Properties of Equilibria in Competitive Routing with Several User Types

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Abstract—**In recent years there has been a growing interest in mathematical models for routing in networks in which the decisions are taken in a non-cooperative way. Instead of a single decision maker (that may represent the network) that chooses the routes so as to maximize a global utility, one considers a number of decision makers having each its own utility to maximize by routing its own flow. This gives rise to the use of non-cooperative game theory and the Nash equilibrium concept for optimality. In the special case in which each decision maker wishes to find a minimal path for each routed object (e.g. a packet) then the solution concept is the Wardrop equilibrium. It is well known that equilibria may exhibit inefficiencies and paradoxical behaviour, such as the famous Braess paradox (in which the addition of a link to a network results in worse performance to all users). This raises the challenge for the network administrator of how to upgrade the network so that it indeed results in improved performance. We present in this paper some guidelines for that.**

Keywords— **Computer Communication Network, Routing, Noncooperative Games, Braess Paradox.**

I. Introduction

In this paper, we consider the problem of routing, in which the performance measure to be minimised is some general cost (which could represent the expected delay). We assume that some objects, are routed over shortest paths computed in terms of that cost. An object could correspond to a whole session in case all packets of a connection are assumed to follow the same path. It could correspond to a single packet if each packet could have its own route. A routing approach in which each packet follows a shortest delay path has been advocated in Ad-hoc networks [10], in which, the large amount of mobility of both users as well as of the routers requires to update the routes frequently; it has further been argued that by minimising the delay of each packet, we minimise re-sequencing delays, that may be harmful in real time applications, but also in data communications (indeed, the throughput of TCP/IP connections may quite deteriorate when packets arrive out of sequence, since the latter is frequently interpreted wrongly as a signal of a loss or of a congestion).

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When the above type of routing approach is used then the expected load at different links in the network can be predicted as an equilibrium which can be computed in a way similar to equilibria that arise in road traffic. The latter is known as a Wardrop equilibrium [24] (it is known to exist and to be unique under general assumptions on the topology and on the cost [22, p. 74-75]). We study in this paper some properties of the equilibrium. In particular, we are interested in the impact of the demand, of link capacities and of the topology on the performance measures at equilibrium. This has a particular significance for the network administrator or designer when it comes to upgrading the network.

A frequently used heuristic approach for upgrading a network is through *Bottleneck Analysis*. A system bottleneck is defined as "a resource or service facility whose capacity seriously limits the performance of the entire system" [15, p. 13]. Bottleneck analysis consists of adding capacity to identified bottlenecks until they cease to be bottlenecks. In a non-cooperative framework, however, this approach may have devastating effects; adding capacity to a link (and in particular, to a bottleneck link) may cause delays of all users to increase; in an economic context in which users pay the service provider, this may further cause a decrease in the revenues of the provider. The first problem has already been identified in the road-traffic context by Braess [4] (see also [8], [23]), and have further been studied in the context of queueing networks [3], [5], [6], [7]. In the latter references both queueing delay as well as rejection probabilities have been considered as performance measure. The focus of Braess paradox on the bottleneck link in a queueing context, as well as the paradoxical impact on the service provider have been studied in [20]. In all the above references, the paradoxical behaviour occurs in models in which the number of users is infinitely large, and the equilibrium concept is that of Wardrop equilibrium [24]. Yet the problem may occur also in models involving finite number of players (e.g. service providers) for which the Nash equilibrium is the optimality concept. This has been illustrated in [16], [18]. The Braess paradox has further been identified and studied in the context of distributed computing [12], [13] where arrivals of jobs may be routed and performed on different processors.

The Braess paradox illustrates that the network designer or service providers, or more generally, whoever is responsible to the network topology and link capacities, have to take into consideration the reaction of non-cooperative users to his decisions. Some upgrading guidelines have been proposed in [16], [17], [18] so as to avoid the Braess paradox or so as to obtain a better performance. They considered not only the framework of the Wardrop equilibrium, but also the Nash-equilibrium concept in which the a finite number of service providers each try to minimise the average delays (or cost) for all the flow generated by its subscribers. The results obtained for the Wardrop equilibrium were restricted to a particular cost representing the delay of an M/M/1 queue at each link. In this paper we extend the above results to general costs. We further consider a more general routing structure (between paths and not just between links) and allow for several classes of users (so that the cost of a path or of a link may depend on the class in some way). Some other guidelines for avoiding Braess paradox in the setting of Nash equilibrium have been obtained in [1], yet in that setting the guidelines turn out to be much more restrictive those we obtain for the setting of Wardrop equilibrium.

The main objective of this present paper is to pursue that direction and to provide new guidelines for avoiding the Braess paradox when upgrading the network. The Braess paradox implies that there is no monotonicity of performance measures with respect to link capacities. Another objective of this paper is to check under what conditions are delays as well as the marginal costs at equilibrium increasing in the demands. The answer to this question turns out to be useful for the analysis of the Braess paradox. Some results on the monotonicity in the demand are already available in [9].

The paper is organised as follows: In the next section (Section II), we present the network model, we define the concept of Wardrop equilibrium, and formulate the problem. In Section III we then present a framework of that equilibrium that allows for different costs for different classes of users (which may reflect, for example, that packets of different users may have different priorities and thus have different delays due to appropriate buffer management schemes). In Section IV we then present a sufficient condition for the monotonicity of performance measures when the demands increase. This allows us then to study in Section V methods for capacity addition. In section 6, we demonstrate the efficiency of the proposed capacity addition by means of a numerical example in **BCMP** queueing network.

II. Problem formulation and notation

We consider an open network model that consists of a set $\mathbb{M}$ containing M nodes and a set $\mathbb{L}$ containing L links. We call the unit that has to be routed a "job". It may stand for a packet (in a packet switched network) or to a whole session (if indeed all packets of a session follow the same route). The network is crossed through by infinitely many jobs that have to choose their routing.

Jobs are classified into K different classes (we will denote $\mathbb{K}$ the set of classes). For example, in the context of road traffic a class may represent the set of a given type of vehicles, such as busses, trucks, car or bicycle. In the context of telecommunications a class may represent the jobs sent by all the users of a given service provider. We assume that jobs do not change their class while passing through the network. We suppose that the jobs of a given class k may arrive in the system at some different possible points, and leave the system at some different possible points. Nevertheless the origin and destination points of a given job are determined when the job arrives in the network, and cannot change while in the system. We call a pair of one origin and one destination points an *O-D* pair.

A job with a given *O-D* pair (od) arrives in the system at node o and leaves it at node d after visiting a series of nodes and links, which we refer to as a *path*, then it leaves the system.

In many previous papers [21], [16], routing could be done at each node. In this paper we follow the approach in which a job of class k with O-D pair (o, d) has to choose one of a given finite set of paths (see also [11], [22]).

In this paper we suppose that the routing decision scheme is completely decentralised: each single job has to decide among a set of possible paths that connect the *O-D* pair of that job. This choice will be made in order to minimise the cost of that job. The solution concept we are thus going to use is the Wardrop equilibrium [24].

Notations regarding the network: We denote,

- $(OD)^k \subset O^k \times D^k$ the set of O-D pairs for the jobs of class k. $O^k \subset \mathbb{M}$ is the set of sources for the jobs of class k, and $D^k \subset \mathbb{M}$ the set of destinations for these jobs. Denote also (OD) the union $(OD) = \bigcup_k (OD)^k$.
- $\Pi^k_{(od)}$ the set of possible paths for jobs of class k with *O-D pair* $(od) \in (OD)^k$, Π^k the set of all paths for jobs of class k, *i*.e. $\Pi^k = \bigcup\limits_{(od)\in(OD)^k} \Pi^k_{(od)}$ and $\Pi_{(od)}$ the set of all paths with *O-D* pair (od), *i*.e. $\Pi_{(od)} = \bigcup\limits_{k\in\mathbb{K}} \Pi^k_{(od)}$.
- $\mathbb{K}_p$ the set of classes that use the path p.

