve the above set of equations.
If $n = -M$, we have

$$\sum_{j \in S} \mu_{ij} P_0^\gamma(j) = 0, \quad i \in S.$$

From Remark 2 , we obtain

$$\left\{ \begin{array}{l} \bar{P}_0^\gamma(i) := P_0^\gamma(\bar{\xi}), \qquad i \in \bar{\xi} \\ P_0^\gamma(i) = \sum_{\bar{\xi} \in \bar{S}} q_{\bar{\xi}}(i) \bar{P}_0^\gamma(\bar{\xi}), \qquad i \in T. \end{array} \right. \tag{13}$$

For $n = 0$, we have

$$\begin{aligned} & A^T(i)P_0^\gamma(i) + P_0^\gamma(i)A(i) + Q(i) - P_0^\gamma(i)\Big(B(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i)\Big)P_0^\gamma(i) \\ & + \sum_{j \in S} \lambda_{ij} P_0^\gamma(j) + \sum_{j \in S} \mu_{ij} P_M^\gamma(j) = 0, \quad i \in S. \end{aligned} \tag{14}$$

Substituting the first expression of (13) in (14) to get:

$$\begin{aligned} A^T(i)\bar{P}_0^\gamma(\bar{\xi}) \quad + \quad & \bar{P}_0^\gamma(\bar{\xi})A(i) + Q(i) - \bar{P}_0^\gamma(\bar{\xi})\Big(B(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i)\Big)\bar{P}_0^\gamma(\bar{\xi}) \\ & + \sum_{j \in S} \mu_{ij} P_M^\gamma(j) + \sum_{j \in S} \lambda_{ij} \bar{P}_0^\gamma(j) = 0, \quad i \in \bar{\xi}. \end{aligned} \tag{15}$$

Multiplying (15) by $m_{\bar{\xi}}(i)$, $i \in \bar{\xi}$ and summing up over $\bar{\xi}$:

$$\sum_{i \in \bar{\xi}} Q(i) m_{\bar{\xi}}(i) + \sum_{i \in \bar{\xi}} m_{\bar{\xi}}(i) A^T(i) \bar{P}_0^\gamma(\bar{\xi}) + \bar{P}_0^\gamma(\bar{\xi}) \sum_{i \in \bar{\xi}} m_{\bar{\xi}}(i) A(i)$$

$$
- \bar{P}_0^{\gamma}(\xi)\left[\sum_{i\in\bar{\xi}} m_{\bar{\xi}}(i)\Big(B(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i)\Big)\right]\bar{P}_0^{\gamma}(\bar{\xi}) + \sum_{j\in S}\left(\underbrace{\sum_{i\in\bar{\xi}} m_{\bar{\xi}}(i)\mu_{ij}}_{=0}\right)P_M^{\gamma}(j)
$$

$$
+ \sum_{\bar{\xi}'\in\bar{S}}\left(\sum_{\substack{i\in\bar{\xi}\\ j\in\bar{\xi}'}} m_{\bar{\xi}}(i)\lambda_{ij}\right)\bar{P}_0^{\gamma}(\bar{\xi}') + \sum_{j\in T}\sum_{i\in\bar{\xi}}\sum_{\bar{\xi}'\in\bar{S}} m_{\bar{\xi}}(i)\lambda_{ij}q_{\bar{\xi}'}(j)\bar{P}_0^{\gamma}(\bar{\xi}') = 0. \tag{16}
$$

We introduce the following notations:

$$
\bar{A}(\bar{\xi}) = \sum_{i\in\bar{\xi}} m_{\bar{\xi}}(i)A(i),, \qquad \bar{Q}(\bar{\xi}) = \sum_{i\in\bar{\xi}} m_{\bar{\xi}}(i)Q(i),
$$

$$
\bar{B}(\bar{\xi}) = \left(\sum_{i\in\bar{\xi}} m_{\bar{\xi}}(i)B(i)B^T(i)\right)^{1/2}, \qquad \bar{D}(\bar{\xi}) = \left(\sum_{i\in\bar{\xi}} m_{\bar{\xi}}(i)D(i)D^T(i)\right)^{1/2}
$$

and

$$
\bar{Q}_0(\bar{\xi}) = \sum_{i\in\bar{\xi}} m_{\bar{\xi}}(i)Q_0(i), \qquad \lambda_{\bar{\xi}\bar{\xi}'} = \sum_{i\in\bar{\xi}}\left(\sum_{j\in\bar{\xi}'}\lambda_{ij} + \sum_{j\in T} q_{\bar{\xi}'}(j)\lambda_{ij}\right).
$$

