An asymptotic simplex method for singularly perturbed linear programs

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Abstract

We study singularly perturbed linear programs. These are linear programs whose constraints and objective coefficients depend on a small perturbation parameter, and furthermore the constraints become linearly dependent when the perturbation parameter goes to zero. Problems like that were studied by Jeroslow in 1970's. He proposed simplex-like method, which works over the field of rational functions. Here we develop an alternative asymptotic simplex method based on Laurent series expansions. This approach appears to be more computationally efficient. In addition, we point out several possible generalizations of our method and provide new simple updating formulae for the perturbed solution.

Key words. asymptotic simplex method, singular perturbations, Laurent series

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Abbreviated title. Asymptotic simplex method

1 Introduction

This paper studies linear programs with matrix and vector coefficients that depend on a (typically small) parameter ε :

$$\min_{x}\{c(\varepsilon)x\} \tag{1}$$

subject to :

$$A(\varepsilon)x = b(\varepsilon), \quad x \geq 0, \tag{2}$$

where x is n-column vector, $c(\varepsilon)$ is n-row vector, $b(\varepsilon)$ is m-column vector and $A(\varepsilon)$ is $m \times n$ -matrix. We denote this LP by the triple $(A(\varepsilon), b(\varepsilon), c(\varepsilon))$. In this paper we consider only linear and polynomial perturbations and hence assume that all functions of ε in $(A(\varepsilon), b(\varepsilon), c(\varepsilon))$ are, at worst, polynomials. This avoids pathological cases that cannot be studied by analytical tools of series expansions. The parameter ε will be called a *perturbation parameter*. We are interested in the determination of an *asymptotically* optimal solution. Namely,

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Definition 1 *The set of basic indices* $B \subset \{1,...,n\}$ *is said to be* asymptotically optimal (*or* a-optimal) *for the perturbed linear program* (1), (2), *if it is optimal for the linear program* (1), (2) *for all* $\varepsilon \in (0, \overline{\varepsilon}]$ *with* $\overline{\varepsilon} > 0$. *The solution* $x_B(\varepsilon) = A_B^{-1}(\varepsilon)b(\varepsilon)$ *corresponding to the a-optimal set of basic indices* B *is called an a-optimal solution.*

Note that the a-optimal set of basic indices uniquely identifies a basis that is "uniformly" optimal for all $\varepsilon \in (0, \overline{\varepsilon}]$. The existence of an a-optimal set of basic indices in the case when $A(\varepsilon), b(\varepsilon)$ and $c(\varepsilon)$ are rational functions was shown in [15, 16, 19]. Note that in [15, 16, 19] a semi-infinite interval $[\overline{t}, +\infty)$ is considered instead of the interval $(0, \overline{\varepsilon}]$. It is clear that the formulation with coefficients that are rational functions on $[\overline{t}, +\infty)$ is equivalent to a formulation with coefficients that are rational functions of ε on $(0, \overline{\varepsilon}]$. Furthermore, by multiplying by a common denominator the coefficients that are rational functions, the problem formulation can be converted to the polynomial setting.

We would like to note that the notion of the asymptotic optimal solution is stronger than the notion of a *limiting* optimal solution [3, 24]. The latter can be formalized by saying that x^{lim} is limiting optimal if for any sequence $\varepsilon_k \to 0$ and $\delta > 0$ there exists $K(\delta)$ such that for all $k > K(\delta)$, x^{lim} is δ-optimal for $(A(\varepsilon_k), b(\varepsilon_k), c(\varepsilon_k))$.

Alternatively, one could have defined x^* as being *sequentially* a-optimal if there exists a sequence $\varepsilon_k \to 0$ such that $x^* = \lim_{k\to\infty} x^*(\varepsilon_k)$, where $x^*(\varepsilon_k)$ is optimal for $(A(\varepsilon_k), b(\varepsilon_k), c(\varepsilon_k))$ for each k. This last definition is restricted by the requirement that the sequence of optimal solutions needs to be selected in such a way as to be convergent. It is easy to check that if we identify an a-optimal set of basic indices B, then $x_B(\varepsilon) = A_B^{-1}(\varepsilon)b(\varepsilon)$ need not even have a limit as ε goes to zero. However, if the expansion of $A_B^{-1}(\varepsilon)b(\varepsilon)$ is a Taylor series rather than a Laurent series, then $\lim_{\varepsilon\to 0} x_B(\varepsilon)$ will exist and be both limit optimal and sequentially a-optimal.

The examples below demonstrate some of the differences between these different notions of asymptotic optimality.

Example 1 This example shows that a-optimality is not only different from limiting optimality but gives the user a solution that is more robust.

$$\min\{10x_1 + 10x_2\}$$

subject to

$$\begin{aligned} \varepsilon x_2 - x_3 &= 0, \\ x_1 + x_2 + x_3 &= 1, \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

There are two basic feasible solutions $[1, 0, 0]$ and $[0, 1/(1+\varepsilon), \varepsilon/(1+\varepsilon)]$ (see Figure 1). Clearly, $x^{a-opt}(\varepsilon) := [0, 1/(1+\varepsilon), \varepsilon/(1+\varepsilon)]$ is an optimal solution for any positive value of ε, that is, $x^{a-opt}(\varepsilon)$ is an a-optimal solution. Note that the point $[0, 1, 0]$ is a *limiting* optimal, that is, the optimal value $10/(1+\varepsilon)$ of the perturbed linear programming $(A(\varepsilon), b(\varepsilon), c(\varepsilon))$ converges to 10 as ε goes to zero. Thus, we can see that the notion of an a-optimal solution is more "robust" than the notion of limiting optimal solution, in the sense that it is optimal for some interval of values of ε. In the present paper we provide a method to find an a-optimal solution. An approach to find a limiting optimal solution was proposed in [9, 24].