Notations regarding arrival and flow rates:

- $r^k_{(od)}$ the rate at which the jobs of class k with O-D pair $(od) \in (OD)^k$ arrive in the system, and $r_{(od)} = \sum_{k \in I\!K} r^k_{(od)}$ the total arrival rate of jobs with *O-D* pair $(od) \in (OD)$.
- x^k_p the rate at which the jobs of class k jobs flow through the path p, x_p the total rate of jobs that flow through path p, *i*.e. $x_p = \sum_k x^k_p$. We denote **x** the vector whose elements are x_p, $p \in \Pi$, *i*.e. $\mathbf{x} = [x_{p_1}, x_{p_2}, ..]^T$, where $[.]^T$ denotes the transposed vector (and is thus a column vector).
- **X** the vector $[x^1_{p_1}, x^2_{p_1}, \ldots x^K_{p_1}, x^1_{p_2}, x^2_{p_2}, \ldots, x^K_{p_2}, \ldots]^T$, referred to as the flow configuration. A flow configuration **X** will be said feasible, if for each *O-D pair*, (od), the following conditions are satisfied: for each $(od) \in (OD)^k$

$$\sum_{p \in \Pi^k_{(od)}} x^k_p = r^k_{(od)} \text{ and } x^k_p \geq 0, \; p \in \Pi^k. \qquad (1)$$

- x^k_l the rate at which the jobs of class k visit the link l, $x^k_l = \sum_{p \in \Pi^k} \delta_{lp} x^k_p$, with $\delta_{lp} = 0$ unless the link l belongs to path p. In that case $\delta_{lp} = 1$.

Notations regarding service and performance values:

- μ^k_l the service rate of class k at link l.
- $\rho^k_l = x^k_l / \mu^k_l$ utilisation of link l for class k jobs, and $\rho_l = \sum_{k=1}^K \rho^k_l$ total utilisation of this link.
- $T^k_p(\mathbf{X})$, $p \in \Pi^k$ the total cost incurred by a job of class k for using the path p if the flow configuration resulting from the routing of each job is **X**. We denote **T** the vector $(T^1_{p_1}, T^2_{p_1}, \ldots, T^K_{p_1}, T^1_{p_2}, T^2_{p_2}, \ldots T^K_{p_2}, \ldots)^T$ of cost functions.

III. Wardrop equilibrium for a multi-class network

Each individual job of class k with *O-D* pair (od), chooses its routing through the system, by means of the choice of a path $p \in \Pi^k_{(od)}$. A flow configuration **X** follows from the choices of each of the infinitely many jobs. A flow configuration **X** will be said to be a *Wardrop equilibrium* or *individually optimal*, if none of the jobs has any incentive to change unilaterally its decision. This equilibrium concept was first introduced by Wardrop [24] in the field of transportation and can be defined through the two principles:

- **Wardrop's first principle:** the cost for crossing the used paths between a source and a destination are equal, the cost for any unused path with same *O-D* pair is larger or equal that that of used ones.
- **Wardrop's second principle:** the cost is minimum for each job.

Formally, in the context of multi-class this can be defined as,

Definition 1. *A feasible flow configuration (i.e. satisfying equation (1))* $\boldsymbol{X}$ *is a Wardrop equilibrium for the multi-class problem if for any class* k*, any* $(od) \in (OD)^k$ *and any path* $p \in \Pi^k_{(od)}$ *we have*

$$\begin{cases} T^k_p(\boldsymbol{X}) \geq \lambda^k_{(od)} \text{ if } x^k_p = 0, \\ T^k_p(\boldsymbol{X}) = \lambda^k_{(od)} \text{ if } x^k_p \geq 0, \end{cases} \qquad (2)$$

where $\lambda^k_{(od)} = \min_{p \in \Pi^k_{(od)}} T^k_p(\boldsymbol{X})$. *The minimal cost* $\lambda^k_{(od)}$ *will be referred to by "the travel cost" associated to class* k *and O-D pair* (od).

We need one of the following assumptions on the cost function:

Assumption A The vector **T** of cost functions $(T^1_{p_1}, T^2_{p_1}, \ldots, T^k_{p_1}, T^1_{p_2}, T^2_{p_2}, \ldots, T^k_{p_1}, \ldots)^T$ is positive, continuous and strictly monotonically increasing, *i*.e. it satisfies for any two distinct flow configurations **X** and **Y**: $(\mathbf{T}(\mathbf{X}) - \mathbf{T}(\mathbf{Y}))^T (\mathbf{X} - \mathbf{Y}) > 0$.

Assumption B

1-There exists a function T_p that depends only upon the total total flow **x** vector (and not on the flow sent by each class), such that the average cost per flow unit for jobs of class k can be written as $T^k_p(\mathbf{X}) = c^k T_p(\mathbf{x}) \; \forall p \in \Pi^k$, where c^k are some class dependent positive constants.

2-T_p is positive, continuous and strictly monotonically increasing. We will denote $\mathbf{T}' = (T_{p_1}, T_{p_2}, ...)^T$ the vector of functions T_p.

Assumption C

1-The average cost per flow unit for jobs of class k that passes through path $p \in \Pi^k$ is:

$$T^k_p(\mathbf{X}) = \sum_{l \in I\!L} \frac{\delta_{lp}}{\mu^k_l} T_l(\rho_l),$$

where $T_l(\rho_l)$ is weighted cost per unit flow in link l (The function T_l does not depend on the class k).

2-$T_l(.)$ is positive, continuous and strictly increasing.

3- μ^k_l can be represented as μ_l / c^k where c^k are some class dependent positive constants, and $0 < \mu_l$ is finite.

We denote $v_{(od)} = \sum_{k \in I\!K} c^k r^k_{(od)}$ the weighted total demand with *O-D* pair $(od) \in (OD)$.

Lemma 1. *(see [22, Thm. 3.2, 3.14]) For any vectorial cost function* $\boldsymbol{T}$ *satisfying Assumption* $\boldsymbol{A}$*, there exists a unique Wardrop flow configuration, i.e. that satisfies equations (1) and (2).*

We make the following observation.

Lemma 2. *Consider a cost functions vector* $\boldsymbol{T}$ *satisfying Assumption* $\boldsymbol{B}$ *or* $\boldsymbol{C}$*. Then the Wardrop equilibrium conditions (1) and (2) become: For all* k*, all* $(od) \in (OD)^k$ *and all* $p \in \Pi^k_{(od)}$*,*

$$\begin{aligned} T^k_p(\boldsymbol{X}) &\geq \lambda^k_{(od)}, \text{ if } x_p = 0, \\ T^k_p(\boldsymbol{X}) &= \lambda^k_{(od)}, \text{ if } x_p > 0. \end{aligned} \tag{3}$$

Moreover, the ratio $\lambda^k_{(od)}/c^k$ *is independent of class* k*. We define* $\lambda_{(od)}$ *by* $\lambda_{(od)} := \lambda^k_{(od)}/c^k$*.*

Proof: Consider first the case of cost function vector that satisfies Assumption **B**. Let $(od) \in (OD)$ and $p \in \Pi^k$. If $x_p = 0$ then $x^k_p = 0\ \forall k \in I\!\!K$. The first part of (3) follows from the first part of (2).
Suppose that $x_p > 0$ and, by contradiction, that there exists $\bar{k} \in I\!\!K$ such that

$$T^{\bar{k}}_p(\mathbf{X}) = T_p(\mathbf{x})c^{\bar{k}} > \lambda^{\bar{k}}_{(od)}. \tag{4}$$

Since $x_p > 0$, there exit $k_0 \in I\!\!K$ such that $x^{k_0}_p > 0$. From the second part of (2), we have

$$T^{k_0}_p(\mathbf{X}) = T_p(\mathbf{x})c^{k_0} = \lambda^{k_0}_{(od)}. \tag{5}$$

Because $r^{\bar{k}}_{(od)} > 0$ there exists $p \in \Pi'^{\bar{k}}_{(od)}$ such that $x^{\bar{k}}_{p'} > 0$. Then, from (2) we get

$$T^{\bar{k}}_{p'}(\mathbf{X}) = T_{p'}(\mathbf{x})c^{\bar{k}} = \lambda^{\bar{k}}_{(od)}. \tag{6}$$

It follows from (4) and (6) that

$$T_p(\mathbf{x}) > T_{p'}(\mathbf{x}). \tag{7}$$

Since $\lambda^{k_0}_{(od)} \leq T_{p'}(\mathbf{x})c^{k_0}$, from (5), we obtain $T_p(\mathbf{x}) \leq T_{p'}(\mathbf{x})$, which contradicts (7). This establishes (3).
For any $(od) \in (OD)$, let $p \in \bigcup_k \Pi^k_{(od)}$ be a path such that $x_p > 0$. From (3), it comes that for any class k such that $p \in \Pi^k_{(od)}$, $T_p(\mathbf{x}) = \frac{T^k_p(\mathbf{X})}{c^k} = \frac{\lambda^k_{(od)}}{c^k}$. The second part of Lemma 2 follows, since the terms in the above equation do not depend on k. The proof for cost function vector satisfying assumption **C** follows along similar lines. ■

IV. Impact of throughput variation on the equilibrium

In this section, we study the impact of a variation of the demands $r^k_{(od)}$ of some class k on the costs $\mathbf{T}(\mathbf{X})$ at the (Wardrop) equilibrium $\mathbf{X}$. The results of this section extend those of [9] who considered a simpler cost structure (where for any class k, the cost for using a path is the sum of link costs along that path, and the link costs do not depend on k).

