Equilibria for Multiclass Routing in Multi-Agent Networks

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Abstract. We study optimal static routing problems in open multiclass networks with state-independent arrival and service rates. Our goal is to study the uniqueness of optimal routing under different scenarios. We consider first the overall optimal policy that is the routing policy whereby the overall mean cost of a job is minimized. We then consider an individually optimal policy whereby jobs are routed so that each job may feel that its own expected cost is minimized if it knows the mean cost for each path. This is related to the Wardrop equilibrium concept in a multiclass framework. We finally study the case of class optimization, in which each of several class of jobs tries to minimize the averaged cost per job within that class; this is related to the Nash equilibrium concept. For all three settings, we show that the routing decisions at optimum need not be unique, but that the utilizations in some large class of links are uniquely determined.

I. Introduction

We consider the problem of optimally routing in networks. Much previous work has been devoted to the routing problem in which at each node one may take new routing decisions. We consider a more general framework in which the sources have to decide how to route their traffic between different existing paths. (These two problems coincide in the case where the set of paths equals to the set of all possible sequences of consecutive directed links which originate at the source and end at the destination.) In ATM (one of the leading architectures for high speed networks) environment, this problem arises when we wish to decide on how to route traffic on a given existing set of virtual paths or virtual connections. Our framework thus allows us to handle routing both in a packet switching as well as in a circuit switching environment. We consider three different frameworks:
(i) Overall optimization criterion, where a single controller makes the routing decisions [9], [10], [12], [17].
(ii) Individual optimality, in which each routed individual chooses its own path so as to minimize its own cost. An individual is assumed to have an infinitesimally small impact on the load in the network and thus on costs of other individuals. This framework has been extensively investigated in transportation science [5], [8], [16], and was also considered in the context of telecommunication [11] and in distributed computing [9], [10], [11]. The suitable optimization concept for this setting is of Wardrop equilibrium [18]; it is defined as a set of routing decisions for all individuals such that a path is followed by an individual if and only if it has the lowest cost for that individual.
(iii) Class optimization; a class may correspond to all the traffic generated by a big organization. It may represent a service provider in a telecommunication in case that it is the service providers that take the routing decisions for their subscribers. A class contains a large amount of individuals and has a nonnegligible impact on the load in the network. Each class wishes to minimize the cost per individual, averaged over all individuals within that class. The suitable optimization concept for this approach is that of Nash equilibrium [8]; it is defined as a set of routing decisions for the different classes such that no class can decrease its own cost by unilaterally deviating from its decision. This approach was used in telecommunication applications in [15], [14], in load balancing problems in distributed computer systems [13], [9], [10] and in transportation science in [8].

An optimization problem does not necessarily have a unique solution. If they are not unique, it is necessary to make clear the range and characteristics of the solutions, in particular, when we calculate numerically the optimal solutions and when we intend to analyse the effects of the system parameters on the optimal solutions. [11] already studied the first two approaches (overall and individual optimization) and characterized the uniqueness for a particular cost structure, that of open BCMP queueing networks [3] with state-independent arrival and service rates. We extend here these results to a fairly general cost function. We also extend substantially results obtained in [15] for the uniqueness of class optimization.

In Section II we provide the notation and some assumptions used in this paper. In Section III we obtain the overall optimal solution, and discuss the uniqueness of the overall optimal solution. In Section IV we show similar results on the uniqueness of the individually optimal solution. Some results on uniqueness for class optimization are presented in Section V. Numerical examples are presented in Section VI, and the paper ends with a concluding section VII.

II. Notation and assumptions

We consider an open network model that consists of a set $\mathcal{M}$ containing M links. We assume that in the network there are pairs of origin and destination points. We call the pair of one origin and one destination points an *O-D pair*. The unit entity that is routed through the network is called a *job*. Each job arrives at one of the origin points and departs from one of the destination points. The origin and destination points of a job are determined when the job arrives in the network. Jobs are classified into J different classes. For the sake of simplicity, we assume that jobs do not change their class while passing through the network. A class k job may have one of several different origin-destination pairs. A class k job with the O-D pair (o, d) originates at node o and destinates for node d through a series of links, which we refer to as a *path*, and then leaves the system. We assume that links are class-dependent directional, i.e. for each class, there is a given direction in which the flow can be sent.

In many previous papers (e.g. [15]), routing could be done at each node. In this paper we follow the more general approach in which a job of class k with O-D pair (o, d) has to choose one of a given finite set of paths (see also [11], [16]). We call this set the *paths* of job class k O-D pair (o, d).

We assume that we can choose the job flow rate of each *path* in order to achieve a performance objective. A path may be a given sequence of links that connect the origin and destination nodes. But we allow path to be some more general object. It may contain a number of subpaths; we assume however that once the job flow rate of a path is given, the job flow rate of each subpath in the path is fully determined (and is not the object of a control decision). That is, the relative flow rate of each subpath in the same path is governed by some fixed transfer proportions (or probabilities) between the links. For example, one may consider paths that include noisy links, where lost packets have to be retransmitted locally over the link. Thus, some given proportion of the traffic in this path use a direct subpath (no losses) whereas other have to loop (this models losses and retransmissions). Another example of a path containing several subpaths is a network in which switches route arriving traffic in some fixed proportions between outgoing links (subpaths); if this proportion is not controlled by the the entity that takes routing decision for the class, then resulting routes from the outgoing links are still considered as a single path. The solution of a routing problem is characterized by the chosen values of job flow rates of all paths.

Notation regarding the network:

$D^{(k)}$ = Set of O-D pairs for class k jobs.

$\Pi_d^{(k)}$ = Set of paths that class k jobs of O-D pair $d \in D^{(k)}$ flow through.

$\Pi^{(k)}$ = Set of all paths for class k jobs, i.e., $\Pi^{(k)} = \bigcup_{d \in D^{(k)}} \Pi_d^{(k)}$.

$$\gamma_{pd}^{kk'} = \begin{cases} 1 & \text{if } p \in \Pi_d^{(k')} \text{ and } k = k', \\ 0 & \text{otherwise.} \end{cases}$$

Notation regarding arrivals to the network and flow rates:

$\phi_d^{(k)}$ = Rate at which class k jobs join O-D pair $d \in D^{(k)}$.

$\phi^{(k)}$ = Total job arrival rate of class k jobs, i.e., $\phi^{(k)} = \sum_{d \in D^{(k)}} \phi_d^{(k)}$.

Φ = System-wide total job arrival rate, i.e., $\Phi = \sum_{k=1}^{J} \phi^{(k)}$.

$x_p^{(k)}$ = Rate at which class k jobs flow through path p.

δ_{lp} = Percentage of the rate $x_p^{(k)}$ that pass through link l, for $p \in \Pi^{(k)}$.
$\lambda_l^{(k)}$ = Rate at which class k jobs visit link l, $\lambda_l^{(k)} = \sum_{p\in\Pi^{(k)}} \delta_{lp} x_p^{(k)}$.
$\phi_d(k)$, $\phi^{(k)}$, Φ and δ_{lp} are given constants (and not decision variables).

Notation regarding service and performance values in the open network:
$\mu_l^{(k)}$ = a constant denoting the service rate of class k jobs at link l.
$\rho_l^{(k)}$ $= \lambda_l^{(k)}/\mu_l^{(k)}$. Utilization of link l for class k jobs.
ρ_l $= \sum_{k=1}^{J} \rho_l^{(k)}$. Total utilization of link l.[1]
$\hat{T}_l^{(k)}$ =Mean cost of class k jobs at link l.
$T_l(\rho_l)$ = Weighted cost per unit flow in link l.
$T_p^{(k)}$ =Average class k cost of path p, $p \in \Pi^{(k)}$, $k = 1, 2, \ldots, J$.[2]
Δ =Overall mean cost of a job (averaged over all classes).
$\Delta^{(k)}$ =Overall mean cost of a job of class k.

