Nash Equilibria for Combined Flow Control and Routing in Networks: Asymptotic Behavior for a Large Number of Users

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Abstract— **We consider a noncooperative game framework for combined routing and flow control in a network of parallel links, where the number of users (players) is arbitrarily large. The utility function of each user is related to the power criterion, and is taken as the ratio of some positive power of the total throughput of that user to the average delay seen by the user. The utility function is nonconcave in the flow rates of the user, for which we introduce a scaling to make it well defined as the number of users, N, becomes arbitrarily large. In spite of the lack of concavity, we obtain explicit expressions for the flow rates of the users and their associated routing decisions, which are in $O(1/N)$ Nash equilibrium. This $O(1/N)$ equilibrium solution, which is symmetric across different users and could be multiple in some cases, exhibits a *delay-equalizing* feature among the links which carry positive flow. The paper also provides the complete optimal solution to the single-user case, and includes several numerical examples to illustrate different features of the solutions in the single- as well as N-user cases, as N becomes arbitrarily large.**

Keywords— **Networks; flow control; routing; nonzero-sum games; noncooperative equilibria; asymptotic Nash equilibria.**

I. Introduction

Flow control and routing are two components of resource and traffic management in today's high-speed networks, such as the Internet and the ATM. Flow control is used by best-effort type traffic in order to adjust the input transmission rates (the instantaneous throughput of a connection) to the available bandwidth in the network. Routing decisions are taken to select paths with certain desirable properties, for example, minimum delays. In many cases, both flow control and routing decisions can be made by the users (rather than by the network) so as to meet some performance criteria. The appropriate framework for modeling this situation is that of noncooperative game theory.

Noncooperative games combining flow and routing decisions have been studied in the past; see, for example, [11] and [14], and references therein. In particular, it is well known that when the objective functions of the players are the sum of link costs plus a reward which is a function of the throughput, then the underlying game can be transformed into one involving only routing decisions. Another recent paper that considers a combined flow control and routing game is [15], where the utility of each player is related to the sum of powers over the links.[1] The part of the utility in [15] that corresponds to the delay is given by the sum of all link capacities minus all link flows, all multiplied by some entropy function. Thus, the utility in this case does not directly correspond to the actual expected delay, but it has the advantage of leading to a computable Nash equilibrium in the case of parallel links.

In this paper, we consider instead the actual power criterion, that is the ratio between (some increasing function of) the total throughput of a user and the average delay experienced by traffic of that user. This power criterion is commonly used in flow control games not involving routing decisions as it enables each user to view the network as a single link with an equivalent cost. (This property holds, under certain assumptions, even in the case of dynamic, state-dependent flow control games; see [12].)

In the paper, we first consider the case of a single user accessing multiple links. Since the utility function we consider is not concave, the optimal solution (which exists) has to be obtained by examining all stationary and boundary points. We show that there is a simple procedure to perform such a search. An interesting feature of the optimal solution is that it could dictate the user not to use all the links in the network. This observation is useful since such a behavior arises even in the case of multiple users attempting to reach a Nash equilibrium.

Following the study of the single-user case, we move on to the case of multiple users and study the asymptotic case when the number of users is very large. Here a user would represent either an individual who is able to split his flow and determine its routes, or a single service provider in the context of noncooperative sharing of the network by many service providers. When the total throughput of all players is fixed, a well-established theory exists for the resulting routing game, and in this case, the solution concept is known as the *Wardrop Equilibrium* [18]. This equilibrium is characterized by the fact that users choose a source-destination path only if it has the smallest delay. A single

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[1] The power criterion is the ratio between some function of the throughput and the delay.

user in that framework is considered to be infinitesimally small, so that it does not have any influence on the costs of other users, and more generally, on the link costs. This equilibrium has often been used in the context of road traffic [6], [11] and it has the appealing feature that under fairly general network topologies and assumptions on the cost, its existence and uniqueness can be established. Moreover, as it has been shown in [11], the Wardrop equilibrium is the unique limit of any sequence of Nash equilibria obtained for a sequence of games in which the number of users is finite and tends to infinity (even in those games where the Nash equilibrium is not unique). For our case here, where the total throughput of the players is not fixed, we consider a similar limit of the Nash equilibrium for a large number of players. We determine all *symmetric* $O(1/N)$ Nash equilibria, and as a byproduct arrive at the conclusion that multiple equilibria do exist in some cases. Another interesting feature of the asymptotic Nash equilibria is that it is possible for only a strict subset of the available links to carry positive flow.

The organization of the paper is as follows. The single-user case is discussed in Section 2, and the multiple-user case in Section 3. Section 4 discusses two numerical examples which illustrate existence, nonexistence, and various features of $O(1/N)$ Nash equilibria. Section 5 includes some discussion of future work and concluding remarks, and the paper ends with an appendix, which proves a *robustness* result for the single-user case, which is also used in the proof of one of the theorems in the multiple-user case.

II. Optimal routing and flow control for a single user

A. Mathematical formulation and the main result

Consider a single user who wishes to send infinitesimally divisible traffic to a destination, by distributing it over M possible links, with a generic link denoted by m, with $m \in \mathcal{M}$. Let c_m and λ_m denote respectively the capacity of and throughput over Link m, and suppose that the links are labeled in such a way that

$$c_1 \geq c_2 \geq \ldots \geq c_M .$$

Thus, the overall throughput of the user is $\sum_{m=1}^{M} \lambda_m$. Assuming an $M/M/1$ queue model for each link, and assuming that $\lambda_m > 0$ for at least one $m \in \mathcal{M}$, the average delay experienced by the user is given by

$$d(\lambda) = \left(1 \Big/ \sum_{m=1}^{M} \lambda_m\right) \sum_{m=1}^{M} \frac{\lambda_m}{c_m - \lambda_m} , \qquad (1)$$

where, also for future use, we have introduced the notation $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_M)$. We note that because of the averaging, the delay function $d(\lambda)$ is not a monotonically increasing function of the λ_m's (whereas it would be if there were only a single link). For mathematical completeness, we define the average delay for the limiting case when $\sum_{m\in\mathcal{M}} \lambda_m = 0$ as

$$d(0) = \frac{1}{M} \sum_{m=1}^{M} \frac{1}{c_m} . \qquad (2)$$

The objective of the user is to maximize the following utility function, which quantifies a trade-off between throughput and delay:

$$U(\lambda) = \left(\sum_{m=1}^{M} \lambda_m\right)^{\beta} \Big/ d(\lambda) , \qquad (3)$$

where $\beta \in (0, 1)$ is a trade-off parameter. Such a utility function is commonly used in the literature in applications that are sensitive to throughput as well as delay (see, for example, [4], [9], [10], [19], and [5]). It consists of the ratio between the expected throughput (or a power of it) and the expected delay. Thus it captures preferences toward higher throughputs and penalizes large delays. Other types of utility functions have been proposed and used in recent years, particularly for voice applications, which however do not take into account delays; see, for example, [16]. Not capturing delays in utility functions has been debated and criticized in [3], where it is argued that a more realistic class of utility functions are those that are in the form of a product of two terms—one a function of the throughput and the other one a function of the delay. The power criterion (3) we have introduced above indeed falls in that category.

Now coming back to (3), for convenience, we prefer to work with the logarithm of this utility function:

$$\begin{aligned} L(\lambda) := \log U(\lambda) \quad &= \quad (\beta + 1) \log \left(\sum_{m=1}^{M} \lambda_m\right) \\ &\quad - \log \left(\sum_{m=1}^{M} \frac{\lambda_m}{c_m - \lambda_m}\right), \qquad (4) \end{aligned}$$

We note that $L(\lambda)$ is not a concave function, but it is differentiable. Because of lack of concavity, to find the optimal solution, it becomes necessary to examine all stationary points of L, as well as its values on the boundary of the set $\mathcal{C} := [0, c_1] \times [0, c_2] \times \ldots [0, c_m]$.[2] The existence of an optimal solution is guaranteed by the fact that $L(\lambda)$ is continuous on $\mathcal{C}$, and $\mathcal{C}$ is compact.

Let us first suppose that the optimal solution is an inner solution, i.e. it is not on the boundary of $\mathcal{C}$. Then, the solution has to be a stationary point of L, determined from the set of equations

$$\frac{\partial L}{\partial \lambda_m} = 0, \; m \in \mathcal{M} ,$$

which leads to

$$\frac{\beta + 1}{\displaystyle\sum_{j=1}^{M} \lambda_j} - \frac{c_m}{(c_m - \lambda_m)^2 \displaystyle\sum_{j=1}^{M} \frac{\lambda_j}{c_j - \lambda_j}} = 0 . \qquad (5)$$

[2]Except at $\lambda_m = 0 \; \forall \, m \in \mathcal{M}$, where $L(\lambda)$ is not well defined, but the fact that $U(0) = 0$ in view of (2) implies that this point cannot be a solution, and hence need not be considered in the analysis.

It follows from (5) that the following relationship should hold, for some positive constant μ:

$$\frac{c_m}{(c_m - \lambda_m)^2} = \frac{1}{\mu^2}\,. \tag{6}$$

Thus,

$$\lambda_m = c_m - \mu\sqrt{c_m}\,, \quad m \in \mathcal{M}\,. \tag{7}$$

Substituting this in (5), we get

$$\frac{\beta+1}{\bar{c} - \mu\bar{c}_{sq}} - \frac{1}{\mu\bar{c}_{sq} - \mu^2 M} = 0\,,$$

which leads to the following quadratic equation for μ :

$$(\beta+1)M\mu^2 - \bar{c}_{sq}(\beta+2)\mu + \bar{c} = 0\,,$$

where

$$\bar{c} := \sum_{m=1}^{M} c_m\,,\ \bar{c}_{sq} := \sum_{m=1}^{M} \sqrt{c_m}\,. \tag{8}$$

Thus,

$$\mu = \frac{(\beta+2)\bar{c}_{sq} \pm \sqrt{(\beta+2)^2\bar{c}_{sq}^2 - 4(\beta+1)\bar{c}M}}{2(\beta+1)M}\,. \tag{9}$$

Expression (7), together with (9), identifies the stationary points of (4) if the following conditions are satisfied:

(i) the discriminant in (9) is nonnegative, i.e.

$$\frac{\bar{c}_{sq}^2}{\bar{c}} \geq \frac{4(\beta+1)M}{(\beta+2)^2}\,, \tag{10}$$

and

(ii) $0 < \mu \leq \sqrt{c_m}$, for all $m \in \mathcal{M}$, which follows from the fact that we require $0 \leq \lambda_m < c_m$.

