Customer Lifetime Value: Stochastic Optimisation Approach *

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Abstract

Since the early 1980's, the concept of relationship marketing has been becoming important in general marketing, especially in the area of direct and interactive marketing. The core of relationship marketing is the maintenance of the long-term relationships with the customers. However, the relationship marketing is costly and therefore the determination of the Customer Lifetime Value (CLV) is an important element in making strategic decisions in both advertising and promotion. In this paper we propose a stochastic dynamic

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programming model with a Markov chain for the optimisation of CLV. Both cases of infinite horizon and finite horizon are discussed. The model is then applied to practical data of a computer service company.

Key Words: Customer Lifetime Value, Relationship Marketing, Stochastic Dynamic Programming, Markov Process.

1 Introduction

We are in the age of relationship marketing, an age in which a sale starts a customer-company relationship [1]. Relationship marketing is a process of making and maintaining value-laden relationships with the customers. Corporations such as banks, computers and telecommunication companies realised that they are facing stiffer competition in the mature markets, the development of relationships with the profitable customers is a crucial factor in staying in the market. The core of the relationship marketing is the development and maintenance of long-term relationship with the customers. In principle, marketing is the art of attracting and keeping customers [2]. A company should strive to identify the "profitable" customers and keep them with an "acceptable effort". Kotler and Armstrong [3] define a profitable customer as "a person, household, or company whose revenues over time exceeds, by an acceptable amount, the company costs consist of attracting, selling, and servicing that customer." This excess is called the Customer Lifetime Value (CLV). In some literatures, CLV is also referred to "customer equity" [4]. In fact, some researchers define CLV as the customer equity less the acquisition cost. Nevertheless, in this paper we define CLV as the present value of the projected net cash flows that a firm expects to receive from the customer over time [5]. Here we use both terms interchangeably, a common practice by researchers [6]. Recognising the importance in decision making, CLV has been successfully applied in the problems of pricing strategy [7], media selection [8] and setting optimal promotion budgets [6].

To calculate the CLV, we first project the net cash flows that a company expects to receive from the customer over time. We then calculate the present value of that stream of cash flows. However, it is a difficult task to estimate the net cash flows to be received from the customer. In fact, one needs to answer, for example, the following questions: How many customers you can attract given a specific advertising budget? What is the probability that the customer will stay with your company? How does this probability change with respect to your budget? To answer the first question, there are a number of advertising models, one can find in the book by Lilien, Kotler and Moorthy [2]. The second and the third questions give rise to an important concept, the retention rate. The retention rate [9] is defined as " the chance that the account will remain with the vendor for the next purchase, provided that the customer has bought from the vendor on each previous purchase". Jackson [9] proposed an estimation method for the retention rate based on historical data. Other retention models can be found in [2, 10]

Blattberg and Deighton [6] proposed a formula for the calculation of CLV (customer equity). The model is simple and deterministic. Using their notations (see also [4, 7]), the CLV is the sum of two net present values: the return from acquisition spending and the return from retention spending. In their model

$$\text{CLV} = \underbrace{am - A}_{acquisition} + \underbrace{\sum_{k=1}^{\infty} a(m - \frac{R}{r})[r(1+d)^{-1}]^k}_{retention} = am - A + a(m - \frac{R}{r}) \times \frac{r}{(1+d-r)}. \tag{1}$$

Here a is the acquisition rate, A is the level of acquisition spending, m is the margin on a transaction, R is the retention spending per customer per year, r is the yearly retention rate (a proportion) and d is the yearly discount rate appropriate for marketing investment. Moreover, they also assume that the acquisition rate a and the retention rate r are functions of A and R respectively, and are given by

$$a(A) = a_0(1 - e^{-K_1 A}) \quad \text{and} \quad r(R) = r_0(1 - e^{-K_2 R})$$

where a_0 and r_0 are the estimated ceiling rates, K_1 and K_2 are two positive constants. By using the above relationships, Berger and Nasr [4] proposed different models for maximising the CLV under a fixed promotion budget. Other deterministic promotion budget allocation models are proposed in [4, 5, 6, 7]. However very few researchers have studied the promotion budget allocation (the number of promotions is fixed) under competitive and stochastic situations.

In this paper we propose to use a stochastic dynamic programming model with the Markov chain to capture the customer behaviour. The advantage of using the Markov chain is that the model can take into the account of the switch of the customers between the company and its competitors. Therefore customer relationships can be described in a probabilistic way, see for instance Pfeifer and Carraway [11]. Stochastic dynamic programming is then applied to solve the optimal allocation of promotion budget for maximising the CLV. To illustrate our model, we apply our model to practical data in a computer services company.

The rest of the paper is organised as follows. In Section 2, we present the Markov chain model for modeling the behavior of the customers. In Section 3, stochastic dynamic programming is then used to calculate the CLV of the customers for three different scenarios: (i) infinite horizon without constraint (without limit in the number of promotions), (ii) finite horizon (with limited number of promotions), and (iii) infinite horizon with constraints (with limited number of promotions). Finally a summary is given to conclude the paper in Section 4.

2 Markov chain models for customers' behavior

In this section, we introduce a Markov chain model for modeling the customers' behavior in a market. According to the usage of the customer, a company customer can be classified into N possible states $\{0, 1, 2, \ldots, N-1\}$. Take for example, a customer can be classified into four states ($N = 4$): low volume user (State 1), medium volume user (State 2) and high volume user (State 3) and in order to classify all customers in the market, we introduce State 0. A customer is said to be in State 0, if he/she is either a customer of the competitor company or he/she did not purchase the service during the period of observation. Therefore at any time a customer in the market belongs to exactly one of the states in $\{0, 1, 2, \ldots, N-1\}$. With these notations, a Markov chain is a good approach to model the transitions of customers among the states in the market. For an introduction to Markov chain, we refer readers to the book by Ross [12].

