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Avoiding paradoxes in multi-agent competitive routing ☆

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Abstract

Strange behavior may occur in networks due to the non-cooperative nature of decision making, when the latter are taken by individual agents. In particular, the well known Braess paradox illustrates that when upgrading a network by adding a link, the resulting equilibrium may exhibit larger delays for all users. We present here some guidelines to avoid the Braess paradox when upgrading a network. We furthermore present conditions for the delays to be monotone increasing in the total demand.

1. Introduction

Service providers or network administrators may often be faced with decisions related to upgrading of the network. For example, where should one add capacity? or where should one add new links? Decisions related to the network capacity and topology have direct influence on the equilibrium that would be attained.

A frequently used heuristic approach for upgrading a network is through *bottleneck analysis*. A system bottleneck is defined as "a resource or service facility whose capacity seriously limits the performance of the entire system" [11, p. 13]. Bottleneck analysis consists of adding capacity to identified bottlenecks until they cease to be bottlenecks. In a non-cooperative framework, however, this approach may have devastating effects; it may cause delays of all users to increase; in an economic context in which users pay the service provider, this may further cause a decrease in the revenues of the provider. The first problem has already been identified in road-traffic context by Braess [4] (see also [8,19]), and has further been studied in networking context in [1,3,5,7,6,12,14]. The focus of Braess paradox on the bottleneck link in a queuing context, as well as the paradoxical impact on the service provider have been studied in [16]. The Braess paradox has further been identified and studied in the context of distributed computing [9,10] where arriving of jobs may be routed and performed on different processors.

Braess paradox illustrates that the network designer or service providers have to take into consideration the reaction of non-cooperative

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users to their decisions. This is in particular important when upgrading the network. Some upgrading guidelines have been proposed in [12–14] so as to avoid the Braess paradox or so as to obtain a better performance. Our first objective is to pursue that direction and to provide new guidelines for avoiding the Braess paradox when upgrading the network. Another related issue is that of monotonicity of the performance measures in the demand. Our second objective is to check under what conditions are delays as well as the marginal costs at equilibrium increasing in the demands.

The paper is organized as follows: In Section 2 we present the model, formulate the problem, and mention some related works. In Section 3 we present a sufficient condition for the monotonicity of performance measures when the demands increase. In Section 4, we present an example of Braess paradox. We propose some methods for adding capacities in a network in Section 5. We illustrate these methods numerically in Section 6.

2. Problem formulation

We consider a network $(\mathcal{N},\mathcal{L})$ where $\mathcal{N}$ is a finite set of nodes and $\mathcal{L}$ is a set of directed links. For simplicity of notation and without loss of generality, we assume that at most one link exists between each pair of nodes (in each direction). For a node $v\in\mathcal{N}$, the sets $\mathrm{In}(v)=\{l\in\mathcal{L}|\exists w\in\mathcal{N}, l=(w,v)\}$ and $\mathrm{Out}(v)=\{l\in\mathcal{L}|\exists w\in\mathcal{N}, l=(v,w)\}$ denote respectively the set of its in-coming links, and the set of its out-going links.

We are given a set $\mathcal{I}=\{1,2,\dots,I\}$ of users that share the network. We assume that all users ship flow from a common source node s to a common destination node d. Each user i has a throughput demand r^i. Denote $r=\sum_{i\in\mathcal{I}} r^i$ the total throughput demand of users.

User i splits its demand r^i among the paths connecting the source to the destination. Let x_l^i denote the flow that user i sends on link l. The user i flow configuration $X^i=(x_l^i)_{l\in\mathcal{L}}$ is called a routing strategy of user i. A user flow configuration is said to be admissible, if it satisfies its demand, and if it preserves its flow at each node. We denote $\mathcal{S}^i$ the set of admissible flows, (or admissible strategies) of user i, i.e. the set defined by $\mathcal{S}^i=\{X^i\in\mathbb{R}^{|\mathcal{L}|}\colon \sum_{l\in\mathrm{Out}(v)} x_l^i=\sum_{l\in\mathrm{In}(v)} x_l^i+r_v^i, v\in\mathcal{V}\}$, where $r_s^i=r^i$, $r_d^i=-r^i$ and $r_v^i=0$ for $v\neq s,d$. A flow configuration profile $\mathbf{X}=(X^1,\dots,X^I)\in\mathcal{S}=\otimes_{i\in\mathcal{I}}\mathcal{S}^i$ is called a routing strategy profile.

The objective of each user i is to find an admissible routing strategy $X^i\in\mathcal{S}^i$ so as to minimize some performance objective, or cost function, J^i, that depends upon X^i but also upon the routing strategies $X^j\in\mathcal{S}^j$ of any other user $j\neq i$, i.e. $J^i:\mathcal{S}\to\mathbb{R}$. Hence $J^i(\mathbf{X})$ is the cost of user i under routing strategy profile $\mathbf{X}$. For example, the user may want to minimize the average delay for its flow to reach the destination from the source. This delay depends upon its routing strategy, but also upon the load of the links used by the user, i.e. upon the routing strategies of the other users.

We will use the following sets of assumptions.

Assumption 2.1
A1. $J^i(\mathbf{X})=\sum_{l\in\mathcal{L}} x_l^i T_l(x_l)$, where $x_l=\sum_{i\in\mathcal{I}} x_l^i$.
A2. $T_l:[0,+\infty[\to[0,+\infty]$.
A3. $T_l(\cdot)$ is positive, strictly increasing and convex.
A4. $T_l(\cdot)$ is continuously differentiable.

Functions that comply with these assumptions are referred to as *type*-**A** functions.

Remark 2.1. In Assumption **A1**, $T_l(x_l)$ is the cost per unit of flow (for example mean delay) on the link l, for the total utilization, $x_l=\sum_{i\in\mathcal{I}} x_l^i$, of that link. Note that if $T_l(x_l)$ is the average delay on link l, it depends only on the total flow on that link. The average delay should be interpreted as a general congestion cost per unit of flow, which encapsulates the dependence of the quality of service provided by a finite capacity resource on the total load x_l offered to it.

Let c_l denote the capacity of link l. The vector $\mathbf{c}=(c_l)_{l\in\mathcal{L}}$ is called the capacity configuration of the network. Although the cost T_l depends also upon the physical capacity c_l of link l, we omit to write this dependency, nevertheless we have the following set of assumptions.

Assumption 2.2
B1. J^i is a *type*-**A** cost function.
B2. T_l and T_l' are strictly decreasing with respect to capacity c_l of link l where $T_l' = \partial T_l / \partial x_l$.

Functions that comply with these assumptions shall be referred to as *type*-**B** functions.

Cost function used in real networks are often related to some performance measure such as expected delay. Frequently, linear link costs, i.e. $T_l(x_l) = a_l x_l + b_l$ are used (see e.g. [15]). Another possibility often used in the literature to represent delay is to assume.

Assumption 2.3
C1. J^i is a *type*-**B** cost function.
C2. $T_l(x_l) = \begin{cases} 1/(c_l - x_l), & x_l < c_l, \\ \infty, & x_l \geqslant c_l, \end{cases}$

where c_l is the capacity of link l.

This represents the expected delay of a M/M/1 queue operating under the FIFO regime (packets are served at arrival order, see [15] or the delay of a M/G/1 queue operating under the processor sharing regime). c_l has the interpretation of the queuing capacity. Other possibilities for the costs related to rejection probabilities can be found in [1]. Functions that comply with these assumptions shall be referred to as *type*-**C** functions.

We note that the above different sets of assumptions on the users' costs have already been introduced in the context of the analysis of uniqueness of equilibria in [17].

Each user of the network strives to find its best routing strategy so as to minimize it own objective function. Nevertheless its objective function depends upon its own choice but also upon the choices of the other users. In this situation, the solution concept widely accepted is the concept of Nash equilibrium.

Definition 2.1. A Nash equilibrium of the routing game is a routing strategy profile $\mathbf{X} = (X^1, X^2, \ldots, X^I) \in \mathscr{S}$ from which no user have any incentive to deviate. More precisely the strategy profile, $\mathbf{X}$ is a Nash equilibrium, if the following holds true for any $i \in 1, 2, \ldots, I$,

$$X^i \in arg \min_{Y^i \in \mathscr{S}^i} J^i(X^1, \ldots, X^{i-1}, Y^i, X^{i+1}, \ldots, X^I).$$

X^i is the best user i can do if the other users choose the routing strategies $\mathbf{X}^{-i} = (X^1, \ldots, X^{i-1}, X^{i+1}, \ldots, X^I)$.