Thus, the equation (16) becomes:

$$
\bar{A}^T(\bar{\xi})\bar{P}_0^{\gamma}(\bar{\xi}) + \bar{P}_0^{\gamma}(\bar{\xi})\bar{A}(\bar{\xi}) + \bar{Q}(\bar{\xi}) - \bar{P}_0^{\gamma}(\bar{\xi})\Big(\bar{B}(\bar{\xi})(\bar{B}^T(\bar{\xi}) - \frac{1}{\gamma^2}\bar{D}(\bar{\xi})\bar{D}^T(\bar{\xi})\Big)\bar{P}_0^{\gamma}(\bar{\xi})
$$

$$
+ \sum_{\bar{\xi}'\in\bar{S}} \lambda_{\bar{\xi}\bar{\xi}'}\bar{P}_0^{\gamma}(\bar{\xi}') = 0, \qquad \bar{\xi}\in\bar{S}. \tag{17}
$$

Now, we define an aggregated Markov chain $\bar{\theta}(t)$ on the state space $\bar{S}$ with the infinitesimal generator matrix $\bar{\Gamma} = (\lambda_{\bar{\xi}\bar{\xi}'})_{m\times m}$

Assumption 3

The pair $(\bar{A}(\bar{\theta}), \bar{B}(\bar{\theta}))$ is stochastically stabilizable and $(\bar{A}(\bar{\xi}), \bar{Q}(\bar{\xi}))$ is observable for each $\bar{\xi}\in\bar{S}$.

Let us introduce a set of spectral radius condition:

$$
\begin{aligned}
\bar{P}_0^{\gamma}(\bar{\xi}) = P_0^{\gamma}(i) &< \gamma^2 Q_0(i), \quad \forall i\in\bar{\xi}, \quad \bar{\xi}\in\bar{S} \\
P_0^{\gamma}(i) = \sum_{\bar{\xi}\in\bar{S}} q_{\bar{\xi}}(i)\bar{P}_0^{\gamma}(\bar{\xi}) &< \gamma^2 Q_0(i), \qquad i\in T.
\end{aligned} \tag{18}
$$

Define the quantity $\bar{\gamma} > 0$:

$$
\bar{\gamma} := \inf \Big\{ \gamma > 0 : \text{ There exists a set of positive definite matrices } \bar{P}^{\gamma}(\bar{\xi}),\ \bar{\xi}\in\bar{S} \text{ solution to (17) such that the spectral radius condition (18) holds}\Big\}
$$

From [8], this threshold level is finite under assumption 3, since for any $\gamma > \gamma^*$, the equation (17) admits positive matrices solutions and from theorem 3.1, we have

$$\sum_{n=0}^{\infty} P_n^{\gamma}(i)\epsilon^{\frac{n}{M}} < \gamma^2 Q_0(i),$$

Hence, for ϵ small enough, we obtain

$$\begin{aligned} \bar{P}_0^{\gamma}(\bar{\xi}) &= P_0^{\gamma}(i) < \gamma^2 Q_0(i) \quad \forall i \in \bar{\xi},\ \bar{\xi} \in \bar{S} \\ P_0^{\gamma}(i) &= \sum_{\bar{\xi}\in\bar{S}} q_{\bar{\xi}}(i)\bar{P}_0^{\gamma}(\bar{\xi}) < \gamma^2 Q_0(i), \qquad i \in T. \end{aligned}$$

Thus there exist $\epsilon_\gamma > 0$ such that for any $\epsilon \in (0, \epsilon_\gamma]$, we have $\gamma^*(\epsilon) \geq \bar{\gamma}$, furthemore

$$\liminf_{\epsilon\rightarrow 0} \gamma^*(\epsilon) \geq \bar{\gamma}, \tag{19}$$

We consider the following form for the aggregated system:

$$\dot{x}_A = \bar{A}(\bar{\theta}(t))x_A + \bar{B}(\bar{\theta}(t))u_A + \bar{D}(\bar{\theta}(t))w_A(t); \quad x_A(0) = x_0 \tag{20}$$