To study whether or not the optimal solution lies on the boundary of $\mathcal{C}$, we first assume that the traffic through each of the links in a given subset of the links, $\mathcal{S}$, is nonzero and the traffic through the remaining links is zero. The following fact on *substitutability* now asserts that we need not consider all possible subsets $\mathcal{S}$: suppose that there exists a link $i \in \mathcal{S}$ and a link $j \in \mathcal{S}^c$ (the complement of $\mathcal{S}$) such that $c_i < c_j$. Then the utility to the user can be strictly improved by swapping i and j in the sets $\mathcal{S}$ and $\mathcal{S}^c$, because any flow through link j results in a smaller delay than the corresponding flow through link i. Thus, the only subsets that we have to examine in the search for the optimal solution are

$$\mathcal{S}_m = \{1, 2, \ldots, m\}\,,\ m \in \mathcal{M}\,.$$

For each of the subsets $\mathcal{S}_m$, the candidates for the optimal solution are given by the following:

$$\lambda_j = \begin{cases} c_j - \mu_m\sqrt{c_j}, & j \in \mathcal{S}_m \\ 0 & j \in \mathcal{S}_m^c \end{cases} \tag{11}$$

where

$$\mu_m = \frac{(\beta+2)\bar{c}_{sq,m} \pm \sqrt{(\beta+2)^2\bar{c}_{sq,m}^2 - 4(\beta+1)\bar{c}_m m}}{2(\beta+1)m} \tag{12}$$

and

$$\bar{c}_m := \sum_{j=1}^{m} c_j\,, \quad \bar{c}_{sq,m} := \sum_{j=1}^{m} \sqrt{c_j}\,. \tag{13}$$

For each m, the candidate solutions should further satisfy the condition $0 < \mu_m < \sqrt{c_i}$ for all $i \in \mathcal{S}_m$, which can be written equivalently as $\mu_m < \sqrt{c_m}$, because of the ordering of the c_i's and because μ_m is positive whenever it exists. Of course, for existence we need the discriminant in (12) to be nonnegative, i.e.

$$\frac{\bar{c}_{sq,m}^2}{m\bar{c}_m} \geq \frac{4(\beta+1)}{(\beta+2)^2}\,. \tag{14}$$

The optimal solution is then obtained by choosing the candidate solution that results in the largest value for the utility function. For each fixed $\mathcal{S}_m$, (12) suggests that there might be two stationary points, corresponding to the positive and negative square roots in the expression for μ_m (whose corresponding values we denote by μ_m^+ and μ_m^-, respectively), which indeed is a real possibility as demonstrated in the context of an example (Example 2) later in this section. Clearly, μ_m^- would be a viable candidate solution (that is satisfy the constraint $\mu_m < \sqrt{c_m}$) whenever μ_m^+ is, but not *vice versa*, and in fact yet another example (Example 1) included later in this section demonstrates that it is possible for only μ_m^- to be a viable solution. What one can actually show is that it is not necessary to consider μ_m^+ at all, since flows corresponding to it (when it is viable) lead to smaller utility than the flows corresponding to μ_m^-. We now provide a proof for this result.

Proposition 1: Let $m \in \mathcal{M}$ be a fixed integer, condition (14) hold with a strict inequality, and μ_m^- and μ_m^+ be given by (12), corresponding to the negative root and positive root, respectively, satisfying the viability condition $\mu_m^+ < \sqrt{c_m}$. Let λ_j^- and λ_j^+ be as given by (11) with μ_m given by μ_m^- and μ_m^+, respectively. Then, with the utility function $U(\lambda)$ as defined by (3), we have

$$U(\lambda^-) > U(\lambda^+)\,.$$

Proof: With $m \in \mathcal{M}$ fixed, let λ be restricted to the structural form

$$\lambda_j = \begin{cases} c_j - \mu\sqrt{c_j}, & j \in \mathcal{S}_m \\ 0 & j \in \mathcal{S}_m^c\,, \end{cases}$$

where μ is a free parameter, which is positive and not exceeding $\sqrt{c_m}$. Evaluating $U(\lambda)$ under this structure, and denoting the resulting function of the single parameter μ by $W_m(\mu)$, we arrive at:

$$W_m(\mu) = \frac{(\bar{c}_m - \mu\bar{c}_{sq,m})^{\beta+1}\mu}{\bar{c}_{sq,m} - m\mu} \tag{15}$$

We know from the hypothesis of the proposition that this function has two stationary points in the interval $(0, \sqrt{c_m})$, at μ_m^- and μ_m^+. On the extended positive real line, the function has two *zeros*, at $\mu = 0$ and $\mu = \bar{c}_m/\bar{c}_{sq,m}$, and a vertical asymptote at $\mu = \bar{c}_{sq,m}/m$, and attains positive values to the left of this asymptote, where both μ_m^-

and μ_m^+ are, since $\bar{c}_{sq,m} \geq mc_m$. The second *zero* is to the right of the asymptote, since $(\bar{c}_{sq,m}/m) < (\bar{c}_m/\bar{c}_{sq,m})$, where the strict inequality follows from Jensen's inequality along with the observation that when $c_1 = \cdots = c_m$, μ_m^+ is not a viable solution (see the discussion later in this section on the equal capacity case). Hence, the function $W_m(\mu)$ is continuous (actually continuously differentiable) in the open interval $(0, \sqrt{c_m})$, is strictly increasing to the left of $\mu = \mu_m^-$, and strictly decreasing in the open interval (μ_m^-, μ_m^+), thus readily leading to the conclusion that it attains a strictly larger value at $\mu = \mu_m^-$ than at $\mu = \mu_m^+$. $\diamond$

We are now in a position to recapitulate in precise terms the complete solution to the single-user problem.

Theorem 1: For the single-user M-link routing/flow-control problem formulated in this subsection,

(i) there exists an optimum solution;

(ii) the optimum solution dictates positive flows on links $1, \ldots, m^*$, and zero flow on the remaining $m^* + 1, \ldots, M$ (if $m^* < M$), where

$$m^* = \arg \max_{m \in \mathcal{M}_f} W_m(\mu_m^-) \,.$$

Here, $\mathcal{M}_f$ is the nonempty set of all integers in $\mathcal{M}$ with the property that $m \in \mathcal{M}_f$ implies that (14) is satisfied and $\mu_m^- < \sqrt{c_m}$. Further, $W_m(\mu_m^-)$ is given by (15), where the expression for μ_m^- is given by (12) with the negative square root;

(iii) the optimum flows are given by (11), with $m = m^*$ and $\mu_m = \mu_{m^*}^-$.

Proof: The result follows readily from the development that preceded the theorem. $\diamond$

B. Two other classes of utility functions

It would be useful to contrast the above solution with the one corresponding to the more common utility function defined as follows:

$$\tilde{U}(\lambda) = \sum_{m=1}^{M} \lambda_m^{\beta}(c_m - \lambda_m) \,,\ \beta \in (0,1) \,. \tag{16}$$

This is simply the sum of the utilities on each individual links, where the individual utility function on a link represents a tradeoff between the throughput and the average delay on that link. An appealing feature of $\tilde{U}(\lambda)$ is that it is strictly concave, and hence optimality of its local solution can readily be ascertained, and in fact the unique optimal solution can easily be obtained as:

$$\lambda_m = \frac{\beta}{\beta+1}\, c_m \; \forall m \in \mathcal{M} \,.$$

Note that the optimal solution in this case puts nonzero flow to every link. However, $\tilde{U}(\lambda)$ no longer captures the trade-off between the overall throughput and the overall average delay, as done by the more realistic utility function $U(\lambda)$ considered in this paper—albeit at the expense of a more complicated (yet explicitly computable) solution.

Yet another utility function that has been considered in the literature before is [15]:

$$\tilde{V}(\lambda) = (\sum_{m=1}^{M} \lambda_m)^{\beta} (\bar{c} - \sum_{m=1}^{M} \lambda_m) \,,$$

where $\bar{c}$ is the total capacity of all the links. This function is concave in λ (not strictly concave, in the multiple link case), and any set of $\{\lambda_m\}$ satisfying

$$\sum_{m=1}^{M} \lambda_m = \frac{\beta}{\beta+1} \bar{c} \,,$$

along with $0 \leq \lambda_m \leq c_m \quad \forall m \in \mathcal{M}$, is an optimal solution. In fact, the optimal solution under $\tilde{U}(\lambda)$ is also optimal for the utility function $\tilde{V}(\lambda)$.

We now recapitulate and list below the observations we have made above regarding the optimal solutions under three different classes of utility functions:

- the utility function $U(\lambda)$ may dictate the flows in some of the links to be zero;
- the utility function $\tilde{U}(\lambda)$ leads to nonzero flow over every link; and
- the utility function $\tilde{V}(\lambda)$ leads to multiple optimal solutions, with some dictating nonzero flow over every link.

C. A special case: Equal link capacities

Later, when we study Nash equilibria with multiple users, the results for the special case of a single user and equal link costs, i.e., $c_m \equiv c \;\; \forall m \in \mathcal{M}$, will turn out to be useful. Thus, we study here this special case, which is also of independent importance and interest. In this case

$$\mu_m \equiv \sqrt{c} \;\text{ or }\; \frac{\sqrt{c}}{\beta+1} \quad \forall m \in \mathcal{M} \,.$$

Since, we require $\mu_m < \sqrt{c}$, the only viable value for μ_m is $\sqrt{c}/(\beta+1)$. Hence, in this case μ^+ does not constitute a viable solution.

Considering any subset $\mathcal{S}_m$, the corresponding optimal flow is

$$\lambda_i = \frac{\beta}{\beta+1}\, c \quad \forall i \leq m \,.$$

Since all the boundary solutions and the inner solution lead to the same flow on each link with nonzero flow, the optimal solution (and the unique one) is easily seen to be the one that uses all M links:

$$\lambda_m^* = \frac{\beta}{\beta+1}\, c \quad \forall m \in \mathcal{M} \,.$$

Note that in this case the optimal solution coincides with the one under the utility function $\tilde{U}$ given by (16).

The optimal single-user solution for the case of equal capacities has the following robustness property which we will require later.

Theorem 2: Consider a system consisting of one user and M links with the capacity of the mth link being $c + \delta_m$.

Then, there exists $\delta > 0$ such that for $|\delta_m| < \delta\ \forall m \in \mathcal{M}$, the utility-maximizing solution is unique and is an inner solution.

Proof: See Appendix I. $\diamond$

Yet another useful robustness property of the single-user solution with equal link capacities, which we will have occasion to use in the next section, is a result that complements that of Theorem 2 in a different direction. It says that starting with an original network of equal capacity links (for which there exists a unique solution which is inner, as already shown), if we add additional links of lower capacity, the original solution remains intact (that is, the utility-maximizing solution dictates zero flow over these additional links) provided that β is sufficiently small. The precise statement is the following.