Remark 2.2. The set of Assumptions **A** guarantees the existence of a Nash equilibrium, see [17]. [1] Note that the proof of existence in [17] is based on [18] that restricted to finite costs. We conclude that if the costs are finite for any strategy, an equilibrium indeed exists.

For the routing strategy profile $\mathbf{X}^{-i} = (X^1, \ldots, X^{i-1}, X^{i+1}, \ldots, X^I)$ of all the users except user i fixed, the best user i can do is to choose the unique solution to the (single-user) optimal routing problem for a network, that is the solution of a constrained minimization problem. The uniqueness follows from the facts that any cost function of *type*-**A**, **B** or **C** is convex with respect to the variable X^i, and the set $\mathscr{S}^i$ is bounded for all $i \in \mathscr{I}$. Note that the uniqueness of best response strategies does not imply the uniqueness of the Nash equilibrium.

The Kuhn–Tucker conditions (for this single user optimization problem) imply that X^i is the optimal response of user i to $\mathbf{X}^{-i}$ if and only if there exist Lagrange multipliers $(\lambda_u^i)_{u \in \mathscr{V}} \geqslant 0$ (that may depend upon X^i and $\mathbf{X}^{-i}$), such that [14,17]

$$\begin{aligned} &\lambda_u^i = x_{uv}^i T_{uv}' + T_{uv} + \lambda_v^i, \quad \text{if } x_{uv}^i > 0, \quad (u,v) \in \mathscr{L}, \\ &\lambda_u^i \leqslant x_{uv}^i T_{uv}' + T_{uv} + \lambda_v^i, \quad \text{if } x_{uv}^i = 0, \quad (u,v) \in \mathscr{L}, \\ &\lambda_d^i = 0. \end{aligned} \tag{1}$$

Therefore, a strategy profile $\mathbf{X} \in \mathscr{S}$ is a Nash equilibrium if and only if there exist λ_u^i, such that the conditions (1) are satisfied for all $i \in \mathscr{I}$.

[1] The following assumption is also made in that reference: for every flow configuration $\mathbf{X}$, if not all cost are finite then at least one user with infinite cost ($J^i(\mathbf{X}) = \infty$) can change its own flow configuration to make its cost finite.

In a constrained optimization problem, the Lagrange multipliers are often interpreted as the impact on the cost function, of the level of constraint at the optimality point. These multipliers are referred to as *marginal costs*. Hence in Eqs. (1) λ_s^i will be interpreted as the marginal cost at the source node for user i at the optimality point. This quantity have been advocated in [14] as another important performance measure for the network. As a matter of fact it accounts for the level of congestion, as seen by users, and are the direct indication of how each user could accommodate fluctuations in the system's state. We will refer to λ_s^i as the *price* for user i.

We consider the problem of the service provider or the network designer that have to distribute some additional capacity among the links of the network. We assume the service provider or network designer is interested in reducing the sum of the costs or the sum of the prices of all users at Nash equilibrium. Hence the problem of the service provider or network designer is to distribute the additional capacity so as to improve the performance (total cost or price) at Nash equilibrium. In particular the service provider does not want any Braess paradox to occur.

Definition 2.2. The evaluation function of the service provider is either the total cost at equilibrium, that is,

$$J = J(\mathbf{X}) = \sum_{i\in\mathscr{I}} J^i(\mathbf{X}),$$

or the total price at equilibrium, that is

$$\lambda = \sum_{i\in\mathscr{I}} \lambda^i,$$

where λ^i are the Lagrange multipliers at equilibrium.

In order to compare Nash equilibria corresponding to different parameters, it may seem desirable to make assumptions on the topology and costs such that under any throughput demand of users or any additional capacity, the equilibrium is unique. Indeed, some results on avoiding the Braess paradox (when adding capacity) have already been obtained in [14] under conditions that imply uniqueness of the equilibria. We have chosen not to make any such assumption since our results are stronger and allow us to compare the performance of any equilibrium in a system, with any other which is obtained by increasing the capacity or the demand appropriately.

In [14] the results were obtained in the restrictive cases where the users are identical (all the users have the same demand), and the users are said simple (i.e. all the flow of a simple user is routed through paths of minimum delay) with link costs corresponding to M/M/1 type queues. We do not have such assumptions. The proposed methods in [14] for adding some capacity in the network were

1. Multiplying the capacity of each link by some constant factor $\alpha > 1$.
2. Adding a link between the source and the destination.

In our work we generalize the first method by multiplying the capacity of each link by a link-dependent factor α_l (Proposition 5.1). We show, in our paper that the second method of upgrading leads to an improvement not only of the price but also of the cost. Only the price was considered in [14].

In the sequel, we shall first study the monotonicity of the total price and the total cost at equilibrium with respect to the demands $(r^i)_{i\in\mathscr{I}}$. This will allow us then to investigate methods of adding links (which can be viewed as adding capacity to a link with zero capacity) to the network so that the total performance is improved.

3. Impact of throughput variation on the equilibrium

In this section, we study the monotonicity with an increase of the total demand, at equilibrium, of the performance measure given by the total price $\lambda_s = \sum_{i\in\mathscr{I}} \lambda_s^i$ and the total cost $J = \sum_{i\in\mathscr{I}} J^i$. Under some assumption, the following establishes that an increase of the total demand of users, results in an increase of the total price.

We suppose that the cost function $J^i(\cdot)$ are some *type*-**A** functions. For a fixed capacity $(c_l)_{l\in\mathscr{L}}$, we consider two throughput demands $(\tilde{r}^i)_{i\in\mathscr{I}}$ and $(\hat{r}^i)_{i\in\mathscr{I}}$ such that $\hat{r} = \sum_{i\in\mathscr{I}} \hat{r}^i < \tilde{r} = \sum_{i\in\mathscr{I}} \tilde{r}^i$. Let $\tilde{\lambda}^i$ and $\hat{\lambda}^i$ (respectively, $\widetilde{J}^i$ and $\widehat{J}^i$) be the prices (respectively, the cost) of user i at the respective Nash equilibria $\widetilde{\mathbf{X}}$ and $\widehat{\mathbf{X}}$. The following holds true:

Lemma 3.1. *There exists some path p^* between the source and destination such that $|\tilde{x}_l - \hat{x}_l| > 0$ for all the links in that path.*

Proof. We construct a directed network $(\mathscr{N}', \mathscr{L}')$, where the set of nodes is the same than in the original network, $\mathscr{N}' = \mathscr{N}$, and the set of links $\mathscr{L}'$ is constructed as follows:

- For each link $l = (u, v) \in \mathscr{L}$, such that $\tilde{x}_l \geqslant \hat{x}_l$, we have a link $l' = (u, v) \in \mathscr{L}'$; to such a link l' we assign a (flow) value $z_{l'} = \tilde{x}_l - \hat{x}_l$.
- For each link $l = (u, v)$, such that $\tilde{x}_l < \hat{x}_l$, we have a link $l' = (v, u) \in \mathscr{L}'$; to such a link we assign a (flow) value $z_{l'} = \hat{x}_l - \tilde{x}_l$.

It is easy to verify that the value $z_{l'}$ constitutes a nonnegative, directed flow in the network $(\mathscr{N}', \mathscr{L}')$. Since $\hat{r} < \tilde{r}$, the network must carry some flow (the amount of $\tilde{r} - \hat{r}$) from the source s to the destination d. This implies that there exists a path p^* from s to d, such that $z_{l'} > 0$ for all $l' \in p^*$. □

Assumption 3.1. For all $l \in p^*$ (defined by Lemma 3.1) for which $\tilde{x}_l > 0$ (resp. $\hat{x}_l > 0$) all users send positive flows in the equilibrium $\widetilde{\mathbf{X}}$ (resp. $\widehat{\mathbf{X}}$), i.e. $\tilde{x}_l^i > 0$ (resp. $\hat{x}_l^i > 0$) for all $i \in \mathscr{I}$.

This assumption is inspired by the (much stronger) assumption in [17] for uniqueness of Nash equilibrium. It states that if at equilibrium a flow on a link is positive then all users have positive flow on that link. The next proposition establishes the monotonicity of the total price in the demand.