A Markov chain model is characterised by an $N \times N$ transition matrix P. Here $P_{ij}(i, j = 0, 1, 2, \ldots, N-1)$ is the transition probability that a customer will move to State j in the next period given that currently he/she is in State i. Hence the retention probability of a customer in State $i(i = 0, 1, \ldots, N-1)$ is given by P_{ii}. If we assume that the underlying Markov chain is irreducible then the stationary distribution $\mathbf{p}$ exists, see for instance [12]. This means that there is an unique

$\mathbf{p} = (p_0, p_1, \ldots, p_{N-1})$ such that

$$\mathbf{p} = \mathbf{p}P, \quad \sum_{i=0}^{N-1} p_i = 1, \quad p_i \geq 0. \tag{2}$$

By making use of the stationary distribution $\mathbf{p}$, one can compute the retention probability of a company customer as follows:

$$\sum_{i=1}^{N-1} \left(\frac{p_i}{\sum_{j=1}^{N-1} p_j} \right) (1 - P_{i0}) = 1 - \frac{1}{1-p_0} \sum_{i=1}^{N-1} p_i P_{i0} = 1 - \frac{p_0(1-P_{00})}{1-p_0}. \tag{3}$$

This is the probability that a company customer will stay and purchase service in the next period. Apart from the retention probability, the Markov model can also help us in computing the CLV. In this case we define c_i to be the revenue obtained from a customer in State i. Then the expected revenue is given by

$$\sum_{i=0}^{N-1} c_i p_i. \tag{4}$$

We remark that the above retention probability and the expected revenue are computed under the assumption that the company makes no promotion (in a non-competitive environment) through out the period. We note that the transition probability matrix P can be significantly different when there is promotion making by the company. We will demonstrate this in the following subsection. Moreover, when promotions are allowed, what is the best promotion strategy such that the expected revenue is maximised ? Similarly, what is the best strategy when there is a fixed budget for the promotions, e.g. the number of promotions is fixed ? We are going to answer these questions by using the stochastic dynamic programming model in the next section.

2.1 Estimation of the transition probabilities

In order to apply the Markov chain model, one has to estimate the transition probabilities from the practical data. In this subsection, we demonstrate this by using an example in the computer service company. In the captured database of the customers, each customer has four important attributes (A, B, C, D). Here A is the "Customer Number", each customer has an unique identity number. B is the "Week", the time (week) when the data was captured. C is the "Revenue" which is the total amount of money the customer spent in the captured week. D is the "Hour", the number of hours the customer consumed in the captured week.

The total number of weeks of data available is 20. Among these 20 weeks, the company has a promotion for 8 consecutive weeks. There is no promotion for other 12 consecutive weeks. We are interested in the behavior of customers in the period of promotion and no-promotion. For each week, all the customers are classified into four states $(1, 2, 3, 0)$ according to the amount of "hours" consumed, see Table 1 below. Here a customer is said to be in State 0, if he/she is a customer of competitor company or he/she did not use the service for the whole week.

State	1	2	3	0
Minutes	$1-20$	$21-40$	> 40	0.00

Table 1: The Four Classes of Customers.

From the data one can estimate two transition probability matrices, one for the promotion period (8 consecutive weeks) and the other one for the no-promotion period (12 consecutive weeks). For each period, we record the number of customers switching from State i to State j. Then, divide it by the total number of customers in the State i, we get the estimations for the one-step transition probabilities. Hence the transition probability matrices under the promotion period $P^{(1)}$ and the no-promotion period $P^{(2)}$ are given respectively below (the states are ordered as follows: $1, 2, 3, 0$):

$$P^{(1)} = \begin{pmatrix} 0.4230 & 0.0992 & 0.0615 & 0.4163 \\ 0.3458 & 0.2109 & 0.2148 & 0.2285 \\ 0.2147 & 0.2034 & 0.4447 & 0.1372 \\ 0.1489 & 0.0266 & 0.0191 & 0.8054 \end{pmatrix}$$

and

$$P^{(2)} = \begin{pmatrix} 0.4146 & 0.0623 & 0.0267 & 0.4964 \\ 0.3837 & 0.1744 & 0.1158 & 0.3261 \\ 0.2742 & 0.2069 & 0.2809 & 0.2380 \\ 0.1064 & 0.0121 & 0.0053 & 0.8762 \end{pmatrix}.$$

We note that $P^{(1)}$ is very different from $P^{(2)}$. We remark that in general there can be more than one type of promotion and hence there can be more than two different transition probability matrices for modeling the behavior of the customers.

2.2 Retention probability and CLV

The stationary distributions of the two Markov chains having transition probability matrices $P^{(1)}$ and $P^{(2)}$ are given respectively by

$$\mathbf{p}^{(1)} = (0.2306, 0.0691, 0.0738, 0.6265) \quad \text{and} \quad \mathbf{p}^{(2)} = (0.1692, 0.0285, 0.0167, 0.7856).$$

The retention probabilities (cf. (3)) in the promotion period and no-promotion period are given respectively by 0.6736 and 0.5461. It is clear that the retention probability is higher when the promotion is carried out.

From the customer data in the database, we obtain the average revenue of a customer in different states in both the promotion period and no-promotion period, see Table 2 below. We remark that in the promotion period, a big discount was given to the customers and therefore the revenue was significantly less than the revenue in the no-promotion period.

State	1	2	3	0
Promotion	6.97	18.09	43.75	0.00
No-promotion	14.03	51.72	139.20	0.00

Table 2: The Average Revenue of the Four Classes of Customers.

From (4), the expected revenue of a customer in the promotion period (assume that the only promotion cost is the discount rate) and no-promotion period are given by 2.42 and 17.09 respectively.

Although one can obtain the CLVs of the customers in the promotion period and the no-promotion period, one would expect to calculate the CLV in a mixture of promotion and no-promotion periods. Especially when there is a limit in promotion budget (the number of promotions is fixed) and one would like to obtain the optimal promotion strategy. Stochastic dynamic programming with Markov process provides a good approach for solving the above problems. Moreover, the optimal stationary strategies for customers in different states can also be obtained by solving the stochastic dynamic programming problem.

3 Stochastic dynamic programming models

The problem of solving the optimal promotion strategy can be fitted into the framework of stochastic dynamic programming models. In this section, we present stochastic dynamic programming models for maximising the CLV under optimal promotion strategy. In the following, we first give the notations for the model.