Theorem 3: Consider a network consisting of one user and M links, with the first $\hat{m}$ links having equal capacity, c, and the remaining $M - \hat{m}$ having capacities less than c (i.e., $c_1 = \cdots = c_{\hat{m}} = c > c_{\hat{m}+1} \geq \cdots \geq c_M$). Then, there exists a $\beta^* > 0$ (depending on c and $c_{\hat{m}+1}$) such that for $\beta \in (0, \beta^*)$, the unique utility-maximizing solution dictates zero flow over the links $\hat{m}+1, \ldots, M$. Equivalently, the unique solution is given by (using the notation introduced earlier)

$$\lambda_i^* = \begin{cases} \big(\beta/(\beta+1)\big)c, & j \in \mathcal{S}_{\hat{m}} \\ 0 & j \in \mathcal{S}_{\hat{m}}^c \end{cases}$$

Proof. First note that the RHS of (14) is less than 1, and can be made arbitrarily close to 1 by picking β sufficiently close to *zero*. The LHS of (14), on the other hand, is 1 when $m = \hat{m}$ (since the first $\hat{m}$ links have equal capacity), and strictly less than 1 if $m > \hat{m}$, as we now show. That is, denoting the LHS of (14), as a function of the integer m, by g_m, we show that $g_m < 1$ for $m > \hat{m}$. This will then immediately lead to the result of the theorem, by picking β^* as the unique positive solution of the quadratic equation $(\beta+2)^2 g_{m^*} - 4(\beta+1) = 0$, where m^* is the $m > \hat{m}$ for which g_m is the largest. This follows because the RHS of (14) is a monotonically decreasing function of β for $\beta > 0$, and for $\beta \in (0, \beta^*)$ the necessary condition for an optimum, (14), does not hold for any $m > \hat{m}$. We should note that one can in fact show (with a little more effort) that g_m is monotonically nonincreasing in m, but we will not do this since it is not needed in the proof of the theorem.

Now, to prove the required auxiliary result, we will proceed by induction (showing that $g_m \leq 1$ for some $m \geq \hat{m}$ implies $g_{m+1} < 1$). Let $m \geq \hat{m}$, and start with the strict inequality $(\bar{c}_m - m c_{m+1})^2 > 0$, which holds because of the ordering of the c_m's. Rewrite this inequality equivalently as

$$4m\bar{c}_m c_{m+1} < (\bar{c}_m + m c_{m+1})^2$$

Since $g_m \leq 1$, the LHS can be bounded from below by $4\bar{c}_{sq,m}^2 c_{m+1}$, leading to (after taking the square roots of both sides):

$$2\bar{c}_{sq,m}\sqrt{c_{m+1}} < \bar{c}_m + m c_{m+1}$$

Now add c_{m+1} to both sides, and the nonpositive quantity $\bar{c}_{sq,m}^2 - m\bar{c}_m$ to the LHS. The resulting strict inequality (after some rearrangement) can be seen to be equivalent to $g_{m+1} < 1$, which proves the desired result. $\diamond$

D. Examples for the single-user case

Example 1: Consider a network of 10 links, with the capacity of the *mth* link being $100 - 10(m-1)$. Let the throughput-delay tradeoff parameter β be 0.6. The optimal solution is given by

$$\lambda_1 = 44.9,\ \lambda_2 = 37.73,\ \lambda_3 = 30.72,\ \lambda_4 = 23.90,\ \lambda_5 = 17.32,\ \lambda_6 = 11.04,\ \lambda_7 = 5.15\,,$$

with the remaining three links having zero flow. The optimal value of the utility function is 1061 units. In this example, only the negative square root in the expression for μ_m, $m \leq 7$, satisfies all the constraints. For $m > 7$, the discriminant in the expression (12) for μ_m is negative, thus there are no candidate optimal solutions for these values of m. $\diamond$

Example 2: Consider a network of M links, with $M \geq 5$, $c_1 = 2.29$, $c_2 = c_3 = c_4 = 1$, $c_m = 0.25, m \geq 5$, $\beta = 0.5$. In this case, condition (14) is not satisfied for $m \geq 5$ (as well as for $m = 2$ and $m = 3$), and hence the optimal solution would dictate zero flow on the links with capacity 0.25. We therefore have to consider only the two subsets $\mathcal{S}_m,\ m = 1, 4$. It turns out that the optimal solution corresponds to $m = 4$, i.e. it dictates use of *four* links, with the corresponding throughputs being:

$$\lambda_1 = 0.9113,\ \lambda_2 = \lambda_3 = \lambda_4 = 0.0889\,.$$

The optimal value of the utility function is 1.455 units. In this example, both positive and negative roots in (12) are viable for $m = 4$, with the corresponding values being $\mu_4^+ = 0.9695$ and $\mu_4^- = 0.9111$, with the utility corresponding to the former case being $U(\lambda^+) = 1.334$, which is of course lower than the optimal utility level of 1.455, consistent with the result of Proposition 1. It is important to underscore two features of the optimal solution as illustrated by this example: First, when there is more than one link at the same capacity level, if the search for an optimum fails when some of these links are used this does not necessarily mean that the search will fail also if additional links with the same capacity are considered. Second, it is possible for both positive and negative roots in the expression (12) for μ_m to lead to viable candidate solutions. $\diamond$

III. Multiple Users

We now return to the original goal of this paper, that is the case of multiple users. To formulate this problem in precise terms, consider a routing and flow control game involving M parallel links and N players (users). Let $\mathcal{N} := \{1, \ldots, N\}$ be the set of players and as before $\mathcal{M} := \{1, \ldots, M\}$ be the set of links. Let $\lambda_{ij} \geq 0$ denote the flow of Player i over link j, and $\lambda_j = \sum_{i=1}^N \lambda_{ij}$ denote total flow on link j.

Each player i chooses $\{\lambda_{ij}\}_{j\in\mathcal{M}}$ to maximize his or her utility function. The total throughput of Player i over all the links is $\sum_{j\in\mathcal{M}}\lambda_{ij}$, and, using an $M/M/1$ queue model as before, the average delay for the generic Player i is given by

$$d_i(\lambda)=\begin{cases}\dfrac{1}{\sum_{j\in\mathcal{M}}\lambda_{ij}}\displaystyle\sum_{j\in\mathcal{M}}\frac{\lambda_{ij}}{c_j-\lambda_j}, & \displaystyle\sum_{j\in\mathcal{M}}\lambda_{ij}>0\\ \dfrac{1}{M}\displaystyle\sum_{j\in\mathcal{M}}\left(\frac{1}{(c_j-\sum_{n\in\mathcal{N},n\neq i}\lambda_{nj})}\right), & \displaystyle\sum_{j\in\mathcal{M}}\lambda_{ij}=0\end{cases}$$

where c_j is the capacity of link j. As in the previous section, we again adopt the labeling $c_1\geq c_2\geq\ldots\geq c_M$. As a natural counterpart of (3), in the single-user case, the utility function of Player i is taken to be in the form:

$$U_i(\lambda)=\begin{cases}\left(\displaystyle\sum_{j\in\mathcal{M}}\lambda_{ij}\right)^{\beta+1}\Big/\displaystyle\sum_{j\in\mathcal{M}}\frac{\lambda_{ij}}{c_j-\lambda_j}, & \displaystyle\sum_{j\in\mathcal{M}}\lambda_{ij}>0\\ 0\ , & \displaystyle\sum_{j\in\mathcal{M}}\lambda_{ij}=0\end{cases}$$

where λ here stands for the collection $\{\lambda_{ij}\}_{i\in\mathcal{N},j\in\mathcal{M}}$, and again $\beta\in(0,1)$. Note that β (the parameter that captures the tradeoff between throughput and delay) is taken here to be player independent. We will, however, take β to depend on N (for reasons to be clear shortly), and write it also as β_N. To indicate the explicit dependence of the utility functions on N, we will also write U_i as U_i^N where necessary, and write the logarithm of U_i^N, except for the case $\sum_{j\in\mathcal{M}}\lambda_{ij}=0$, by L_i^N, where

$$L_i^N(\lambda)=(\beta_N+1)\log\sum_{j\in\mathcal{M}}\lambda_{ij}-\log\sum_{j\in\mathcal{M}}\frac{\lambda_{ij}}{c_j-\lambda_j}\qquad(17)$$

A few observations and remarks are in place here. First note that since Player i will be maximizing $U_i(\lambda)$, and $U_i(\lambda)=0$ when $\sum_{j\in\mathcal{M}}\lambda_{ij}=0$, (17) can be considered as the utility of Player i without any loss of generality. Second, a decision maker need not correspond to a single connection, so that the utility need not be considered at a packet level, that is as a power of throughput divided by the delay as experienced by a single connection. Instead, a decision maker could correspond to a service provider that generates through its subscribers a flow of calls. The throughput of a call can then be taken to be constant, and then the throughput determined by the decision maker will correspond to the average number of calls that can be generated by his subscribers. The utility part that corresponds to the throughput of a service provider could then also be viewed as a result of some pricing mechanism; a detailed study of this, however, is beyond the scope of this paper.

We now seek a Nash equilibrium solution [2] for the N-player game introduced above; that is, an N-tuple $\{\lambda_{ij}^*\}_{i\in\mathcal{N},j\in\mathcal{M}}$ that satisfies for all $\{\lambda_{ij}\}_{j\in\mathcal{M}}$, and for all $i\in\mathcal{N}$:

$$\begin{aligned}&U_i^N(\{\lambda_{ij}^*\}_{j\in\mathcal{M}},\{\lambda_{kj}^*\}_{k\in\mathcal{N},k\neq i,j\in\mathcal{M}})\\&\qquad\geq U_i^N(\{\lambda_{ij}\}_{j\in\mathcal{M}},\{\lambda_{kj}^*\}_{k\in\mathcal{N},k\neq i,j\in\mathcal{M}})\,.\end{aligned}$$

Of course, the same set of inequalities can equivalently be written in terms of L_i^N instead of U_i^N.

To obtain closed-form expressions for such a solution seems to be out of reach for the most general case, particularly in view of the non-concave nature of the individual utility functions. In view of this, we will focus here on the asymptotic case where the number of users is large. We will show that, under some appropriate conditions, symmetric Nash equilibrium takes on a simple form in the limit as $N\to\infty$, and the limiting solution exhibits an appealing *delay-equalizing* property as in Wardrop equilibrium [11], [18].

Before delving into the derivation of asymptotic equilibrium policies of the users, let us make it precise what we mean by an *asymptotic Nash equilibrium*.