Proposition 3.1. *Suppose the cost functions of users are type-***A** *function. Suppose we have two throughput demands $(\tilde{r}^i)_{i\in\mathscr{I}}$ and $(\hat{r}^i)_{i\in\mathscr{I}}$ such that $\hat{r} = \sum_{i\in\mathscr{I}} \hat{r}^i < \tilde{r} = \sum_{i\in\mathscr{I}} \tilde{r}^i$. Denote $\widetilde{\mathbf{X}}$ and $\widehat{\mathbf{X}}$ the two respective Nash equilibria. Suppose that Assumption* 3.1 *holds. Then the total price at $\widetilde{\mathbf{X}}$ is larger than the total price at $\widehat{\mathbf{X}}$. That is $\hat{\lambda}_s < \tilde{\lambda}_s$.*

Proof. Consider now a link $l' = (u, v) \in p^*$ (defined in Lemma 3.1). Since $z_{l'} > 0$ (defined in the proof of Lemma 3.1), either $\tilde{x}_{uv} > \hat{x}_{uv} \geqslant 0$ or $\hat{x}_{vu} > \tilde{x}_{vu}$.

- In the case where $\tilde{x}_{uv} > \hat{x}_{uv}$, under Assumption 3.1, $\tilde{x}^i_{uv} > 0$, $\forall i$. Hence, from the Kuhn–Tucker conditions (1), we have, for all $i \in \mathscr{I}$,

$$\tilde{\lambda}^i_u - \tilde{\lambda}^i_v = \tilde{x}^i_{uv}\widetilde{T}'_{uv} + \widetilde{T}_{uv}, \quad \text{and}$$
$$\hat{\lambda}^i_u - \hat{\lambda}^i_v \leqslant \hat{x}^i_{uv}\widehat{T}'_{uv} + \widehat{T}_{uv},$$

where $\widetilde{T}$, $\widehat{T}$, $\widetilde{T}'$, and $\widehat{T}'$ stands for T_{uv} and T'_{uv} computed at $\tilde{x}_{uv}$ and $\hat{x}_{uv}$.
Summing over $i \in \mathscr{I}$, we obtain: $\tilde{\lambda}_u - \tilde{\lambda}_v = \tilde{x}_{uv}\widetilde{T}'_{uv} + I\widetilde{T}_{uv}$, and $\hat{\lambda}_u - \hat{\lambda}_v \leqslant \hat{x}_{uv}\widehat{T}'_{uv} + I\widehat{T}_{uv}$, where $\tilde{\lambda}_w = \sum_{i\in\mathscr{I}} \tilde{\lambda}^i_w$ and $\hat{\lambda}_w = \sum_{i\in\mathscr{I}} \hat{\lambda}^i_w$ for all $w \in \mathscr{N}$. Since $\tilde{x}_{uv} > \hat{x}_{uv}$ and $\widehat{T}_{uv} = T_{uv}(\hat{x}_{uv})$, then $\widetilde{T}_{uv} > \widehat{T}_{uv}$ and $\widetilde{T}'_{uv} > \widehat{T}'_{uv}$ (Assumption 2.1-**A3**), we deduce that $\tilde{\lambda}_u - \tilde{\lambda}_v = \tilde{x}_{uv}\widetilde{T}'_{uv} + I\widetilde{T}_{uv} > \hat{x}_{uv}\widehat{T}'_{uv} + I\widehat{T}_{uv} \geqslant \hat{\lambda}_u - \hat{\lambda}_v$. Thus,

$$\tilde{\lambda}_u - \hat{\lambda}_u > \tilde{\lambda}_v - \hat{\lambda}_v. \tag{2}$$

- In the case $\hat{x}_{vu} > \tilde{x}_{vu}$, we have by symmetry that $\hat{\lambda}_v - \hat{\lambda}_u > \tilde{\lambda}_v - \tilde{\lambda}_u$, thus we obtain (2).

Define more precisely the path p^*, by $p^* = (s, u_1, u_2, \ldots, u_{n^*}, d)$, where u_k, $k = 1, 2, \ldots, n^*$, is the kth node after the source s on the path p^* and n^* is the number of nodes between the source s and the destination d. Hence, from (2) we have $\tilde{\lambda}_s - \hat{\lambda}_s > \tilde{\lambda}_{u_1} - \hat{\lambda}_{u_1} > \cdots > \tilde{\lambda}_{u_{n^*}} - \hat{\lambda}_{u_{n^*}} > \tilde{\lambda}_d - \hat{\lambda}_d = 0$ $(\tilde{\lambda}_d = \hat{\lambda}_d = 0)$, and we conclude that $\tilde{\lambda}_s > \hat{\lambda}_s$. □

In the sequel, we will give sufficient conditions for obtaining the monotonicity of the total cost, when the total throughput demand of the users increases, in the case where the cost functions of the users are of *type*-**C**. We first need the following definition.

Definition 3.1. Users are said to be consistent (for a given capacity configuration) if, at the Nash equilibrium, they all use same set of links.

In the following proposition we obtain relations between costs under two different demands assuming only that users are consistent at Nash equilibrium $\widehat{\mathbf{X}}$. This assumption weakens the much stronger "all-positive flows" assumption of [12].

Proposition 3.2. *Suppose the cost functions of the users are of type* **C**. *Suppose we have two throughput demands* $(\tilde{r}^i)_{i\in\mathscr{I}}$ *and* $(\hat{r}^i)_{i\in\mathscr{I}}$ *such that* $\tilde{r}/\hat{r} \geqslant I$, *with* $\hat{r} = \sum_{i\in\mathscr{I}} \hat{r}^i < \tilde{r}$ *and* $\tilde{r}^i = \sum_{i\in\mathscr{I}} \tilde{r}^i$. *Denote* $\widetilde{\mathbf{X}}$ *and* $\widehat{\mathbf{X}}$ *the two respective Nash equilibria. Suppose that users are consistent at the Nash equilibrium* $\widehat{\mathbf{X}}$.

Then the total cost at $\widetilde{\mathbf{X}}$ *is larger than the total cost at* $\widehat{\mathbf{X}}$. *That is* $\widehat{J} = J(\widehat{\mathbf{X}}) < J(\widetilde{\mathbf{X}}) = \widetilde{J}$.

Proof. Let $\widehat{\mathscr{L}}_1$ be a subset of $\mathscr{L}$ defined by $\widehat{\mathscr{L}}_1 = \{l \in \mathscr{L} \mid \hat{x}_l > 0\}$. Consider a link l in $\widehat{\mathscr{L}}_1$. Since the users are consistent at Nash equilibrium $\widehat{\mathbf{X}}$, it follows that $\hat{x}^i_l > 0$, $\forall i \in \mathscr{I}$. This implies, by Kuhn–Tucker conditions (1), that $\forall i \in \mathscr{I}$, $\hat{\lambda}^i_u = \hat{x}^i_l \widehat{T}'_{uv} + \widehat{T}_{uv} + \hat{\lambda}^i_v$, $\forall l = (u, v) \in \mathscr{L}_1$, and we always have $\forall i \in \mathscr{I}$, $\hat{\lambda}^i_u \leqslant \hat{x}^i_l \widehat{T}'_{uv} + \widehat{T}_{uv} + \hat{\lambda}^i_v$. By summing up over $i \in \mathscr{I}$, we obtain $\hat{\lambda}_u = \hat{x}_l \widehat{T}'_{uv} + I\widehat{T}_{uv} + \hat{\lambda}_v$ if $\hat{x}_l > 0$, and $\hat{\lambda}_u \leqslant \hat{x}_l \widehat{T}'_{uv} + I\widehat{T}_{uv} + \hat{\lambda}_v$ if $\hat{x}_l = 0$. Thus,

$$\hat{\lambda}_u = \frac{\hat{x}_l}{(c_l - \hat{x}_l)^2} + \frac{I}{c_l - \hat{x}_l} + \hat{\lambda}_v \quad \text{if } \hat{x}_l > 0,$$
$$\hat{\lambda}_u \leqslant \frac{\hat{x}_l}{(c_l - \hat{x}_l)^2} + \frac{I}{c_l - \hat{x}_l} + \hat{\lambda}_v, \quad \text{if } \hat{x}_l = 0. \tag{3}$$

Define the function V by

$$V((y_l)_{l\in\mathscr{L}}) = \sum_{l\in\mathscr{L}} \frac{y_l}{c_l - y_l} - (I-1)\sum_{l\in\mathscr{L}} \ln(c_l - y_l), \tag{4}$$

where $(y_l)_{l\in\mathscr{L}} \in \widehat{S} := \{(t_l)_{l\in\mathscr{L}} \in \mathbb{R}^{|\mathscr{L}|}\colon \sum_{l\in\mathrm{Out}(v)} t_l = \sum_{l\in\mathrm{In}(v)} t_l + \hat{r}_v, v \in \mathscr{V}\}$, where $\hat{r}_s = \hat{r}$, $\hat{r}_d = -\hat{r}$ and $\hat{r}_v = 0$ for $v \neq s, d$.