Notation regarding vectors and matrices:
$\boldsymbol{\rho} = [\rho_1, \rho_2, \ldots, \rho_M]^{\mathrm{T}}$ where T means 'transpose'. We call this the utilization vector.
$\boldsymbol{\lambda} = [\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_M^{(1)}, \ldots, \lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_M^{(k)}, \ldots,]^{\mathrm{T}}$, i.e, the vector of total flows over all links.
$\boldsymbol{\phi}$ $=[\phi_1^{(1)}, \phi_2^{(1)}, \ldots, \phi_1^{(2)}, \phi_2^{(2)}, \ldots]^{\mathrm{T}}$, i.e., the arrival rate vector.
$\mathbf{x}$ $=[x_1^{(1)}, x_2^{(1)}, \ldots, x_1^{(2)}, x_2^{(2)}, \ldots]^{\mathrm{T}}$, i.e., the path flow rate vector.
$\boldsymbol{\alpha}$ $=[\alpha_1^{(1)}, \alpha_2^{(1)}, \ldots, \alpha_1^{(2)}, \alpha_2^{(2)}, \ldots]^{\mathrm{T}}$, i.e., the vector whose elements are $\alpha_d^{(k)}$, $d \in D^{(k)}$, $k = 1, 2, \ldots, J$; the elements $\alpha_d^{(k)}$ will correspond to some Lagrange multipliers.
$\boldsymbol{\xi}$ $=[\xi_1^{(1)}, \xi_2^{(1)}, \ldots, \xi_1^{(2)}, \xi_2^{(2)}, \ldots]^{\mathrm{T}}$, i.e., the vector whose elements are $\xi_l^{(k)}$, $l \in \mathcal{M}$, $k = 1, 2, \ldots, J$; the elements $\xi_l^{(k)}$ will correspond to some Lagrange multipliers.
$\mathbf{T}$ $=[T_1^{(1)}, T_2^{(1)}, \ldots, T_1^{(2)}, T_2^{(2)}, \ldots]^{\mathrm{T}}$, i.e., the vector whose elements are $T_p^{(k)}, p \in \Pi^{(k)}$, $k = 1, 2, \ldots, J$.
$\mathbf{x}^{(k)}$ $=[x_1^{(k)}, x_2^{(k)}, \ldots]^{\mathrm{T}}$, i.e., the path flow rate vector for class k jobs.
$\boldsymbol{\phi}^{(k)}$, $\boldsymbol{\alpha}^{(k)}$, and $\mathbf{T}^{(k)}$ are defined similarly.
$\mathbf{x}^{-k}$ $=[x_1^{(1)}, x_2^{(1)}, \ldots, x_1^{(k-1)}, \ldots, x_1^{(k+1)}, \ldots]^{\mathrm{T}}$, i.e., the path flow rate vector for jobs of the classes other than class k.

$$\boldsymbol{\Gamma} = \begin{bmatrix} \gamma_{11}^{11} & \gamma_{12}^{11} & \cdots & \gamma_{11}^{12} & \gamma_{12}^{12} & \cdots \\ \gamma_{21}^{11} & \gamma_{22}^{11} & \cdots & \gamma_{21}^{12} & \gamma_{22}^{12} & \cdots \\ \vdots & & \ddots & \vdots & & \ddots \\ \gamma_{11}^{21} & \gamma_{12}^{21} & \cdots & \gamma_{11}^{22} & \gamma_{12}^{22} & \cdots \\ \gamma_{21}^{21} & \gamma_{22}^{21} & \cdots & \gamma_{21}^{22} & \gamma_{22}^{22} & \cdots \\ \vdots & & \ddots & \vdots & & \ddots \end{bmatrix}$$ i.e., the incident matrix

whose (i,j) element is $\gamma_{pd}^{kk'}$, $p \in \Pi^{(k)}$, $k = 1, 2, \ldots, J$, $d \in D^{(k')}$, $k' = 1, 2, \ldots, J$, where $i = p + \sum_{\kappa=1}^{k-1} |\Pi^{(\kappa)}|$ and $j = d + \sum_{\kappa=1}^{k'-1} |D^{(\kappa)}|$.
$\mathbf{x} \cdot \mathbf{y}$ $= \sum_i x_i y_i$, i.e., the inner product of vectors $\mathbf{x} = [x_1, x_2, \ldots]^{\mathrm{T}}$ and $\mathbf{y} = [y_1, y_2, \ldots]^{\mathrm{T}}$.

We make the following assumptions on the cost:

B1: The cost over a path is given by as a weighted sum of link-by-link costs over the path: associated with each link $l \in \mathcal{M}$ there is a cost $T_l(\rho_l)$ per flow unit, that depends on the utilization of the link (The function T_l does not depend on the class k!). There is further a class dependent weight factor $\mu_l^{(k)}$ per link l. Thus, the cost per unit flow of class k on link l is $\hat{T}_l^{(k)} = T_l/\mu_l^{(k)}$. Thus the average cost per unit flow of class k job that passes through path $p \in \Pi^{(k)}$ is

$$T_p^{(k)} = \sum_{l\in\mathcal{M}} \delta_{lp} \hat{T}_l^{(k)} = \sum_{l\in\mathcal{M}} \frac{\delta_{lp}}{\mu_l^{(k)}} T_l(\rho_l). \tag{1}$$

(For examples of such costs, see [11].)

B2: $T_l : [0, \infty) \to [0, \infty]$, and $T_l(0)$ is finite.

B3: The set $\mathcal{M}$ is composed of two disjoint sets of links:
(i) $\mathcal{M}_{\mathcal{I}}$, for which $T_l(\rho_l)$ are convex and strictly increasing (in the interval where they are finite),
(ii) $\mathcal{M}_{\mathcal{C}}$, for which $T_l(\rho_l) = T_l$ are constant (independent of ρ_l).

B4: $T_l(\rho_l)$ are continuous. Moreover, they are continuously differentiable whenever they are finite.

$B1 - B4$ cover in particular the cost that is mostly used in networking games in telecommunications, which is the expected queueing delay in BCMP queueing networks [3] with state-independent arrival and service rates. Denote
$\boldsymbol{\rho}_U$ $= \boldsymbol{\rho}_U(\mathbf{x}) = \boldsymbol{\rho}|_{\rho_l=0, l\in\mathcal{M}_{\mathcal{C}}}$. This is the same as $\boldsymbol{\rho}$ except that $\rho_l = 0$ for all $l \in \mathcal{M}_{\mathcal{C}}$.