Definition 1: For the N player game defined above, with N arbitrarily large, let $\lambda_{ij}^*(N)$, $i\in\mathcal{N},j\in\mathcal{M}$ be a set of policies (flow rates) for the users, which are defined for all positive integers N, and show possible dependence on N. We say that these constitute an *asymptotic Nash equilibrium*, or are *asymptotic equilibrium policies* if, for all $i\in\mathcal{N}$,

$$\begin{aligned}&\lim_{N\to\infty}L_i^N(\{\lambda_{ij}^*(N)\}_{j\in\mathcal{M}},\{\lambda_{kj}^*(N)\}_{k\in\mathcal{N},k\neq i,j\in\mathcal{M}})=\\&\lim_{N\to\infty}\max_{\{\lambda_{ij}\}_{j\in\mathcal{M}}}L_i^N(\{\lambda_{ij}\}_{j\in\mathcal{M}},\{\lambda_{kj}^*(N)\}_{k\in\mathcal{N},k\neq i,j\in\mathcal{M}})\,.\quad(18)\end{aligned}$$

$\diamond$

Of course, (18) could have been expressed also in terms of U_i^N, or any continuous monotonically increasing function of U_i^N, without affecting the definition. Now, a further refinement can be brought in to an asymptotic Nash equilibrium by specifying how close the two expressions in (18) are for finite but arbitrarily large N. In this case one has to work with a specific structure for the utility function, which we choose to be the logarithmic utility function L_i^N. This further refinement is now captured by the definition below.

Definition 2: For the N player game defined earlier, with the logarithmic utility functions and with N arbitrarily large, let $\lambda_{ij}^*(N)$, $i\in\mathcal{N},j\in\mathcal{M}$ be a set of asymptotic equilibrium policies (flow rates) for the users. We say that these constitute an $O(1/N)$ *Nash equilibrium*, or are $O(1/N)$ *equilibrium policies*, *with exponent* κ, if there exists a nonpositive scalar κ, independent of N, such that, for all $i\in\mathcal{N}$,

$$\begin{aligned}&L_i^N(\{\lambda_{ij}^*(N)\}_{j\in\mathcal{M}},\{\lambda_{kj}^*(N)\}_{k\in\mathcal{N},k\neq i,j\in\mathcal{M}})=\\&\max_{\{\lambda_{ij}\}_{j\in\mathcal{M}}}L_i^N(\{\lambda_{ij}\}_{j\in\mathcal{M}},\{\lambda_{kj}^*(N)\}_{k\in\mathcal{N},k\neq i,j\in\mathcal{M}})\qquad(19)\\&\qquad+\frac{\kappa}{N}+o(1/N)\,.\end{aligned}$$

$\diamond$

Now, toward obtaining the asymptotic equilibrium solution (which we will also show to be in $O(1/N)$ Nash equilibrium), let us first assume that for each N the Nash equilibrium exists and is an inner solution, i.e., $\lambda_{ik}^*(N)\neq 0$ $\forall i\in\mathcal{N},k\in\mathcal{M}$, and consider the first-order necessary conditions given by

$$\frac{\partial L_i}{\partial\lambda_{ik}}=0\quad\forall\,i\in\mathcal{N},k\in\mathcal{M}\,,\qquad(20)$$

which are equivalent to: $\forall\, i \in \mathcal{N},\, k \in \mathcal{M}$,

$$\frac{\beta_N + 1}{\sum_{j\in\mathcal{M}} \lambda_{ij}(N)} - \frac{c_k - \lambda_k(N) + \lambda_{ik}(N)}{(c_k - \lambda_k(N))^2 \sum_{j\in\mathcal{M}} \frac{\lambda_{ij}(N)}{c_j - \lambda_j(N)}} = 0\,. \quad (21)$$

In order to obtain nontrivial solutions to (21) in the limit as $N \to \infty$, we will scale the parameter β_N appropriately with respect to N, specifically as

$$\beta_N = \frac{\alpha}{N} \quad \forall\, N\,, \quad (22)$$

where α is a positive constant. This scaling is reminiscent of heavy-traffic limit results for certain stochastic models of communication networks where appropriate scaling is invoked on the problem parameters to obtain meaningful limiting solutions; see [17], [8], [13], [7] and references therein. To motivate this particular scaling in the context of our model here, consider a simpler version of this model where there is only one link, i.e., $M = 1$. It is easy to see from (21) that in this case the Nash equilibrium solution is unique[3], symmetric, and given by[4]

$$\lambda_{i1}(N) = \frac{\beta_N}{N\,\beta_N + 1} c_1 =: \lambda^*(N)\,, \quad i \in \mathcal{N}\,.$$

Note that the corresponding total flow over the link is $\lambda_1 = N\lambda^*(N)$, and hence to ascertain finiteness of the resulting delay in the limit as $N \to \infty$, we have to require β_N to be of the order of $1/N$. Under this scaling (22) for β_N, we first present an informal derivation of the limiting value of the solution that satisfies the first-order necessary conditions given by (21), and then make the result precise.

We henceforth restrict our attention only to solutions that are symmetric across the users, which is a reasonable assumption given that the players (users) enter the game symmetrically:

$$\lambda_{ij}(N) \;=\; \lambda_j(N)/N \qquad \forall i \in \mathcal{N},\ \text{and each } j \in \mathcal{M}\,.$$

In view of this assumption, (21) can equivalently be written as: $\forall\, k \in \mathcal{M}$,

$$\frac{N(\beta_N + 1)}{\overline{\lambda}(N)} - \frac{c_k - \lambda_k(N) + \lambda_k(N)/N}{(c_k - \lambda_k(N))^2 \sum_{j\in\mathcal{M}} \frac{\lambda_j(N)/N}{c_j - \lambda_j(N)}} = 0\,, \quad (23)$$

where

$$\overline{\lambda}(N) \;:=\; \sum_{j=1}^{M} \lambda_j(N)\,.$$

Note that by our "inner solution" assumption, $\overline{\lambda}(N) > 0\;\forall\, N$.[5] Now, using $\beta_N = \alpha/N$ in (23) yields $\forall\, k \in \mathcal{M}$,

$$\frac{\alpha + N}{\overline{\lambda}(N)} - \frac{N(c_k - \lambda_k(N)) + \lambda_k(N)}{(c_k - \lambda_k(N))^2 \sum_{j\in\mathcal{M}} \frac{\lambda_j(N)}{c_j - \lambda_j(N)}} = 0\,,$$

which can be written as

$$\frac{N}{\overline{\lambda}(N)}\left[1 - \frac{1}{c_k - \lambda_k(N)} \cdot \frac{\overline{\lambda}(N)}{\sum_{j\in\mathcal{M}} \frac{\lambda_j(N)}{c_j - \lambda_j(N)}}\right] + \frac{\alpha}{\overline{\lambda}(N)} - \frac{\lambda_k(N)}{(c_k - \lambda_k(N))^2 \sum_{j\in\mathcal{M}} \frac{\lambda_j(N)}{c_j - \lambda_j(N)}} = 0 \quad (24)$$

We are interested in the solution to this set of M equations for large N.

Consider any convergent subsequence of $\{\lambda_k(N)\}$ and denote its limit by λ_k. Assume that $\lambda_k < c_k$, $k \in \mathcal{M}$, and further that $\bar{\lambda} := \sum_{k\in\mathcal{M}} \lambda_k > 0$. Then, from (24), the quantity

$$\lim_{N\to\infty} \frac{N}{\overline{\lambda}(N)}\left[1 - \frac{1}{c_k - \lambda_k(N)} \cdot \frac{\overline{\lambda}(N)}{\sum_{j\in\mathcal{M}} \frac{\lambda_j(N)}{c_j - \lambda_j(N)}}\right]$$

is finite for each $k \in \mathcal{M}$; that is, for some constants $\{h_k\}$,

$$1 - \frac{1}{c_k - \lambda_k(N)}\, p(N) = \frac{1}{N} h_k + o(1/N)\,, \quad k \in \mathcal{M}\,,$$

where (along the subsequence identified earlier),

$$\lim_{N\to\infty} p(N) = \bar{p} = \bar{\lambda} \Big/ \sum_{j\in\mathcal{M}} \frac{\lambda_j}{c_j - \lambda_j}\,.$$

This leads to

$$c_k - \lambda_k(N) = \left[1 + \frac{1}{N} h_k\right] p(N) + o(1/N)\,.$$

Thus, to an $o(1/N)$ approximation,

$$c_j - \lambda_j(N) \;=\; c_1 - \lambda_1(N) + \frac{\mu_j}{N},\ j > 1, \quad (25)$$

for some constants $\mu_j := (h_j - h_1)\bar{p}$, with $\mu_1 = 0$. Introduce

$$f(N) := \frac{N}{\overline{\lambda}(N)}\left[1 - \frac{1}{c_1 - \lambda_1(N)} \cdot \frac{\overline{\lambda}(N)}{\sum_{j\in\mathcal{M}} \frac{\lambda_j(N)}{c_j - \lambda_j(N)}}\right]. \quad (26)$$

Substituting (25) into (26) and ignoring the $o(1/N)$ terms,

$$\begin{aligned} f(N) &= \frac{N}{\overline{\lambda}(N)}\Bigg[1 - \frac{1}{c_1 - \lambda_1(N)} \cdot \\ &\qquad \cdot \frac{\overline{\lambda}(N)}{\sum_{j\in\mathcal{M}} \frac{\lambda_j(N)}{c_1 - \lambda_1(N)}(1 - \frac{\mu_j}{(c_1 - \lambda_1(N))N})}\Bigg] \\ &= \frac{N}{\overline{\lambda}(N)}\left[1 - \frac{\overline{\lambda}(N)}{\sum_{j\in\mathcal{M}} \lambda_j(N)(1 - \frac{\mu_j}{(c_1 - \lambda_1(N))N})}\right] \end{aligned}$$

[3] *Uniqueness* follows because (21) is also sufficient in this case.

[4] For consistency in notation, we still use a subscript "1" to designate the link, even though there is only one link.

[5] Even without the "inner solution" assumption, this property holds because $U_i(\lambda) = 0$ when the total throughput of Player i is *zero*, $i \in \mathcal{N}$.