The control input and the disturbance input are generated by the following causal mapping

$$\begin{aligned} u_A(t) &= \varphi_A(t, x_{A[0,T]}, \bar{\theta}_{[0,T]}) \\ (x_0, w_A(t)) &= (\delta_{A0}(\bar{\theta}(0)), \delta_{A1}(t, x_{A[0,T]}, \bar{\theta}_{[0,T]}) \end{aligned}$$

The general cost function associated with this system is defined to be

$$J_A(x, \bar{\xi}, \varphi_A, \delta_A) = E_{x,\bar{\xi}}^{\bar{\gamma}}\{\int_t^T (x_A^T \bar{Q}(\bar{\theta}(t))x_A + u_A^T u_A - \gamma^2 w_A^T w_A)dt - \gamma^2 |x_0|^2_{\bar{Q}_0(\bar{\theta}(0))}\}. \tag{21}$$

where $\delta_A = (\delta_{A0}, \delta_{A1})$. The game problem, related to the aggregate jump linear system (20) is

$$\hat{J}_A(t, x_A, \bar{\xi}) = \inf_{\varphi_A} \sup_{\delta_A} J_A(t, x_A, \bar{\xi}, \gamma). \tag{22}$$

We consider any $\gamma > \gamma^*(\epsilon)$ where $\epsilon \in (0, \epsilon_\gamma]$. The optimal control law for the minimizer is

$$\varphi_A(t, x_A, \bar{\xi}) = -\bar{B}^T(\bar{\xi})\bar{P}(\bar{\xi})x_A,$$

and the following jump linear system is mean-square stable:

$$\dot{x}(t) = F(\bar{\theta}(t))x(t), \tag{23}$$

where $F(\bar{\xi}) = \bar{A}(\bar{\xi}) - \Big(\bar{B}(\bar{\xi})\bar{B}^T(\bar{\xi}) - \frac{1}{\gamma^2}\bar{D}(\bar{\xi})\bar{D}^T(\bar{\xi})\Big)\bar{P}_0^{\gamma}(\bar{\xi})$.
If $n = 1$, we have

$$\begin{aligned} &A^T(i)P_1^{\gamma}(i) + P_1^{\gamma}(i)A(i) - P_0^{\gamma}(i)\Big(B(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i)\Big)P_1^{\gamma}(i) \\ &-P_1^{\gamma}(i)\Big(B(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i)\Big)P_0^{\gamma}(i) + \sum_{j\in S}\lambda_{ij}P_1^{\gamma}(j) + \sum_{j\in S}\mu_{ij}P_{M+1}^{\gamma}(j) = 0, \;\; i \in S, \end{aligned} \tag{24}$$

From (12), we have:

$$\sum_{j\in S}\mu_{ij}P_1^{\gamma}(j)=0. \tag{25}$$

By using (12) and Remark 2, we obtain:

$$\begin{cases} \bar{P}_1^{\gamma}(\bar{\xi})=P_1^{\gamma}(i), & i\in\bar{\xi} \\ P_1^{\gamma}(i)=\sum_{\bar{\xi}\in\bar{S}}q_{\bar{\xi}}(i)\bar{P}_1^{\gamma}(\bar{\xi}), & i\in T. \end{cases} \tag{26}$$

Substituting the first expression of (26) in (24) to get

$$\begin{aligned} &A^T(i)\bar{P}_1^{\gamma}(\bar{\xi})+\bar{P}_1^{\gamma}(\bar{\xi})A(i)-\bar{P}_0^{\gamma}(\bar{\xi}))\Big(B(i)B^T(i)-\frac{1}{\gamma^2}D(i)D^T(i)\Big)\bar{P}_1^{\gamma}(\bar{\xi}) \\ &-\bar{P}_1^{\gamma}(\bar{\xi}))\Big(B(i)B^T(i)-\frac{1}{\gamma^2}D(i)D^T(i)\Big)\bar{P}_0^{\gamma}(\bar{\xi})+\sum_{\bar{\xi}'\in\bar{S}}\sum_{j\in\bar{\xi}'}\lambda_{ij}\bar{P}_1^{\gamma}(\bar{\xi}')+\sum_{j\in S}\mu_{ij}P_{M+1}^{\gamma}(j)=0. \end{aligned} \tag{27}$$