N	=	the total number of states (indexed by $i = 0, 1, \ldots, N-1$).
M	=	the total number of promotion plans (indexed by $j = 1, \ldots, M$).
T	=	number of months in the planning horizon (indexed by $t = 1, \ldots, T$).
d_j	=	the resources required for carrying out promotion plan j in each period.
$c_i^{(j)}$	=	the revenue obtained from a customer in State i with the jth promotion plan in each period.
$p_{ik}^{(j)}$	=	the transition probability for customer to move from State i to State k under the jth promotion plan in each period.
α	=	discount rate.

We define $v_i(t)$ to be the total expected revenue obtained in the stochastic dynamic programming model with t months remained for a customer in State i at the beginning of the $(T-t)$th period for $i = 0, 1, \ldots, N-1$ and $t = 1, 2, \ldots, T$. Then we have the following recursive relation for maximising the revenue:

$$v_i(t) = \max_{j=1,\ldots,M} \left\{ c_i^{(j)} - d_j + \alpha \sum_{k=0}^{N-1} p_{ik}^{(j)} v_k(t-1) \right\}. \qquad (5)$$

In the following subsections, we will consider three different CLV models based on the above recursive relation for infinite horizon and finite horizon cases.

3.1 Infinite horizon without constraints

We first consider the problem as an infinite horizon stochastic dynamic programming. From the standard results in stochastic dynamic programming [13], for each i, the optimal values v_i for the discounted infinite horizon Markov decision process satisfy the relationship

$$v_i \geq \max_{j=1,\ldots,M} \left\{ c_i^{(j)} - d_j + \alpha \sum_{k=0}^{N-1} p_{ik}^{(j)} v_k \right\}.$$

Therefore we have

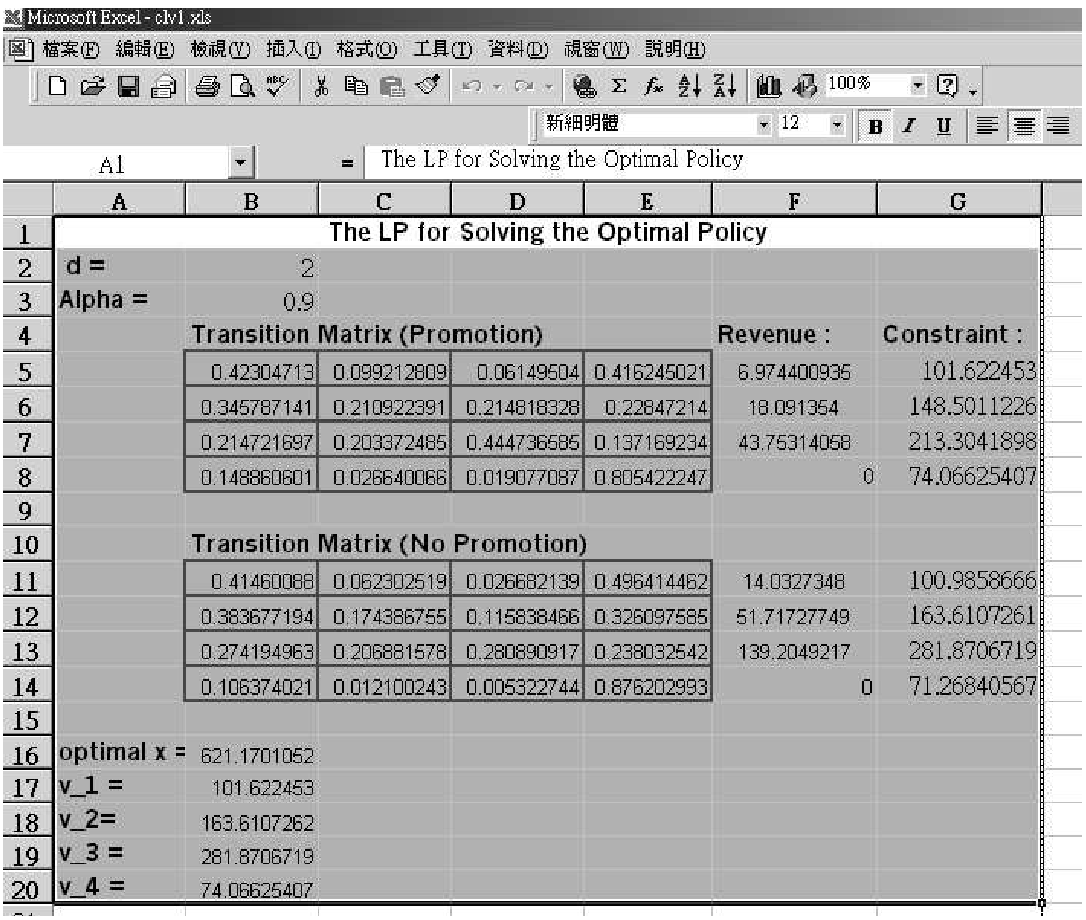
$$v_i \geq c_i^{(j)} - d_j + \alpha \sum_{k=0}^{N-1} p_{ik}^{(j)} v_k.$$

for each i and $j = 1, ..., M$. In fact, the optimal values v_i are the smallest numbers (the least upper bound over all possible policy values) that satisfy these inequalities.

This suggests that the problem of determining the v_i's can be transformed into the following linear programming problem [13]:

$$
\left\{
\begin{array}{l}
\qquad\qquad \min \quad x_0 = \displaystyle\sum_{i=0}^{N-1} v_i \\
\text{subject to} \\
\qquad\qquad v_i \geq c_i^{(j)} - d_j + \alpha \displaystyle\sum_{k=0}^{N-1} p_{ik}^{(j)} v_k, \quad \text{for} \quad i = 0, \ldots, N-1; \quad j = 1, \ldots, M. \\
\qquad\qquad v_i \geq 0 \quad \text{for} \quad i = 0, \ldots, N-1.
\end{array}
\right.
$$

The above linear programming problem can be solved easily by using spreadsheet EXCEL. A demonstration EXCEL file is available at "http://hkumath.hku.hk/~wkc/clv1.zip", see also Figure 1. Return to our model for the computer service company, we have $M = 2$ (either (1) promotion or (2) no-promotion) and $N = 4$ (possible states of a customer are $1, 2, 3, 0$).