Theorem 1. *Consider a cost function vector* $\boldsymbol{T}$ *satisfying Assumption* $\boldsymbol{A}$*, and two throughput demand profiles* $\tilde{\mathbf{r}} = (\tilde{r}^k_{(od)})_{((od),k)\in(OD)\times I\!\!K}$ *and* $\hat{\mathbf{r}} = (\hat{r}^k_{(od)})_{((od),k)\in(OD)\times I\!\!K}$*. Let* $\tilde{\boldsymbol{X}}$ *and* $\hat{\boldsymbol{X}}$ *be the Wardrop equilibria associated to these throughput demands, and* $\tilde{\lambda}^k_{(od)}$ *and* $\hat{\lambda}^k_{(od)}$ *the class k's travel costs associated for* $(od) \in OD^k$ *computed respectively at* $\tilde{\boldsymbol{X}}$ *and* $\hat{\boldsymbol{X}}$*. Then we have:*

$$\sum_{k\in I\!\!K} \sum_{(od)\in(OD)} (\hat{r}^k_{(od)} - \tilde{r}^k_{(od)})(\hat{\lambda}^k_{(od)} - \tilde{\lambda}^k_{(od)}) > 0$$

Proof: From (2), we have

$$\begin{aligned} \hat{\lambda}^k_{(od)} &= T^k_p(\hat{\mathbf{X}}), \text{ if } \hat{x}^k_p > 0, \\ \hat{\lambda}^k_{(od)} &\leq T^k_p(\hat{\mathbf{X}}), \text{ if } \hat{x}^k_p = 0, \end{aligned} \quad \text{and} \quad \begin{aligned} \tilde{\lambda}^k_{(od)} &= T^k_p(\tilde{\mathbf{X}}), \text{ if } \tilde{x}^k_p > 0, \\ \tilde{\lambda}^k_{(od)} &\leq T^k_p(\tilde{\mathbf{X}}), \text{ if } \tilde{x}^k_p = 0. \end{aligned}$$

Thus

$$\begin{aligned} \hat{x}^k_p\hat{\lambda}^k_{(od)} &= T^k_p(\hat{\mathbf{X}})\hat{x}^k_p, \\ \tilde{x}^k_p\hat{\lambda}^k_{(od)} &\leq T^k_p(\hat{\mathbf{X}})\tilde{x}^k_p, \end{aligned} \quad \text{and} \quad \begin{aligned} \tilde{x}^k_p\tilde{\lambda}^k_{(od)} &= T^k_p(\tilde{\mathbf{X}})\tilde{x}^k_p, \\ \hat{x}^k_p\tilde{\lambda}^k_{(od)} &\leq T^k_p(\tilde{\mathbf{X}})\hat{x}^k_p. \end{aligned}$$

By summing up over $p \in \Pi^k_{(od)}$, we obtain:

$$\begin{aligned} \hat{r}^k_a\hat{\lambda}^k_{(od)} &= \textstyle\sum_{p\in\Pi^k_{(od)}} T^k_p(\hat{\mathbf{X}})\hat{x}^k_p, \\ \tilde{r}^k_a\hat{\lambda}^k_{(od)} &\leq \textstyle\sum_{p\in\Pi^k_{(od)}} T^k_p(\hat{\mathbf{X}})\tilde{x}^k_p, \end{aligned}$$

and

$$\begin{aligned} \tilde{r}^k_a\tilde{\lambda}^k_{(od)} &= \textstyle\sum_{p\in\Pi^k_{(od)}} T^k_p(\tilde{\mathbf{X}})\tilde{x}^k_p, \\ \hat{r}^k_a\tilde{\lambda}^k_{(od)} &\leq \textstyle\sum_{p\in\Pi^k_{(od)}} T^k_p(\tilde{\mathbf{X}})\hat{x}^k_p, \end{aligned}$$

and by summing up over $k \in K$ and over $(od) \in (OD)$, it comes

$$\begin{aligned} &\sum_{k\in I\!\!K} \sum_{(od)\in(OD)} (\hat{r}^k_{(od)} - \tilde{r}^k_{(od)})(\hat{\lambda}^k_{(od)} - \tilde{\lambda}^k_{(od)}) \\ &\geq (\mathbf{T}(\hat{\mathbf{X}}) - \mathbf{T}(\tilde{\mathbf{X}}))(\hat{\mathbf{X}} - \tilde{\mathbf{X}}) > 0 \end{aligned}$$

The last inequality comes from the strict monotony of $\mathbf{T}(.)$ (Assumption **A**). This completes the proof. ■

Theorem 1 states that an increase (*resp.* decrease) of the demands of class k for some $(od) \in (OD)^k$ and class k leads to a increase (*resp.* decrease) of the associated cost $\lambda^k_{(od)}$. Nevertheless an increase of the demand of class k for some (od) may lead to a decrease of the cost associated to an other class or an other *O-D* pair.

Corollary 1. *Let* $\boldsymbol{T}$ *a cost function vector satisfying Assumption* $\boldsymbol{A}$*, two throughput demand profiles* $(\tilde{r}^k_{(od)})_{((od),k)}$ *and* $(\hat{r}^k_{(od)})_{((od),k)}$*. Let* $\tilde{\mathbf{X}}$ *and* $\hat{\mathbf{X}}$ *be the Wardrop equilibria associated to these throughput demands,* $\tilde{\lambda}^k_{(od)}$ *and* $\hat{\lambda}^k_{(od)}$ *the cost for the jobs of class k with O-D pair* $(od) \in$

$(OD)^k$*, respectively at* $\tilde{\boldsymbol{X}}$ *and* $\hat{\boldsymbol{X}}$*. If* $\hat{r}^{\bar{k}}_{(\overline{od})} < \tilde{r}^{\bar{k}}_{(\overline{od})}$ *for some* $((\overline{od}), \bar{k}) \in (OD) \times I\!K$ *and* $\hat{r}^{k}_{(od)} = \tilde{r}^{k}_{(od)}$ $\forall((od), k) \neq ((\overline{od}), \bar{k})$*, then* $\hat{\lambda}^{\bar{k}}_{(\overline{od})} < \tilde{\lambda}^{\bar{k}}_{(\overline{od})}$

The following theorem, states that under Assumption **B** or **C**, an increase in the demands associated with a particular O-D pair (od) always leads to an increase of the cost associated to (od) for all class k.

Theorem 2. *Consider two throughput demand profiles* $(\tilde{r}^k_{(od)})_{((od),k)}$ *and* $(\hat{r}^k_{(od)})_{((od),k)}$*. Let* $\tilde{\boldsymbol{X}}$ *and* $\hat{\boldsymbol{X}}$ *be the Wardrop equilibria associated to these throughput demands, and let* $\tilde{\lambda}^k_{(od)}$ *and* $\hat{\lambda}^k_{(od)}$ *be class k's travel cost associated to these two equilibria.*

1. For the cost function vector $\boldsymbol{T}$ *satisfying Assumption* ***B****, if* $\hat{r}_{(\overline{od})} < \tilde{r}_{(\overline{od})}$*, for some* $(\overline{od}) \in (OD)$ *and* $\hat{r}_{(od)} = \tilde{r}_{(od)}$ *for all* $(od) \neq (\overline{od})$*, then* $\hat{\lambda}^k_{(\overline{od})} < \tilde{\lambda}^k_{(\overline{od})}$ $\forall k \in I\!K$.

2. For the cost function vector $\boldsymbol{T}$ *satisfying Assumption* ***C****, if* $\hat{v}_{(\overline{od})} < \tilde{v}_{(\overline{od})}$*, for some* $(\overline{od}) \in (OD)$ *and* $\hat{v}_{(od)} = \tilde{v}_{(od)}$ *for all* $(od) \neq (\overline{od})$*, then* $\hat{\lambda}^k_{(\overline{od})} < \tilde{\lambda}^k_{(\overline{od})}$ $\forall k \in I\!K$.