Thus,

$$\lim_{N\to\infty} f(N) = -\sum_{j\in\mathcal{M}} \lambda_j\mu_j / \big[(c_1-\lambda_1)\overline{\lambda}^2\big] \,.$$

Therefore, as $N\to\infty$, (24) becomes (for $k=1$):

$$\frac{\alpha}{\overline{\lambda}} - \frac{\sum_{j\in\mathcal{M}}\lambda_j\mu_j}{\overline{\lambda}^2(c_1-\lambda_1)} - \frac{\lambda_1}{\overline{\lambda}(c_1-\lambda_1)} = 0\,.$$

Thus, for $k=1$, we obtain, in the limit as $N\to\infty$,

$$\alpha \;=\; \frac{1}{c_1-\lambda_1}\left[\lambda_1 + \frac{1}{\overline{\lambda}}\sum_{j\in\mathcal{M}}\lambda_j\mu_j\right]\,.$$

For a general k, we have

$$\alpha \;=\; \frac{1}{c_k-\lambda_k}\left[\lambda_k + \frac{1}{\overline{\lambda}}\sum_{j\in\mathcal{M}}\lambda_j(\mu_j-\mu_k)\right]. \qquad (27)$$

Since, by (25), $(c_k-\lambda_k)$ is independent of k, we have $\forall\, k,m$,

$$\lambda_k + \frac{1}{\overline{\lambda}}\sum_{j\in\mathcal{M}}\lambda_j(\mu_j-\mu_k) \;=\; \lambda_m + \frac{1}{\overline{\lambda}}\sum_{j\in\mathcal{M}}\lambda_j(\mu_j-\mu_m)\,,$$

which leads to

$$\lambda_k-\lambda_m \;=\; \frac{1}{\overline{\lambda}}\sum_{j\in\mathcal{M}}\lambda_j(\mu_k-\mu_m) \;=\; \mu_k-\mu_m.$$

Therefore,

$$\alpha = \frac{1}{c_k-\lambda_k}\underbrace{\left[\lambda_k + \frac{1}{\overline{\lambda}}\sum_{j\in\mathcal{M}}\lambda_j(\lambda_j-\lambda_k)\right]}_{\lambda_k - \frac{\overline{\lambda}\lambda_k}{\overline{\lambda}} + \frac{1}{\overline{\lambda}}\sum_{j\in\mathcal{M}}\lambda_j^2}$$

$$= \frac{1}{c_k-\lambda_k}\cdot\frac{1}{\overline{\lambda}}\sum_{j\in\mathcal{M}}\lambda_j^2\,,\quad k\in\mathcal{M}\,. \qquad (28)$$

Next, we explicitly compute λ_k. In order to solve for the λ_j's, we first introduce

$$\bar{c} := \sum_{j\in\mathcal{M}} c_j\,,\quad \overline{\lambda^2} := \sum_{j\in\mathcal{M}}\lambda_j^2\,,\quad \overline{c^2} := \sum_{j\in\mathcal{M}} c_j^2.$$

Then, from (28), we obtain

$$\alpha\overline{\lambda}\sum_k(c_k-\lambda_k) = \sum_k\sum_j\lambda_j^2 \Leftrightarrow \alpha\overline{\lambda}\bar{c} - \alpha\overline{\lambda}^2 = M\overline{\lambda^2}. \quad (29)$$

Let $\zeta := c_k-\lambda_k$. Summation over $k\in\mathcal{M}$ yields

$$M\zeta = \bar{c}-\overline{\lambda} \qquad (30)$$

This time starting with the relationship $\lambda_k = c_k-\zeta$, squaring both sides, summing over $k\in\mathcal{M}$, multiplying by M, and using (30), we arrive at the following relationship:

$$\begin{aligned} M\overline{\lambda^2} &= M\overline{c^2} - 2M\bar{c}\zeta + M^2(\zeta)^2 \\ &= M\overline{c^2} - 2\bar{c}(\bar{c}-\overline{\lambda}) + (\bar{c}-\overline{\lambda})^2 \\ &= M\overline{c^2} - \bar{c}^2 + \overline{\lambda}^2\,. \end{aligned} \qquad (31)$$

Using this in (29), we obtain the following quadratic equation for $\overline{\lambda}$:

$$(\alpha+1)\overline{\lambda}^2 - \alpha\bar{c}\overline{\lambda} + M\overline{c^2} - \bar{c}^2 \;=\; 0\,,$$

which admits the solution(s)

$$\overline{\lambda} \;=\; \frac{\alpha \pm \sqrt{(\alpha+2)^2 - 4(\alpha+1)M\nu}}{2(\alpha+1)}\,\bar{c} \qquad (32)$$

where $\nu = \overline{c^2}/\bar{c}^2$. For all this to be valid, at least one of the solutions above should satisfy the bounds

$$0 \le \overline{\lambda} \le \bar{c}\,, \qquad (33)$$

in which case the corresponding individual λ_j's (that is, total flows over the individual links) become

$$\lambda_j = c_j - \frac{\bar{c}-\overline{\lambda}}{M},\; j\in\mathcal{M}\,, \qquad (34)$$

provided that

$$\lambda_j \ge 0 \;\forall\, j\in\mathcal{M}\,. \qquad (35)$$

Theorem 4: Suppose that there exist $\{\lambda_j\}_{j=1}^M$ satisfying (32)-(35). Then,

$$\lambda_{ij} = \frac{\lambda_j}{N}\,,\quad i\in\mathcal{N},\; j\in\mathcal{M}\,, \qquad (36)$$

constitute an $O(1/N)$ Nash equilibrium with exponent

$$\kappa = \alpha\log\frac{2\overline{\lambda}}{\sqrt{\gamma\overline{Q}} + \alpha M\gamma} < 0\,, \qquad (37)$$

where

$$\overline{Q} := \frac{\overline{\lambda}^2 - M\overline{\lambda^2}}{\gamma} + \alpha^2M^2\gamma > 0\,, \qquad (38)$$

and

$$\gamma := \frac{\overline{\lambda^2}}{\alpha\overline{\lambda}}\,. \qquad (39)$$

Proof: Let us fix the flows of all users except those of a generic user, Player i, as given by (32)-(35). Let the flows of Player i, arbitrary at this point, be denoted by η_{im}, $m\in\mathcal{M}$. Then, what Player i faces is a single-user problem of the type studied in Section 2, with the capacity of link m, $m\in\mathcal{M}$, as seen by Player i being

$$c_m^i = c_m - \frac{N-1}{N}\lambda_m \equiv \frac{1}{\alpha\overline{\lambda}}\overline{\lambda^2} + \frac{1}{N}\lambda_m\,,\; m\in\mathcal{M}\,. \quad (40)$$

Note that $\lambda_m, \overline{\lambda^2}, \overline{\lambda}$ all being independent of N, c_m^i above is a $(1/N)$-perturbation around a nominal constant capacity $(\overline{\lambda^2}/\alpha\overline{\lambda})$ per link (independent also of the user), and hence the generic user sees an almost equal link capacity network. By Theorem 2, there exists an N^*, sufficiently large, such that for all $N > N^*$ Player i's response to (32)-(35) is an inner solution. Further, for each such N, the solution is unique, and obtained as the unique stationary point of

U_i^N, with all other users' policies fixed as given, that is of the function

$$V_i^N(\{\eta_{ij}\}_{j\in\mathcal{M}}) = \left(\sum_{m=1}^{M}\eta_{im}\right)^{1+\frac{\alpha}{N}} \bigg/ \left(\sum_{m=1}^{M}\frac{\eta_{im}}{c_m^i-\eta_{im}}\right) \tag{41}$$

where c_m^i is given by (40). It now readily follows from Theorem 1, and particularly expression (15), that the unique inner maximizing solution for V_i^N, alluded to above, yields the value:

$$\max_{\{\eta_{ij}\}_{j\in\mathcal{M}}} V_i^N(\{\eta_{ij}\}_{j\in\mathcal{M}}) = \frac{(\overline{c^i}-\mu\,\overline{c_{sq}^i})^{1+\frac{\alpha}{N}}}{\overline{c_{sq}^i}-M\mu}\mu =: V_i^{N^*} \tag{42}$$

where $\overline{c^i}$ and $\overline{c_{sq}^i}$ are given by

$$\begin{aligned}\overline{c^i} &:= \sum_{m=1}^{M} c_m^i = \overline{c}-\overline{\lambda}+\frac{\overline{\lambda}}{N} \equiv M\gamma+\frac{\overline{\lambda}}{N}\,,\\ \overline{c_{sq}^i} &:= \sum_{m=1}^{M}\sqrt{c_m^i} = \sum_{m=1}^{M}\sqrt{\gamma+\frac{\lambda_m}{N}}\,,\end{aligned}$$

with γ defined by (39), and μ given by (9) with the negative sign, and with $\overline{c_{sq}}$ and $\overline{c}$ replaced by $\overline{c_{sq}^i}$ and $\overline{c^i}$, respectively. Equivalently,

$$\mu = \left(\left(2+\frac{\alpha}{N}\right)\overline{c_{sq}^i}-\sqrt{Q}\right)\bigg/\left(2M\left(1+\frac{\alpha}{N}\right)\right),$$

with

$$\begin{aligned}Q &:= \left(2+\frac{\alpha}{N}\right)^2\overline{c_{sq}^i}^2-4M\left(1+\frac{\alpha}{N}\right)\overline{c^i}\\ &\equiv \frac{\alpha^2}{N^2}\overline{c_{sq}^i}^2+4\left(1+\frac{\alpha}{N}\right)\left(\overline{c_{sq}^i}^2-M\overline{c^i}\right).\end{aligned}$$

Our goal now is to compute $V_N^{i\,*}$ for N sufficiently large, and particularly as $N\to\infty$. Toward this end, we start with the fact that for any real positive number x,

$$\sqrt{1+x} = 1+\frac{x}{2}-\frac{x^2}{8}+O(x^3)\,,$$

and use it to obtain the expansion:

$$\sqrt{\gamma+\frac{\lambda_k}{N}} = \sqrt{\gamma}\left(1+\frac{\lambda_k}{2\gamma N}-\frac{1}{8}\left(\frac{\lambda_k}{\gamma N}\right)^2+O\left(\frac{1}{N^3}\right)\right),$$

which further leads to

$$\overline{c_{sq}^i} = \sqrt{\gamma}\left(M+\frac{\overline{\lambda}}{2\gamma N}-\frac{\overline{\lambda^2}}{8\gamma^2}\cdot\frac{1}{N^2}+O\left(\frac{1}{N^3}\right)\right), \tag{43}$$

and

$$\overline{c_{sq}^i}^2-M\overline{c^i} = \frac{\overline{\lambda}^2-M\overline{\lambda^2}}{4\gamma}\cdot\frac{1}{N^2}+O\left(\frac{1}{N^3}\right). \tag{44}$$

In view of these expansions, we have

$$\begin{aligned}Q &= \frac{\overline{\lambda}^2-M\overline{\lambda^2}}{4\gamma}\cdot\frac{4}{N^2}+\frac{\alpha^2M^2\gamma}{N^2}+O\left(\frac{1}{N^3}\right)\\ &\equiv \overline{Q}\cdot\frac{1}{N^2}+O\left(\frac{1}{N^3}\right),\end{aligned}$$

where $\overline{Q}$ is precisely the one given by (38), which can also be written as

$$\overline{Q} \equiv \frac{\overline{c}^2-M\overline{c^2}+\alpha^2(\overline{c}-\overline{\lambda})^2}{\overline{c}-\overline{\lambda}}M\,.$$

Note also that

$$\mu = \frac{\overline{c_{sq}^i}}{M}-\frac{1}{2MN}\left(\alpha\overline{c_{sq}^i}+\sqrt{\overline{Q}}\right)+o(1/N). \tag{45}$$

For this expression to be valid, it is of course necessary that $\overline{Q}$ be positive, which however readily follows from the positivity of Q for all sufficiently large N, which itself follows from the single-user result embodied in Theorem 2. It is also possible to show positivity of $\overline{Q}$ directly, which we quickly do here for the sake of completeness: Start with the obvious inequality

$$\left(M(\overline{\lambda^2})-\overline{\lambda}^2\right)^2 > -M\overline{\lambda^2}(\overline{\lambda})^2\,,$$

which is equivalent to

$$M^2(\overline{\lambda^2})^2-M\overline{\lambda^2}(\overline{\lambda})^2+(\overline{\lambda}^2)^2 > 0\,.$$

Dividing throughout by $\overline{\lambda}^2$ and using the definition of γ, leads to $\overline{Q}>0$.