Denote $(\hat{x}_l)_{l\in\mathscr{L}}$ the vector of total link flows at the Nash equilibrium $\widehat{\mathbf{X}}$. The condition (3) can be interpreted as Kuhn–Tucker condition for a single-user minimization of the function V, under the constraints $(y)_{l\in\mathscr{L}} \in \widehat{S}$. This shows that the vector $(\hat{x}_l)_{l\in\mathscr{L}}$ is the unique minimum of the function V.

Let $(\bar{x}_l)_{l\in\mathscr{L}} \in \mathbb{R}^{|\mathscr{L}|}$ defined by $\bar{x}_l = (\hat{r}/\tilde{r})\tilde{x}_l$, hence $(\bar{x}_l)_{l\in\mathscr{L}} \in \widehat{S}$, and since $(\hat{x}_l)_{l\in\mathscr{L}}$ minimizes the function V, we have

$$\sum_{l\in\mathscr{L}} \frac{\hat{x}_l}{c_l - \hat{x}_l} - (I-1)\sum_{l\in\mathscr{L}} \ln(c_l - \hat{x}_l) \leqslant \sum_{l\in\mathscr{L}} \frac{\bar{x}_l}{c_l - \bar{x}_l} - (I-1)\sum_{l\in\mathscr{L}} \ln(c_l - \bar{x}_l). \tag{5}$$

It follows that

$$\sum_{l\in\mathscr{L}} \frac{\hat{x}_l}{c_l - \hat{x}_l} \leqslant \sum_{l\in\mathscr{L}} \frac{(\hat{r}/\tilde{r})\tilde{x}_l}{c_l - (\hat{r}/\tilde{r})\tilde{x}_l} - (I-1)\sum_{l\in\mathscr{L}} \ln\left(\frac{c_l - (\hat{r}/\tilde{r})\tilde{x}_l}{c_l - \hat{x}_l}\right) < \sum_{l\in\mathscr{L}} \frac{(\hat{r}/\tilde{r})\tilde{x}_l}{c_l - (\hat{r}/\tilde{r})\tilde{x}_l} - (I-1)\sum_{l\in\mathscr{L}} \ln\left(1 - \frac{(\hat{r}/\tilde{r})\tilde{x}_l}{c_l}\right).$$

Hence, in order to prove that $\widehat{J} = J(\widehat{\mathbf{X}}) = \sum_{l\in\mathscr{L}} (\hat{x}_l/(c_l - \hat{x}_l)) < \widetilde{J} = J(\widetilde{\mathbf{X}}) = \sum_{l\in\mathscr{L}} (\tilde{x}_l/(c_l - \tilde{x}_l))$, it is enough to show that

$$\frac{\tilde{x}_l}{c_l - \tilde{x}_l} - \frac{(\hat{r}/\tilde{r})\tilde{x}_l}{c_l - (\hat{r}/\tilde{r})\tilde{x}_l} + (I-1)\ln\left(1 - \frac{(\hat{r}/\tilde{r})\tilde{x}_l}{c_l}\right) > 0,$$

which holds if $\tilde{r}/\hat{r} \geqslant I$ (see Appendix A with $\alpha = \tilde{r}/\hat{r}$). □

If we assume that the "all-positive flows" assumption of [17] holds true, or equivalently that the users are consistent at Nash equilibria $\widetilde{\mathbf{X}}$ and $\widehat{\mathbf{X}}$, then each link of the network satisfies the assumptions in Propositions 3.1 and 3.2, and we have the result.

Corollary 3.1. *Assume that users are consistent at Nash equilibria* $\widetilde{\mathbf{X}}$ *and* $\widehat{\mathbf{X}}$. *Then*

1. *For the cost functions of type*-**A**, *if* $\hat{r} < \tilde{r}$ *then* $\hat{\lambda}_s < \tilde{\lambda}_s$.
2. *For the cost functions of type*-**C**, *if* $\tilde{r}/\hat{r} \geqslant I$ *then* $\widehat{J} < \widetilde{J}$.

4. Braess paradox

In this section, we present an example of Braess paradox. It demonstrates that addition of capacity may, in general, increase both the total price and total cost. To obtaining we present a simple computation approach that allows us to compute the equilibrium.

Consider the network depicted in Fig. 1. In this example we choose the capacity c_2 large enough so that the delays on link (2,4) or on link (1,3) have low sensitivities to flow changes. Such constructions are necessary so as to represent the infinite-service queue (the average delay is independent of the flow) in order to produce an example of a queuing network setting [6]. In this example adding capacity may lead to an increase in the mean transit time.

We set the capacity of the links (1,2) and (3,4) to $c_1 = 2.7$. Link (1,3) and link (2,4) are made of n tandem links, each with capacity $c_2 = 27$. Similarly the link (2,3) is a path of n consecutive links having each capacity $c_3 = \Delta$, while Δ varies from 0 (absence of the link) to infinity. We suppose that the network is used by I users, each sending a flow r^i from node 1 to node 4.

We study here the scenario where the capacity Δ is added to path (2,3). We take $n = 54$, $I = 2$, $r^1 = 0.8$ and $r^2 = 1.2$. Denote p_l the left path using link (1,2) and link (2,4), p_r the right path using link (1,3) and link (2,4), and p_z the zigzag path using link (1,2), links (2,3) and (2,4).

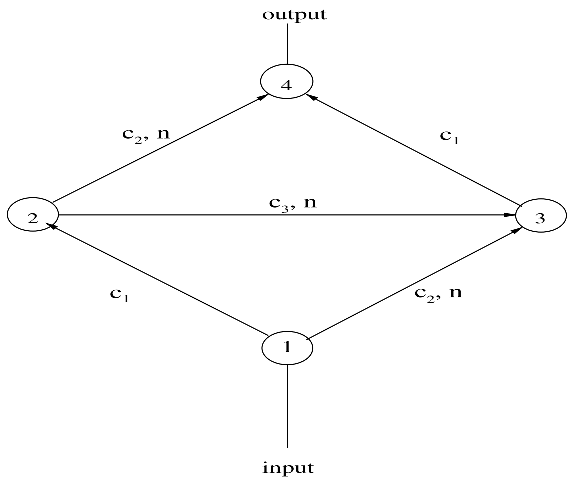

Fig. 1. Network example.

The total marginal cost for a path is defined by $D_p = \sum_{i\in\mathscr{I}} \partial J^i/\partial x_p^i$, where x_p^i is the flow of user i on path p. Note that the function D_p is exactly the total price λ at Nash equilibrium if path p is used. We have computed the equilibrium iteratively with the following relaxation method. This method has been proved to converge to the Nash equilibrium for some topologies, see [2].

1. Define an initial candidate $\{x(0)\}_{l\in\mathscr{L}}$ for the equilibrium *total link flows* profile. $\{x(0)\}_{l\in\mathscr{L}}$ is obtained by minimizing the function V defined in (4). The flow of each user i in the initial iteration is then defined as $x_l^i(0) = x_l(0)r^i[\sum_{j\in\mathscr{I}} r^j]^{-1}$.
2. At iteration $n > 0$, we first compute the best responses $\{x_l^i(n)\}$ for each user i when all the other users use $\{x_l^j(n-1)\}_{j\neq i,l}$.
3. The approximation of the equilibrium at step n is then given by $x_l^i(n) = \alpha x_l^i(n-1) + (1-\alpha)x_l^i(n)$, for all $i \in \mathscr{I}$ and $l \in \mathscr{L}$.
4. The procedure ends when $x(n)$ is sufficiently close to $x(n-1)$.

Remark 4.1. Note that if the "all-positive flows" condition (defined before Corollary 3.1) holds at equilibrium, then $\{x_l(0)\}_l$ will already be the total link flow at equilibrium, and only the individual link flows have to be defined. Note also that if the users were identical, then by [17], there would be a unique equilibrium and it would be symmetric. Hence the condition of "all-positive flows" would indeed hold, and $\{x_l^i(0)\}_{i,l}$ would already correspond to the equilibrium. No further iteration would be needed.