The effect of perturbations (for small values of ε) can be either small or large. The mathematical reasons for this difference underlie the classification of problems into either *regular* or *singular* perturbation problems. More precisely, we have the following definition.

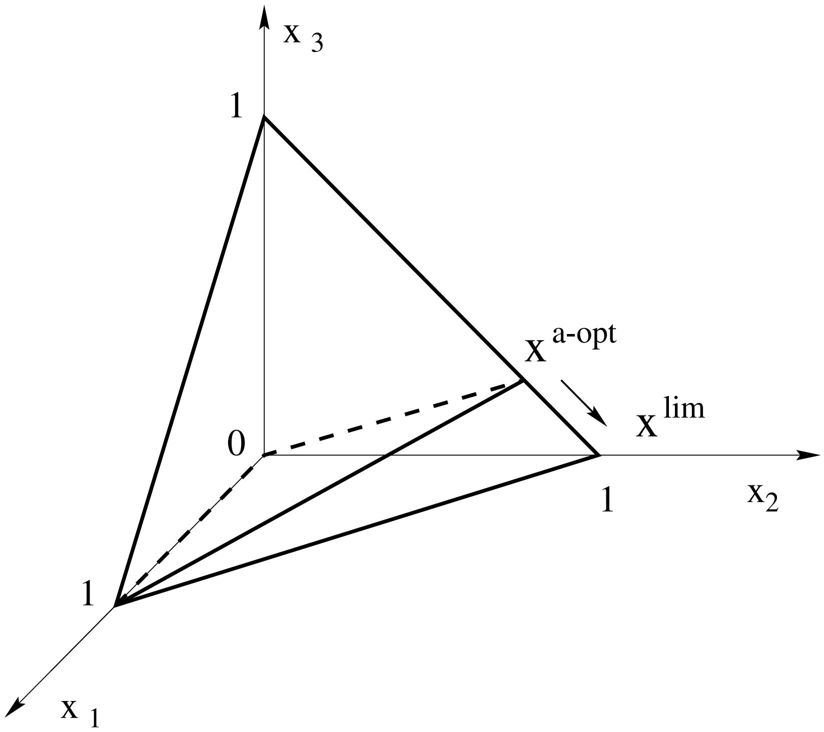

Figure 1: Comparison between a-optimal and limiting optimal solutions

Definition 2 *Let $B = \{j_1, j_2, ..., j_m\}$ be a subset of m indices selected from $\{1, 2, ..., n\}$, and $A_B(\varepsilon)$ be the $m \times m$ sub matrix of $A(\varepsilon)$, whose columns correspond to the index set B, for $\varepsilon \geq 0$. In all cases it will be assumed that $rank[A(\varepsilon)] = m$ for $\varepsilon > 0$ and sufficiently small. There are now three cases:*

(i) Regular perturbation: *$A_B^{-1}(0)$ exists whenever $A_B^{-1}(\varepsilon)$ exists for $\varepsilon > 0$ and sufficiently small.*

(ii) Weakly singular perturbation: *$rank[A(0)] = m$ but there exits at least one B such that $rank[A_B(0)] < m$ and $rank[A_B(\varepsilon)] = m$ for $\varepsilon > 0$ and sufficiently small.*

(iii) Strongly singular perturbation: *$rank[A(0)] < m$.*

It was shown [24] that an a-optimal solution of the regularly perturbed LOP is always an optimal solution of the original unperturbed LOP ($\varepsilon = 0$). However, in the case of singular perturbations it is often not true. Let us demonstrate this phenomenon with the help of the following elegant example [24].

Example 2

$$\max_{x_1, x_2} \{x_2\}$$

subject to

$$\left\{ \begin{array}{l} x_1 + x_2 = 1, \\ (1+\varepsilon)x_1 + (1+2\varepsilon)x_2 = 1+\varepsilon, \end{array} \right. \tag{3}$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