Now, first expanding the numerator of the expression (42) for $V_i^{N^*}$, but without the power $(1+\alpha/N)$, we obtain:

$$\overline{c^i}-\mu\overline{c_{sq}^i} = \left(\frac{\gamma\alpha M}{2}+\frac{\sqrt{\gamma\overline{Q}}}{2}\right)\frac{1}{N}+o(1/N)$$

and next expanding the denominator of (42), we get:

$$\overline{c}_{sq}^i-M\mu = \frac{M\alpha\sqrt{\gamma}+\sqrt{\overline{Q}}}{2N}+o(1/N).$$

Hence,

$$\begin{aligned}\lim_{N\to\infty} V_i^{N^*} &= \lim_{N\to\infty}\left[\left(\sqrt{\overline{Q}}+\alpha M\sqrt{\gamma}\right)^{\dfrac{\alpha}{N}}\left(\frac{1}{2N}\right)^{\dfrac{\alpha}{N}}\cdot\right.\\ &\qquad\left.\cdot\gamma^{1+\dfrac{\alpha}{2N}}\right] = \gamma \equiv \frac{\overline{c}-\overline{\lambda}}{M}\,.\end{aligned} \tag{46}$$

On the other hand, with $\eta_{ij}=\lambda_{ij}$ as given in Theorem 4 substituted into V_i^N given by (41), yields

$$V_i^N(\{\lambda_j/N\}) = \frac{(\overline{\lambda}/N)^{1+\alpha/N}}{\frac{\alpha(\overline{\lambda})^2}{N\overline{\lambda^2}}} = \gamma\left(\frac{\overline{\lambda}}{N}\right)^{\frac{\alpha}{N}} = \frac{\overline{c}-\overline{\lambda}}{M}\left(\frac{\overline{\lambda}}{N}\right)^{\frac{\alpha}{N}}. \tag{47}$$

This tends to $(\bar{c}-\bar{\lambda})/M$ as N tends to infinity, which is identical with (46). Hence, the policies in Theorem 4 indeed constitute an asymptotic Nash equilibrium.

To prove that they are also in $O(1/N)$ equilibrium, it will be sufficient to compute, using the expansions already obtained,

$$\log V_i^N(\{\lambda_j/N\}) - \log V_i^{N*} = \frac{\alpha}{N}\log\frac{2\bar{\lambda}}{\sqrt{\gamma\overline{Q}}+\alpha M\gamma} + o(1/N)$$

which is exact to the $(1/N)$ term. The exponent κ also readily follows from the above expression, which is also negative as the following sequence of simple steps shows:

$$\begin{aligned}
M\overline{\lambda^2} > \overline{\lambda}^2 &\iff 4M\overline{\lambda^2} > 3\overline{\lambda}^2 + M\overline{\lambda^2}\\
&\iff 4\alpha M\overline{\lambda}\gamma > 3\overline{\lambda}^2 + M\overline{\lambda^2}\\
\iff \quad \overline{\lambda}^2 - M\overline{\lambda^2} + \alpha^2M^2\gamma^2 &> 4\overline{\lambda}^2 - 4\alpha M\overline{\lambda}\gamma + \alpha^2M^2\gamma^2\\
\iff \quad \gamma\overline{Q} > \left(2\overline{\lambda}-\alpha M\gamma\right)^2 &\implies \sqrt{\gamma\overline{Q}}+\alpha M\gamma > 2\overline{\lambda}
\end{aligned}$$

$\diamond$

Theorem 4 has provided a characterization of a symmetric $O(1/N)$ Nash equilibrium provided that (32)-(35) admit a solution. This inevitably requires that flows on *all* M links be positive, which may not always be the case. In other words, a symmetric $O(1/N)$ Nash equilibrium with nonzero flows on all the links may not exist, even though there may still exist a symmetric $O(1/N)$ Nash equilibrium that uses only a subset of the links, that is $\mathcal{S}_m := \{1,\dots,m\}$ for some m. Such a solution would be obtained by simply substituting m for M in (32)-(35). Now, even if a solution exists to (32)-(35) with $m < M$ replacing M, it does not necessarily follow from Theorem 4 that the resulting policies, extended by assigning zero flows to links in $\mathcal{S}_m^c$ will be in $O(1/N)$ Nash equilibrium. They would be in $O(1/N)$ Nash equilibrium when restricted to $\mathcal{S}_m$ (this follows readily from Theorem 4), but this property may not hold in the extended game with all M links. What needs to be shown is that when all players except one use the flows dictated by the result of Theorem 4 with $m < M$ replacing M, and zero flows on the remaining $M-m$ links, then the remaining player does not have any incentive (in terms of maximizing his utility) in using any of the links outside $\mathcal{S}_m$. Assuring this will inevitably impose some restriction on the capacities of the links belonging to $\mathcal{S}_m^c$ relative to the excess capacities left for that player on the links belonging to $\mathcal{S}_m$, which is what the next theorem does. It provides testable necessary and sufficient conditions for a set of positive flows on only a subset of the links to constitute a symmetric $O(1/N)$ Nash equilibrium for the original M-link network.

Theorem 5: For some $\mathcal{S}_m$, suppose that there exists a solution to (32)-(35), with the corresponding total flow on Link j denoted by $\lambda_j^{(m)}$. Then, the set of flows

$$\lambda_{ij}(N) = \begin{cases} \lambda_j^{(m)}/N, & j \le m \\ 0 & j > m\,, \end{cases}\;, \; j\in\mathcal{M},\; i\in\mathcal{N}\,, \quad (48)$$

provides an $O(1/N)$ Nash equilibrium if

$$c_{m+1} < c_1 - \lambda_1^{(m)}\,. \quad (49)$$

In this case, the exponent κ in the $O(1/N)$ approximation is given by (37) with M replaced by m.

Conversely, if $c_{m+1} \ge c_1 - \lambda_1^{(m)}$, then the set of flows (48) is not in $O(1/N)$ Nash equilibrium nor do they constitute an asymptotic Nash equilibrium.

Proof: Under the hypothesis of the theorem, fix the flows of all users except Player 1, as given by (48), and consider the optimal flow allocation for Player 1. Assume that the condition (49) holds. Then, the problem faced by Player 1 is a single-user problem (as in the proof of Theorem 4) with link capacities $c_i^1,\ i\in\mathcal{M}$, where

$$c_i^1 = c_1 + O(1/N),\ i = 2,3,\dots,m,$$

and

$$c_j^1 = c_j < c_m^1,\ j > m.$$

We will now show that this single-user problem does not admit a solution for any $\mathcal{S}_n$, $n > m$. Since $c_{m+1}^1 < c_m^1$, we have

$$\left(\frac{1}{n}\sum_{i=1}^{n}\sqrt{c_i^1}\right)^2 < \frac{1}{n}\left(\sum_{i=1}^{n}c_i^1\right) \iff \bar{c^1}_{sq,n} < n\bar{c^1}_n\,, \quad (50)$$

where the notation is that of (13) with only superscript ^{1} added. In view of the strict inequality in (50), the discriminant in the expression (12) is negative when $\beta = 0$, and by continuity for all positive values of β sufficiently close to *zero* (and hence for all sufficiently large N). This implies that μ_n given by (12) does not exist for sufficiently large N. Thus, there is no incentive for Player 1 to use the links outside the set $\mathcal{S}_n$, and therefore the result of Theorem 4) applies with just M replaced by m. Hence, the set of policies (48) provides a symmetric $O(1/N)$ Nash equilibrium under the given condition $c_{m+1} < c_1 - \lambda_1^{(m)}$.

Next, suppose that $c_{m+1} > c_1 - \lambda_1^m$. Considering the set of links $\mathcal{S}_{m+1}$, by the argument above, for large N, the optimal response of Player 1 is to allocate all its flow to Link $m+1$. Thus, the solution to (32)-(35) cannot be in $O(1/N)$ Nash equilibrium, nor in asymptotic Nash equilibrium, because the optimal response of Player 1 (which uses Link $m+1$) with the flows of all other players fixed as given results in an $O(1)$ improvement in his logarithmic utility. If $c_{m+1} = c_1 - \lambda_1^m$, then by our earlier discussion of the single-user problem, Player 1 will distribute its total flow among all $m+1$ links and again the solution does not provide an $O(1/N)$ Nash equilibrium, nor an asymptotic Nash equilibrium. $\diamond$

The two theorems included in this section provide, in a sense, the complete solution to the combined routing and flow control problem with multiple links and an arbitrarily large number of users. They provide testable conditions for a characterization of the entire set of symmetric $O(1/N)$

Nash equilibria, where the number of tests is equal the number of links, M. Each test involves checking the existence of a solution to (32)-(35) interpreted for a general m, and if $m < M$ also checking condition (49). If all these conditions fail, for all $m \in \mathcal{M}$, then the network game does not admit any symmetric $O(1/N)$ Nash equilibrium, but we should note that this does not rule out the existence of an $O(1/N)$ Nash equilibrium which is not symmetric across players.

Even though the conditions embodied in the two theorems are easily testable through numerical computation, it may still be desirable to translate these conditions into ones that involve simple regions in the space of all parameters defining the network game. One set of such conditions is provided in Appendix II for the case corresponding to full use of all links of the network (that is, the situation covered by Theorem 4), but can easily be extended to cover the set-up of Theorem 5 by simply replacing M with m and requiring also satisfaction of (49).