In our experimentation below (and in the experimentation of Section 6), it turned out that the condition of "all-positive flows" was satisfied. Consequently, we could check that our algorithm provided the correct value for the total link flows at equilibrium. The number of iterations that were required was around 20 (which leads to a difference between $x(n)$ and $x(n+1)$ of less than 10^{-5}). We used $\alpha = 1/2$, as relaxation coefficient.

In Figs. 2 and 3, we observe that there is no traffic on the zigzag path for $0 \leqslant \Delta < 39.9$. For $39.9 \leqslant \Delta$, the three paths, p_l, p_r and p_z are used. Fig. 2, shows that, for $39.9 \leqslant \Delta \leqslant 58.1$, the total cost is, paradoxically, worse than it would be without the zigzag path, i.e., eliminating the zigzag path would lead to an improvement of performance. The total cost diminishes to 4.6966 as Δ goes to infinity. Fig. 3 shows that the total price increases when the additional capacity is more than 39.9. More surprisingly, it can be verified that this paradoxical behavior persists even if $\Delta = \infty$ (this is possible, if node 2 and node 3 are merged into a single node). The total price increases to 8.111 when Δ goes to infinity.

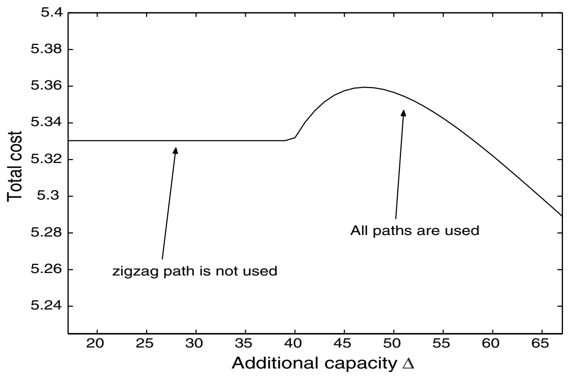

Fig. 2. Total cost as a function of the added capacity in path (2,3).

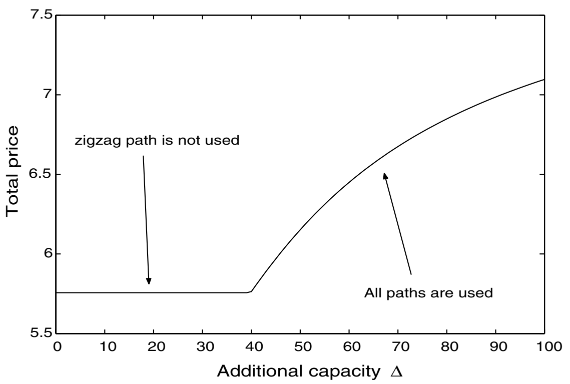

Fig. 3. Total price as a function of the added capacity in path (2,3).

When the additional capacity Δ is greater than 39.9, first traffic in the zigzag path does benefit a smaller marginal cost, while the marginal costs in paths p_l and p_r increase because of congestion in the shared links (1,2) and (3,4).

As traffic approaches equilibrium, the marginal cost in the zigzag path increases and becomes larger that the marginal cost when $\Delta = 0$. The marginal costs in the right and left paths increase even more. Thus, users continue to choose the zigzag path to the extent that, when equilibrium is reached, marginal cost (total prices) have increased along the old paths. However total cost (respectively, average delay) increases since the delay of the three paths have increased along the old paths.

This example have shown that in this network one have to be careful when adding some capacity, since this may result in an increase of both the price and the cost of every single user. This indicates that the total price and total cost may increase when upgrading (in term of capacity or addition of links) a general network. This counter-intuitive result is referred to as Braess paradox.

5. Impact of extra capacity on the equilibrium

The example presented in Section 4, demonstrates that adding capacity to a network may result in an increase of both the price and the cost of every single user. In this section, we propose some methods for adding resources to general network that guarantee an improvement in performance so that the Braess paradox does not occur. We study several ways to upgrade a general network:

1. Multiplying the capacity of some specific links ($l \in \mathscr{L}$) by a constant factor $\alpha_l > 1$.
2. Adding some capacity to an existing direct link from the source to the destination or adding a new direct link from the source to the destination.

5.1. Multiplying the capacity of each link

We first consider an upgrade achieved by multiplying the capacity of each link $l \in \mathscr{L}$ by a factor

$\alpha_l \geqslant 1$. For a fixed throughput demands $(r^i)_{i\in\mathscr{I}}$, we consider two capacity configurations $\tilde{\mathbf{c}}$ and $\hat{\mathbf{c}}$ such that $\hat{c}_l = \alpha_l \tilde{c}_l$ where $\alpha_l \geqslant 1$. Let $\widetilde{\mathbf{X}}$ and $\widehat{\mathbf{X}}$ be the Nash equilibria under capacity configurations $\tilde{\mathbf{c}}$ and $\hat{\mathbf{c}}$, respectively. Consider the set $\widetilde{\mathscr{L}}_1$ defined by $\widetilde{\mathscr{L}}_1 = \{l \in \mathscr{L} \mid \tilde{x}_l > 0\}$, i.e., $\widetilde{\mathscr{L}}_1$ is the set of links such that at Nash equilibrium $\widetilde{\mathbf{X}}$, at least on user sends some flow through those links.

We have the following proposition,

Proposition 5.1. *Suppose that the cost functions are of type-***C***; assume that users are consistent at Nash equilibrium* $\widehat{\mathbf{X}}$. *If* $\alpha_l \geqslant I$ *for all* $l \in \widetilde{\mathscr{L}}_1$ *then the configuration* $\hat{\mathbf{c}}$ *is total cost efficient relative to configuration* $\tilde{\mathbf{c}}$, *i.e.* $J(\widehat{\mathbf{X}}) \leqslant J(\widetilde{\mathbf{X}})$, *where* $\widehat{\mathbf{X}}$ *and* $\widetilde{\mathbf{X}}$ *are the respective Nash equilibria for capacity configuration* $\tilde{\mathbf{c}}$ *and* $\hat{\mathbf{c}}$.

Proof. We start the proof as the proof of Proposition 3.2, we obtain the following equation (instead of Eq. (3) in the proof of Proposition 3.2):

$$\hat{\lambda}_u = \frac{\hat{x}_l}{(\alpha_l\tilde{c}_l - \hat{x}_l)^2} + \frac{I}{\alpha_l\tilde{c}_l - \hat{x}_l} + \hat{\lambda}_v \quad \text{if } \hat{x}_l > 0,$$
$$\hat{\lambda}_u \leqslant \frac{\hat{x}_l}{(\alpha_l\tilde{c}_l - \hat{x}_l)^2} + \frac{I}{\alpha_l\tilde{c}_l - \hat{x}_l} + \hat{\lambda}_v \quad \text{if } \hat{x}_l = 0. \tag{6}$$

We define the function $\widehat{V}$ by

$$\widehat{V}((y_l)_{l\in\mathscr{L}}) = \sum_{l\in\mathscr{L}} \frac{y_l}{\alpha_l\tilde{c}_l - y_l} - (I-1)\sum_{l\in\mathscr{L}} \ln(\alpha_l\tilde{c}_l - y_l),$$

where $(y_l)_{l\in\mathscr{L}} \in \widehat{S}$ with

$$\widehat{S}_\alpha = \left\{ (t_l)_{l\in\mathscr{L}} \in \mathbb{R}^{|\mathscr{L}|}:\ 0 \leqslant t_l \leqslant \alpha_l\tilde{c}_l, l \in \mathscr{L}; \sum_{l\in \mathrm{Out}(v)} t_l = \sum_{l\in \mathrm{In}(v)} t_l + r_v, v \in \mathscr{V} \right\},$$

and $r_s = r$, $r_d = -r$ and $r_v = 0$ for $v \neq s, d$.