	A	B	C	D	E	F	G
1	The LP for Solving the Optimal Policy						
2	d =	2					
3	Alpha =	0.9					
4		Transition Matrix (Promotion)				Revenue :	Constraint :
5		0.42304713	0.099212809	0.06149504	0.416245021	6.974400935	101.622453
6		0.345787141	0.210922391	0.214818328	0.22847214	18.091354	148.5011226
7		0.214721697	0.203372485	0.444736585	0.137169234	43.75314058	213.3041898
8		0.148860601	0.026640066	0.019077087	0.805422247	0	74.06625407
9							
10		Transition Matrix (No Promotion)					
11		0.41460088	0.062302519	0.026682139	0.496414462	14.0327348	100.9858666
12		0.383677194	0.174386755	0.115838466	0.326097585	51.71727749	163.6107261
13		0.274194963	0.206881578	0.280890917	0.238032542	139.2049217	281.8706719
14		0.106374021	0.012100243	0.005322744	0.876202993	0	71.26840567
15							
16	optimal x =	621.1701052					
17	v_1 =	101.622453					
18	v_2=	163.6107262					
19	v_3 =	281.8706719					
20	v_4 =	74.06625407					

Figure 1: EXCEL File for Solving Infinite Horizon Problem Without Constraint.

Table 3 presents the optimal stationary policies (i.e., to have promotion $D_i = 1$ or no promotion $D_i = 2$ depends on State i of the customer) and the corresponding revenues for different discount factors α and fixed promotion costs d. For instance, when the promotion cost is 0 and the discount factor is 0.99 , then the optimal strategy is that when the current state is 1 or 4, the promotion should be done i.e. $D_1 = D_4 = 1$, and when the current state is 2 or 3, no promotion is required, i.e. $D_2 = D_3 = 2$, (see the first column of the upper left hand box of Table 3). The other values can be interpreted similarly. From the numerical examples, we have the following findings.

- When the fixed promotion cost d is large, the optimal strategies are that we should not conduct any promotion on our active customers and we should only conduct promotion scheme to both inactive (purchase no service) customers and customers of the competitor company. However, when d is small, we should take care of the low-volume customers to avoid this group of customers being churned to the competitor companies.

- It is also clear that the CLV of a high-volume user is larger than the CLV of other groups.

- The CLVs of each group of customers depend on the discount rate α significantly. Here the discount rate can be viewed as the technology depreciation of the computer services in the company. Therefore, in order to generate the revenue of the company, new technology and services should be provided.

	$d=0$			$d=1$			$d=2$		
	$\alpha=0.99$	$\alpha=0.95$	$\alpha=0.90$	$\alpha=0.99$	$\alpha=0.95$	$\alpha=0.90$	$\alpha=0.99$	$\alpha=0.95$	$\alpha=0.90$
x_0	4791	1149	687	4437	1080	654	4083	1012	621
v_1	1144	234	119	1054	216	110	965	198	101
v_2	1206	295	179	1118	278	171	1030	261	163
v_3	1328	415	296	1240	399	289	1153	382	281
v_0	1112	204	92	1023	186	83	934	168	74
D_1	1	1	1	1	1	1	1	1	1
D_2	2	2	2	2	2	2	2	2	2
D_3	2	2	2	2	2	2	2	2	2
D_0	1	1	1	1	1	1	1	1	1
	$d=3$			$d=4$			$d=5$		
	$\alpha=0.99$	$\alpha=0.95$	$\alpha=0.90$	$\alpha=0.99$	$\alpha=0.95$	$\alpha=0.90$	$\alpha=0.99$	$\alpha=0.95$	$\alpha=0.90$
x_0	3729	943	590	3375	879	566	3056	827	541
v_1	877	181	94	788	164	88	707	151	82
v_2	942	245	156	854	230	151	775	217	145
v_3	1066	366	275	978	351	269	899	339	264
v_0	845	151	65	755	134	58	675	119	51
D_1	1	1	2	1	2	2	2	2	2
D_2	2	2	2	2	2	2	2	2	2
D_3	2	2	2	2	2	2	2	2	2
D_0	1	1	1	1	1	1	1	1	1

Table 3: The Optimal Stationary Policies and Their CLVs.

3.2 Finite horizon with hard constraints

In the computer service and telecommunication industry, the product life cycle is short, e.g., it is usually one year. Therefore, we consider the case of finite horizon with limited budget constraint, the problem can also be solved efficiently by using stochastic dynamic programming. For this problem, we used the optimal revenues obtained in Section 3.1 as the boundary conditions. We define

w = Number of weeks remaining;
p = Number of possible promotions remaining.

The recursive relation for the problem is given as follows:

$$v_i(w,p) = \max\left\{c_i^{(1)} - d_1 + \alpha\sum_{k=0}^{N-1} p_{ik}^{(1)} v_k(w-1,p-1),\, c_i^{(2)} - d_2 + \alpha\sum_{k=0}^{N-1} p_{ik}^{(2)} v_k(w-1,p)\right\}$$

for $w = 1,\ldots,w_{max}$ and $p = 1,\ldots,p_{max}$ and

$$v_i(w,0) = c_i^{(2)} - d_2 + \alpha\sum_{k=0}^{N-1} p_{ik}^{(2)} v_k(w-1,0)$$

for $w = 1, \ldots, w_{max}$. The above dynamic programming problem can be solved easily by using spreadsheet EXCEL. A demonstration EXCEL file can be found at the following site: "http://hkumath.hku.hk/~wkc/clv2.zip", see also Figure 2. In our numerical experiment we assume that the length of planning period is $w_{max} = 52$ and the maximum number of promotions is $p_{max} = 4$. By solving the dynamic programming problem we obtain Table 4 for the optimal values and promotion strategies. In Table 4 we present the optimal solution