Proof: Consider first the case of cost function vector that satisfies Assumption **B**.

1. From (3) and Assumption **B** we have

$$\hat{\lambda}_{(od)} = T_p(\hat{x}) \text{ if } \hat{x}_p > 0, \quad \hat{\lambda}_{(od)} \leq T_p(\hat{x}), \text{ if } \hat{x}_p = 0$$

and

$$\tilde{\lambda}_{(od)} = T_p(\tilde{x}), \text{ if } \tilde{x}_p > 0, \quad \tilde{\lambda}_{(od)} \leq T_p(\tilde{x}), \text{ if } \tilde{x}_p = 0.$$

Thus

$$\begin{array}{l} \hat{x}_p \hat{\lambda}_{(od)} = T_p(\hat{\mathbf{x}})\hat{x}_p, \\ \tilde{x}_p \hat{\lambda}_{(od)} \leq T_p(\hat{\mathbf{x}})\tilde{x}_p, \end{array} \quad \text{and} \quad \begin{array}{l} \tilde{x}_p \tilde{\lambda}_{(od)} = T_p(\tilde{\mathbf{x}})\tilde{x}_p, \\ \hat{x}_p \tilde{\lambda}_{(od)} \leq T_p(\tilde{\mathbf{x}})\hat{x}_p. \end{array}$$

Now by summing up over $p \in \Pi_{(od)}$, we obtain

$$\hat{r}_{(od)}\hat{\lambda}_{(od)} = \textstyle\sum_{p\in\Pi_{(od)}} T_p(\hat{\mathbf{x}})\hat{x}_p, \quad \tilde{r}_{(od)}\hat{\lambda}_{(od)} \leq \textstyle\sum_{p\in\Pi_{(od)}} T_p(\hat{\mathbf{x}})\tilde{x}_p,$$

and

$$\tilde{r}_{(od)}\tilde{\lambda}_{(od)} = \textstyle\sum_{p\in\Pi_{(od)}} T_p(\tilde{\mathbf{x}})\tilde{x}_p, \quad \hat{r}_{(od)}\tilde{\lambda}_{(od)} \leq \textstyle\sum_{p\in\Pi_{(od)}} T_p(\tilde{\mathbf{x}})\hat{x}_p,$$

By summing up over $(od) \in (OD)$, it comes

$$\sum_{(od)\in(OD)} (\tilde{r}_{(od)} - \hat{r}_{(od)})(\tilde{\lambda}_{(od)} - \hat{\lambda}_{(od)}) \geq (\mathbf{T}'(\hat{\mathbf{x}}) - \mathbf{T}'(\tilde{\mathbf{x}}))(\hat{\mathbf{x}} - \tilde{\mathbf{x}}) > 0. \tag{8}$$

The last inequality follows from assumption **B**. Since $\hat{r}_{(od)} = \tilde{r}_{(od)}$ for $(od) \neq (\overline{od})$, inequality (8) yields $(\tilde{r}_{(\overline{od})} - \hat{r}_{(\overline{od})})(\tilde{\lambda}_{(\overline{od})} - \hat{\lambda}_{(\overline{od})}) > 0$, which implies since $\tilde{r}_{(\overline{od})} > \hat{r}_{(\overline{od})}$, that $\tilde{\lambda}_{(\overline{od})} > \hat{\lambda}_{(\overline{od})}$, it follows that $\tilde{\lambda}^k_{(\overline{od})} > \hat{\lambda}^k_{(\overline{od})}$ for all $k \in I\!K$.

Consider now the case of cost function vector that satisfies Assumption **C**.

2. From (3) and Assumption **C** we have

$$\hat{\lambda}_{(od)} = \textstyle\sum_{l\in I\!L} \frac{\delta_{lp}}{\mu_l} T_l(\hat{\rho}_l) \text{ if } \hat{x}_p > 0,$$

$$\hat{\lambda}_{(od)} \leq \textstyle\sum_{l\in I\!L} \frac{\delta_{lp}}{\mu_l} T_l(\hat{\rho}_l), \text{ if } \hat{x}_p = 0$$

and

$$\tilde{\lambda}_{(od)} = \textstyle\sum_{l\in I\!L} \frac{\delta_{lp}}{\mu_l} T_l(\tilde{\rho}_l), \text{ if } \tilde{x}_p > 0,$$

$$\tilde{\lambda}_{(od)} \leq \textstyle\sum_{l\in I\!L} \frac{\delta_{lp}}{\mu_l} T_l(\tilde{\rho}_l), \text{ if } \tilde{x}_p = 0.$$

Let $y_p = \sum_{k\in I\!K_p} c^k x^k_p$, the above equations become

$$\hat{y}_p\hat{\lambda}_{(od)} = \textstyle\sum_{l\in I\!L} \frac{\delta_{lp}}{\mu_l} T_l(\hat{\rho}_l)\hat{y}_p,$$

$$\tilde{y}_p\hat{\lambda}_{(od)} \leq \textstyle\sum_{l\in I\!L} \frac{\delta_{lp}}{\mu_l} T_l(\hat{\rho}_l)\tilde{y}_p,$$

and

$$\tilde{y}_p\tilde{\lambda}_{(od)} = \textstyle\sum_{l\in I\!L} \frac{\delta_{lp}}{\mu_l} T_l(\tilde{\rho}_l)\tilde{y}_p,$$

$$\hat{y}_p\tilde{\lambda}_{(od)} \leq \textstyle\sum_{l\in I\!L} \frac{\delta_{lp}}{\mu_l} T_l(\tilde{\rho}_l)\hat{y}_p.$$

By summing up over $p \in \Pi_{(od)}$, and $(od) \in (OD)$, we obtain

$$\textstyle\sum_{(od)\in(OD)} \hat{v}_{(od)}\hat{\lambda}_{(od)} = \sum_{l\in I\!L} \hat{\rho}_l T_l(\hat{\rho}_l), \quad \sum_{(od)\in(OD)} \tilde{v}_{(od)}\hat{\lambda}_{(od)} \leq \sum_{l\in I\!L} \hat{\rho}_l T_l(\hat{\rho}_l),$$

and

$$\textstyle\sum_{(od)\in(OD)} \tilde{v}_{(od)}\tilde{\lambda}_{(od)} = \sum_{l\in I\!L} \tilde{\rho}_l T_l(\tilde{\rho}_l), \quad \sum_{(od)\in(OD)} \hat{v}_{(od)}\tilde{\lambda}_{(od)} \leq \sum_{l\in I\!L} \tilde{\rho}_l T_l(\tilde{\rho}_l).$$

Indeed, we have from (1), $r^k_{(od)} = \sum_{p\in\Pi^k_{(od)}} x^k_p$, multiplying by c^k and summing up over $k \in I\!K$, we obtain

$$v_{(od)} = \sum_{k\in I\!K} \sum_{p\in\Pi^k_{(od)}} c^k x^k_p = \sum_{p\in\Pi_{(od)}} \sum_{k\in I\!K_p} c^k x^k_p = \sum_{p\in\Pi_{(od)}} y_p$$

And

$$\sum_{(od)\in(OD)} (\tilde{v}_{(od)} - \hat{v}_{(od)})(\tilde{\lambda}_{(od)} - \hat{\lambda}_{(od)}) \geq \sum_{l\in\mathbb{L}} (\tilde{\rho}_l - \hat{\rho}_l)(T_l(\tilde{\rho}) - T_l(\hat{\rho}_l)) > 0. \quad (9)$$

The last inequality follows from assumption **C**. Proceeding as in the first part of the proof, we obtain $\tilde{\lambda}^k_{\overline{(od)}} > \hat{\lambda}^k_{\overline{(od)}}$ for all $k \in \mathbb{K}$.

■

Remark 1. *In the case where all classes ship flow from a common source s to a common destination d i.e. $\Pi^k = \{(sd)\}, \forall k$, Theorem 2 establishes the monotonicity of performance (given by travel cost $\lambda^k_{(sd)}$) at Wardrop equilibrium for all $k \in \mathbb{K}$ when the demands of classes increases.*

V. AVOIDING BRAESS PARADOX

The purpose of this section is to provide some methods for adding resources to a general network with one source s and one destination d that guarantee improvement in performance. This would guarantee in particular that the well known Braess paradox (in which adding a link results in deterioration of performance for all users) does not occur.