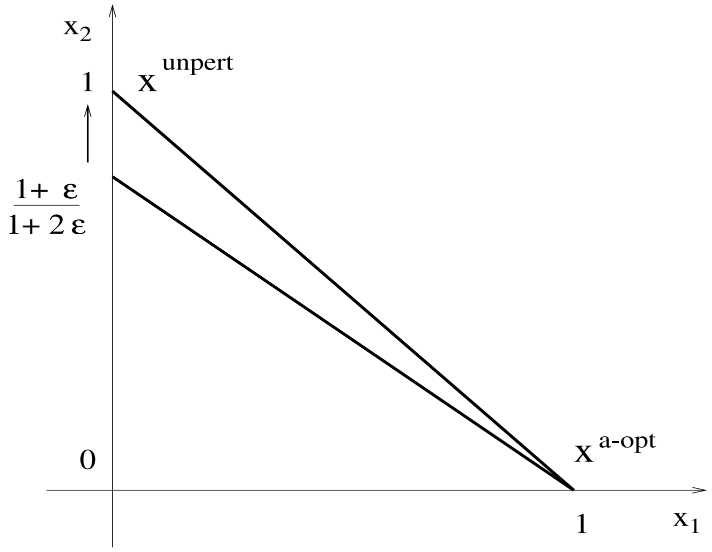

Figure 2: An example of singularly perturbed linear program

It is obvious that the system of constraints (3) has the unique feasible solution $x^{a-opt} = [1, 0]$, when $\varepsilon > 0$, which is also an a-optimal solution (see Figure 2). However, the optimal solution of the original unperturbed ($\varepsilon = 0$) problem is $x^{unpert} = [0, 1]$, which is not anywhere near the previous solution.

Note that most papers on sensitivity analysis and parametric programming (e.g. see [8, 10, 11]) concern with the perturbation of the objective function and the right hand side. Past researches on the perturbation of the entire coefficient matrix are quite limited. Moreover, practically all authors restrict themselves to the case of regular perturbations. Jeroslow [15, 16] was, perhaps, the first who studied the general case. He considered the elements of matrices $A(\varepsilon), b(\varepsilon)$ and $c(\varepsilon)$ as arbitrary rational functions. To deal with such perturbed LPs, Jeroslow [15, 16] proposed simplex-like method which works directly over the field of rational functions. The main disadvantage of this method is that the polynomials involved in calculation can have a high degree. For instance, the inversion of a basis matrix takes $O(m^4 log(m))$ flops. Jeroslow's method can be viewed as an instance of a more general theory of extending algorithms for parametric problems (see Eaves and Rothblum [7]). Another application of this general theory would be a method which works in the field of series expansions. We will call this method the *asymptotic simplex method*. It uses only initial terms of asymptotic expansions and hence can significantly reduce computational effort.

Our method is related to the techniques introduced by Lamond [19, 20] and Huang [14], however it exploits the structure of Laurent series to a greater degree. In particular, Lamond [19] proposed a method for the expansion of the inverse of the basis matrix which demands $O(\bar{s}m^3)$ flops, where $\bar{s}$ is the maximal order of poles of the basis matrices. Huang [14] improved further the expansion of the perturbed basis matrix by proposing an algorithm which demands only $O(m^3)$ flops. In another paper [20] Lamond proposed to update the asymptotic expansion for the inverse of the perturbed basis matrix rather than to compute it anew. However, his approach applies only to some particular cases, that is when the inverse of the perturbed basis matrix has the pole of order one. This updating procedure demands $O(m^2)$ operations, which is comparable with the standard simplex method. In this paper we propose an updating procedure which deals with the general case and demands only $O(\bar{s}m^2)$. Moreover, our procedure is simpler than the inversion technique of Huang [14] and the updating algorithm of Lamond [20]. It is based on the elegant recurrent formulae of

Langenhop [21] and Schweitzer and Stewart [26].

The above mentioned authors [24, 19, 20, 14] who worked on the asymptotic simplex method restrict consideration to the most common and natural type of perturbation, namely, they consider the case of a linear perturbation

$$\min_x\{(c^{(0)}+\varepsilon c^{(1)})x\} \tag{4}$$

subject to

$$(A^{(0)}+\varepsilon A^{(1)})x = b^{(0)}+\varepsilon b^{(1)}, \quad x \geq 0. \tag{5}$$

We also present our method in this linear setting. However, in contrast to the other papers, our method can be readily generalized to the case of polynomial perturbations of the form (1),(2) with

$$A(\varepsilon) = A^{(0)} + \varepsilon A^{(1)} + \ldots + \varepsilon^p A^{(p)},$$

and similarly for $b(\varepsilon)$ and $c(\varepsilon)$. In fact, we outline the adaptation of our method for these polynomial perturbations in Section 4.

Since the asymptotic simplex method is based on a generalisation of the revised simplex method to the field of asymptotic expansions, we shall briefly outline the classical revised simplex method in the next section.

2 The revised simplex method

To ensure that this paper is self-contained and has consistent notation we briefly outline the now classical, revised simplex method. More detailed explanations and proofs can be found in any standard book on linear programming (e.g., [6, 22]). Consider the following linear program

$$\min_x\{cx\} \tag{6}$$

subject to

$$Ax = b, \quad x \geq 0, \tag{7}$$

where $\text{rank}(A) = m$. Denote by B the index set of the current basic variables. Let A_B be the corresponding basis and $x_B = A_B^{-1}b$ be the vector of basic variables. Suppose that all elements of this vector are positive (non-degeneracy assumption).

Recall that when applying the revised simplex method the following quantities need to be computed or updated:

(i) Simplex multipliers:

$$\lambda = c_B A_B^{-1}.$$

(ii) Reduced cost coefficients:

$$r_N = \lambda A_N - c_N,$$

where N corresponds to the set of non-basic column indices, namely $N = \{1,2,...,n\}\backslash B$.