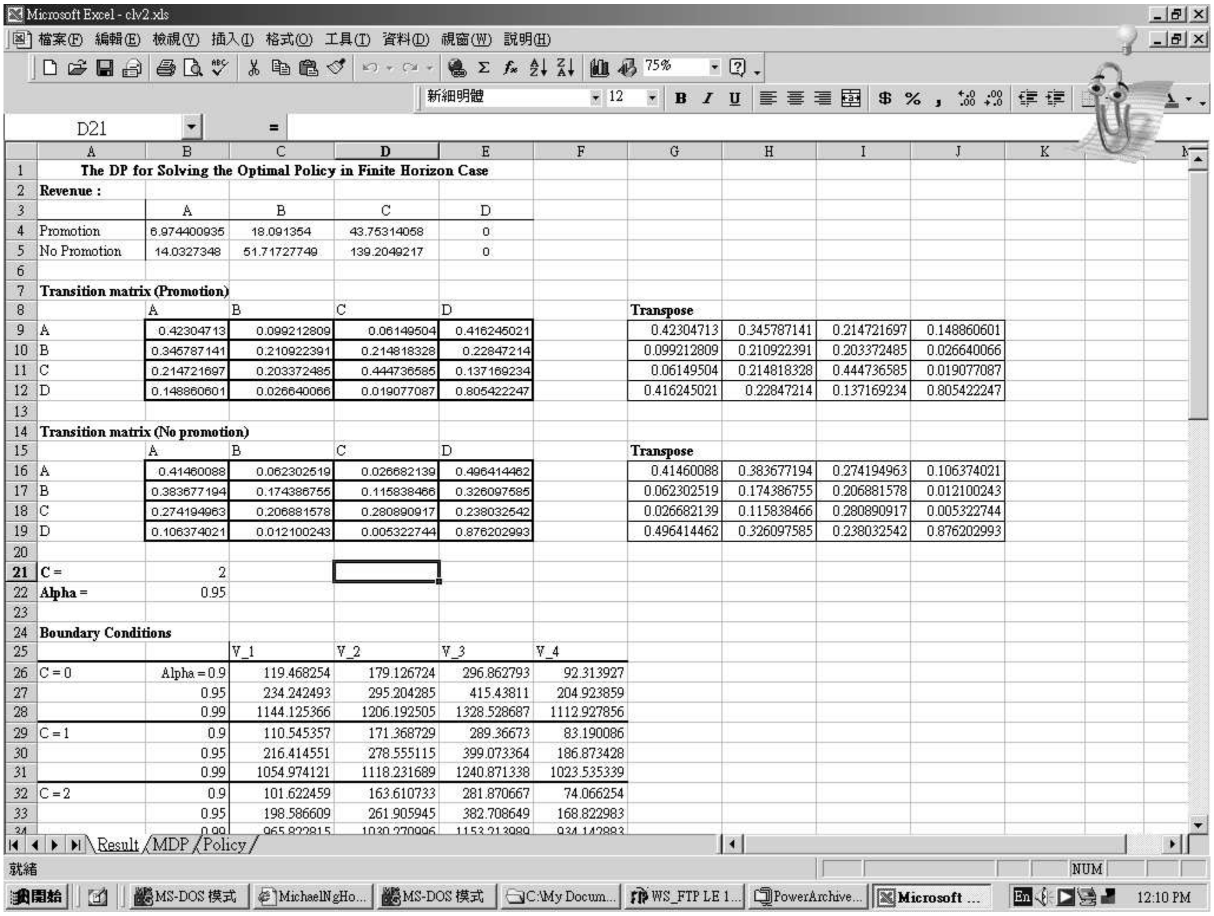
$$(t_1, t_2, t_3, t_4, r^*),$$

where r^* is the optimal expected revenue, and t_i is the promotion week of the optimal promotion strategy and "-" means no promotion. We summarise our findings as follows:

- For different values of the fixed promotion cost d, the optimal strategy for the customers in States 2 and 3 is to conduct no promotion.
- While for those in State 0, the optimal strategy is to conduct all the four promotions to them as early as possible.
- In State 1, the optimal strategy depends on the value of d. If d is large, then there is no promotion. However, when d is small, the promotion is carried out and the strategy is put the promotion as late as possible.

	α	State 1	State 2	State 3	State 0
$d = 0$	0.9	(1, 45, 50, 52, 95)	(-,-,-,-,158)	(-,-,-,-,276)	(1, 2, 3, 4, 67)
	0.95	(45, 48, 50, 51, 169)	(-,-,-,-,234)	(-,-,-,-,335)	(1, 2, 3, 4, 138)
	0.99	(47, 49, 50, 51, 963)	(-,-,-,-,1031)	(-,-,-,-,1155)	(1, 2, 3, 4, 929)
$d = 1$	0.9	(47, 49, 51, 52, 92)	(-,-,-,-,155)	(-,-,-,-,274)	(1, 2, 3, 4, 64)
	0.95	(47, 49, 51, 52, 164)	(-,-,-,-,230)	(-,-,-,-,351)	(1, 2, 3, 4, 133)
	0.99	(47, 49, 51, 52, 906)	(-,-,-,-,974)	(-,-,-,-,1098)	(1, 2, 3, 4, 872)
$d = 2$	0.9	(49, 50, 51, 52, 89)	(-,-,-,-,152)	(-,-,-,-,271)	(1, 2, 3, 4, 60)
	0.95	(48, 50, 51, 52, 160)	(-,-,-,-,225)	(-,-,-,-,347)	(1, 2, 3, 4, 128)
	0.99	(48, 49, 51, 52, 849)	(-,-,-,-,917)	(-,-,-,-,1041)	(1, 2, 3, 4, 815)
$d = 3$	0.9	(-,-,-,-,87)	(-,-,-,-,150)	(-,-,-,-,269)	(1, 2, 3, 4, 60)
	0.95	(49, 50, 51, 52, 155)	(-,-,-,-,221)	(-,-,-,-,342)	(1, 2, 3, 4, 123)
	0.99	(48, 50, 51, 52, 792)	(-,-,-,-,860)	(-,-,-,-,984)	(1, 2, 3, 4, 758)
$d = 4$	0.9	(-,-,-,-,84)	(-,-,-,-,147)	(-,-,-,-,266)	(1, 2, 3, 4, 54)
	0.95	(-,-,-,-,151)	(-,-,-,-,217)	(-,-,-,-,338)	(1, 2, 3, 4, 119)
	0.99	(49, 50, 51, 52, 736)	(-,-,-,-,804)	(-,-,-,-,928)	(1, 2, 3, 4, 701)
$d = 5$	0.9	(-,-,-,-,81)	(-,-,-,-,144)	(-,-,-,-,264)	(1, 2, 3, 4, 50)
	0.95	(-,-,-,-,147)	(-,-,-,-,212)	(-,-,-,-,334)	(1, 2, 3, 4, 114)
	0.99	(-,-,-,-,684)	(-,-,-,-,752)	(-,-,-,-,876)	(1, 2, 3, 4, 650)

Table 4: The Optimal Promotion Strategies and Their CLVs.