IV. Examples of symmetric $O(1/N)$ Nash equilibria

We present in this section two numerical examples which illustrate various features of symmetric $O(1/N)$ Nash equilibria.

Example 3: Consider the same network of links as in Example 1 of Section II-D, with $\alpha = 0.9$. There is no solution to (32)-(35) if we consider $\mathcal{S}_m$ for $m \geq 7$. For $m = 6$, the positive square root in the expression (32) leads to a feasible solution, with the corresponding value for (32) being $\bar{\lambda}^+ = 182.95$. The negative one, $\bar{\lambda}^- = 30.206$, however does not, since it leads to a negative λ_6. The flows corresponding to $\bar{\lambda}^+$ are, from (33):

$$\lambda_1 = 55.49,\ \lambda_2 = 45.49,\ \lambda_3 = 35.49,$$

$$\lambda_4 = 25.49,\ \lambda_5 = 15.49,\ \lambda_6 = 5.49.$$

It is easy to verify that the condition given in Theorem 5 is satisfied in this case, and thus, we have a symmetric $O(1/N)$ Nash equilibrium. The exponent κ in the $O(1/N)$ approximation is $\kappa = -0.0875$.

For $m < 6$, (32)-(35) provide a solution, but the condition given in Theorem 5 is not satisfied, and thus, there is no symmetric asymptotic Nash equilibrium where the users send positive flow on only the first *five* (or fewer) links. $\diamond$

Example 4: This example will demonstrate the existence of multiple $O(1/N)$ Nash equilibria. Specifically, consider a network of 10 links with

$$c_1 = 100,\ \text{and}\ c_m = 50,\ m = 2, 3, \ldots, 10,$$

and with $\alpha = 0.9$. In this case, it turns out that all 10 links can be used, and hence Theorem 4 directly applies, with both positive and negative square roots in the expression (32) for $\bar{\lambda}$ yielding feasible solutions. Hence we have *two* symmetric $O(1/N)$ Nash equilibria. The one corresponding to the positive square root, $\bar{\lambda}^+ = 201.9$, is:

$$\lambda_1 = 65.19;\ \lambda_m = 15.19,\ m = 2, 3, \ldots, 10\,,$$

and the one corresponding to the negative square root, $\bar{\lambda}^- = 58.7$, is:

$$\lambda_1 = 50.87;\ \lambda_m = 0.87,\ m = 2, 3, \ldots, 10\,.$$

Again we see the delay-equalizing property in both cases. Finally, the exponents for the $O(1/N)$ approximation for these two sets of policies are

$$\kappa^+ = -0.1472\,,\ \kappa^- = -0.7773\,.$$

To explore the possibility for other Nash equilibria (with fewer links used with positive flow), we computed $\bar{\lambda}$ for all $m < 10$. For $\mathcal{S}_m$, $m = 6, 7, 8, 9$, both the positive and negative square roots provide $\{\lambda_i\}$ that satisfy (32)-(35), but the condition in Theorem 5 for an $O(1/N)$ Nash equilibrium is not satisfied for any of these. Thus, none of these solutions constitute an $O(1/N)$ Nash equilibrium. For $\mathcal{S}_m$, $m = 2, 3, 4, 5$, there exists no $\{\lambda_i\}$ that satisfies (32)-(35). For $\mathcal{S}_1$, the negative square root yields solution to (32)-(35) given by $\lambda_1 = 47.37$. Since $c_2 = 50 < 52.63 = c_1 - \lambda_1^{(1)}$, the condition of Theorem 5 is satisfied, and hence there is indeed an $O(1/N)$ Nash equilibrium where all users use only Link 1. The exponent κ in this case is $\kappa = -0.6904$. Note that the total flow over the network under this $O(1/N)$ Nash equilibrium (which is 47.37) is less than the ones under the other $O(1/N)$ Nash equilibria above which dictate use of all 10 links (which are 201.9 and 58.7). $\diamond$

An interesting feature exhibited by this last example is that it is possible for $O(1/N)$ Nash equilibria for a particular network game to dictate use of *all* available links or *only one* link for all users, and nothing in between.

V. Conclusions

We have obtained explicit expressions for asymptotic and $O(1/N)$ Nash equilibria in a network with M parallel links and a large number, N, of players who attempt to choose routes and flows to maximize their individual utility functions, which are taken as the ratio of some positive power of the total throughput of that user to the average delay seen by the user. We have focused only on the symmetric equilibria, which turned out to have the appealing property that as the number of players, N, becomes arbitrarily large, the delays over all links with positive flow become equal. It would be interesting to explore whether the problem also admits nonsymmetric $O(1/N)$ Nash equilibria, and what their properties would be. It would also be interesting to study the more general network game where different users or different groups of users have different delay-throughput tradeoff parameters, β's (or α's), in which case it will be necessary to consider nonsymmetric equilibria. Other extensions one can envision are: (i) studying the existence and characterization of Nash equilibria in the case of a finite number of users; (ii) developing distributed dynamic algorithms for the users' myopic response behavior to evolve (converge) to the $O(1/N)$ Nash equilibrium; (iii) developing the counterparts of these results for general topology networks, such as [1]; and (iv) exploring the possibilities of other types of scaling (of β with respect to N) and their implications on existence of Nash equilibria.

VI. Appendix I: Proof of Theorem 2

In this appendix we provide a proof for Theorem 2 given in Section 2. Let us first introduce some notation and state some properties of the utility function to be maximized by the generic user.

Let $f(\lambda; c)$ denote the utility function (3), where we show here its explicit dependence on the link capacities vector $c := (c_1, \ldots, c_M)$, in addition to the throughput vector $\lambda := (\lambda_1, \ldots, \lambda_M)$. The domain of definition for f as a function of λ is $[0, c_1] \times \cdots \times [0, c_M]$. Note that on this domain, f is a continuous function of λ for each fixed c. Furthermore, as a function of c, for fixed $\lambda \geq 0$, it is continuous in the domain $[\lambda_1, \infty) \times \cdots \times [\lambda_M, \infty)$. Moreover, $f(\lambda; c) \geq 0$, with equality holding if and only if either $\lambda = 0$ or $\lambda_m = c_m$ for at least one $m = 1, \ldots, M$. Now consider, for fixed $c > 0$, the maximization problem:

$$\max_{\lambda_m \in [0, c_m], m=1,\ldots,M} f(\lambda; c) \tag{51}$$

Since f is continuous (in λ) and the constraint set is closed and bounded, a maximum exists (which we denote by $\lambda^*(c)$, and the maximum value by $f^*(c)$), and furthermore since f takes the value *zero* when $\lambda_m = c_m$ for any m, we have $\lambda_m^*(c) < c_m$.

Now let c^o denote the vector of link capacities, whose components are all equal, with (by a slight abuse of notation) c^o also denoting the common value of these individual components. Then, we know that

$$\lambda_m^*(c^o) = \frac{\beta}{\beta + 1}\, c^o\,, \quad m \in \mathcal{M} := \{1, \ldots, M\}$$

is the unique solution to (51), which is clearly an inner solution.

Two further properties of f will be useful in the development below:

$$f(\lambda_{-m}, \lambda_m = 0; c^o) < f^*(c^o)\,,\ \forall\, \lambda_i \in [0, c^o],\, i \neq m,\, m \in \mathcal{M} \tag{52}$$

and for any $c' := (c_1', \ldots, c_M')$ such that $c_i' \geq c_i\ \forall\, i$,

$$f(\lambda; c) \leq f(\lambda; c')\,,\ \forall\, \lambda_m \in [0, c_m], \quad m \in \mathcal{M}\,. \tag{53}$$

Consider now the following class of *perturbed* optimization problems:

$$\max_{\lambda_i \in [0, c_i^o + \delta_i], i=1,\ldots,m} f_m(\lambda; c^o + \Delta)\,, \quad m \in \mathcal{M} \tag{54}$$

where $\Delta := (\delta_1, \ldots, \delta_M)$, and f_m is f with $\lambda_{m+1} = \cdots = \lambda_M = 0$, that is with only the first m links.

The following lemma now says that the maximum value of f_m in (54) for each fixed m can be made sufficiently close to $f_m^*(c^o)$ by picking δ_i's sufficiently close to *zero*.

Lemma A.1: Given $\epsilon > 0$, $\exists\ \delta > 0$ such that $\forall\ \delta_i$, $|\delta_i| < \delta$, $i \leq m$,

$$\begin{aligned} -\epsilon + f_m^*(c^o) \quad &< \max_{\lambda_i \in [0, c_i^o + \delta_i], i=1,\ldots,m} f_m(\lambda; c^o + \Delta) \\ &< \quad f_m^*(c^o) + \epsilon \end{aligned}$$

Proof: Define $\delta_i^+ := \max(\delta_i, 0)$, $\Delta^+ := (\delta_1^+, \ldots, \delta_M^+)$, and note that in view of property (53),

$$\begin{aligned} &\max_{\lambda_i \in [0, c_i^o + \delta_i], i \in \mathcal{M}} f(\lambda; c^o + \Delta) \\ &\qquad \leq \max_{\lambda_i \in [0, c_i^o + \delta_i], i \in \mathcal{M}} f(\lambda; c^o + \Delta^+) \\ &\qquad \leq \max_{\lambda_i \in [0, c_i^o + \delta_i^+], i \in \mathcal{M}} f(\lambda; c^o + \Delta^+) \\ &\qquad \leq \max_{\lambda_i \in [0, c_i^o + \delta], i \in \mathcal{M}} f(\lambda; c^o + \delta 1_M) \end{aligned} \tag{55}$$

where 1_M is the M-dimensional vector with all entries 1. Here the first inequality follows because the function to the maximized on the RHS is no smaller than the one on the LHS; the second inequality follows because the constraint set is no smaller; and the third inequality follows for both reasons above.