(iii) Entering column:

$$y_k = A_B^{-1}a_k,$$

where

$$k = \arg\max_{j\in N}\{r_j | r_j > 0\}.$$

(iv) Exiting column index:

$$p = \arg\min_{l}\{\frac{[x_B]_l}{[y_k]_l}|y_l > 0\}.$$

Of course, most of the above depend on the knowledge of A_B^{-1} and it is a well known fact that this inverse need not be computed from scratch but can be updated at each iteration.

In particular, if B' is the next set of basic variable indices and we construct the following vector and matrix

$$\xi = [-\frac{y_1}{y_p}, ..., -\frac{y_{p-1}}{y_p}, \frac{1}{y_p}, -\frac{y_{p+1}}{y_p}, ..., -\frac{y_m}{y_p}]^T,$$

$$E = [e_1, ..., e_{p-1}, \xi, e_{p+1}, ..., e_m],$$

where e_i is a vector whose all elements are zeros except the i-th element which is one. Then the next inverse basis is given by

$$A_{B'}^{-1} = EA_B^{-1}.$$

3 Asymptotic simplex method

In this section we present the asymptotic simplex method for obtaining an a-optimal solution to the perturbed linear program (4),(5).

At first let us introduce some rather mild assumptions, which will be relaxed in Section 4. Let M_ε denote the feasible region of (4),(5) and M_0 be the feasible region of the unperturbed problem.

Assumption 1 *The region M_0 is bounded.*

The above assumption ensures that basic feasible solutions of the perturbed program (4),(5) can be expanded as Taylor series (see Lemma 2 for details).

Assumption 2 *The perturbed problem is non-degenerate, namely, every element of the basic feasible vector $x_B(\varepsilon) = A_B^{-1}(\varepsilon)b(\varepsilon), \varepsilon \in (0, \bar{\varepsilon}]$ is positive.*

From the preceding description of the classical revised simplex method we see that everything depends on the inverse basis matrix A_B^{-1}. In the perturbed case it is necessary to realise that the analogous matrix $A_B^{-1}(\varepsilon)$ is a Laurent series whose structure and coefficients determine the asymptotic behaviour, as $\varepsilon \to 0$. In particular,

$$A_B^{-1}(\varepsilon) = \frac{1}{\varepsilon^s}U^{(-s)} + ... + \frac{1}{\varepsilon}U^{(-1)} + U^{(0)} + ... \; . \tag{8}$$

If $A_B(\varepsilon)$ becomes singular at $\varepsilon = 0$, then the above series will have a pole of order s at $\varepsilon = 0$ and will contain a nontrivial *singular part*; defined by

$$A_B^S(\varepsilon) = \frac{1}{\varepsilon^s}U^{(-s)} + ... + \frac{1}{\varepsilon}U^{(-1)} \; . \tag{9}$$

Similarly, a *regular part* of (8) is defined by

$$A_B^R(\varepsilon) = U^{(0)} + \varepsilon U^{(1)} + ... \; . \tag{10}$$

Clearly, if $A_B^S(\varepsilon) \neq 0$ and ε is small, standard simplex operations could result in unstable numerical behaviour. The methods developed in this section overcome this difficulty by working with the coefficients of (8).

At first sight it might appear that computations involving the series expansion (8) would be too difficult. Fortunately, recursive formulae developed by Langenhop [21] and Schweitzer and Stewart [26] provide tools that can be adapted to the revised simplex method. A key observation here is that if $U^{(0)}$ and $U^{(-1)}$ are known, then every coefficient of (8) can be obtained according to (see [21], [26] and the Appendix)

$$U^{(k)} = (-U^{(0)}A_B^{(1)})^{k-1}U^{(0)} = U^{(0)}(-A_B^{(1)}U^{(0)})^{k-1}, \quad k = 0, 1, ... \tag{11}$$

$$U^{(-k)} = (-U^{(-1)}A_B^{(0)})^{k-1}U^{(-1)} = U^{(-1)}(-A_B^{(0)}U^{(-1)})^{k-1}, \quad k = 1, ..., s, \tag{12}$$

where $A_B^{(i)}$ are the coefficients of ε^i in the basis matrix $A_B(\varepsilon)$. Consequently, if $U^{(0)}$ and $U^{(-1)}$ can be efficiently updated, when moving from one basis to another, then any coefficient of (8) can also be calculated for the next basis via (11),(12), if it is needed.

However, we still need to compute $U^{(0)}$ and $U^{(-1)}$ for the initial step of the asymptotic simplex method. There are two solutions to this problem. The first solution is to compute the singular part and the first regular coefficient of asymptotic expansion (8) by using methods presented in [1, 13, 14, 19, 26]. The other solution is to start the asymptotic simplex method with an analog of phase 1 method [22]. Note that when we introduce artificial variables in phase 1, the linear program becomes weakly singular, even if it was strongly singular before modification. The latter enables us to start from a basis matrix which possesses a Taylor series expansion instead of a Laurent series. This significantly facilitates computations. In addition, if we use the phase 1 method, we need not be concerned about the choice of an initial basic feasible solution.