Multiplying (27) by $m_{\bar{\xi}}(i)$, $i\in\bar{\xi}$ and summing over $\bar{\xi}$, we obtain:

$$F^T(\bar{\xi})\bar{P}_1^{\gamma}(\bar{\xi})+\bar{P}_1^{\gamma}(\bar{\xi})F(\bar{\xi})+\sum_{\bar{\xi}'\in\bar{S}}\lambda_{\bar{\xi}\bar{\xi}'}\bar{P}_1^{\gamma}(\bar{\xi}')=0. \tag{28}$$

Since the model (23) is mean-square stable, one applies Proposition 2 in [2] to show that equation (28) has a unique solution. Since 0 is solution to (28), then 0 is the unique solution to (28). Thus, it follows from $P_1^{\gamma}(i)=0$, $i\in S$ and (24) becomes:

$$\sum_{j\in S}\mu_{ij}P_{M+1}^{\gamma}(j)=0,\ i\in S. \tag{29}$$

If $M\neq 1$, then for $0<n<M$, we have

$$\begin{aligned} &A^T(i)P_n^{\gamma}(i)+P_n^{\gamma}(i)A(i)-P_0^{\gamma}(i)(B(i)B^T(i)-\frac{1}{\gamma^2}D(i)D^T(i))P_n^{\gamma}(i) \\ &-P_n^{\gamma}(i)(B(i)B^T(i)-\frac{1}{\gamma^2}D(i)D^T(i))P_0^{\gamma}(i)+\sum_{j\in S}\mu_{ij}P_{n+M}^{\gamma}(j)+\sum_{j\in S}\lambda_{ij}P_n^{\gamma}(j)=0. \end{aligned}$$

From (12), it follows that $\sum_{j\in S}\mu_{ij}P_n^{\gamma}(j)=0$. Then we can derive $P_1^{\gamma}(i)=..=P_{M-1}^{\gamma}=0$, and

$$\sum_{j\in S}\mu_{ij}P_{n+M}^{\gamma}(j)=0 \qquad n=0,1,...,M-1. \tag{30}$$

Similarly, we have that:

$$\begin{aligned} P_0^{\gamma}(i) &> 0, \quad i\in S \\ P_n^{\gamma}(i) &= 0, \quad n\neq kM, \quad k\in IN^*. \end{aligned}$$

Now, we can state the following result.

Theorem 3.2
Under the sssumption 1-3 , the coupled algebraic Riccati equation (9) admits a unique positive solution, which can be expressed as a Taylor series.

The next theorem is about the computation of the terms in the expansion of P^{γ}_{ϵ}.

Theorem 3.3

1. *$P^{\gamma}_{\epsilon}(i)$, $i \in S$ has the expansion $\sum_{n=0}^{+\infty} P^{\gamma}_n(i)\epsilon^n$, where $(P^{\gamma}_n)_{kl} = (\tilde{P^{\gamma}}_n)_{kl} + (\bar{P}^{\gamma}_n)_{kl}$ and $(\tilde{P^{\gamma}}_n)_{kl} := ((\tilde{P^{\gamma}}_n(i))_{kl})_{i\in S} \in Im(C)$ and $(\bar{P}^{\gamma}_n)_{kl} = ((\bar{P}^{\gamma}_n(i))_{kl})_{i\in S} \in Ker(C)$.*
   The sequence $((\tilde{P^{\gamma}}_n), (\bar{P}^{\gamma}_n))$ is uniquely determine by:

   (a) *$\bar{P}^{\gamma}_0$ can be computed by:*

   $$\bar{A}^T(\bar{\xi})\bar{P}^{\gamma}_0(\bar{\xi}) + \bar{P}^{\gamma}_0(\bar{\xi})\bar{A}(\bar{\xi}) + \bar{Q}(\bar{\xi}) - \bar{P}^{\gamma}_0(\bar{\xi})\Big(\bar{B}(\bar{\xi})\bar{B}^T(\bar{\xi}) - \frac{1}{\gamma^2}\bar{D}(\bar{\xi})\bar{D}^T(\bar{\xi})\Big)\bar{P}^{\gamma}_0(\bar{\xi})$$
   $$+\sum_{\bar{\xi}'\in\bar{S}} \lambda_{\bar{\xi}\bar{\xi}'}\bar{P}^{\gamma}_0(\bar{\xi}') = 0, \qquad \bar{\xi}\in\bar{S}.$$