The overall mean cost of a job, Δ, can be written as

$$\Delta = \sum_{k=1}^{J} \sum_{p\in\Pi^{(k)}} \frac{x_p^{(k)}}{\Phi} T_p^{(k)} = \frac{1}{\Phi} \sum_{l\in\mathcal{M}} \rho_l T_l(\rho_l).$$

The mean cost of a job of class k, $\Delta^{(k)}$, can be written as

$$\Delta^{(k)} = \sum_{p\in\Pi^{(k)}} \frac{x_p^{(k)}}{\phi^{(k)}} T_p^{(k)} = \frac{1}{\phi^{(k)}} \sum_{l\in\mathcal{M}} \rho_l^{(k)} T_l(\rho_l).$$

Note that the following conditions should be satisfied

$$\text{for each } k = 1, 2, \ldots, J, \quad \sum_{p\in\Pi_d^{(k)}} x_p^{(k)} = \phi_d^{(k)}, \ d \in D^{(k)}, \tag{2}$$

$$x_p^{(k)} \geq 0, \ p \in \Pi^{(k)}. \tag{3}$$

We can express (2) as $\sum_{k'=1}^{J} \sum_{p\in\Pi^{(k')}} \gamma_{pd}^{k'k} x_p^{(k')} = \phi_d^{(k)}$, $d \in D^{(k)}$, or, equivalently,

$$\boldsymbol{\Gamma}^{\mathrm{T}} \mathbf{x} = \boldsymbol{\phi}. \tag{4}$$

III. Overall optimal solution

The problem of minimizing the overall mean cost is

$$\text{minimize:} \quad \Delta \quad = \quad \frac{1}{\Phi} \sum_{l\in\mathcal{M}} \rho_l T_l(\rho_l) \tag{5}$$

$$\text{with respect to } \mathbf{x} \text{ s.t. } \boldsymbol{\Gamma}^{\mathrm{T}} \mathbf{x} = \boldsymbol{\phi}, \quad \mathbf{x} \geq 0, \tag{6}$$

where $\rho_l = \sum_{k=1}^{J} \lambda_l^{(k)}/\mu_l^{(k)}$ and $\lambda_l^{(k)} = \sum_{p\in\Pi^{(k)}} \delta_{lp} x_p^{(k)}$. Note that (6) are the same as (2) and (3), respectively. We call the above problem the *overall optimization problem*, and its solution the *overall optimal solution*. Define
$t_p^{(k)}$ $= \partial(\Phi\Delta)/\partial x_p^{(k)}$, i.e., class k marginal cost of path p, $p \in \Pi^{(k)}, k = 1, 2, \ldots, J$.
$\mathbf{t}$ $=[t_1^{(1)}, t_2^{(1)}, \ldots, t_1^{(2)}, t_2^{(2)}, \ldots]^{\mathrm{T}}$ is the gradient vector of the function $\Phi\Delta$, i.e., the vector whose elements are $t_p^{(k)}, p \in \Pi^{(k)}, k = 1, 2, \ldots, J$.

An application of Kuhn-Tucker conditions yields the following [2]:

Lemma III.1: $\bar{\mathbf{x}}$ is optimal for problem (5) if and only if

$$\mathbf{t}(\bar{\mathbf{x}}) \cdot (\mathbf{x} - \bar{\mathbf{x}}) \quad \geq \quad 0, \quad \text{for all } \mathbf{x} \tag{7}$$
$$\text{such that } \boldsymbol{\Gamma}^{\mathrm{T}} \mathbf{x} \quad = \quad \boldsymbol{\phi} \text{ and } \mathbf{x} \geq 0.$$

From condition B1 we see that Δ depends only on the utilization of each link, ρ_l, which results from the path flow rate matrix. It is possible, therefore, that different values of the path flow rate matrix result in the same utilization of each link and the same minimum mean cost.

We define below the concept of monotonicity of vector-valued functions with vector-valued arguments.

Definition Let $\mathbf{F}(\bullet)$ be a vector-valued function that is defined on a domain $S \subseteq R^n$ and that has values $\mathbf{F}(\mathbf{x})$ in R^n. This function is *monotone* in S if for every pair $\mathbf{x}, \mathbf{y} \in S$ $(\mathbf{x} - \mathbf{y}) \cdot [\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})] \geq 0$. It is *strictly monotone* if, for every pair $\mathbf{x}, \mathbf{y} \in S$ with $\mathbf{x} \neq \mathbf{y}$, $(\mathbf{x} - \mathbf{y}) \cdot [\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})] > 0$.

We need the following property:

[1] Each term in the sum is positive even if the directions of flows are not the same.
[2] $T_l^{(k)}$, T_l and $\hat{T}_l^{(k)}$ are defined at the end of the section (Assumption B1)

Lemma III.2: Assume $B1 - B4$, and let $l \in \mathcal{M}_{\mathcal{I}}$. Then
(i) $T_l(\rho_l)$ is finite if and only if its derivative $T_l'(\rho_l)$ is finite.
(ii) If $T_l'(\rho_l)$ is infinite then for any $\mathbf{x}$ for which the load on link l is ρ_l, the corresponding cost $\Delta(\mathbf{x})$ is infinite.

Proof. (i) Due to the convexity of T_l, $T_l(\rho_l) = \int_0^{\rho_l} T_l'(\zeta_l)d\zeta_l \leq \rho_l T_l'(\rho_l)$. By B2, if $T_l(\rho_l) = \infty$ then $\rho_l > 0$, which implies by the latter equation that $T_l'(\rho_l)$ is infinite.
For the converse, assume that $T_l(\rho_l)$ is finite. Then by continuity, $\exists \epsilon > 0$ such that $T_l(\rho_l + \epsilon)$ is finite. Since T_l is convex, $T_l'(\rho_l) \leq \epsilon^{-1}(T_l(\rho_l + \epsilon) - T_l(\rho_l))$ and is thus finite as well. (ii) If $T_l'(\rho_l)$ is infinite then by (i), $T_l(\rho_l)$ is infinite; moreover, $\rho_l > 0$ by assumption B2, so that $\Delta(\rho) = \infty$ (by (5)). ∎

For the function $\mathbf{t}(\mathbf{x})$ we have the following.

Lemma III.3: Assume $B1 - B4$. Whenever finite, $\mathbf{t}(\mathbf{x})$ is monotone but is not strictly monotone, i.e., for arbitrary $\mathbf{x}$ and $\mathbf{x}'$ ($\mathbf{x} \neq \mathbf{x}'$), if $\Delta(\mathbf{x})$ or $\Delta(\mathbf{x}')$ is finite then

$$(\mathbf{x} - \mathbf{x}') \cdot [\mathbf{t}(\mathbf{x}) - \mathbf{t}(\mathbf{x}')] > 0 \quad \text{if} \quad \boldsymbol{\rho}_U \neq \boldsymbol{\rho}_U', \tag{8}$$
$$= 0 \quad \text{if} \quad \boldsymbol{\rho}_U = \boldsymbol{\rho}_U', \tag{9}$$

where $\boldsymbol{\rho}_U = \boldsymbol{\rho}_U(\mathbf{x})$ and $\boldsymbol{\rho}_U' := \boldsymbol{\rho}_U(\mathbf{x}')$ are the utilization vectors that $\mathbf{x}$ and $\mathbf{x}'$ result in, respectively.
Proof. Assume that $\boldsymbol{\rho}_U \neq \boldsymbol{\rho}_U'$. Then

$$\begin{aligned}
&(\mathbf{x} - \mathbf{x}') \cdot [\mathbf{t}(\mathbf{x}) - \mathbf{t}(\mathbf{x}')] \\
&= \sum_{k=1}^{J} \sum_{p \in \Pi^{(k)}} (x_p^{(k)} - x_p^{\prime(k)})[t_p^{(k)}(\mathbf{x}) - t_p^{(k)}(\mathbf{x}')] \\
&= \sum_{k=1}^{J} \sum_{p \in \Pi^{(k)}} \sum_{l \in \mathcal{M}} (x_p^{(k)} - x_p^{\prime(k)}) \times \qquad (10)\\
&\quad \frac{\delta_{lp}}{\mu_l^{(k)}} \left\{ \left[T_l(\rho_l) - T_l(\rho_l') \right] + \left[\rho_l \frac{dT_l(\rho_l)}{d\rho_l} - \rho_l' \frac{dT_l(\rho_l')}{d\rho_l'} \right] \right\} \\
&= \sum_{l \in \mathcal{M}_{\mathcal{I}}} (\rho_l - \rho_l') \left\{ \left[T_l(\rho_l) - T_l(\rho_l') \right] \right. \qquad (11)\\
&\quad \left. + \left[\rho_l \frac{dT_l(\rho_l)}{d\rho_l} - \rho_l' \frac{dT_l(\rho_l')}{d\rho_l'} \right] \right\} > 0 \qquad (12)
\end{aligned}$$