In asymptotic simplex method we need to compare two functions in some small neighbourhood of zero. This will be done with the help of asymptotic expansion (8) and the concept of lexicographical ordering. Before proceeding further, let us recall the definition of *lexicographical ordering* [27].

Definition 3 *A vector a (possibly infinite) is called* lexicographically nonnegative, *written $a \succeq 0$, if the first nonzero element (if any) in a is positive, and a is called* lexicographically positive, *written $a \succ 0$, if $a \succeq 0$ and $a \neq 0$. For two vectors a and b we say that a is* lexicographically greater (strictly) *than b, if $a - b \succeq 0$ ($a - b \succ 0$).*

Suppose that we have an analytic function $g(\varepsilon)$ which is expanded as a Laurent series at $\varepsilon = 0$ with a finite singular part

$$g(\varepsilon) = \frac{1}{\varepsilon^{-s}}g^{(-s)} + ... + \frac{1}{\varepsilon}g^{(-1)} + g^{(0)} + \varepsilon g^{(1)} + ... \ .$$

We construct from the coefficients of the above series the infinite vector

$$\gamma = [g^{(-s)}, ..., g^{(0)}, g^{(1)}, ...].$$

It is easy to see that $g(\varepsilon) > 0$ for ε sufficiently small and positive $\Leftrightarrow \gamma \succ 0$. Moreover, if $g(\varepsilon)$ is a rational function then only a finite number of elements in γ needs to be checked (see Lemma 1). The comparison (in a neighbourhood of 0) between two functions $g(\varepsilon)$ and $h(\varepsilon)$ possessing Laurent series expansions with finite order poles can be carried out similarly, by considering $g(\varepsilon) - h(\varepsilon)$.

Now let us formally present the *asymptotic simplex method* and comment on its possible implementation. The proof of its finite convergence is given at the end of the section.

The asymptotic simplex method:

Let the initial set of basic indices B be given.

Step 1: Obtain or update only the singular and the first regular coefficients of the Laurent series expansion (8) for the inverse basis matrix $A_B^{-1}(\varepsilon)$. The implementation of this step is discussed at the end of this section.

Step 2 (basic idea): Next we have to decide which column enters the basis. Namely, among the non-basic elements of the reduced cost vector

$$r_N(\varepsilon) := \lambda(\varepsilon)A_N(\varepsilon) - c_N(\varepsilon), \quad \text{where} \quad \lambda(\varepsilon) := c_B(\varepsilon)A_B^{-1}(\varepsilon), \tag{13}$$

we need to find such k that

$$k \in \arg\max_{j\in N}\{r_j(\varepsilon)|r_j(\varepsilon) > 0, \varepsilon \in (0, \bar{\varepsilon}]\}.$$

Substituting (8) into (13), we obtain the next asymptotic expansion

$$r_N(\varepsilon) = \frac{1}{\varepsilon^s} r_N^{(-s)} + \frac{1}{\varepsilon^{s-1}} r_N^{(-s+1)} + \ldots \quad ,$$

where

$$r_N^{(i)} := \lambda^{(i)}A_N^{(0)} + \lambda^{(i-1)}A_N^{(1)} - \delta_{0i}c_N^{(0)} - \delta_{1i}c_N^{(1)} \quad \text{and} \quad \lambda^{(i)} := c_B^{(0)}U^{(i)} + c_B^{(1)}U^{(i-1)}.$$

Let us construct the following (possibly semi-infinite) matrix

$$\rho = \begin{bmatrix} r_N^{(-s)} \\ r_N^{(-s+1)} \\ \ldots \end{bmatrix}$$

and denote its i-th column by ρ_i.

As mentioned above, the lexicographical ordering can be used to compare functions in the "small" neighbourhood of zero. In particular, it is easy to see that

$$\arg\max_{j\in N}\{r_j(\varepsilon)|r_j(\varepsilon) > 0, \varepsilon \in (0, \bar{\varepsilon}]\} = \arg\text{lex-}\max_{j\in N}\{\rho_j|\rho_j \succ 0\},$$

where "lex-max" is a maximum with respect to the lexicographical ordering and "arg lex-max" is an index at which "lex-max" is attained. Note that to compare two reduced cost coefficients $r_i(\varepsilon)$ and $r_j(\varepsilon)$ for sufficiently small ε we need only to check a *finite* number of elements of the vectors ρ_i and ρ_j. This follows from the fact that $r_i(\varepsilon)$ and $r_j(\varepsilon)$ are rational functions (see Lemma 1). A practical implementation of the lexicographical entering rule is as follows:

Step 2 (implementation): Set $i := -s$ and $\lambda^{(-s-1)} := 0$.

Step 2a) Calculate the i-th term of the Laurent expansion for the vector of simplex multipliers

$$\lambda^{(i)} := c_B^{(0)}U^{(i)} + c_B^{(1)}U^{(i-1)}.$$

Step 2b) Calculate the i-th term of the Laurent expansion for the vector of non-basic reduced cost coefficients

$$r_N^{(i)} := \lambda^{(i)}A_N^{(0)} + \lambda^{(i-1)}A_N^{(1)} - \delta_{0i}c_N^{(0)} - \delta_{1i}c_N^{(1)}$$

where δ_{0i} and δ_{1i} are the Kronecker deltas. Let $N^{(-s-1)} := N$ and $N^{(i)} = \{j \in N^{(i-1)} | r_j^{(i)} = 0\}$. If $r_j^{(i)} < 0$ for $j \in N^{(i-1)}$, STOP; the current solution is a-optimal. If there is an index k such that

$$k := \arg \max_{j \in N^{(i-1)}} \{r_j^{(i)} | r_j^{(i)} > 0\},$$

then k identifies the entering non-basic variable; go to Step 3.