Denote $(\hat{x}_l)_{l\in\mathscr{L}}$ the vector of total link flows at the Nash equilibrium $\widehat{\mathbf{X}}$. The condition (6) can be interpreted as Kuhn–Tucker condition for a single-user minimization of function $\widehat{V}$ under constraints $(y)_{l\in\mathscr{L}} \in \widehat{S}_\alpha$. Then we can deduce that the vector $(\hat{x}_l)_{l\in\mathscr{L}}$ is the unique minimum of the function V, and since $(\tilde{x}_l)_{l\in\mathscr{L}} \in \widehat{S}_\alpha$, we have

$$\begin{aligned}
\sum_{l\in\mathscr{L}} \frac{\hat{x}_l}{\alpha_l\tilde{c}_l - \hat{x}_l}
&\leqslant \sum_{l\in\mathscr{L}} \frac{\tilde{x}_l}{\alpha_l\tilde{c}_l - \tilde{x}_l} - (I-1)\sum_{l\in\mathscr{L}} \ln\left(\frac{\alpha_l\tilde{c}_l - \tilde{x}_l}{\alpha_l\tilde{c}_l - \hat{x}_l}\right)\\
&< \sum_{l\in\mathscr{L}} \frac{\tilde{x}_l}{\alpha_l\tilde{c}_l - \tilde{x}_l} - (I-1)\sum_{l\in\mathscr{L}} \ln\left(1 - \frac{\tilde{x}_l}{\alpha_l\tilde{c}_l}\right)\\
&= \sum_{l\in\widetilde{\mathscr{L}}_1} \frac{\tilde{x}_l}{\alpha_l\tilde{c}_l - \tilde{x}_l} - (I-1)\sum_{l\in\widetilde{\mathscr{L}}_1} \ln\left(1 - \frac{\tilde{x}_l}{\alpha_l\tilde{c}_l}\right).
\end{aligned}$$

Hence, in order to prove that $\widehat{J} = J(\widehat{\mathbf{X}}) = \sum_{l\in\mathscr{L}} (\hat{x}_l/(\alpha_l\tilde{c}_l - \hat{x}_l)) < J(\widetilde{\mathbf{X}}) = \sum_{l\in\mathscr{L}} (\tilde{x}_l/(\tilde{c}_l - \tilde{x}_l))$, it is enough to show that for all $l \in \widetilde{\mathscr{L}}_1$:

$$\frac{\tilde{x}_l}{\tilde{c}_l - \tilde{x}_l} - \frac{\tilde{x}_l}{\alpha_l\tilde{c}_l - \tilde{x}_l} + (I-1)\ln\left(1 - \frac{\tilde{x}_l}{\alpha_l\tilde{c}_l}\right) > 0$$

which holds if $\alpha_l \geqslant I$ (see Appendix A). □

5.2. Adding a direct link

We now consider an upgrade achieved by adding a new link $\hat{l}$ connecting directly the source to the destination. That direct path could be in fact a whole new network, provided that it is disjoint with the previous network.

We consider the throughput demands $(r^i)_{i\in\mathscr{I}}$ fixed, and we examine the two capacity configuration $\hat{c}$ and $\tilde{c}$. These two capacity configurations differ only by the addition of the new link, $\hat{l}$, in capacity configuration $\hat{c}$. Denote $\widehat{\mathbf{X}}$ and $\widetilde{\mathbf{X}}$ the Nash equilibria corresponding to the two capacity configurations capacity configurations $\hat{\mathbf{c}}$ and $\tilde{\mathbf{c}}$. The following result shows that adding a link between the source and the destination, may lead to a decrease of both the total price and the total cost.

Proposition 5.2

1. *Suppose that the cost functions are type-***A** *functions. Suppose that Assumption* 3.1 *holds.*
   If $\hat{x}_{\hat{l}} > 0$, *then the total price* $\hat{\lambda}_s$ *computed at equilibrium for configuration* $\hat{\mathbf{c}}$, *is smaller than the total price* $\tilde{\lambda}_s$ *computed at equilibrium for configuration* $\tilde{\mathbf{c}}$. *That is* $\hat{\lambda}_s < \tilde{\lambda}_s$.

2. *Suppose that the cost functions are type-***C** *functions. Assume that users are consistent at the Nash equilibrium* $\hat{\mathbf{X}}$, *corresponding to capacity configuration* $\hat{\mathbf{c}}$.
   If $\hat{x}_{\hat{l}} \geqslant \hat{c}_{\hat{l}}(1 - \prod_{l\in\mathscr{L}}(1 - \tilde{x}_l/\tilde{c}_l))$, *where* $\hat{x}_{\hat{l}}$ *is the total flow on link* $\hat{l}$ *at Nash equilibrium* $\widehat{\mathbf{X}}$. *Then the total cost* $J(\widehat{\mathbf{X}})$ *computed at equilibrium for configuration* $\hat{\mathbf{c}}$ *is smaller than the total cost* $J(\widetilde{\mathbf{X}})$ *computed at equilibrium for configuration* $\tilde{\mathbf{c}}$. *That is* $J(\widehat{\mathbf{X}}) < J(\widetilde{\mathbf{X}})$.

Proof

1. Consider the initial network $(\mathscr{N},\mathscr{L})$ with the initial capacity configuration $\tilde{c}$ and throughput demand $(\bar{r}^i)_{i\in\mathscr{I}}$ where $\bar{r}^i = r^i - \hat{x}^i_{\hat{l}}$ for $i\in\mathscr{I}$. Let $\bar{\mathbf{X}}$ the Nash equilibrium associated to the throughput demand $(\bar{r}^i)_{i\in\mathscr{I}}$. From conditions (1) we deduce that $\bar{x}^i_l = \hat{x}^i_l$ for all $i\in\mathscr{I}$ and that $l\in\mathscr{L}$, and that we have the equality between the Lagrange multipliers $\bar{\lambda}^i_u = \hat{\lambda}^i_u$, for all $i\in\mathscr{I}$ and $u\in\mathscr{N}$. In other word, if $\hat{x}_{\hat{l}} > 0$, then $\bar{r} < r$, and we deduce from Proposition 3.1 that $\bar{\lambda}_s < \tilde{\lambda}_s$. Consequently we have $\hat{\lambda}_s < \tilde{\lambda}_s$.
2. If $\hat{x}_{\hat{l}} = 0$, by the above analysis, we show that $\hat{x}_l = \tilde{x}_l$, $\forall l\in\mathscr{L}$. Hence $\widehat{J} = \widetilde{J}$.
   If $\hat{x}_{\hat{l}} > 0$, then, using same procedure in the proof of Proposition 5.1, we obtain that the vector $(\hat{x}_l)_{l\in\mathscr{L}'}$ where $\mathscr{L}' = \mathscr{L}\cup\{\hat{l}\}$ is the unique minimum of the function:

$$\widetilde{V}((y_l)_{l\in\mathscr{L}'}) = \sum_{l\in\mathscr{L}'}\frac{y_l}{\hat{c}_l - y_l} - (I-1)\sum_{l\in\mathscr{L}'}\ln(\hat{c}_l - y_l),$$

   where $(y_l)_{l\in\mathscr{L}'}\in\widehat{S}' := \{(t_l)_{l\in\mathscr{L}'}\in\mathbb{R}^{|\mathscr{L}|+1}: 0\leqslant t_l\leqslant\hat{c}_l,\ l\in\mathscr{L}'; \sum_{l\in\mathrm{Out}(v)}t_l = \sum_{l\in\mathrm{In}(v)}t_l + r_v,\ v\in\mathscr{V}\}$, where $r_s = r$, $r_d = -r$ and $r_v = 0$ for $v\neq s,d$.
   Let $(\bar{x}_l)_{l\in\mathscr{L}'}\in\mathbb{R}^{|\mathscr{L}|+\infty}$ defined by: $\bar{x}_l = \tilde{x}_l$ for $l\in\mathscr{L}$ and $\bar{x}_{\hat{l}} = 0$. Clearly $(\bar{x}_l)_{l\in\mathscr{L}}\in\widehat{S}'$. Since $(\hat{x}_l)_{l\in\mathscr{L}}$ minimizes the function $\widetilde{V}$, we have

$$\sum_{l\in\mathscr{L}'}\frac{\hat{x}_l}{\hat{c}_l - \hat{x}_l} - (I-I)\sum_{l\in\mathscr{L}'}\ln(\hat{c}_l - \hat{x}_l) \leqslant \sum_{l\in\mathscr{L}'}\frac{\bar{x}_l}{\hat{c}_l - \bar{x}_l} - (I-1)\sum_{l\in\mathscr{L}'}\ln(\hat{c}_l - \bar{x}_l). \quad (7)$$