For some given network with one source and one destination, the designer problem is to distribute some additional capacity among the links of the network so as to improve the performances at the (Wardrop) equilibrium.
Adding capacity in the network can be done by several way. Among them,
(1) by adding a new direct path from the source s to the destination d,
(2) by improving an existing direct path,
(3) by improving all the paths connecting s to d.

We first consider (1), i.e. the addition of a direct path from s to d that can be used by the jobs of all classes. That direct path could be in fact a whole new network, provided that it is disjoint with the previous network; it may also have new sources and destinations in addition to s and d and new traffic from new classes that use these new sources and destinations. The next theorem shows that this may lead to a decrease of the costs of all paths used at equilibrium.

Theorem 3. *Consider a cost function vector that satisfies Assumption **B** or **C**. Let $\hat{\mathbf{X}}$ and $\tilde{\mathbf{X}}$ the Wardrop equilibria after and before the addition of a direct path $\hat{p}$ from s to d. Consider $\tilde{\lambda}^k_{(sd)}$ and $\hat{\lambda}^k_{(sd)}$ the travel cost for class k respective at $\tilde{\mathbf{X}}$ and $\hat{\mathbf{X}}$. Then, $\hat{\lambda}^k_{(sd)} \leq \tilde{\lambda}^k_{(sd)}, \forall k \in \mathbb{K}$, moreover the last inequality is strict if $\hat{x}_{\hat{p}} > 0$.*

Proof: Consider the same network $(\mathbb{M}, \mathbb{L})$ with the initial service rate configuration $\tilde{\mu}$ and throughput demand $(\bar{r}^k_{(sd)})_{k\in\mathbb{K}}$ where $\bar{r}^k_{(sd)} = r^k_{(sd)} - \hat{x}^k_{\hat{p}}$ for all class $k \in \mathbb{K}$, and let $\bar{\mathbf{X}}$ represent the Wardrop equilibrium associated to this new throughput demand and $\bar{\lambda}^k_{(sd)}$ the travel cost for class k at Wardrop equilibrium $\bar{\mathbf{X}}$. From Conditions (1)-(2) the travel cost $\bar{\lambda}^k_{(sd)} = \hat{\lambda}^k_{(sd)}, \forall k \in \mathbb{K}$. If $x_{\hat{p}} = 0$, $\bar{\lambda}^k_{(sd)} = \tilde{\lambda}^k_{(sd)}$, which implies that $\hat{\lambda}^k_{(sd)} = \tilde{\lambda}^k_{(sd)}$.
Assume, then, that $\hat{x}_{\hat{p}} > 0$, then we have $\bar{r}_{(sd)} < r_{(sd)}$ (which will be used for Assumption **B**) and $\bar{v}_{(sd)} < v_{(sd)}$ (which will be used for Assumption **C**), following the Theorem 2, we conclude that $\hat{\lambda}^k_{(sd)} = \bar{\lambda}^k_{(sd)} < \tilde{\lambda}^k_{(sd)}$, for all $k \in \mathbb{K}$ and this completes the proof. ■

Now, we consider a network $(\mathbb{M}, \mathbb{L})$ that contains a direct path, $\hat{p}$, from s to d that can be used by the jobs of all classes. We derive sufficient conditions that guarantee an improvement in the performance when we increase the capacity of this direct path.

Theorem 4.
*Let **T** a cost function vector satisfying Assumptions **B**. We consider an improvement of the path $\hat{p}$ so that the cost associated to this path is smaller for all classes, i.e. $\hat{T}_{\hat{p}}(\boldsymbol{x}) < \tilde{T}_{\hat{p}}(\boldsymbol{x})$. Let $\hat{\boldsymbol{X}}$ and $\tilde{\boldsymbol{X}}$, respectively, the Wardrop equilibria after and before this improvement. Consider $\tilde{\lambda}^k_{(sd)}$ and $\hat{\lambda}^k_{(sd)}$ the travel cost of class k at the equilibria. Then $\hat{\lambda}^k_{(sd)} \leq \tilde{\lambda}^k_{(sd)}, \forall k \in \mathbb{K}$. Moreover the inequality is strict if $\hat{x}_{\hat{p}} > 0$ or $\tilde{x}_{\hat{p}} > 0$.*

Proof: From Lemma 2 we have

$$\begin{aligned} \lambda_{(sd)} &= T_p(\mathbf{x}), \quad x_p > 0, \\ \lambda_{(sd)} &\leq T_p(\mathbf{x}), \quad x_p = 0, \\ \sum_{p\in\Pi_{(sd)}} x_p &= r_{(sd)}, \quad x_p \geq 0, \; p \in \Pi^k. \end{aligned} \quad (10)$$

We know form Theorem 3.2 and 3.14 in [22], $\hat{\mathbf{x}}$ and $\tilde{\mathbf{x}}$ must satisfy the variational inequalities

$$\hat{\mathbf{T}}'(\hat{\mathbf{x}})^T(\mathbf{x} - \hat{\mathbf{x}}) \;\geq\; 0 \quad \forall \mathbf{x} \text{ that satisfies } (10) \quad (11)$$
$$\tilde{\mathbf{T}}'(\tilde{\mathbf{x}})^T(\mathbf{x} - \tilde{\mathbf{x}}) \;\geq\; 0 \quad \forall \mathbf{x} \text{ that satisfies } (10) \quad (12)$$

By adding (11) with $\mathbf{x} = \tilde{\mathbf{x}}$ and (12) with $\mathbf{x} = \hat{\mathbf{x}}$, we obtain $[\hat{\mathbf{T}}'(\hat{\mathbf{x}}) - \tilde{\mathbf{T}}'(\tilde{\mathbf{x}})][\hat{\mathbf{x}} - \tilde{\mathbf{x}}] \leq 0$, thus

$$[\hat{\mathbf{T}}'(\hat{\mathbf{x}}) - \tilde{\mathbf{T}}'(\hat{\mathbf{x}}) + \tilde{\mathbf{T}}'(\hat{\mathbf{x}}) - \tilde{\mathbf{T}}'(\tilde{\mathbf{x}})][\hat{\mathbf{x}} - \tilde{\mathbf{x}}] \leq 0,$$

and

$$[\hat{\mathbf{T}}'(\hat{\mathbf{x}}) - \tilde{\mathbf{T}}'(\hat{\mathbf{x}})][\hat{\mathbf{x}} - \tilde{\mathbf{x}}] \leq [\tilde{\mathbf{T}}'(\hat{\mathbf{x}}) - \tilde{\mathbf{T}}'(\tilde{\mathbf{x}})][\hat{\mathbf{x}} - \tilde{\mathbf{x}}] < 0. \quad (13)$$

Since the costs of other paths are unchanged, *i.e.* $\hat{T}_p = \tilde{T}_p$ for all $p \neq \hat{p}$, (13) becomes $(\hat{T}_{\hat{p}}(\hat{\mathbf{x}}) - \tilde{T}_{\hat{p}}(\hat{\mathbf{x}})(\hat{x}_{\hat{p}} - \tilde{x}_{\hat{p}}) < 0$ if $\hat{\mathbf{x}} \neq \tilde{\mathbf{x}}$. Since $\hat{T}_{\hat{p}}(\hat{\mathbf{x}}) < \tilde{T}_{\hat{p}}(\hat{\mathbf{x}})$, then we have

$$\hat{x}_{\hat{p}} > \tilde{x}_{\hat{p}} \text{ if } \hat{\mathbf{x}} \neq \tilde{\mathbf{x}}. \tag{14}$$

Now we have two cases:

- If $\hat{\mathbf{x}} = \tilde{\mathbf{x}}$ and since $\hat{T}_{\hat{p}}(\mathbf{x}) \neq \tilde{T}_{\hat{p}}(\mathbf{x})$, it follows that $\hat{x}_{\hat{p}} = \tilde{x}_{\hat{p}} = 0$, which implies that $\hat{\lambda}^k_{(sd)} = \tilde{\lambda}^k_{(sd)}$.
- If $\hat{\mathbf{x}} \neq \tilde{\mathbf{x}}$, then from (14) we have $\hat{x}_{\hat{p}} > \tilde{x}_{\hat{p}}$. Consider now two networks that differ only by the presence or absence of the direct path $\hat{p}$ from s to d. In both networks we have the same initial capacity configuration and the same set $I\!K$ of classes, with respectively demands $\check{r}^k_{(sd)} = r^k_{(sd)} - \hat{x}^k_{\hat{p}}$ and $\bar{r}^k_{(sd)} = r^k_{(sd)} - \tilde{x}^k_{\hat{p}}$. Let $\check{\lambda}^k_{(sd)}$ and $\bar{\lambda}^k_{(sd)}$ the travel cost of class k associated to these throughput demands. Since $\hat{x}_{\hat{p}} > \tilde{x}_{\hat{p}}$ then $\check{r}_{(sd)} < \bar{r}_{(sd)}$, and from Theorem 2 we have

$$\forall k \in I\!K, \quad \check{\lambda}^k_{(sd)} < \bar{\lambda}^k_{(sd)} \tag{15}$$