	A	B	C	D	E	F	G	H	I	J
1	The DP for Solving the Optimal Policy in Finite Horizon Case									
2	Revenue :									
3		A	B	C	D					
4	Promotion	6.974400935	18.091354	43.75314058	0					
5	No Promotion	14.0327348	51.71727749	139.2049217	0					
6										
7	Transition matrix (Promotion)									
8		A	B	C	D		Transpose			
9	A	0.42304713	0.099212809	0.06149504	0.416245021		0.42304713	0.345787141	0.214721697	0.148860601
10	B	0.345787141	0.210922391	0.214818328	0.22847214		0.099212809	0.210922391	0.203372485	0.026640066
11	C	0.214721697	0.203372485	0.444736585	0.137169234		0.06149504	0.214818328	0.444736585	0.019077087
12	D	0.148860601	0.026640066	0.019077087	0.805422247		0.416245021	0.22847214	0.137169234	0.805422247
13										
14	Transition matrix (No promotion)									
15		A	B	C	D		Transpose			
16	A	0.41460088	0.062302519	0.026682139	0.496414462		0.41460088	0.383677194	0.274194963	0.106374021
17	B	0.383677194	0.174386755	0.115838466	0.326097585		0.062302519	0.174386755	0.206881578	0.012100243
18	C	0.274194963	0.206881578	0.280890917	0.238032542		0.026682139	0.115838466	0.280890917	0.005322744
19	D	0.106374021	0.012100243	0.005322744	0.876202993		0.496414462	0.326097585	0.238032542	0.876202993
20										
21	C =	2								
22	Alpha =	0.95								
23										
24	Boundary Conditions									
25			V_1	V_2	V_3	V_4				
26	C = 0	Alpha = 0.9	119.468254	179.126724	296.862793	92.313927				
27		0.95	234.242493	295.204285	415.43811	204.923859				
28		0.99	1144.125366	1206.192505	1328.528687	1112.927856				
29	C = 1	0.9	110.545357	171.368729	289.36673	83.190086				
30		0.95	216.414551	278.555115	399.073364	186.873428				
31		0.99	1054.974121	1118.231689	1240.871338	1023.535339				
32	C = 2	0.9	101.622459	163.610733	281.870667	74.066254				
33		0.95	198.586609	261.905945	382.708649	168.822983				

Figure 2: EXCEL File for Solving Finite Horizon Problem Without Constraint.

3.3 Infinite horizon with constraints

For comparisons, we extend the model in 3.2 to the infinite horizon case. In this model we have a finite budget for promotions $p_{\max}$. Then the value function $v_i(p)$, which represents the optimal discounted utility when starting at State i and there are p number of promotions remaining, is the unique fixed point of the equations:

$$v_i(p) = \max \left\{ c_i^{(1)} - d_1 + \alpha \sum_{k=0}^{N-1} p_{i,k}^{(1)} v_k(p-1), c_i^{(2)} - d_2 + \alpha \sum_{k=0}^{N-1} p_{i,k}^{(2)} v_k(p) \right\}, \tag{6}$$