(The second equality above follows from (1). The last inequality follows from the strict monotonicity of $T_l(\rho_l)$, as well as the fact that its derivative is increasing in ρ_l, and the derivative remains increasing when multiplied by ρ_l. Due to Lemma III.2, if $\Delta(\rho)$ is finite then $T_l'(\rho_l)$ is finite for all links $l \in \mathcal{M}$ (and similarly for $\Delta(\rho')$). The last inequality follows since by condition $B3$, $T_l(\rho_l)$ are strictly monotone and $\rho_l dT_l(\rho_l)/d\rho_l$ are increasing for $l \in \mathcal{M}_{\mathcal{I}}$. Therefore we have the relations (8) and (9). ∎

Theorem III.4: Assume $B1 - B4$ and that there exists some finite feasible solution. Then the utilization in each link $k \in \mathcal{M}_{\mathcal{I}}$ is uniquely determined and is the same for all overall optimal solutions.
Proof. Suppose that the overall optimal policy has two distinct solutions $\hat{\mathbf{x}} =$ and $\tilde{\mathbf{x}}$, which result in the utilization vectors $\hat{\boldsymbol{\rho}}_U := \boldsymbol{\rho}_U(\hat{\mathbf{x}})$ and $\tilde{\boldsymbol{\rho}}_U := \boldsymbol{\rho}_U(\tilde{\mathbf{x}})$, respectively, and $\hat{\boldsymbol{\rho}}_U \neq \tilde{\boldsymbol{\rho}}_U$. Then we have from Lemma III.1, $\mathbf{t}(\hat{\mathbf{x}}) \cdot (\tilde{\mathbf{x}} - \hat{\mathbf{x}}) \geq 0$, $\mathbf{t}(\tilde{\mathbf{x}}) \cdot (\hat{\mathbf{x}} - \tilde{\mathbf{x}}) \geq 0$.
Hence $(\hat{\mathbf{x}} - \tilde{\mathbf{x}}) \cdot [\mathbf{t}(\hat{\mathbf{x}}) - \mathbf{t}(\tilde{\mathbf{x}})] \leq 0$. From Lemma III.3 we have $(\hat{\mathbf{x}} - \tilde{\mathbf{x}}) \cdot [\mathbf{t}(\hat{\mathbf{x}}) - \mathbf{t}(\tilde{\mathbf{x}})] > 0$, since $\hat{\boldsymbol{\rho}}_U \neq \tilde{\boldsymbol{\rho}}_U$. This leads to a contradiction. That is, if there exist two distinct optimal solutions, the utilization vectors of both the solutions must be the same. Note that the utilization of link $l \in \mathcal{M}_{\mathcal{C}}$ is considered always zero. Naturally, in that case, $\sum_{l \in \mathcal{M}_{\mathcal{C}}} \rho_l$ must be unique but each of $\rho_l,\ l \in \mathcal{M}_{\mathcal{C}}$, need not be unique. ∎

Note that even when the utilization in each link is unique, the overall optimal solution may not be unique. This is due to the fact that T depends only on $\boldsymbol{\rho}$ (see (5)) (thus if $\mathbf{x}$ is overall optimal then any solution $\mathbf{x}'$ that gives rise to the same value of $\boldsymbol{\rho}$ will be optimal as well). In Section 5 of [11] there is an example of the cases where more than one optimal solution exists.

Now let us consider the range of the optimal solutions. From the above, we obtain the following relations that characterize the range of the optimal solutions.

$$\sum_{k=1}^{J} \sum_{p \in \Pi^{(k)}} \delta_{lp} \frac{x_p^{(k)}}{\mu_l^{(k)}} = \rho_l, \quad l \in \mathcal{M}_{\mathcal{I}}, \tag{13}$$

$$\sum_{l \in \mathcal{M}_{\mathcal{C}}} \sum_{k=1}^{J} \sum_{p \in \Pi^{(k)}} \delta_{lp} \frac{x_p^{(k)}}{\mu_l^{(k)}} = \sum_{l \in \mathcal{M}_{\mathcal{C}}} \rho_l,$$

and for $k = 1, 2, \ldots, J$,

$$\sum_{p \in \Pi_d^{(k)}} x_p^{(k)} = \phi_d^{(k)},\ d \in D^{(k)}, \tag{14}$$

$$x_p^{(k)} \geq 0,\ p \in \Pi^{(k)}, \tag{15}$$

where the value of each ρ_l is what an optimal solution $\mathbf{x}$ results in. From the relations (13)-(15) we see that optimal path flow rates belong to a convex polyhedron. Then we have the following proposition about the uniqueness of the optimal solutions.

Corollary III.5: The overall optimal solution is unique if and only if the total number of elements in $\mathbf{x}$ does not exceed the number of linearly independent equations in the set of linear equations (13)–(14).

IV. Individually optimal solution

By the individually optimal policy we mean that jobs are scheduled so that each job may feel that its own mean cost is minimum if it knows the mean cost $T_p^{(k)}(\mathbf{x})$ of each path of O-D pair $d, p \in \Pi_d^{(k)}$, $k = 1, ..., J$. By the *individual optimization problem* we mean the problem of obtaining the routing decision that achieves the objective of the individually optimal policy. We call the solution of the individual optimization problem the *individually optimal solution* or the *equilibrium*. In the equilibrium, no user has any incentive to make a unilateral decision to change his route. Wardrop [18] considered this equilibrium for a transportation network and defined it through two principles: a policy is equilibrium if for each individual of a class, the delay along paths which are actually used between the source and destination are (i) the same, and (ii) they are smaller than or equal to the delays along paths not used. It is well known that the solution of Wardrop equilibrium can be obtained by a single mathematical problem that is obtained by a transformation of the cost [16].

We assume that there is a routing decision and that $\mathbf{x}$ is the path flow rate matrix which results from the routing decision. The individually optimal policy requires that a class k job of O-D pair d should follow a path $\hat{p}$ that satisfies

$$T_{\hat{p}}^{(k)}(\mathbf{x}) = \min_{p \in \Pi_d^{(k)}} T_p^{(k)}(\mathbf{x}) \tag{16}$$

for all $d \in D^{(k)},\ k = 1, 2, \ldots, J$. If a routing decision satisfies the above condition we say the routing decision realizes the individually optimal policy.