Step 2c) If the algorithm does not stop and k has not been identified in Step 2b, and $i < m+1$, then $N^{(i)}$ is not empty. Increment the index i and return to Step 2a to consider the higher order approximation of $r_N(\varepsilon)$.

Remark 1 *Lemma 1, proved in the sequel, ensures that if index $i = m+1$ is reached in Step 2c and $N^{(m+1)}$ is still non-empty, then $r_j(\varepsilon) \equiv 0$ for $j \in N^{(m+1)}$ and ε sufficiently small. The latter implies that $r_N(\varepsilon) \leq 0$ for all ε sufficiently small, that is, the current solution is a-optimal.*

Step 3 (basic idea): Now, as in the revised simplex method, we have to find out which elements of the vector $y_k(\varepsilon) = A_B^{-1}(\varepsilon)a_k(\varepsilon)$ are positive for $\varepsilon > 0$ and sufficiently small. Namely, we have to identify the set of indices $P := \{l | [y_k(\varepsilon)]_l > 0, \varepsilon \in (0, \bar{\varepsilon}]\}$. Towards this, as in Step 2, we first expand $y_k(\varepsilon)$ as a Laurent series

$$y_k(\varepsilon) = \frac{1}{\varepsilon^s} y_k^{(-s)} + \frac{1}{\varepsilon^{s-1}} y_k^{(-s+1)} + \ldots$$

then define an auxiliary semi-infinite matrix

$$v = [y_k^{(-s)}, y_k^{(-s+1)}, \ldots].$$

Let v_l denote the l-th row of matrix v. The set P is given by $P = \{l | v_l \succ 0\}$. Here is a practical procedure for obtaining P:

Step 3 (implementation): Set $i := -s$, $U^{(-s-1)} := 0$ and $P = \emptyset$.

Step 3a) Calculate the i-th term of the Laurent expansion for $y_k(\varepsilon)$

$$y_k^{(i)} = U^{(i)} a_k^{(0)} + U^{(i-1)} a_k^{(1)}.$$

Step 3b) Let $Q^{(-s-1)} := \{1, \ldots, m\}$ and $Q^{(i)} := \{j \in Q^{(i-1)} | [y_k^{(i)}]_j = 0\}$. Add the index $j \in Q^{(i-1)}$ to the set P if $[y_k^{(i)}]_j > 0$. If $Q^{(i)} = \emptyset$, then go to Step 3d.

Step 3c) If $Q^{(i)} \neq \emptyset$ and $i < m$, then increment i by one and return to Step 3a. If $i = m$, go to the next Step 3d. Lemma 1 guarantees that $[y_k(\varepsilon)]_j \equiv 0, j \in Q^{(m)}$.

Step 3d) Stop. At this point the set P is determined.

Remark 2 *From Lemma 2 one can easily deduce that the feasible region for the perturbed problem is bounded. The latter implies that in our setting the set P is always non-empty.*

Step 4 (basic idea): Now we have to choose a basic variable which exits the basis, namely we have to find

$$p \in \arg \min_l \left\{ \frac{[x_B(\varepsilon)]_l}{[y_k(\varepsilon)]_l} \quad | \quad l \in P, \varepsilon \in (0, \bar{\varepsilon}] \right\}.$$

According to the previous step the functions $[y_k(\varepsilon)]_l, l \in P$ are expressed as a Laurent series

$$[y_k(\varepsilon)]_l = \varepsilon^{q_l} [y_k^{(q_l)}]_l + \varepsilon^{q_l+1} [y_k^{(q_l+1)}]_l + \ldots, \tag{14}$$

with $y_l^{(q_l)} > 0$. Due to the non-degeneracy Assumption 2 and Lemma 2 $[x_B(\varepsilon)]_l$ is expressed as a power series with a positive leading coefficient

$$[x_B(\varepsilon)]_l = \varepsilon^{t_l} x_{Bl}^{(t_l)} + \varepsilon^{t_l+1} x_{Bl}^{(t_l+1)} + ..., \qquad x_{Bl}^{(t_l)} > 0. \tag{15}$$

Then by Lemma 3 the quotient $\Delta_l(\varepsilon) := [x_B(\varepsilon)]_l / [y_k(\varepsilon)]_l$ is written in terms of Laurent series

$$\Delta_l(\varepsilon) = \varepsilon^{t_l - q_l}(\Delta_l^{(0)} + \varepsilon \Delta_l^{(1)} + \varepsilon^2 \Delta_l^{(2)} + ...), \tag{16}$$

where the coefficients $\Delta_l^{(i)}$ are calculated by simple recurrent formulae (see Lemma 3). Thus, similarly to the previous steps the lexicographical ordering is implemented as follows:

Step 4 (implementation): Set $i := 0$.