  On the other hand, for the network with demands $(\check{r}^k_{(sd)})_{k \in K}$, it is easy to see that the equilibria conditions (1) and (2) are satisfied by the system flow configuration $\check{\mathbf{X}}$, with $\check{\lambda}^k_{(sd)} = \hat{\lambda}^k_{(sd)}$. Similarly we conclude that the network with demands $(\check{r}^k_{(sd)})_{k \in I\!K}$ has the system flow configuration $\bar{\mathbf{X}}$, with $\bar{\lambda}^k_{(sd)} = \tilde{\lambda}^k_{(sd)}$. Hence from (15) we obtain $\hat{\lambda}^k_{(sd)} < \tilde{\lambda}^k_{(sd)}$. ∎

Theorem 5.
*Consider a cost function vector that satisfies Assumptions **C**. Let $\hat{\mu}^k_l$ and $\tilde{\mu}^k_l$, respectively, be the service rate configurations after and before the addition the capacity of the path $\hat{p}$, i.e $\hat{\mu}_l > \tilde{\mu}_l$ for $l \in \hat{p}$ and $\hat{\mu}_l = \tilde{\mu}_l$ for $l \notin \hat{p}$. Let $\hat{\mathbf{X}}$ and $\tilde{\mathbf{X}}$, respectively, the Wardrop equilibria after and before this improvement. Consider $\tilde{\lambda}^k_{(sd)}$ and $\hat{\lambda}^k_{(sd)}$ the travel cost of class k at the equilibria. Then $\hat{\lambda}^k_{(sd)} \leq \tilde{\lambda}^k_{(sd)}$, $\forall k \in I\!K$. Moreover the inequality is strict if $\hat{x}_{\hat{p}} > 0$ or $\tilde{x}_{\hat{p}} > 0$.*

Proof: Note that if there exists a link l_1 belongs to the path $\hat{p}$, such that $\hat{\rho}_{l_1} < \tilde{\rho}_{l_1}$, then $\hat{\rho}_l < \tilde{\rho}_l$ for each link belongs to the path $\hat{p}$.
Assume, then, that $\hat{\rho}_l \leq \tilde{\rho}_l$ for $l \in \hat{p}$, we have two possibilities. First, if $\tilde{x}_{\hat{p}} = 0$, then $\hat{\lambda}^k_{(sd)} = \tilde{\lambda}^k_{(sd)}$ for all $k \in I\!K$. Second, if $\tilde{x}_{\hat{p}} > 0$, then we have

$$\tilde{\lambda}_{(sd)} = \sum_{l \in \hat{p}} \frac{\tilde{T}_l(\tilde{\rho}_l)}{\tilde{\mu}_l} \text{ and } \hat{\lambda}_{(sd)} \leq \sum_{l \in \hat{p}} \frac{\hat{T}_l(\hat{\rho}_l)}{\hat{\mu}_l}$$

Since $T_l(.)$ is strictly increasing and $\tilde{\mu}_l < \hat{\mu}_l$ for all $l \in \hat{p}$ then we have $\hat{\lambda}_{(sd)} \leq \sum_{l \in \hat{p}} \frac{\hat{T}_l(\hat{\rho}_l)}{\hat{\mu}_l} < \sum_{l \in \hat{p}} \frac{\tilde{T}_l(\tilde{\rho}_l)}{\tilde{\mu}_l} = \tilde{\lambda}_{(sd)}$, it follows that $\hat{\lambda}^k_{(sd)} < \tilde{\lambda}^k_{(sd)}$ for all $k \in I\!K$.
Now assume that $\hat{\rho}_l > \tilde{\rho}_l$ for $l \in \hat{p}$. Let us consider the two networks that differ only by the presence or absence of the direct path $\hat{p}$ from s to d. In both networks we have the same initial capacity configuration and the same set $I\!K$ of classes, with respectively demands $\check{r}^k_{(sd)} = r^k_{(sd)} - \hat{x}^k_{\hat{p}}$ and $\bar{r}^k_{(sd)} = r^k_{(sd)} - \tilde{x}^k_{\hat{p}}$. Let $\check{\lambda}^k_{(sd)}$ and $\bar{\lambda}^k_{(sd)}$ the travel cost of class k associated to these throughput demands. Since $\hat{\rho}_l > \tilde{\rho}_l$ and $\hat{\mu}_l > \tilde{\mu}_l$ for $l \in \hat{p}$, then we have

$$\begin{aligned}
\bar{v}_{(sd)} - \check{v}_{(sd)} &> \sum_{k \in I\!K} c^k \bar{r}^k_{(sd)} - c^k \check{r}^k_{(sd)} \\
&= \sum_{k \in I\!K} c^k (r^k_{(sd)} - \tilde{x}^k_{\hat{p}}) - c^k (r^k_{(sd)} - \hat{x}^k_{\hat{p}}) \\
&= \sum_{k \in I\!K} c^k (\hat{x}^k_{\hat{p}}) - c^k (\tilde{x}^k_{\hat{p}}) \\
&= \sum_{l \in \hat{p}} (\hat{\mu}_l \hat{\rho}_{\hat{l}} - \tilde{\mu}_l \tilde{\rho}_l) > 0.
\end{aligned}$$

From Theorem 2, we conclude that $\check{\lambda}^k_{(sd)} < \bar{\lambda}^k_{(sd)}$ for all $k \in I\!K$. Proceeding as in the proof of Theorem 4, we obtain $\hat{\lambda}^k_{(sd)} < \tilde{\lambda}^k_{(sd)}$ for all $k \in I\!K$. ∎

Consider a network $(I\!M, I\!L)$ and a cost function vector $\mathbf{T}$ that satisfies Assumption **B** or **C**. We consider the improving of the capacity of all path so that the following holds:

$$\hat{T}^k_p(\mathbf{X}) = \frac{1}{\alpha} \tilde{T}^k_p(\frac{\mathbf{X}}{\alpha}), \text{ with } \alpha > 1. \tag{16}$$

We observe that for any $\alpha > 1$, $\hat{T}^k_p(\mathbf{X}) = \frac{1}{\alpha} \tilde{T}^k_p(\frac{\mathbf{X}}{\alpha}) < \tilde{T}^k_p(\frac{\mathbf{X}}{\alpha}) < \tilde{T}^k_p(\mathbf{X})$.

Theorem 6. *Consider a cost function vector that satisfies Assumption **B** or **C**. Let $\tilde{\mathbf{X}}$ and $\hat{\mathbf{X}}$, respectively, the Wardrop equilibria associated with cost functions $\tilde{T}^k_p$ and $\hat{T}^k_p$. Consider $\tilde{\lambda}^k_{(sd)}$ and $\hat{\lambda}^k_{(sd)}$ the travel cost of class k at the respective Wardrop equilibria $\tilde{\mathbf{X}}$ and $\hat{\mathbf{X}}$. Then $\hat{\lambda}^k_{(sd)} < \tilde{\lambda}^k_{(sd)}$, $\forall k \in I\!K$.*

Proof: We consider now the network $(I\!M, I\!L)$, with travel costs $\tilde{T}^k_p$ and throughput demands $\bar{r}^k_{(sd)} = r^k_{(sd)}/\alpha$, $k \in I\!K$. Let $\bar{\lambda}^k_{(sd)}$ the travel cost of class k associated to these throughput demands. At equilibrium $\hat{\mathbf{X}}$, by redefining the cost and path flows as $\alpha \hat{\lambda}^k_{(sd)}$ and $(1/\alpha)\hat{x}^k_p$, respectively, it is straightforward to show that the demand variation is equivalent to variation the cost function instead to $\tilde{\mathbf{T}}^k_p(\mathbf{X}/\alpha)$. Hence the corresponding travel cost are $\bar{\lambda}^k_{(sd)} = \alpha \hat{\lambda}^k_{(sd)}$. On the other hand, we have $\bar{r}_{(sd)} = r_{(sd)}/\alpha <$

$r_{(sd)}$ and $\bar{v}_{(sd)} = v_{(sd)}/\alpha < v_{(sd)}$ hence from Theorem 2, $\bar{\lambda}^k_{(sd)} < \tilde{\lambda}^k_{(sd)}$, $\hat{\lambda}^k_{(sd)} = \bar{\lambda}^k_{(sd)}/\alpha < \bar{\lambda}^k_{(sd)} < \tilde{\lambda}^k_{(sd)}$, which concludes the proof. ■

VI. AN OPEN **BCMP** QUEUEING NETWORK

In this section we study an example of such Braess paradox in networks consisting entirely of **BCMP** [2] queueing networks (BCMP stand for the initial of the authors) (see also [14]).