   (b) *$\bar{P}^{\gamma}_n$, $n > 0$ can be computed by:*

   $$F^T(\bar{\xi})\bar{P}^{\gamma}_n(\bar{\xi}) + \bar{P}^{\gamma}_n(\bar{\xi})F(\bar{\xi}) + \sum_{\bar{\xi}'\in\bar{S}} \lambda_{\bar{\xi}\bar{\xi}'}\bar{P}^{\gamma}_n(\bar{\xi}') + T_n(\bar{\xi}) = 0.$$

   where,

   $$\begin{aligned} T_n(\bar{\xi}) &= \sum_{i\in\bar{\xi}} m_{\bar{\xi}}(i)[A^T(i)\tilde{P}_n(i) + \tilde{P}_n(i)A(i) - P_0(i)(B(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i))\tilde{P}_n(i) \\ &\quad -\tilde{P}_n(i)(B(i)R^{-1}(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i))P_0(i) + \sum_{j\in S}\lambda_{ij}\tilde{P}_n(j) - f_n(i)], \end{aligned}$$

   with $f_n(i) = \sum_{k=1}^{n-1} P^{\gamma}_k(i)(B(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i))P^{\gamma}_{n-1-k}(i)$.

   (c) *$\tilde{P}^{\gamma}_n$, $n > 0$ is the unique solution to the linear system:*

   $$C(\tilde{P}^{\gamma}_n)_{kl} = (\alpha_{n-1})_{kl}, \tag{31}$$

   where,

   $$\begin{aligned} \alpha_0(i) &= -Q(i) - A^T(i)P^{\gamma}_0(i) - P^{\gamma}_0(i)A(i) + (P^{\gamma}_0(i)B(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i))P^{\gamma}_0(i) \\ &\quad -\sum_{j\in S}\lambda_{i f_{ij}} P^{\gamma}_0(j), \end{aligned}$$

   and

   $$\begin{aligned} \alpha_n(i) &= -A^T(i)P^{\gamma}_n(i) - P^{\gamma}_n(i)A(i) + (P^{\gamma}_0(i)B(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i))P^{\gamma}_n(i) \\ &+ P^{\gamma}_n(i)(B(i)(i)B^T(i) - \frac{1}{\gamma^2}D(i)D^T(i))P^{\gamma}_0(i) \\ &- \sum_{j\in S}\lambda_{i f_{ij}} P^{\gamma}_n(j) + f_n(i), \quad n > 0, \end{aligned}$$

2. The optimal performance level $\gamma^*(\epsilon)$ satisfies

$$\lim_{\epsilon\to 0^+} \gamma^*(\epsilon) = \bar{\gamma}$$

Proof

1. The proof of the first part is similar to that in Theorem 4.1 in [5].

2. For (19), we have
$$\liminf_{\epsilon\to 0} \gamma^*(\epsilon) \geq \bar{\gamma}$$
then we show only that
$$\limsup_{\epsilon\to 0^+} \gamma^*(\epsilon) \leq \bar{\gamma}$$
Under assumption 3, for any $\gamma > \bar{\gamma}$, the Riccati equation (17) admits a minimal positive solution and the jump linear system (23) is mean-square stable, then the equations in first part of theorem admits solutions. Thus for ϵ small enough, we can construct one positive solution of perturbed Riccati equation which admits Taylor series and this terms is calculated by the first part of theorem 3.3 and we conclude that for ϵ small enough $\gamma \geq \gamma^*(\epsilon)$, furthermore, we have that:
$$\limsup_{\epsilon\to 0^+} \gamma^*(\epsilon) \leq \bar{\gamma}$$
This completes the proof of the second part of Theorem 3.3. ∎

Appendix

In this appendix, we complete the verification of the fact that the spectral radius condition (10) can be taken to be strict inequality.

Lemma 3.1
Assume that $\gamma > \gamma^*$ then there exists a unique minimal solution of (9) that satisfies the spectral radius condition (10),i.e.,

$$\gamma^2 Q_0(i) - P_\epsilon^\gamma(i) > 0, \quad i \in S,$$

Proof
We suppose that there exist at least one $i_0 \in S$ and a nonzero x_0 such that

$$x_0^T(\gamma^2 Q_0(i_0) - P^\gamma(i_0))x_0 = 0.$$

Let (x_0, i_0) the initial state of (1), It 's easy to see that the joint process $\{\theta(t), x(t), t \in [0, +\infty)\}$ determined by setting

$$u(t) = \vartheta(t, x_{[0,t]}, \theta_{[0,t]}) \quad w(t) = \delta_1(t, x_{[0,t]}, \theta_{[0,t]}),$$