   Since $\bar{x}_l = \tilde{x}_l$ and $\hat{c}_l = \tilde{c}_l$ for $l\in\mathscr{L}$, and $\bar{x}_{\hat{l}} = 0$ Eq. (7) becomes

$$\begin{aligned}\sum_{l\in\mathscr{L}'}\frac{\hat{x}_l}{\hat{c}_l - \hat{x}_l} &\leqslant \sum_{l\in\mathscr{L}}\frac{\tilde{x}_l}{\tilde{c}_l - \tilde{x}_l} - (I-1)\sum_{l\in\mathscr{L}}\ln\left(\frac{\tilde{c}_l - \tilde{x}_l}{\tilde{c}_l - \hat{x}_l}\right) + (I-1)\ln\left(\frac{\hat{c}_{\hat{l}} - \hat{x}_{\hat{l}}}{\hat{c}_{\hat{l}}}\right)\\ &< \sum_{l\in\mathscr{L}}\frac{\tilde{x}_l}{\tilde{c}_l - \tilde{x}_l} - (I-1) \times\left[\sum_{l\in\mathscr{L}}\ln\left(1 - \frac{\tilde{x}_l}{\tilde{c}_l}\right) - \ln\left(1 - \frac{\hat{x}_{\hat{l}}}{\hat{c}_{\hat{l}}}\right)\right].\end{aligned}$$

   To prove that $\widehat{J} = \sum_{l\in\mathscr{L}'}(\hat{x}_l/(\hat{c}_l - \hat{x}_l)) < \widetilde{J} = \sum_{l\in\mathscr{L}}(\tilde{x}_l/(\tilde{c}_l - \tilde{x}_l))$, it suffices to show that

$$\sum_{l\in\mathscr{L}}\ln\left(1 - \frac{\tilde{x}_l}{\tilde{c}_l}\right) - \ln\left(1 - \frac{\hat{x}_{\hat{l}}}{\hat{c}_{\hat{l}}}\right) \geqslant 0.$$

   This is equivalent to $\hat{x}_{\hat{l}} \geqslant \hat{c}_{\hat{l}}(1 - \prod_{l\in\mathscr{L}}(1 - \tilde{x}_l/\tilde{c}_l))$, which is true by assumption. This ends the proof of the proposition. □

5.3. Increasing the capacity of a direct link

Now, we consider a network $(\mathscr{N},\mathscr{L})$ such that there exists a direct link connecting the source to the destination. We derive sufficient conditions that guarantee an improvement in the performance when we increase the capacity of the link that connects the source s to the destination d. Denote by $\hat{l}$ the link connecting s and d.

Consider $\tilde{\mathbf{c}}$ and $\hat{\mathbf{c}}$ be two capacity configurations such that $\hat{c}_l = \tilde{c}_l$ for $l\neq\hat{l}$ and $\hat{c}_{\hat{l}} = \alpha\tilde{c}_{\hat{l}}$ where $\alpha\in\mathbb{R}^+$. Let $\widetilde{\mathbf{X}}$ and $\widehat{\mathbf{X}}$ be the Nash equilibria under capacity configurations $\tilde{\mathbf{c}}$ and $\hat{\mathbf{c}}$, respectively.

Proposition 5.3

1. *Suppose that the users cost functions are type-***B** *functions. Suppose that Assumption* 3.1 *holds true.*
   If $\alpha > 1$ *and* $\tilde{x}_{\hat{l}} > 0$. *then the total price* $\hat{\lambda}_s$ *computed at equilibrium for configuration* $\hat{\mathbf{c}}$, *is smaller than the total price* $\tilde{\lambda}_s$ *computed at equilibrium for configuration* $\tilde{\mathbf{c}}$. *That is* $\hat{\lambda}_s < \tilde{\lambda}_s$.

2. *Suppose that the users cost functions are type-***C** *functions. Suppose that the users are consistent at the Nash equilibrium* $\widehat{\mathbf{X}}$ (*corresponding to capacity configuration* $\hat{\mathbf{c}}$). *If* $\hat{x}_{\hat{l}} \geqslant \alpha\tilde{c}_{\hat{l}} \ (1 - \prod_{l\in\mathscr{L}}(1 - \tilde{x}_l/\tilde{c}_l))$. *Then the total cost* $J(\widehat{\mathbf{X}})$ *computed at equilibrium for configuration* $\hat{\mathbf{c}}$ *is smaller than the total cost* $J(\widetilde{\mathbf{X}})$ *computed at equilibrium for configuration* $\tilde{\mathbf{c}}$. *That is* $J(\widehat{\mathbf{X}}) < J(\widetilde{\mathbf{X}})$.

Proof

1. Assume first that $\hat{x}_{\hat{l}} \leqslant \tilde{x}_{\hat{l}}$. Since $\tilde{x}_{\hat{l}} > 0$, and since the users are consistent at the Nash equilibrium $\widetilde{\mathbf{X}}$, we have $\forall i \in \mathscr{I}$, $\tilde{\lambda}^i_s = \tilde{x}^i_{\hat{l}}\widetilde{T}'_{\hat{l}} + \widetilde{T}_{\hat{l}}$. By summing up over $i \in \mathscr{I}$, we obtain that $\tilde{\lambda}_s = \tilde{x}_{\hat{l}}\widetilde{T}'_{\hat{l}} + I\widetilde{T}_{\hat{l}}$. On the other hand, we have that $\hat{\lambda}_s \leqslant \hat{x}_{\hat{l}}\widehat{T}'_{\hat{l}} + I\widehat{T}_{\hat{l}}$. Finally, by the fact that $\hat{x}_{\hat{l}} \leqslant \tilde{x}_{\hat{l}}$ and $\alpha > 1$, we deduce that $\hat{\lambda}_s < \tilde{\lambda}_s$.
   Now we assume that $\hat{x}_{\hat{l}} > \tilde{x}_{\hat{l}}$. Let us consider the two networks that differ only by the presence or absence of the link $\hat{l}$ that connects the source s to the destination d. In both networks we have the same initial capacity configuration $\tilde{\mathbf{c}}$, the same set $\mathscr{I}$ of users, but respective demands $\check{r}^i = r^i - \hat{x}^i_{\hat{l}}$ and $\bar{r}^i = r^i - \tilde{x}^i_{\hat{l}}$.
   Since $\hat{x}_{\hat{l}} > \tilde{x}_{\hat{l}}$ we have $\check{r} = \sum_{i\in\mathscr{N}}\check{r}^i < \bar{r} = \sum_{i\in\mathscr{N}}\bar{r}^i$. Hence from Proposition 3.1 we obtain $\check{\lambda}_s < \bar{\lambda}_s$. On the other hand, for the network with demands $(\check{r}^i)_{i\in\mathscr{I}}$, it is easy to see that the conditions (1) are satisfied by the system flow configuration $\check{\mathbf{X}}$, with $\check{x}^i_l = \hat{x}^i_l$, $(\forall i \in \mathscr{I}, \forall l \in \mathscr{L})$, and $\check{\lambda}^i_u = \hat{\lambda}^i_u$ $(\forall u \in \mathscr{N})$. Similarly we conclude that the network with demands $\bar{r}^i$ has the system flow configuration $\bar{\mathbf{X}}$, with $\bar{x}^i_l = \tilde{x}^i_l$ $(\forall i \in \mathscr{I}, \forall l \in \mathscr{L})$, and $\bar{\lambda}^i_{\underline{u}} = \tilde{\lambda}^i_u$, $(\forall u \in \mathscr{N})$. Hence from the fact that $\check{\lambda}_s < \bar{\lambda}_s$, we obtain $\hat{\lambda}_s < \tilde{\lambda}_s$.
2. The proof for the second part follows the same reasoning than the proof of the second part of Proposition 5.2. □

Remark 5.1

1. For the first part of Proposition 5.3, we can obtain some results in the case where $\tilde{x}_{\hat{l}} = 0$ by assuming that $\hat{x}_{\hat{l}} > 0$ (Proposition 5.2).
2. For the second part of Propositions 5.2 and 5.3, we can obtain some results in the case where the following condition is verified $\prod_{l\in\mathscr{L}'}(\hat{c}_l - \hat{x}_l) \leqslant \hat{c}_{\hat{l}}\prod_{l\in\mathscr{L}}(\tilde{c}_l - \tilde{x}_l)$. This condition weakens the much stronger condition presented in Propositions 5.2 and 5.3 and it is always verified when the capacity of direct link is big enough. We can use this condition when the total capacity is much larger than the total demand, i.e. $c \gg r$.

6. Experimental result

In this section, we illustrate the methods proposed in Section 5.1. We consider the same example as in Section 4 where we have shown that a Braess paradox (increase of both price and cost) may occur when adding some capacity. Now, to avoid paradox, the previous section advocates the addition of capacities either to all links or directly from the source node to the destination node.