Step 4a) Form the set of indices corresponding to the maximal powers of the leading coefficients in (16).

$$R^{(-1)} := \{j | j = \arg\max_l \{t_l - q_l | l \in P\}\}$$

Step 4b) Calculate the $(q_l + i)$-th and $(t_l + i)$-th terms of the expansions (14),(15) respectively

$$[y_k^{(q_l+i)}]_l = [U^{(q_l+i)}]_l a_k^{(0)} + [U^{(q_l+i+1)}]_l a_k^{(1)}, \quad l \in R^{(i-1)},$$

and

$$x_{Bl}^{(t_l+i)} = [U^{(t_l+i)}]_l b^{(0)} + [U^{(t_l+i+1)}]_l b^{(1)}, \quad l \in R^{(i-1)},$$

Step 4c) Calculate the i-th coefficient of expansion (16)

$$\Delta_l^{(i)} = (x_{Bl}^{(t_l+i)} - \sum_{j=0}^{i-1} [y_k^{(q_l+i-j)}]_l \Delta_l^{(j)}) / [y_k^{(q_l)}]_l, \quad l \in R^{(i-1)}.$$

Step 4d) Form the following set of indices

$$R^{(i)} := \{j | j = \arg\min_l \{\Delta_l^{(i)} | l \in R^{(i-1)}\}\}.$$

If $R^{(i)}$ consists of a unique index p, go to Step 5. If $R^{(i)}$ is not a singleton and $i < 2m+1$, we should take into account the higher order approximation of $\Delta_l(\varepsilon)$. Namely, increment i by one and return to Step 4b. However, if $i = 2m+1$, choose any $p \in R^{(i)}$ and go to Step 5.

Remark 3 *Again Lemma 1 guarantees that $\Delta_p(\varepsilon) \equiv \Delta_q(\varepsilon)$, if $p, q \in R^{(2m+1)}$.*

Step 5: Construct a new basis $A_{B'}(\varepsilon)$ obtained from $A_B(\varepsilon)$ by replacing $a_p(\varepsilon)$ with $a_k(\varepsilon)$. Go to Step 1.

This completes the algorithm.

Remark 4 *Note that if we know the first regular and the first singular terms of the Laurent expansion (8), then the computation of Laurent series coefficients for simplex quantities $\lambda(\varepsilon), x_B(\varepsilon)$ and $y(\varepsilon)$ is easily performed by the following recurrent formulae*

$$\lambda^{(t)} = \lambda^{(t-1)} D_1, \quad y_k^{(t)} = D_2 y_k^{(t-1)},$$

$$x_B^{(t)} = D_2 x_B^{(t-1)}, \quad t \geq 2,$$

and

$$\lambda^{(-t)} = \lambda^{(-t+1)} F_1, \quad y_k^{(-t)} = F_2 y_k^{(-t+1)},$$

$$x_B^{(-t)} = F_2 x_B^{(-t+1)}, \quad t \geq 3,$$

where $D_1 := -A_B^{(1)} U^{(0)}, D_2 := -U^{(0)} A_B^{(1)}$ and $F_1 := -A_B^{(0)} U^{(-1)}, F_2 := -U^{(-1)} A_B^{(0)}$.

As in the revised simplex method, it is possible to update the expansion (8) for the inverse of the new basis matrix $A_{B'}(\varepsilon)$ via the multiplication of the series by

$$E(\varepsilon) = [e_1, ..., e_{p-1}, \xi(\varepsilon), e_{p+1}, ..., e_m],$$

where $\xi(\varepsilon) = [-\frac{y_1(\varepsilon)}{y_p(\varepsilon)}, ..., -\frac{y_{p-1}(\varepsilon)}{y_p(\varepsilon)}, \frac{1}{y_p(\varepsilon)}, -\frac{y_{p+1}(\varepsilon)}{y_p(\varepsilon)}, ..., -\frac{y_m(\varepsilon)}{y_p(\varepsilon)}]^T$. Since the division of two Laurent series in the scalar case is not a problem (see Lemma 3), one can easily obtain the Laurent series for $E(\varepsilon)$

$$E(\varepsilon) = \frac{1}{\varepsilon^t}E^{(-t)} + ... + \frac{1}{\varepsilon}E^{(-1)} + E^{(0)} + ... \tag{17}$$

Consequently, the coefficients $U'^{(k)}, k = -s', -s'+1, ...$ of the Laurent series for $A_{B'}^{(-1)}(\varepsilon)$ is calculated by the following formula

$$U'^{(k)} = \sum_{i+j=k} E^{(i)}U^{(j)}, \quad k = -s', -s'+1, ... \tag{18}$$

However, we would like to emphasize that we need to update only the coefficients $U'^{(-1)}, U'^{(0)}$ by the above formula. The other coefficients, if needed, can be restored by iterative formulae (11),(12) in a more efficient way. The computational complexity for this updating procedure is given in the next proposition.