A. *BCMP queueing network*

We consider an open **BCMP** queueing network model that consists of L service links. Each service centre contains either a single-server queue with the processor-sharing (**PS**). We assume that the service rate of each single server is state independent. Jobs are classified into K different classes. The arrival process of jobs of each class forms a Poisson process and is independent of the state of the system.

Let us denote the state of the network by $\mathbf{n} = (n_1, n_2, ..., n_L)$ where $\mathbf{n}_l = (n_l^1, n_l^2, .., n_l^K)$ and $n_l = \sum_{l \in L} n_l^k$ where n_l^k denotes the total number of jobs of class k visiting link l. For an open queueing network [14], [2], the equilibrium probability of the network state $\mathbf{n}$ is obtained as follows:

$$p(\mathbf{n}) = \prod_{l \in L} \frac{p_l(n_l)}{G_l},$$

where $p_l(n_l) = n_l! \prod_{l \in L} (\rho_l^k)/n_l^k$ and $G_l = 1/(1-\rho_l)$. Let $E[n_l^k]$ be the average number of class k jobs at link l. We have $E[n_l^k] = \rho_l^k/(1-\rho_l)$. By using Little's formula, we have

$$T_l^k = \frac{E[n_l^k]}{x_l^k} = \frac{1/\mu_l^k}{(1-\rho_l)}.$$

from which the average delay of a class k job that passes through path-class $p \in \Pi^k$ is given by

$$T_p^k = \sum_{l \in L} \delta_{lp} T_l^k = \sum_{l \in L} \delta_{lp} \frac{1/\mu_l^k}{(1-\rho_l)}.$$

We assume that μ_l^k can be represented as μ_l/c^k, hence the average delays satisfy assumption **C**.

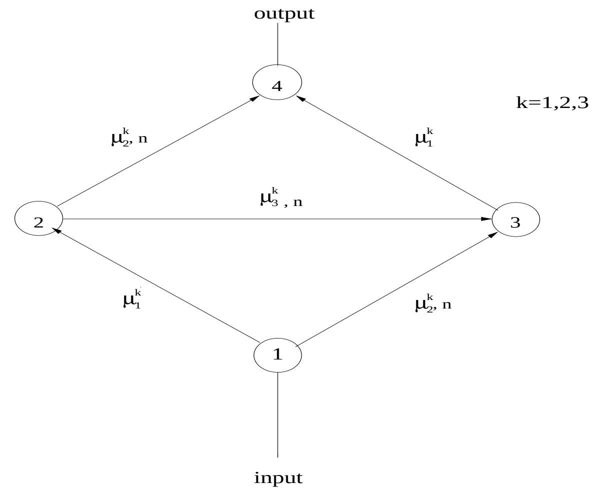

Fig. 1. Network

B. *Braess paradox*

Consider the networks shown in Figure 1. Packets are classified into $three$ different classes. Links (1,2) and (3,4) have each the following service rates: $\mu_1^k = \mu_1$, $\mu_1^2 = 2\mu_1$ and $\mu_1^3 = 3\mu_3$ where $\mu_1 = 2.7$. Link (1,3) represents a path of n tandem links, each with the service rates: $\mu_2^1 = \mu_2$, $\mu_2^2 = 2\mu_2$ and $\mu_2^3 = 3\mu_2$ with $\mu_2 = 27$. Similarly link (2,4) is a path made of n consecutive links, each with service rates: $\mu_2^1 = 27$, $\mu_2^2 = 54$ and $\mu_2^3 = 81$. Link (2,3) is path of n consecutive links each with service rate of each class $\mu_3^1 = \mu$, $\mu_3^2 = 2\mu$ and $\mu_3^2 = 3\mu$ where μ varies from 0 (absence of the link) to infinity. We denote $x_{p_1}^k$ the left flow of class k using links (1,2) and (2,4), $x_{p_2}^k$ the right flow of class k using links (1,3) and (3,4), and x_{p_3} the zigzag flow of class k using links (1,2), (2,3) and (3,4). The total cost for each class is given by

$$\mathbf{T}^k = x_{p_1}^k T_{p_1}^k + x_{p_2}^k T_{p_2}^k + x_{p_3}^k T_{p_3}^k,$$

where $x_{p_1}^k + x_{p_1}^k + x_{p_1}^k = r^k$.

We first consider the scenario where additional capacity μ is added to path (2,3), for $n = 54$, $r^1 = 0.6$, $r^2 = 1.6$ and $r^3 = 1.8$. In figure 2 we observe that no traffic uses the zigzag path for $0 \le \mu \le 36.28$. For $36.28 \le \mu \le 96.49$, all three paths are used. For $\mu > 96.49$, all traffic uses the zigzag path. For μ between 36.28 and 96.49, the delay is, paradoxically, worse than it would be without the zigzag path. The delay of class 1 (*resp.* 2,3) decreases to 2.85 (*resp.* 1.42, 0.95) as μ goes to infinity.

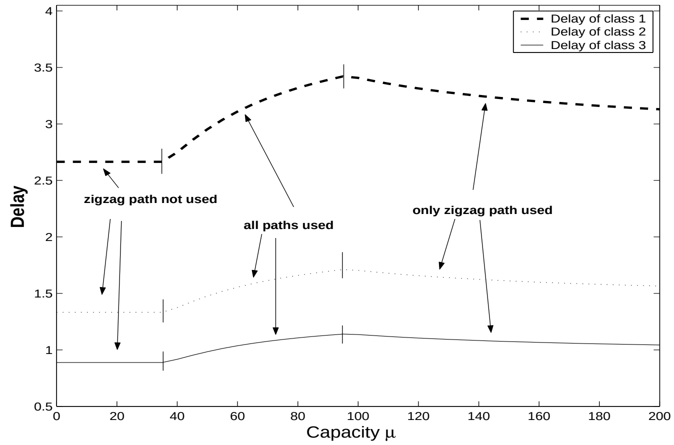

Fig. 2. Delay of each class as a function of the added capacity in path (2,3)

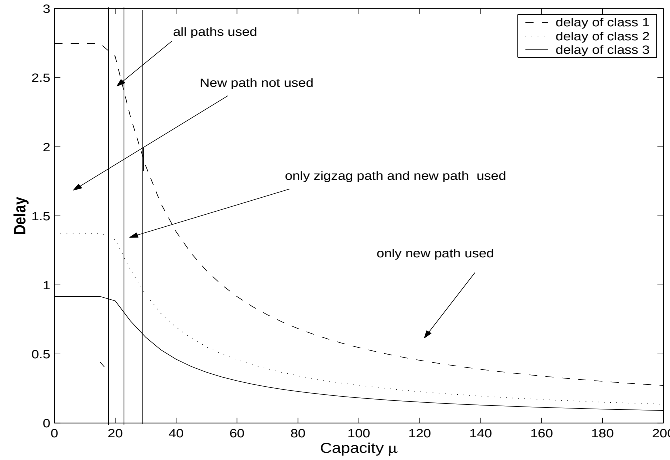

Fig. 4. Delay as a function of the added capacity in path (1,4)

C. Adding a direct path between source and destination

Now we use the method proposed in Theorem 3 and Theorem 4, i.e., the upgrade achieved by adding a direct path connecting source 1 and destination 4.