is Markov (see [12]). Consider the generator $\mathcal{F}$ of joint process Markovien $\{\theta(t), x(t), t \in [0, +\infty)\}$ and let $V(x, i) = x^T P^\gamma(i) x$, so we have

$$\begin{aligned}
\mathcal{F}V(x(t), i) &= \lim_{\Delta \to 0} \frac{1}{\Delta}[E\{V(x(t+\Delta), \theta(t+\Delta)/x(t), \theta(t) = i) - V(x(t), i) \\
&= (A(i)x + B(i)u + D(i)w)^T P^\gamma(i)x + x^T P^\gamma(i)(A(i)x + B(i)u + D(i)w) \\
&\quad + x^T \sum_{j \in S} \lambda_{ij} P^\gamma(j)x \\
&= -x^T(Q(i) + P^\gamma(i)B(i)B^T(i)P^\gamma(i) - \frac{1}{\gamma^2}P^\gamma(i)D(i)D^T(i)P^\gamma(i))x \\
&\quad + u^T B^T(i)P^\gamma(i)x + w^T D^T(i)P^\gamma(i)x + x^T P^\gamma(i)B(i)u + x^T P^\gamma(i)D(i)w \\
&= -x^T Q(i)x - u^T u + \gamma^2 w^T w + [u + B^T(i)P^\gamma(i)x]^T[u + B^T(i)P^\gamma(i)x] \\
&\quad - [\gamma w - \frac{1}{\gamma}B^T(i)P^\gamma(i)x][\gamma w - \frac{1}{\gamma}B^T(i)P^\gamma(i)x].
\end{aligned}$$

Now set $w(t) = \frac{1}{\gamma^2}B^T(i)P^\gamma(i)x(t)$, $x(0) = x_0$, $\theta(0) = i_0$ and integrating the above equation from 0 to ∞, we obtain

$$\begin{aligned}
&E\{\int_0^{+\infty} \mathcal{F}V(x(t), \theta(t))dt\} \\
&\quad = -E\{\int_0^{+\infty} (x^T(t)Q(\theta(t))x(t) + u^T(t)u(t) - \gamma^2 w^T(t)w(t))dt\} + \|u + B^T P^\gamma x\|^2. \quad (32)
\end{aligned}$$

By using assumption 2, we obtain

$$E\{\int_0^{+\infty} \mathcal{F}V(x(t), \theta(t))dt\} = -x_0^T P^\gamma(i_0)x_0.$$

Then (32) becomes

$$\begin{aligned}
&x_0^T P^\gamma(i_0)x_0 \\
&\quad = E\{\int_0^{+\infty} (x^T(t)Q(\theta(t))x(t) + u^T(t)u(t) - \gamma^2 w^T(t)w(t))dt\} - \|u + B^T P^\gamma x\|^2.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&E\{\int_0^{+\infty} (x^T(t)Q(\theta(t))x(t) + u^T(t)u(t) - \gamma^2 w^T(t)w(t))dt\} - \gamma^2 x_0^T Q_0(i_0)x_0 \\
&= -\gamma^2 x_0^T Q_0(i_0)x_0 + x_0^T P^\gamma(i_0)x_0 + \|u + B^T P^\gamma x\|^2 \\
&\geq 0.
\end{aligned}$$

It follows that for any $u(t) = \varphi(t, x_{[0,t]}, \theta_{[0,t]}) \in \mathcal{M}$ and $\delta = (\delta_0, \delta_1) \in \mathcal{D}$, such that:

$$x_0 = \delta_0(i_0), \quad w(t) = \delta_1(t, x_{[0,t]}, \theta_{[0,t]})$$

$$\frac{\sqrt{EL(\varphi, \delta)}}{(E\{\int_0^{+\infty} |w(t)|^2 dt + |x_0|^2_{Q_0(\theta(0))}\})^{\frac{1}{2}}} \geq \gamma$$

which is in contradiction with

$$\inf_{\varphi\in\mathcal{M}}\sup_{\delta\in\mathcal{D}}\frac{\sqrt{EL(\varphi,\delta)}}{(E\{\int_0^{+\infty}|w(t)|^2dt+|x_0|^2_{Q_0(\theta(0))}\})^{\frac{1}{2}}}=\gamma^*<\gamma.$$

This completes the strict inequality of the spectral radius condition (10). ∎

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