Definition The path flow rate vector $\mathbf{x}$ is said to satisfy the equilibrium conditions for a multi-class open network if the following relations are satisfied for all $d \in D^{(k)},\ k = 1, 2, \ldots, J$,

$$\begin{aligned}
T_p^{(k)}(\mathbf{x}) &\geq A_d^{(k)},\ x_p^{(k)} = 0, &(17)\\
T_p^{(k)}(\mathbf{x}) &= A_d^{(k)},\ x_p^{(k)} > 0, &(18)\\
\sum_{p \in \Pi_d^{(k)}} x_p^{(k)} &= \phi_d^{(k)}, &(19)\\
x_p^{(k)} &\geq 0,\ p \in \Pi_d^{(k)}, &(20)
\end{aligned}$$

$$\text{where } A_d^{(k)} = \min_{p \in \Pi_d^{(k)}} T_p^{(k)}(\mathbf{x}),\ \ d \in D^{(k)}, k = 1, \ldots, J.$$

Note that (17)-(20) are identical to the relations

$$\begin{aligned}
[\mathbf{T}(\mathbf{x}) - \boldsymbol{\Gamma}\mathbf{A}] \cdot \mathbf{x} &= 0, &(21)\\
\mathbf{T}(\mathbf{x}) - \boldsymbol{\Gamma}\mathbf{A} &\geq 0, &(22)\\
\boldsymbol{\Gamma}^{\mathrm{T}}\mathbf{x} - \boldsymbol{\phi} &= 0, &(23)\\
\mathbf{x} &\geq 0, &(24)
\end{aligned}$$

where $\mathbf{A} = [A_1^{(1)}, A_2^{(1)}, \ldots, A_1^{(2)}, A_2^{(2)}, \ldots]^{\mathrm{T}}$, i.e., the vector whose elements are $A_d^{(k)},\ d \in D^{(k)},\ k = 1, 2, \ldots, J$. The above definition is the natural extension of the notion of Wardrop [18] equilibrium to our setting.

Theorem IV.1: Assume $B1-B4$. There exists an individually optimal solution $\mathbf{x}$ which satisfies the relations (21)-(24).
Proof. Define $\tilde{T}(\mathbf{x})$ by

$$\tilde{T}(\mathbf{x})=\frac{1}{\Phi}\Big[\sum_{l\in\mathcal{M}_{\mathcal{I}}}\int_0^{\rho_l}T_l(s)ds+\sum_{l\in\mathcal{M}_{\mathcal{C}}}\rho_l T_l\Big],$$

Note that $\tilde{T}(\mathbf{x})$ is a convex increasing function of $\mathbf{x}$. Then by (1),

$$T_p^{(k)}(\mathbf{x})=\frac{\partial}{\partial x_p^{(k)}}(\Phi\tilde{T}(\mathbf{x})).$$

Introduce the following convex nonlinear program: minimize $\tilde{T}(\mathbf{x})$ with respect to $\mathbf{x}$ s.t. (23)-(24). The Kuhn-Tucker conditions are the same as (21)-(24). Therefore, the program should have an optimal solution which must satisfy relations (21)-(24). ∎

Corollary IV.2: Assuming $B1-B4$, $\bar{\mathbf{x}}$ is an individually optimal solution if and only if it is feasible and $\mathbf{T}(\bar{\mathbf{x}})\cdot(\mathbf{x}-\bar{\mathbf{x}})\geq 0$, for all $\mathbf{x}$ s.t. $\mathbf{\Gamma}^{\mathrm{T}}\mathbf{x}=\boldsymbol{\phi}$ and $\mathbf{x}\geq 0$.
Proof. Similar to the proof of Lemma III.1. ∎

Lemma IV.3: Assume $B1-B4$. Whenever finite, the function $\mathbf{T}(\mathbf{x})$ is monotone but is not strictly monotone. That is, for arbitrary $\mathbf{x}$ and $\mathbf{x}'$ $(\mathbf{x}\neq\mathbf{x}')$, if $\mathbf{T}(\mathbf{x})$ are finite or $\mathbf{T}(\mathbf{x}')$ are finite then

$$(\mathbf{x}-\mathbf{x}')\cdot[\mathbf{T}(\mathbf{x})-\mathbf{T}(\mathbf{x}')]>0 \text{ if } \boldsymbol{\rho}_U\neq\boldsymbol{\rho}'_U, \tag{25}$$
$$=0 \text{ if } \boldsymbol{\rho}_U=\boldsymbol{\rho}'_U \tag{26}$$

where $\boldsymbol{\rho}_U$ and $\boldsymbol{\rho}'_U$ are the utilization vectors that $\mathbf{x}$ and $\mathbf{x}'$ result in, respectively.
Proof. This Lemma can be proved by the same way as that for the Lemma III.3. Assume that $\boldsymbol{\rho}_U\neq\boldsymbol{\rho}'_U$. Then

$$\begin{aligned}&(\mathbf{x}-\mathbf{x}')\cdot[\mathbf{T}(\mathbf{x})-\mathbf{T}(\mathbf{x}')]\\ =&\sum_{k=1}^{J}\sum_{p\in\Pi^{(k)}}\sum_{l\in\mathcal{M}_{\mathcal{I}}}(x_p^{(k)}-x_p^{'(k)})\times\frac{\delta_{lp}}{\mu_l^{(k)}}(T_l(\rho_l)-T_l(\rho'_l))\\ =&\sum_{l\in\mathcal{M}_{\mathcal{I}}}(\rho_l-\rho'_l)(T_l(\rho_l)-T_l(\rho'_l))>0\end{aligned}$$

The last inequality follows since by B3, $T_l(\rho_l)$ are strictly monotone for $l\in\mathcal{M}_{\mathcal{I}}$. Therefore we have the relations (25) and (26). ∎

Theorem IV.4: Assume $B1-B4$. Then all equilibria, for which all users have finite cost, have the same utilization on links $l\in\mathcal{M}_{\mathcal{I}}$.
Proof. We can prove this in the same way as Theorem III.4. ∎

Here again, individually optimal solution may not be unique. The range of the individually optimal solutions (related to finite costs) is given by the same set of relations as (13)-(15) but with possibly different values of $\rho_l,\ l=1,2,\ldots,M$.

Next, we illustrate the uniqueness of the utilization is indeed restricted to equilibria with finite cost. Consider the following network. There are 4 nodes: $\{1,2,3,4\}$ and 1 class. The set of links is $\{(12),(13),(24),(34),(23)\}$. There is an amount of flow of $\phi=\phi^{(1)}=1$ to ship between the source node 1 and the destination node 4. The cost per link is given by $T_l(\rho_l)=(1-\rho_l)^{-1}$. The strategy in which all the flow goes along the path $(1\rightarrow 2\rightarrow 3\rightarrow 4)$ is individually optimal. Indeed, given that all users follow this path, no individual can decrease his cost by choosing another path. This gives rise to infinite cost for all individuals. However, there exists another individual optimal strategy: to route half of the flow along the path $1\rightarrow 2\rightarrow 4$ and the other half through the path $1\rightarrow 3\rightarrow 4$. This is the unique equilibrium that has finite cost for all users.

V. Class optimal solution

We present below equivalent characterizations of the class optimal solution and obtain new uniqueness results. The question of uniqueness for the class optimal solution has only been treated for some special cases [1], [15]. A counter example in [15] shows that different class optimal solutions may exist, with different utilizations. The following assumption will be made throughout:
G: If not all classes have finite cost then at least one of the classes which has infinite cost can change its own flow to make this cost finite.

A. Problem formulation

By the class optimal policy we mean that jobs are scheduled so that the expected cost of each class may be minimum under the condition that the scheduling decisions on jobs of the other classes are given and fixed. By the *class optimization problem* we mean the problem of obtaining the routing decision $\mathbf{x}$ that achieves the objective of the class optimal policy. We call the solution of the class optimization problem the *class optimal solution* or the *Nash equilibrium*. In the Nash equilibrium, no class has any incentive to make a unilateral decision to change the decision on the routes of the jobs of the class.