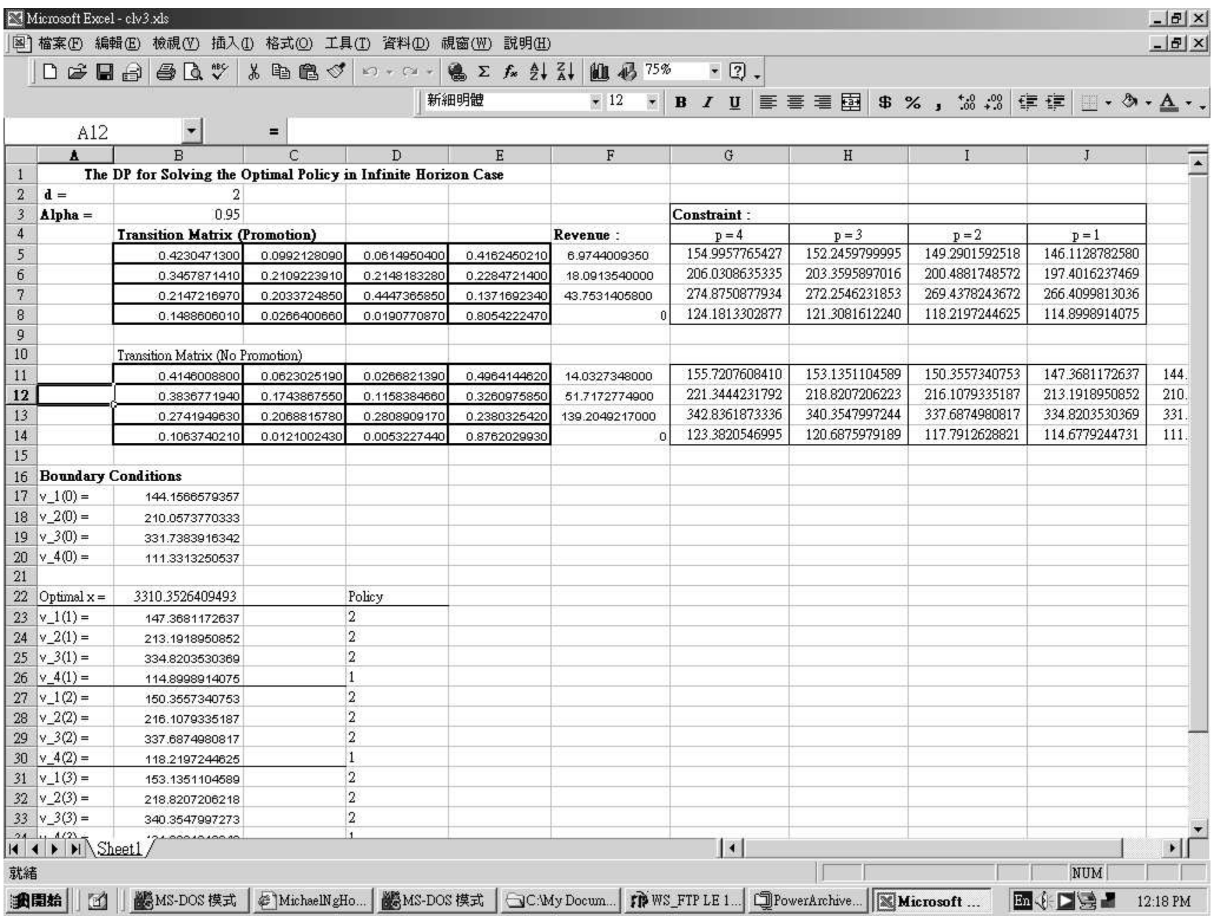
for $p = 1, \ldots, p_{\max}$, and

$$v_i(0) = c_i^{(2)} - d_2 + \alpha \sum_{k=0}^{N-1} p_{i,k}^{(2)} v_k(0). \tag{7}$$

We note that since $\{p_{i,j}^{(k)}\}$ is a probability matrix, the set of linear equations (7) with four unknowns has a unique solution. Now, (6) can be computed by the value iteration algorithm, i.e. as the limit of $v_i(w,p)$ (computed in Section 3.2) as w tends to infinity. Alternatively, it can be solved by linear programming ([14]):

$$
\begin{cases}
\min \quad x_0 = \displaystyle\sum_{i=0}^{N-1} \sum_{p=1}^{p_{\max}} v_i(p) \\
\text{subject to} \\
v_i(p) \geq c_i^{(1)} - d_1 + \alpha \displaystyle\sum_{k=0}^{N-1} p_{ik}^{(1)} v_k(p-1), \quad \text{for} \quad i = 0, \ldots, N-1, p = 1, \ldots, p_{\max}; \\
v_i(p) \geq c_i^{(2)} - d_2 + \alpha \displaystyle\sum_{k=0}^{N-1} p_{ik}^{(2)} v_k(p), \quad \text{for} \quad i = 0, \ldots, N-1, p = 1, \ldots, p_{\max}.
\end{cases}
$$

Note that we do not have to include in the linear programming constraints that correspond to $v_i(0)$ nor do we have to include it in the objective function; $v_i(0)$ is solved beforehand using (7). A demonstration EXCEL file can be found at the following site: "http://hkumath.hku.hk/∼wkc/clv3.zip", see also Figure 3.



	A	B	C	D	E	F	G	H	I	J	
1	The DP for Solving the Optimal Policy in Infinite Horizon Case										
2	d =	2									
3	Alpha =	0.95					Constraint :				
4		Transition Matrix (Promotion)				Revenue :	p = 4	p = 3	p = 2	p = 1	
5		0.4230471300	0.0992128090	0.0614950400	0.4162450210	6.9744009350	154.9957765427	152.2459799995	149.2901592518	146.1128782580	
6		0.3457871410	0.2109223910	0.2148183280	0.2284721400	18.0913540000	206.0308635335	203.3595897016	200.4881748572	197.4016237469	
7		0.2147216970	0.2033724850	0.4447365850	0.1371692340	43.7531405800	274.8750877934	272.2546231853	269.4378243672	266.4099813036	
8		0.1488606010	0.0266400660	0.0190770870	0.8054222470	0	124.1813302877	121.3081612240	118.2197244625	114.8998914075	
9											
10		Transition Matrix (No Promotion)									
11		0.4146008800	0.0623025190	0.0266821390	0.4964144620	14.0327348000	155.7207608410	153.1351104589	150.3557340753	147.3681172637	144.
12		0.3836771940	0.1743867550	0.1158384660	0.3260975850	51.7172774900	221.3444231792	218.8207206223	216.1079335187	213.1918950852	210.
13		0.2741949630	0.2068815780	0.2808909170	0.2380325420	139.2049217000	342.8361873336	340.3547997244	337.6874980817	334.8203530369	331.
14		0.1063740210	0.0121002430	0.0053227440	0.8762029930	0	123.3820546995	120.6875979189	117.7912628821	114.6779244731	111.
15											
16	Boundary Conditions										
17	v_1(0) =	144.1566579357									
18	v_2(0) =	210.0573770333									
19	v_3(0) =	331.7383916342									
20	v_4(0) =	111.3313250537									
21											
22	Optimal x =	3310.3526409493		Policy							
23	v_1(1) =	147.3681172637		2							
24	v_2(1) =	213.1918950852		2							
25	v_3(1) =	334.8203530369		2							
26	v_4(1) =	114.8998914075		1							
27	v_1(2) =	150.3557340753		2							
28	v_2(2) =	216.1079335187		2							
29	v_3(2) =	337.6874980817		2							
30	v_4(2) =	118.2197244625		1							
31	v_1(3) =	153.1351104589		2							
32	v_2(3) =	218.8207206218		2							
33	v_3(3) =	340.3547997273		2							

Figure 3: EXCEL File for Solving Infinite Horizon Problem With Constraints.

Table 5 gives the optimal values and promotion strategies. For instance, when the promotion cost is 0 and the discount factor is 0.99 , then the optimal strategy is that when the current state is 1, 2 or 3, the promotion should be done when there are some available promotions, i.e. $D_1(p) = D_2(p) = D_3(p) = 1$ for $1 \leq p \leq 4$, and when the current state is 0, no promotion is required, i.e. $D_0(p) = 2$ for $1 \leq p \leq 4$. Their corresponding CLVs $v_i(p)$ for different states and different numbers of remaining promotion are also listed (see the first column in the left hand side of Table 5(a)).