We used the same algorithm than previously to compute Nash equilibrium. As previously we obtain convergence after around 20 iterations.

6.1. Increasing the capacity of some specific links

Here we add capacity to all links of the network so that $c_1 = 2.7 + \Delta/5$, $c_2 = 27 + \Delta/4$ and $c_3 = 36 + \Delta/10$ (see Fig. 4).

With these parameters, there is no traffic in the zigzag path for $0 \leqslant \Delta \leqslant 2.1$ and $53.3 \leqslant \Delta$. For $2.1 < \Delta < 53.3$, the three paths are used. Fig. 5 displays the total cost as a function of additional capacity Δ.

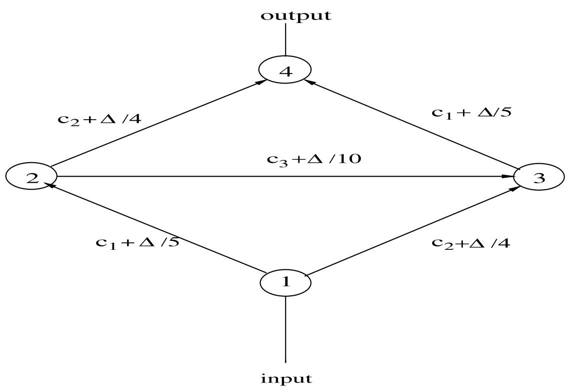

Fig. 4. New network.

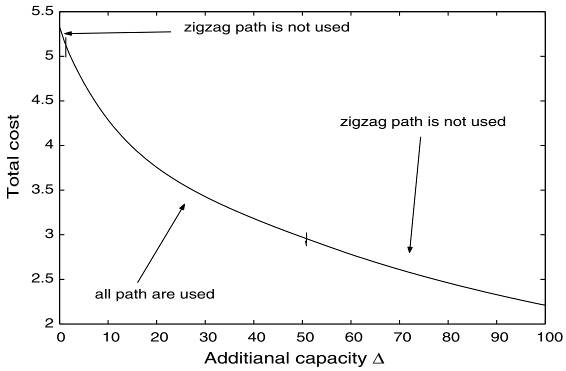

Fig. 5. Total cost as a function of the added capacity in all links.

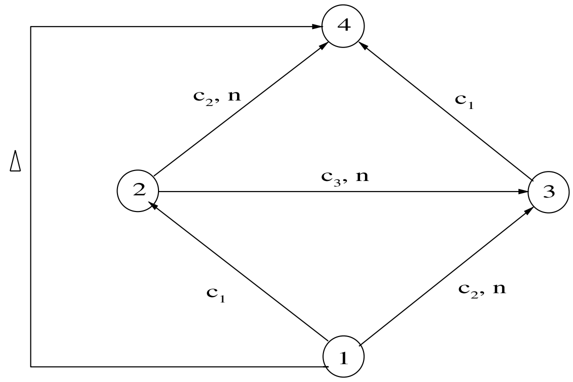

Fig. 6. New network.

For $\Delta \leqslant 2.1$ the zigzag path is not used. Nevertheless the increase of the capacity in the other links leads to a decrease of the total cost. We observed furthermore that the Nash equilibrium did not change (each user send the same proportion of flow in every link). When Δ increases up to 53.3, then all the paths are used, in particular the zigzag path is used. The total cost decreases. In that case the capacity of all links were increased. In particular the capacity of the links (1,2) and (3,4) have been increased, and then these links do not suffer from congestion as it was the case in Section 4 where only the capacity of the path (2,3) have been increased. Hence the total cost decreases as the capacity Δ increases, as Proposition 5.1 suggested. Hence the Braess paradox is indeed avoided.

6.2. Adding capacity or link directly from the source and destination

Propositions 5.2 and 5.3 suggest to upgrade, or add a direct link from the source to the destination. Direct link is made of $n = 54$ tandem links with capacity Δ, while Δ varies from 0 (absence of the direct link) to infinity. We illustrate these methods (see Fig. 6).

For values of Δ less than 18.52, the users do not have any advantage to use the direct path. We can observe that there is no traffic on that path. When Δ increases above 18.74 but below 28 then the direct path start to be attractive, and we observe some traffic in that path. The consequence of that is that there is less traffic on the old network (i.e. the network without the direct link). From Propositions 5.2 and 5.3, it comes that the total cost and total price decreases in the old network. We observe in our numerical example that the total cost in the new network (i.e. with the direct link) decrease. Note that this is not a consequence of Propositions 5.2 and 5.3 since at the beginning the assumption $\hat{x}_{\hat{l}}$ may no be greater that $\Delta(1 - \prod_{l\in\mathscr{L}}(1 - \tilde{x}_l/\tilde{c}_l))$. This is consequence of Remark 5.1, in which the condition $\prod_{l\in\mathscr{L}'}(\hat{c}_l - \hat{x}_l) \leqslant \Delta\prod_{l\in\mathscr{L}}(\tilde{c}_l - \tilde{x}_l)$ is verified when the direct link is attractive.

Then the addition of the direct path never leads to an increase of the delay which is illustrated in

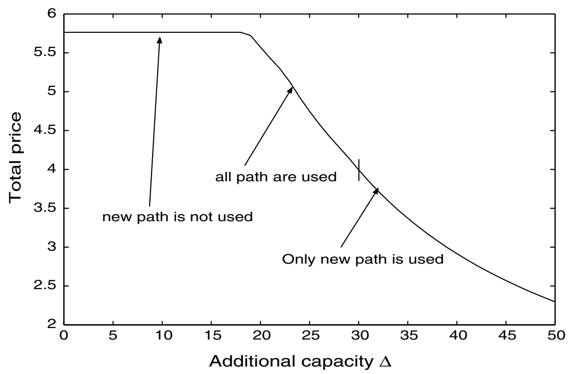

Fig. 7. Total price as a function of the added capacity in link (1,4).

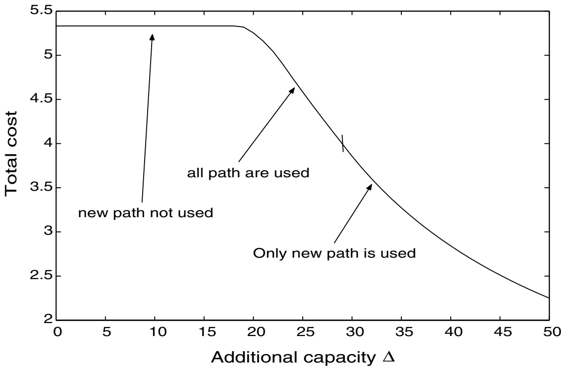

Fig. 8. Total cost as a function of the added capacity in link (1,4).

Figs. 7 and 8, where we can observe that the total costs and total prices decrease.

Appendix A

In this appendix, we study the function $H_\alpha : [0, \alpha\bar{c}) \to \mathbb{R}$ defined by

$$H_\alpha(x) = \frac{x}{\bar{c} - x} - \frac{x}{\alpha c - x} + (I - 1)\ln\left(1 - \frac{x}{\alpha}c\right), \tag{A.1}$$

where $\alpha \geqslant 1$ and $\bar{c}$ is a positive constant. More precisely, we wish to determine α such that H_α is positive for every x in $[0, \alpha\bar{c})$. By remarking that $H(0) = 0$, for all α, it is enough to determine α such that

$$\frac{\partial H_\alpha}{\partial x} = \frac{c}{(c-x)^2} - \frac{\alpha c}{(\alpha c - x)^2} - (I-1)\frac{1}{\alpha c - x} > 0.$$

This last inequality is equivalent to

$$\frac{c}{(c-x)^2} > \frac{I\alpha c - (I-1)x}{(\alpha c - x)^2}$$

which is equivalent to $(\alpha^2 - I\alpha)c^3 + cx^2$ c$(3 - I\alpha - 2I) > c^2x(2\alpha - 2I\alpha - I + 1) - (I-1)x^3$. If $\alpha \geqslant I$, it is enough to show that $cx(3 - I\alpha - 2I) > c^2(2\alpha - 2I\alpha - I + 1) - (I-1)x^2$. Since $x^2 + c^2 \geqslant 2cx$, it is sufficient to verify that $c^2(1 + 3I\alpha - 4\alpha) > x^2(I\alpha - 1)$, or $c^2(\alpha(I-2) + 1) > 0$, which trivially holds.

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