Assumption G above implies that in any Nash equilibrium, all classes have finite costs.

We assume that there is a routing decision and that $\mathbf{x}$ is the path flow rate matrix which results from the routing decision. The class optimal policy requires that

$$\Delta^{(k)}(\mathbf{x}^{(k)},\mathbf{x}^{-k})=\min_{\mathbf{x}'^{(k)}}\Delta^{(k)}(\mathbf{x}'^{(k)},\mathbf{x}^{-k}) \tag{27}$$

for all $k=1,2,\ldots,J$ ($\Delta^{(k)}(\mathbf{x}'^{(k)},\mathbf{x}^{-k})$ is the overall mean cost of a job of class k given that other classes use flow rate $\mathbf{x}^{-k}$, and class k uses $\mathbf{x}'^{(k)}$). If $\mathbf{x}$ satisfies the above condition we say that it realizes the class optimal policy. The problem of minimizing the mean cost for jobs of class k is:

$$\text{minimize: }\Delta^{(k)}=\frac{1}{\phi^{(k)}}\sum_{l\in\mathcal{M}}\rho_l^{(k)}T_l(\rho_l) \tag{28}$$

with respect to $\mathbf{x}^{(k)}$ with $\mathbf{x}^{-k}$ being fixed subject to $\mathbf{\Gamma}^{\mathrm{T}}\mathbf{x}=\boldsymbol{\phi},\ \mathbf{x}\geq 0$.

B. Variational inequalities and Kuhn-Tucker conditions

As in the previous sections we can get the variational inequalities form by using the same reasoning as before. First we define
$\tilde{t}_p^{(k)}\ =\partial(\phi^P(k)\Delta^{(k)})/\partial x_p^{(k)}$, i.e., class k marginal class-cost of path $p,\ p\in\Pi^{(k)}, k=1,2,\ldots,J$.
$\tilde{\mathbf{t}}\ =[\tilde{t}_1^{(1)},\tilde{t}_2^{(1)},\ldots,\tilde{t}_1^{(2)},\tilde{t}_2^{(2)},\ldots]^{\mathrm{T}}$, i.e., the vector whose elements are $\tilde{t}_p^{(k)}, p\in\Pi^{(k)}, k=1,2,\ldots,J$.
$\tilde{\mathbf{t}}^{(k)}\ =[\tilde{t}_1^{(k)},\tilde{t}_2^{(k)},\ldots]^{\mathrm{T}}$.

The following characterization of an optimal solution can be obtained by applying Kuhn-Tucker conditions (see [2] for details):

Lemma V.1: $\mathbf{x}$ is an optimal solution of problem (27) if and only if $\mathbf{x}$ is feasible and it satisfies the following conditions

$$[\tilde{\mathbf{t}}(\mathbf{x})-\mathbf{\Gamma}\boldsymbol{\alpha}]\cdot\mathbf{x}\ =\ 0, \tag{29}$$
$$\tilde{\mathbf{t}}(\mathbf{x})-\mathbf{\Gamma}\boldsymbol{\alpha}\ \geq\ 0, \tag{30}$$
$$\mathbf{\Gamma}^{\mathrm{T}}\mathbf{x}-\boldsymbol{\phi}\ =\ 0, \tag{31}$$
$$\mathbf{x}\ \geq\ 0. \tag{32}$$

We can express the class optimal solution in the variational inequality form by using the same way as that for the overall optimal solution as follows.

Corollary V.2: Assume $B1-B4$. $\bar{\mathbf{x}}$ is a class optimal solution if and only if it is feasible and $\tilde{\mathbf{t}}(\bar{\mathbf{x}})\cdot(\mathbf{x}-\bar{\mathbf{x}})\geq 0$, for all $\mathbf{x}$, s.t. $\mathbf{\Gamma}^{\mathrm{T}}\mathbf{x}=\boldsymbol{\phi}$ and $\mathbf{x}\geq 0$.
Proof. Similar to the proof of Lemma III.1. ∎

C. All positive flows

We make the following assumptions:
(i) $\mu_l^{(k)}$ can be represented as $a^{(k)}\mu_l$, and $0<\mu_l^{(k)}$ is finite.
(ii) At each node, each class may re-route all the flow that it sends through that node to any of the out-going links of that node. Thus the set of paths for class k equals to the set of all possible sequences of consecutive directed links which originate at a source s and end at the destination d, $sd\in D^{(k)}$.
(iii) The rate of traffic of class k that enters the network at node v is given by $\phi_v^{(k)}$; if this quantity is negative this means that traffic of class k leaves node v at a rate of $|\phi_v^{(k)}|$. We assume that $\sum_v\phi_v^{(k)}=0$.

For each node u and class k, denote by $In(u,k)$ the set of its ingoing links, and denote by $Out(u,k)$ the set of its out-going links. Due to the second assumption, we may work directly with the decision variables $\lambda_l^{(k)}$ instead of working with the path flows. For each node

v we can then replace (4) by: $\sum_{l\in Out(v,k)} \lambda_l^{(k)} = \sum_{l\in In(v,k)} \lambda_l^{(k)} + \phi_v^{(k)}$. Define the Lagrangian

$$L^{(k)}(\boldsymbol{\lambda}, \boldsymbol{\xi}^{(k)}) = \sum_{l\in\mathcal{M}} \rho_l^{(k)} T_l - \sum_u \xi_u^{(k)} \left[\sum_{l\in Out(u,k)} \lambda_l^{(k)} - \sum_{l\in In(u,k)} \lambda_l^{(k)} - \phi_u^{(k)}\right].$$

Here, $\boldsymbol{\xi}^{(k)} = [\xi_1^{(k)}, \xi_2^{(k)}, \ldots, \xi_M^{(k)}]^{\mathrm{T}}$ is the vector of Lagrange multipliers for class k. An assignment $\boldsymbol{\lambda}^*$ is class-optimal if and only if the following Kuhn-Tucker conditions hold. There exists some $\boldsymbol{\xi}^{(k)} = [\xi_u^{(k)}]$ such that

$$\frac{\partial L^{(k)}(\boldsymbol{\lambda}^*, \boldsymbol{\xi}^{(k)})}{\partial \lambda_l^{(k)}} \geq 0, \tag{33}$$

$$\frac{\partial L^{(k)}(\boldsymbol{\lambda}^*, \boldsymbol{\xi}^{(k)})}{\partial \lambda_l^{(k)}} = 0 \text{ if } , \lambda_l^{(k)} > 0; \tag{34}$$

$$\lambda_l^{(k)} \geq 0, \quad \sum_{l\in Out(v,k)} \lambda_l^{(k)} = \sum_{l\in In(v,k)} \lambda_l^{(k)} + \phi_v^{(k)}.$$

Define $K_l^{(k)}(\rho_l^{(k)}, \rho_l) = \frac{\partial \rho_l^{(k)} T_l(\rho_l)}{\partial \lambda_l^{(k)}}$.. Then

$$K_l^{(k)}(\rho_l^{(k)}, \rho_l) = \frac{1}{\mu_l^{(k)}} \left(\rho_l^{(k)} \frac{\partial T_l(\rho_l)}{\partial \rho_l} + T_l(\rho_l)\right)$$

Conditions (33)-(34) can be rewritten as

$$K_l^{(k)}(\rho_l^{(k)}, \rho_l) \geq \xi_u^{(k)} - \xi_v^{(k)}, \tag{35}$$

with equality if $\lambda_l^{(k)} > 0$ and $l = (u, v)$. Note that condition B3 implies that $K_l^{(k)}(\rho_l^{(k)}, \rho_l)$ is strictly monotonically increasing in both arguments.