From Table 5, we find that for different values of the fixed promotion cost d, the optimal strategy for the customers in States 1, 2 and 3 is to conduct no promotion. These results are slightly different from those for the finite horizon case. However, the optimal strategy is to conduct all the four promotions to customer with State 0 as early as possible.

	$d=0$			$d=1$			$d=2$		
	$\alpha=0.99$	$\alpha=0.95$	$\alpha=0.90$	$\alpha=0.99$	$\alpha=0.95$	$\alpha=0.90$	$\alpha=0.99$	$\alpha=0.95$	$\alpha=0.90$
x_0	11355	3378	2306	11320	3344	2277	11277	3310	2248
$v_1(1)$	645	149	85	644	148	84	643	147	84
$v_2(1)$	713	215	149	712	214	148	711	213	147
$v_3(1)$	837	337	267	836	336	267	845	335	266
$v_0(1)$	610	117	55	609	116	54	608	115	53
$v_1(2)$	650	154	89	648	152	87	647	150	86
$v_2(2)$	718	219	152	716	218	151	714	216	149
$v_3(2)$	842	341	271	840	339	269	839	338	268
$v_0(2)$	616	122	60	614	120	58	612	118	56
$v_1(3)$	656	158	92	654	156	90	650	153	88
$v_2(3)$	724	224	155	722	221	153	718	219	151
$v_3(3)$	848	345	273	846	343	271	842	340	270
$v_0(3)$	622	127	63	619	124	61	616	121	58
$v_1(4)$	662	162	95	658	159	92	654	158	89
$v_2(4)$	730	228	157	726	225	155	722	221	152
$v_3(4)$	854	349	276	850	346	273	846	343	271
$v_0(4)$	628	131	67	624	128	63	620	124	60
$D_1(1)$	2	2	2	2	2	2	2	2	2
$D_2(1)$	2	2	2	2	2	2	2	2	2
$D_3(1)$	2	2	2	2	2	2	2	2	2
$D_0(1)$	1	1	1	1	1	1	1	1	1
$D_1(2)$	2	2	2	2	2	2	2	2	2
$D_2(2)$	2	2	2	2	2	2	2	2	2
$D_3(2)$	2	2	2	2	2	2	2	2	2
$D_0(2)$	1	1	1	1	1	1	1	1	1
$D_1(3)$	2	2	2	2	2	2	2	2	2
$D_2(3)$	2	2	2	2	2	2	2	2	2
$D_3(3)$	2	2	2	2	2	2	2	2	2
$D_0(3)$	1	1	1	1	1	1	1	1	1
$D_1(4)$	2	2	2	2	2	2	2	2	2
$D_2(4)$	2	2	2	2	2	2	2	2	2
$D_3(4)$	2	2	2	2	2	2	2	2	2
$D_0(4)$	1	1	1	1	1	1	1	1	1

Table 5(a): The Optimal Promotion Strategies and Their CLVs.

	$d=3$			$d=4$			$d=5$		
	$\alpha=0.99$	$\alpha=0.95$	$\alpha=0.90$	$\alpha=0.99$	$\alpha=0.95$	$\alpha=0.90$	$\alpha=0.99$	$\alpha=0.95$	$\alpha=0.90$
x_0	11239	3276	2218	11200	3242	2189	11161	3208	2163
$v_1(1)$	641	146	83	641	146	82	640	145	81
$v_2(1)$	710	212	146	709	211	145	708	211	145
$v_3(1)$	834	334	265	833	333	264	832	332	264
$v_0(1)$	607	114	52	606	113	51	605	112	50
$v_1(2)$	645	149	84	643	147	83	641	145	81
$v_2(2)$	713	214	148	711	213	146	709	211	145
$v_3(2)$	837	336	266	835	334	265	833	333	264
$v_0(2)$	610	116	54	608	114	52	606	112	50
$v_1(3)$	647	151	86	645	148	83	642	146	81
$v_2(3)$	715	216	149	713	214	147	710	211	145
$v_3(3)$	839	338	268	837	336	266	834	333	264
$v_0(3)$	613	119	56	610	116	53	607	113	50
$v_1(4)$	650	152	87	646	149	84	643	146	81
$v_2(4)$	718	218	150	714	215	147	711	212	145
$v_3(4)$	842	340	269	838	337	266	835	334	265
$v_0(4)$	616	121	57	612	117	54	608	113	50
$D_1(1)$	2	2	2	2	2	2	2	2	2
$D_2(1)$	2	2	2	2	2	2	2	2	2
$D_3(1)$	2	2	2	2	2	2	2	2	2
$D_0(1)$	1	1	1	1	1	1	1	1	1
$D_1(2)$	2	2	2	2	2	2	2	2	2
$D_2(2)$	2	2	2	2	2	2	2	2	2
$D_3(2)$	2	2	2	2	2	2	2	2	2
$D_0(2)$	1	1	1	1	1	1	1	1	1
$D_1(3)$	2	2	2	2	2	2	2	2	2
$D_2(3)$	2	2	2	2	2	2	2	2	2
$D_3(3)$	2	2	2	2	2	2	2	2	2
$D_0(3)$	1	1	1	1	1	1	1	1	1
$D_1(4)$	2	2	2	2	2	2	2	2	2
$D_2(4)$	2	2	2	2	2	2	2	2	2
$D_3(4)$	2	2	2	2	2	2	2	2	2
$D_0(4)$	1	1	1	1	1	1	1	1	1

Table 5(b): The Optimal Promotion Strategies and Their CLVs.

4 Summary

In this paper we propose a stochastic dynamic programming model for the optimisation of CLV. Both cases of infinite horizon and finite horizon with budget constraints are discussed. The former case can be solved by using linear programming techniques, the later problem can be solved by using dynamic programming approach. For both cases, they can be implemented easily in an EXCEL spreadsheet.

The models are then applied to practical data of a computer service company. The company makes use of the proposed CLV model to make and maintain value-laden relationships with the customers.

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