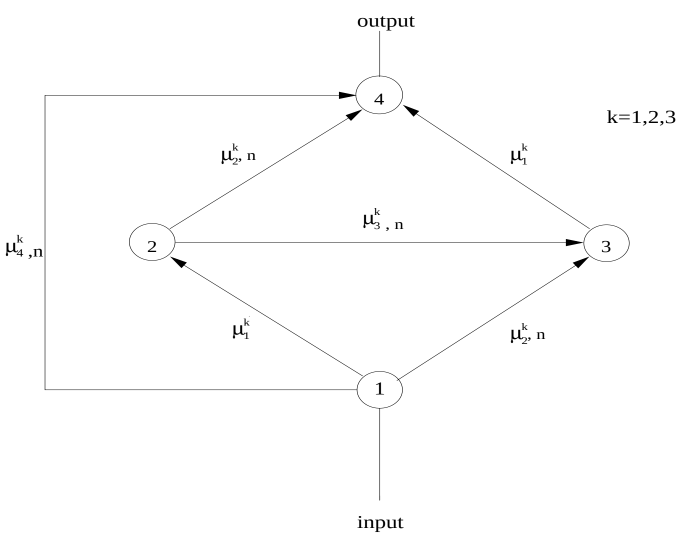

Fig. 3. New network

The results in Theorem 3 and Theorem 4 suggest that yet another good design practice is to focus the upgrades on direct connections between source and destination; and figure 4 illustrates that indeed this approach decreases the delay of each class.

D. Multiplying the capacity of all links $(l \in \mathbb{L})$ by a constant factor $\alpha > 1$.

Now we use the method proposed in Theorem 6 for efficiently adding resources to this network.

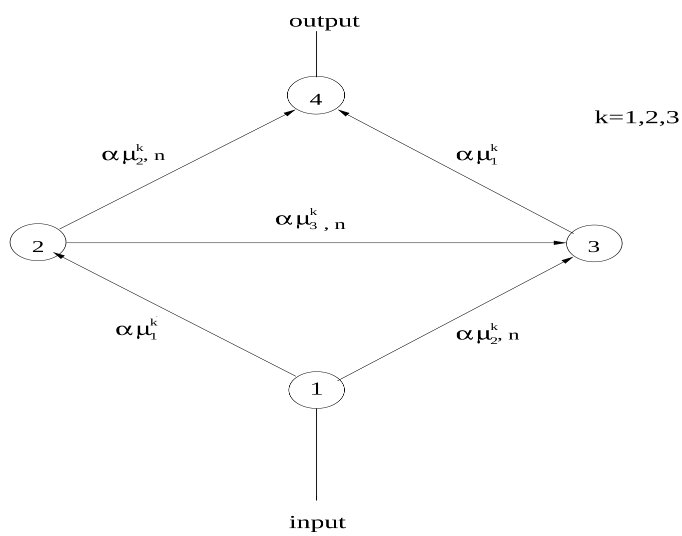

Fig. 5. New network

Figure 6 shows the delay of each class as a function of the additional capacity μ where $\mu = (\alpha - 1)(2\mu_1 + 2\mu_2 + \mu 3)$ with $\mu_1 = 2.7$, $\mu_2 = 27$ and $\mu_3 = 40$. Figure 6 indicates that the delay of each class decreases when the additional capacity μ increases. Hence the Braess paradox is indeed avoided.

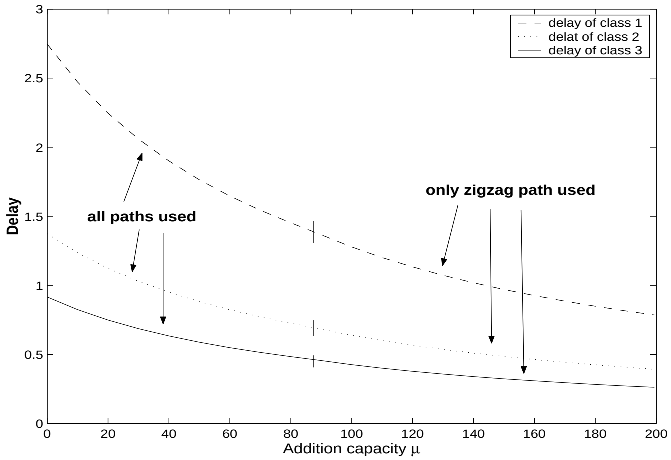

Fig. 6. Delay of each class as a function of the added capacity in all links

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REFERENCES

[1] E. Altman, R. El Azouzi and O. Pourtallier, "Avoiding Paradoxes in Routing Games", Proceedings of the 17th International Teletraffic Congress, Salvador da Bahia, Brazil, September 24-28, 2001.

[2] F. Baskett, K. M. Chandy, R. R. Muntz and F. Palacios, *O*pen, Closed and Mixed Networks of Queues with Different classes of Customers, J. ACM 22, No 2, 248-260, 1975.

[3] N. G. Beans, F. P. Kelly and P. G. Taylor, *B*raess's paradox in a loss network, J. Appl. Prob., Vol. 34, pp155–159, 1997.

[4] D. Braess, ¨Uber ein Paradoxen aus der Verkehrsplanung, Unternehmensforschung 12, pp258-268, 1968.

[5] B. Calvert, W. Solomon and I. Ziedins, *B*raess's Paradox in a Queueing Network with State-Dependent Routing, J. Appl. Prob., Vol. 34, pp134–154, 1997.

[6] J. E. Cohen and C. Jeffries, *C*ongestion Resulting from Increased Capacity in Single-Server Queueing Networks, IEEE/ACM Trans. on Networking, Vol.5, No 2, pp1220–1225, April 1997.

[7] J. E. Cohen and F. P. Kelly *A* Paradox of Congestion in a Queueing Network, J. Appl. Prob., Vol 27, pp730–734,1990.

[8] Dafermos and A. Nagurney, *O*n some traffic equilibrium theory paradoxes, Transportation Research B, Vol.18, pp101–110, 1984.

[9] Dafermos and A. Nagurney, *S*ensitivity analysis for the asymmetric network equilibrium problem , Mathematical Programming 28, pp174-184, 1984.

[10] P. Gupta and P. R. Kumar,*A* system and traffic dependent adaptive routing algorithm for Ad Hoc networks, Proceedings of the 37th IEEE Conference on Decision and Control, Tampa, Florida, USA, Dec. 1998.

[11] H. Kameda and Y. Zhang, *U*niqueness of the solution for optimal static routing in open BCMP queueing networks, Math. Comput. Modelling, Vol.22, No. 10-12, pp119-130, 1995.

[12] H. Kameda, E. Altman and T. Kozawa, *A* case where a paradox like Braess's occurs in the Nash equilibrium but does not occur in the Wardrop equilibrium - a situation of load balancing in distributed computer systems, Proc. of the 38th IEEE Conference on Decision and Control, Phoenix, Arizona, USA, Dec. 1999.

[13] H. Kameda, E. Altman, T. Kozawa and Y. Hosokawa, *B*raess-like paradoxes of Nash Equilibria for load balancing in distributed computer systems, IEEE Trans. on Automatic Control, Vol.45, No. 9, 1687-1691, 2000.

[14] F. P. Kelly, *R*eversibility and stochastic networks, John Wiley & Sons, Ltd., New York, 1979.

[15] H. Kobayashi *M*odelling and Analysis, An Introduction to system Performance Evaluation Methodology, Addison Wesley, 1978.

[16] Korilis, Y. A., Lazar, A.A. and Orda, A.,*A*rchitecting non-cooperative networks, IEEE JSAC, Vol.13, pp1241-1251, 1995.

[17] Y. A. Korilis, A. A. Lazar and A. Orda, *C*apacity Allocation under Non-Cooperative Routing, IEEE AC, Vol.42, No.3, pp309–325, March 1997.

[18] Y. A. Korilis, A. A. Lazar and A. Orda, *A*voiding the Braess Paradox in Non-cooperative Networks, JAP, Vol.36, pp211–222, 1999.

[19] F.P. Kelly, *R*eversibility and Stochastic Networks, John Wiley & Sons, Ltd., New York, 1979.

[20] Y. Massuda,*B*raess's paradox and capacity management in decentralised network, manuscript, 1999.

[21] A. Orda, R. Rom and N. Shimkin, *C*ompetitive routing in multi-user environments, IEEE/ACM Trans. on Networking, pp510-521, 1993.

[22] M. Patriksson *T*he Traffic Assignment Problem: Models and Methods VSP BV, Topics in transportation, The Netherlands, 1994.

[23] M. J. Smith, *T*he marginal cost taxation of a transportation network, Transportation Research-B, Vol.13, pp237–242, 1979.

[24] J.G. Wardrop, *S*ome theoretical aspects of road traffic research, Proc. Inst. Civ. Eng. Part 2, pp325–378, 1952.