Lemma V.3: Assume $B1 - B4$. Assume that $\boldsymbol{\lambda}$ and $\hat{\boldsymbol{\lambda}}$ are two class-optimal solutions with finite costs. If $\rho_l = \hat{\rho}_l$ for all links l of type $\mathcal{M}_{\mathcal{I}}$ then $\lambda_l^{(k)} = \hat{\lambda}_l^{(k)}$, $k = 1, ..., J$.

Proof: Assume that under the assumptions of the Lemma the conclusions do not hold. Then $\exists l \in \mathcal{M}_{\mathcal{I}}$ and some k such that

$$\hat{\lambda}_l^{(k)} > \lambda_l^{(k)}. \tag{36}$$

We now construct another network with the same nodes and links as the original one, with the flow on a link l between two points u and v given by $|\hat{\lambda}_l^{(k)} - \lambda_l^{(k)}|$, its direction is (uv) if and only if $\hat{\lambda}_{uv}^{(k)} - \lambda_{uv}^{(k)} > 0$ and is otherwise (vu). In this network there are no inputs and outputs. It follows from (36) that the network contains a cycle $\mathcal{C}$ with strictly positive flow.

We now consider any link $(uv) \in \mathcal{C}$. Then in the original network either (uv) is the direction of the flow of class k and $\hat{\lambda}_{(uv)}^{(k)} > \lambda_{(uv)}^{(k)}$, or the direction is (vu) and $\hat{\lambda}_{(vu)}^{(k)} < \lambda_{(vu)}^{(k)}$. In the first case we have by Kuhn-Tucker conditions

$$\hat{\xi}_u^{(k)} - \hat{\xi}_v^{(k)} = K_{(uv)}^{(k)}(\hat{\rho}_{(uv)}^{(k)}, \hat{\rho}_{(uv)}) \geq K_{(uv)}^{(k)}(\rho_{(uv)}^{(k)}, \rho_{(uv)}) = \xi_u^{(k)} - \xi_v^{(k)} \tag{37}$$

In the second case, we have

$$\begin{aligned} \xi_v^{(k)} - \xi_u^{(k)} &= K_{(vu)}^{(k)}(\rho_{(vu)}^{(k)}, \rho_{(vu)}) \\ &\geq K_{(vu)}^{(k)}(\hat{\rho}_{(vu)}^{(k)}, \hat{\rho}_{(vu)}) = \hat{\xi}_v^{(k)} - \hat{\xi}_u^{(k)} \end{aligned} \tag{38}$$

Due to the strict monotonicity of K for $l \in \mathcal{M}_{\mathcal{I}}$, there is at least one link in $\mathcal{C}$ for which a strict inequality holds in the corresponding inequality among (37) and (38). This implies that

$$0 = \sum_{i=1}^{|\mathcal{C}|} (\hat{\xi}_i^{(k)} - \hat{\xi}_{i-1}^{(k)}) > \sum_{i=1}^{|\mathcal{C}|} (\xi_i^{(k)} - \xi_{i-1}^{(k)}) = 0$$

which is a contradiction. Thus the Lemma is established. ∎

Theorem V.4: Assume $B1 - B4$. Denote by $\mathcal{M}_1(\boldsymbol{\lambda})$ the sets of links l such that $\lambda_l^{(k)} > 0, \forall k = 1, ..., J$ for an assignment $\boldsymbol{\lambda}$. Assume that $\boldsymbol{\lambda}$ and $\hat{\boldsymbol{\lambda}}$ are two class-optimal solutions with finite costs for all players. Assume that $\lambda_l^{(k)} = 0, \forall k, \forall l \notin \mathcal{M}_1(\boldsymbol{\lambda})$, $\hat{\lambda}_l^{(k)} = 0, \forall k, \forall l \notin \mathcal{M}_1(\hat{\boldsymbol{\lambda}})$. Then $\lambda_l^{(k)} = \hat{\lambda}_l^{(k)}$ for all $l \in \mathcal{M}_{\mathcal{I}}$.

Proof. Denote $\xi_u = \sum_{k=1}^J a^{(k)} \xi_u^{(k)}$, and

$$S_l(\rho_l) = \sum_k \mu_l^{(k)} K_l^{(k)}(\rho_l^{(k)}, \rho_l) = \rho_l \frac{\partial T_l(\rho_l)}{\partial \rho_l} + J T_l(\rho_l).$$

Note that the assumption that costs are finite and Lemma III.2 imply that $S_l(\rho_l)$ are finite and Assumption B3 implies that $S_l(\rho_l)$ is strictly monotone. Let $\hat{\boldsymbol{\xi}}$ denote the vector of the Lagrange multipliers corresponding to $\hat{\boldsymbol{\lambda}}$. (35) implies that

$$\mu_{uv}^{-1} S_{uv}(\rho_{uv}) \geq \xi_u - \xi_v, \tag{39}$$

with equality for $(u, v) \in \mathcal{M}_1(\boldsymbol{\lambda})$. A similar relation holds for $\hat{\boldsymbol{\lambda}}$. We obtain that

$$\begin{aligned} 0 &\leq \sum_{(u,v)\in\mathcal{M}} (\rho_{uv} - \hat{\rho}_{uv})(S_{uv}(\rho_{uv}) - S_{uv}(\hat{\rho}_{uv})) \\ &\leq \sum_{(u,v)\in\mathcal{M}} \mu_{uv}(\rho_{uv} - \hat{\rho}_{uv})\Big((\xi_u - \hat{\xi}_u) - (\xi_v - \hat{\xi}_v)\Big) = 0 \end{aligned} \tag{40}$$

The first inequality follows from the strict monotonicity of $S_l(\rho_l)$ for $l \in \mathcal{M}_{\mathcal{I}}$; for $l \in \mathcal{M}_{\mathcal{C}}$ this relation is trivial. The second inequality holds in fact for each pair u, v (and not just for the sum). Indeed, for $(u, v) \in \mathcal{M}_1(\boldsymbol{\lambda}) \cap \mathcal{M}_1(\hat{\boldsymbol{\lambda}})$ this relation holds with equality due to (39). This is also the case for $(u, v) \notin \mathcal{M}_1(\boldsymbol{\lambda}) \cup \mathcal{M}_1(\hat{\boldsymbol{\lambda}})$, since in that case $\rho_{uv} = \hat{\rho}_{uv} = 0$. Consider next the case $(u, v) \in \mathcal{M}_1(\boldsymbol{\lambda}), (u, v) \notin \mathcal{M}_1(\hat{\boldsymbol{\lambda}})$. Then we have

$$\begin{aligned} &(\rho_{uv} - \hat{\rho}_{uv})(S_{uv}(\rho_{uv}) - S_{uv}(\hat{\rho}_{uv})) \\ &\quad= \rho_{uv}(S_{uv}(\rho_{uv}) - S_{uv}(\hat{\rho}_{uv})) \\ &\quad\leq \mu_{uv}\rho_{uv}\Big((\xi_u - \hat{\xi}_u) - (\xi_v - \hat{\xi}_v)\Big). \end{aligned}$$

A symmetric argument establishes the case $(u, v) \in \mathcal{M}_1(\hat{\boldsymbol{\lambda}}), (u, v) \notin \mathcal{M}_1(\boldsymbol{\lambda})$. We finally establish the last equality in (40).