Now consider the maximization problem

$$\max_{\lambda_i \in [0, c^o + \delta''], i \in \mathcal{M}} f(\lambda; c^o + \delta' 1_M) \tag{56}$$

where $\delta'' \leq \delta'$. Since $f(\lambda; c) \to 0$ as λ_i approaches the upper limit of its constraint, for any i, $\exists\, \delta''' \leq \delta'$ such that $\forall\, \delta''$, $0 < \delta'' < \delta'''$, the maximization problem (56) is equivalent to the one below:

$$\max_{\lambda_i \in [0, c^o], i \in \mathcal{M}} f(\lambda; c^o + \delta' 1_M) \tag{57}$$

Since $f(\lambda; c^o + \delta' 1_M)$ is continuous in $\delta' \geq 0$ for each $\lambda \in [0, c^o] \times \cdots \times [0, c^o]$, where the latter is a closed and bounded subset of $\mathbf{R}^M$, given $\epsilon > 0$, $\exists\, \tilde{\delta}$ such that $\forall\, \delta'$, $0 \leq \delta' < \tilde{\delta}$, and $\forall\, \lambda_i \in [0, c^o]$, $i \in \mathcal{M}$,

$$f(\lambda; c^o + \delta' 1_M) < f(\lambda; c^o) + \epsilon\,.$$

This property immediately leads to the bound:

$$\begin{aligned} \max_{\lambda_i \in [0, c^o], i \in \mathcal{M}} f(\lambda; c^o + \delta' 1_M) \quad &< \quad \max_{\lambda_i \in [0, c^o], i \in \mathcal{M}} f(\lambda; c^o) + \epsilon \\ &= \quad f^*(c^o) + \epsilon\,. \end{aligned}$$

In view of this bound, picking $\delta = \min(\delta''', \tilde{\delta})$ in (55), provides the bound

$$\max_{\lambda_i \in [0, c^o + \delta_i], i=1,\ldots,m} f_m(\lambda; c^o + \Delta) < f_m^*(c^o) + \epsilon\,,$$

which is the RHS inequality of the Lemma, for m=M. One can go through the same steps for any $m < M$ as well, where δ could actually depend on m. But since there is only a finite number of them, taking the smallest one proves the result (RHS only) for arbitrary m.

To prove the LHS inequality of the Lemma, we go through similar steps, but this time obtaining lower bounds instead of upper. Letting $\delta_i^- := min(\delta_i, 0)$, $\Delta^- := (\delta_1^-, \ldots, \delta_M^-)$, the first set of inequalities in the proof above is now replaced by

$$\begin{aligned}
&\max_{\lambda_i \in [0, c^o+\delta_i], i\in\mathcal{M}} f(\lambda; c^o + \Delta) \geq \\
&\qquad \geq \max_{\lambda_i \in [0, c^o+\delta_i^-], i\in\mathcal{M}} f(\lambda; c^o + \Delta) \\
&\qquad \geq \max_{\lambda_i \in [0, c^o+\delta_i^-], i\in\mathcal{M}} f(\lambda; c^o + \Delta^-) \\
&\qquad \geq \max_{\lambda_i \in [0, c^o-\delta], i\in\mathcal{M}} f(\lambda; c^o - \delta 1_M) \\
&\qquad \geq f^*(c^o) - \epsilon
\end{aligned}$$

and this completes the proof of the Lemma. ⋄

We now use the result of Lemma A.1 above to prove Theorem 2 of Section 2. What we need to show is that there exists an open neighborhood of $c^o \in \mathbf{R}^M$ such that in that neighborhood the solution to (51) still requires all M links to be used, as $\lambda^*(c^o)$ does.

Toward this end, let the maximum of $f(\lambda; c)$ with $\lambda_M = 0$ be $f^*_{M-1}(c)$, and without this constraint be $f^*_M(c)$, which we had denoted earlier as $f^*(c)$. Since $f(\lambda; c^o)$ has a unique maximum, we already know that

$$\alpha := f^*_M(c^o) - f^*_{M-1}(c^o) > 0 \,.$$

Then, picking $\epsilon = \alpha/3$ on the RHS inequality of Lemma A.1, with $m = M-1$, leads to:

$$\begin{aligned}
&\max_{\lambda_i \in [0, c^o+\delta_i], i=1,\ldots,M-1} f(\lambda_{-M}, \lambda_M = 0; c^o + \Delta) \\
&\qquad < f^*_{M-1}(c^o) + \epsilon = f^*_M(c^o) + \epsilon - 3\epsilon = f^*_M(c^o) - 2\epsilon
\end{aligned}$$

whereas doing the same on the LHS inequality of Lemma A.1, with $m = M$, leads to:

$$\max_{\lambda_i \in [0, c^o+\delta_i], i=1,\ldots,M} f(\lambda; c^o + \Delta) > f^*_M(c^o) - \epsilon$$

which (when compared with the previous inequality) shows that there is loss of performance if the M'th link is dropped. This completes the proof of Theorem 2.

VII. Appendix II: Existence Analysis

We provide in this Appendix direct conditions on the parameters of the network game under which the discriminant in (32) is nonnegative and at least one of the solutions $\bar{\lambda}^+$ and $\bar{\lambda}^-$ satisfy conditions (33) and (35), equivalently conditions on the parameters for the network game to admit a symmetric $O(1/N)$ Nash equilibrium with positive flow on all M links. The lengthy but fairly straightforward analysis that has led to these results has not been included here.

Let us first introduce some notation:

$$\begin{aligned}
\widehat{c} &:= c_M/\bar{c}\,, \\
\nu &:= \overline{c^2}/\bar{c}^2 && (58) \\
p &:= 2\left[M\nu - 1 + \sqrt{(M\nu-1)M\nu}\right] && (59) \\
q &:= \frac{\nu}{(1-M\widehat{c})\widehat{c}} - \frac{2-M\widehat{c}}{1-M\widehat{c}}, \quad \text{if } \widehat{c} < \frac{1}{M}, && (60) \\
&:= 0\,, \quad \text{if } \widehat{c} = \frac{1}{M} \\
r &:= (1-M\widehat{c})/(M\widehat{c} - \frac{1}{2}) && (61)
\end{aligned}$$

We now consider two complementary regions for the parameter values (which do not depend on α):

Region A: $M\,\widehat{c} < 1/2$

If $\alpha \geq q$, (32) with the positive square root, $\bar{\lambda}^+$, satisfies both conditions (33) and (35).
Otherwise, there is no feasible solution to (32)-(35) when parameters lie in Region A.

Region B: $M\,\widehat{c} > 1/2$

Subcase 1. $\alpha \geq \max\{p, r\}$.
Again $\overline{\lambda}^+$ satisfies both conditions (33) and (35).

Subcase 2.

$$\max\{p, q\} \quad \leq \quad \alpha \leq r \quad \text{and} \quad \nu \leq \frac{M\widehat{c}^2}{2M\widehat{c}-1}$$

Under these two conditions, again $\overline{\lambda}^+$ satisfies both conditions (33) and (35).

Subcase 3.

$$\max\{p, r\} \leq \alpha \leq q \quad \text{and} \quad \nu \geq \frac{M\widehat{c}^2}{2M\widehat{c}-1}\,.$$

The condition of subcase 1 is subsumed by these two conditions, and hence $\overline{\lambda}^+$ is a feasible solution here. However, under these two more restrictive conditions, (32) with the negative square root, $\bar{\lambda}^-$, also provide a feasible solution, satisfying both conditions (33) and (35). Hence, in this case we have two solutions.

Subcase 4.

If the conditions of the three subcases above fail, then there is no feasible solution to (32)-(35) when parameters lie in Region B.

We now make a few useful observations based on the results above:

1. The maximum possible value of $M\widehat{c}$ is 1, which is attained when all the link capacities are equal. What distinguishes Region A from Region B is, roughly speaking, whether the difference between the largest and smallest link capacities is relatively large (the former) or relatively small (the latter). On the boundary between the two regions, that is in the limiting case $M\widehat{c} = 1/2$, a solution (and a unique one) exists if and only if $\alpha \geq 4M\nu - 3$, and the solution is $\overline{\lambda}^+$. In this case, $q > p$, and Region A solution and Region B Subcase 2 lead to the same asymptotic result.

2. In Subcase 1 of Region B, it is possible for $\max\{p, r\}$ to be p as well as r.

• If $M\widehat{c}$ is closer to 1, then p dominates (e.g., $M = 3$, $c_1 = \frac{3}{2}$, $c_2 = c_3 = 1 \Rightarrow M\widehat{c} = \frac{6}{7}$).

• If $M\widehat{c}$ is closer to $1/2$, then p is the smaller quantity (e.g., $M = 3$, $c_1 = 3$, $c_2 = c_3 = 1 \Rightarrow M\widehat{c} = \frac{3}{5}$).
• For $M = 3$, $c_1 = 2$, $c_2 = c_3 = 1$ makes them equal.
3. If all c_i's are equal, from Subcase 1, no restriction is imposed on α, and the discriminant in (32) is zero, leading to the solution $\overline{\lambda} = (\alpha/\alpha + 1)\,\bar{c}$, and hence

$$\lambda_i = \frac{\alpha}{\alpha+1}\, c_i \quad \forall\, i \in \mathcal{M}\,.$$

4. For the network game of Example 3 in Section 4,

$$\hat{c} = \frac{1}{55}\,, \quad M\hat{c} = \frac{2}{11} < \frac{1}{2}\,, \quad \nu = \frac{7}{55}\,, \quad q = \frac{57}{9}\,,$$

which puts us in Region A. Since $q > \alpha = 0.9$, the condition fails, and there is no feasible solution—consistent with the result of Example 3 that there is no symmetric $O(1/N)$ Nash equilibrium with positive flow on all links.
5. For the network game of Example 4 in Section 4,

$$\begin{aligned} \hat{c} &= \frac{1}{11}\,,\ M\hat{c} = \frac{10}{11} > \frac{1}{2}\,,\ \nu = 0.1074\,,\ r = 0.1010\,, \\ p &= 0.7141\,,\ q = 1\,, \end{aligned}$$

and since $\alpha = 0.9$, we have the strict ordering $r < p < \alpha < q$, which puts us in Region B, Subcase 3, implying that both $\bar{\lambda}^+$ and $\bar{\lambda}^-$ are feasible—consistent with what was reported in Section 4.
6. Replacing M with $m < M$ in the conditions presented above would provide a set of direct conditions applicable to networks where only m out of M links carry positive flow (the situation covered by Theorem 5). Of course in this case one also have condition (49) to check, but at least the direct conditions above would help to eliminate infeasible cases. For example, in the network game of Example 3, one can show that these conditions fail not only for $m = M = 10$, as shown in item 4 above, but also for all $m \geq 7$. For example, for $m = 7$, $m\hat{c}_m = (4/7) > (1/2)$, which puts us in Region B. However, the ordering

$$p = 0.7575 < \alpha = 0.9 < q = 1.083 < r = 6\,,$$

tells us that there is no feasible solution. For $m = 6$, on the other hand, $m\hat{c}_m = (2/3) > 1/2)$, which puts us again in Region B. The ordering in this case is:

$$p = 0.5707 < r = 0.6667 < q = 0.7333 < \alpha = 0.9\,,$$

which is Subcase 1, telling us that $\bar{\lambda}^{(}m)^+$ is feasible for $m = 6$. This is of course all consistent with what was reported in Section 4 for Example 3.

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