$$\begin{aligned} &\sum_{(u,v)\in\mathcal{M}} \mu_{uv}(\rho_{uv} - \hat{\rho}_{uv})\Big((\xi_u - \hat{\xi}_u) - (\xi_v - \hat{\xi}_v)\Big) \\ &= \sum_j (\xi_j - \hat{\xi}_j) \sum_w (\rho_{jw} - \hat{\rho}_{jw})\mu_{jw} \\ &\quad - \sum_j (\xi_j - \hat{\xi}_j) \sum_w (\rho_{wj} - \hat{\rho}_{wj})\mu_{wj} \\ &= \sum_{k=1}^J \Big(\sum_j (\xi_j - \hat{\xi}_j) \sum_w (\rho_{jw}^{(k)} - \hat{\rho}_{jw}^{(k)})\mu_{jw} \\ &\quad - \sum_j (\xi_j - \hat{\xi}_j) \sum_w (\rho_{wj}^{(k)} - \hat{\rho}_{wj}^{(k)})\mu_{wj}\Big) \\ &= \sum_{k=1}^J \frac{1}{a^{(k)}} \Big[\sum_j (\xi_j - \hat{\xi}_j)\Big(\sum_{l\in Out(j,k)} (\lambda_l^{(k)} - \hat{\lambda}_l^{(k)}) \\ &\quad - \sum_{l\in In(j,k)} (\lambda_l^{(k)} - \hat{\lambda}_l^{(k)})\Big)\Big] = 0 \end{aligned}$$

(we used the fact that the sum of $\phi_v^{(k)}$ over all nodes v equals zero so that the difference of ingoing and outgoing lambda's is also zero). We conclude from (40) that $\rho_l = \hat{\rho}_l$ for all links in $\mathcal{M}_{\mathcal{I}}$. The proof follows from Lemma V.3. ∎

Remark V.1: The Theorem and its proof are substantial extensions of [15] who considered the special case where μ_l^i do not depend on l and i, where there is a *single source-destination* pair which is the same for all users (all paths and all classes), and where $\mathcal{M}_1(\boldsymbol{\lambda}) = \mathcal{M}_1(\hat{\boldsymbol{\lambda}})$. Moreover, the costs of all links are assumed in [15] to be strictly increasing.

Next, we present an example of load balancing [13] that occurs in distributed computing, in which different classes have different sources and where our uniqueness result may apply.

Example V.1: There are two processors and a single communication means that connects them. Nodes are numbered 1 and 2. We associate a class to each node (and thus have two players in the game). Node i has the external arrival of jobs to process with rate ϕ_i and it has to decide what fraction of the arriving jobs would be processed locally and what fraction should be forwarded to the other node. Delay is incurred at each node (processing delay) as well as in the communication buss (communication delay), and the goal of each class is to minimize the average delay of jobs of that class. The delay at each network element (nodes and communication buss) is an increasing function of the total job rates that use that element (thus the decisions of one class also influence the cost for the other class). This load balancing problem can be modelled as a network game that consists of three nodes and three links:

- Nodes: s_1, s_2, d, where s_i is the source of jobs of class i, and d is a common destination.
- Links: $s_i d$, $i = 1, 2$ represent the processor i, and $s_1 s_2$ represents the communication bus.
- Paths: Class i has two paths, $s_i \to d$ (corresponding to local processing) and path $s_i \to s_j \to d$, that corresponds to forwarding jobs to the other processor.

This network model is depicted in Fig. 1. We conclude that for the

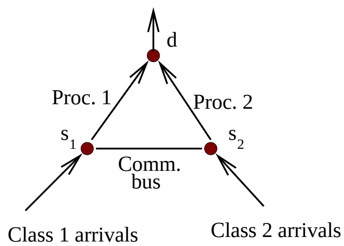

Fig. 1. A network representation of the load balancing problem

above problem, there is at most one equilibrium (under the appropriate assumptions on the delay functions) at which each class splits its arrival flows: a fraction is processed locally and a fraction is forwarded. Numerical examples can be found in [13] (in which the problem of the uniqueness of the equilibrium was not addressed).

VI. Numerical Examples

Consider a simple example of a network composed of two parallel links $\mathcal{M} = \{a, b\}$ and J identical classes: Each link can be identified with a path. We consider for simplicity $\mu_a^{(k)} = 1, \mu_b^{(k)} = 2, k = 1, ..., J$. Consider an M/M/1 type cost, i.e. $\hat{T}_l = 1/(\mu_l^{(k)}(1-\rho_l))$, $l = a, b,\ k = 1, ..., J$ (T_l is infinite for $\rho_l \geq 1$). Let $\phi^k = 2/J$. We note that $\rho_a = \sum_j \lambda_a^{(j)} = 2 - \sum_j \lambda_b^{(j)} = 2(1-\rho_b)$. Hence, $\rho_a < 1$ implies that $\rho_b > 0.5$. We have:

$$J\Delta = \sum_{l=a,b} \rho_l T_l(\rho_l) = \sum_{l=a,b} \frac{\rho_l}{1-\rho_l} = \frac{2(1-\rho_b)}{2\rho_b - 1} + \frac{\rho_b}{1-\rho_b}.$$

The overall optimal solution is obtained at $\rho_b^* = \sqrt{1/2}$, which gives $\rho_a^* = 2 - \sqrt{2}$ and $\Delta(\rho) = (2\sqrt{2}+1)/J$. In order to obtain the individual optimization, we note that $T_l^{(k)} = 1/(\mu_l(1-\rho_l)),\ k = 1, ..., J$. This gives $T_a^{(k)} = (2\rho_b - 1)^{-1}$, $T_b^{(k)} = (2(1-\rho_b))^{-1}$, $k = 1, ..., J$. The individual optimum is obtained at $\bar{\rho}_a = 1/4$, $\bar{\rho}_b = 3/4$, which gives delays along the two links of $T_a^{(k)} = T_b^{(k)} = 2$.

In both cases the solution in terms of the ρ_l's is unique. Any choice of rates $x_l^{(k)}$ that gives the corresponding ρ_l is optimal, and it is clearly not unique. For example, if $J = 2$, $x_l^{(k)} = \mu_l \rho_l^*/2,\ l = a, b,\ k = 1, 2$ is an overall optimal solution and $x_l^{(k)} = \mu_l \bar{\rho}_l/2,\ l = a, b,\ k = 1, 2$ is an individually optimal solution. Another overall optimal solution is $x_a^{(1)} = \rho_a^{(1)} = \rho_a^*,\ x_b^{(2)} = 1 - x_a^{(1)},\ x_a^{(2)} = 0, x_b^{(2)} = 1$, and another individually optimal solution is $x_a^{(1)} = \rho_a^{(1)} = \bar{\rho}_a,\ x_b^{(2)} = 1 - x_a^{(1)},\ x_a^{(2)} = 0, x_b^{(2)} = 1$. Unlike the overall and individually optimal solutions, the class optimal solution for this problem is indeed unique, as has been shown in [15].

VII. Concluding remarks and perspectives

We studied *multiclass* static routing problems with several types of optimization concepts in networks: the overall optimization, individual optimization and class optimization. The routing problem is of the type studied in [11], where one has to determine the assignment of the flow rates among different paths. Our flow allocation model is a simplification of the most general ones expected to be encountered in actual communication networks. In particular, we considered a single cost per decision maker which is based on additive link costs. This model covers costs such as expected delays, but may fall short of covering other types of costs such as loss probabilities or call rejection rates. We should mention that in practice network conditions may change frequently; this means that one should update the routing decisions from time to time. We believe that our static optimization could be a starting point for the design of future distributed adaptive routing protocol (see e.g. [7]).

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