Proposition 1 *The updating procedure for terms $U^{(-1)}$ and $U^{(0)}$ of the Laurent series expansion (8) requires $O(\bar{s}m^2)$ operations, where $\bar{s}$ is the maximal order of poles of the Laurent expansions for basis matrices.*

PROOF: Note that for our updating procedure we need to compute $\bar{s}$ terms of the Laurent series (17). To calculate the Laurent series for $E(\varepsilon)$, we need to calculate m scalar Laurent expansions for elements of $\xi(\varepsilon)$. This can be done by applying the recursive formulae from Lemma 3. Since the computation of each scalar expansions takes $O(\bar{s}^2)$ flops, the computation of first $\bar{s}$ terms of Laurent series (17) requires $O(\bar{s}^2 m)$ operations. Note that since matrix $E^{(i)}$ has a special structure, the matrix multiplication $E^{(i)}U^{(j)}$ takes only $O(m^2)$ operations. Then the calculation of $U^{(-1)}$ and $U^{(0)}$ by formula (18) demands $O(\bar{s}m^2)$ flops. Consequently, the whole updating procedure requires $O(\bar{s}^2 m + \bar{s}m^2)$ operations, which is equivalent to $O(\bar{s}m^2)$, since $\bar{s} \le m$.

□

Remark 5 *If $\bar{s} << m$ (which is often the case), then the estimated number of operations $O(\bar{s}m^2)$ needed in our updating procedure could be significantly less than $O(m^3)$, which is required in Huang's method [14].*

Now we prove the finite convergence of the asymptotic simplex method.

Theorem 1 *Let the assumptions 1 and 2 hold. Then the asymptotic simplex method finds the a-optimal basic index set for perturbed LP (4),(5), $0 < \varepsilon < \bar{\varepsilon}$, in a finite number of steps. Furthermore, let B^* denote this a-optimal basic index set, then the basic variables of the a-optimal solution are expressed by the power series*

$$x_{B^*}(\varepsilon) = x_{B^*}^{(0)} + \varepsilon x_{B^*}^{(1)} + ... \ , \qquad \varepsilon < \min\{\bar{\varepsilon}, 1/||D_2||\} \tag{19}$$

where $x_{B^*}^{(0)} = U^{(0)}b^{(0)} + U^{(-1)}b^{(1)}, x_{B^*}^{(1)} = U^{(1)}b^{(0)} + U^{(0)}b^{(1)}$ and the subsequent coefficients are calculated by the recurrent formula $x_{B^*}^{(k)} = D_2 x_{B^*}^{(k-1)}, \quad k \ge 2$, with D_2 as in Remark 4. The matrices $U^{(0)}$ and $U^{(-1)}$ are coefficients of the Laurent series (8) corresponding to the basic set B^*.

PROOF: At each iteration of the asymptotic simplex method it is necessary to determine the column that enters the basis and the column that exits it. Namely, it is necessary to determine the following numbers

$$k \in \arg\max_{j\in N}\{r_j(\varepsilon)|r_j(\varepsilon)>0\}, \quad p \in \arg\min_{l}\{\frac{x_{Bl}(\varepsilon)}{y_l(\varepsilon)}|y_l(\varepsilon)>0\}.$$

According to Lemma 1, in the asymptotic simplex method k and p are determined in a finite number of steps by a recursive procedure analogous to lexicographic ordering of the coefficients of Laurent/power series expansions.

Next let us show that the asymptotic simplex method has a finite number of iterations. Note that after each iteration the objective function $c(\varepsilon)x$ is "decreased" by the subtraction of the function

$$r_k(\varepsilon)\frac{x_{Bp}(\varepsilon)}{y_p(\varepsilon)}$$

for all $\varepsilon \in (0,\bar{\varepsilon}]$. Since all quantities in the above expression are positive for small $\varepsilon > 0$ (in particular, $x_{Bp}(\varepsilon) > 0$ for $\varepsilon \in (0,\bar{\varepsilon}]$ due to the nondegeneracy assumption), after each iteration the objective function is strictly decreased for all $\varepsilon \in (0,\bar{\varepsilon}]$. Hence, cycling is impossible and the asymptotic simplex method converges in a finite number of iterations.

The series expansion (19) is obtained by substituting the Laurent expansion (8) into $x_{B^*}(\varepsilon) = A_{B^*}^{-1}(\varepsilon)(b^{(0)}+\varepsilon b^{(1)})$ and noticing that $x_{B^*}(\varepsilon)$ cannot have a singular part because of Lemma 2. The inequality $\varepsilon < 1/||D_2||$ is the standard convergence condition for Neumann series [17, p. 32].

□

Once an a-optimal basis is found by the asymptotic simplex method, one can exactly calculate the optimal solution for any sufficiently small value of the perturbation parameter. Namely,

Corollary 1 *There is a precise updating formula for the optimal solution of the perturbed linear program*

$$x_{B^*}(\varepsilon) = x_{B^*}^{(0)} + \varepsilon[I-\varepsilon D_2]^{-1}x_{B^*}^{(1)}, \quad \varepsilon < \min\{\bar{\varepsilon}, \frac{1}{||D_2||}\} \tag{20}$$

where $x_{B^*}^{(0)}$, *and* $x_{B^*}^{(1)}$ *are as in Theorem 1.*

Note that this updating formula is computationally stable even in the case of singular perturbations, since one needs to invert the matrix which is close to the identity.

Computational aid:

Finally, we would like to present a practical modification of the asymptotic simplex method. Note that the main computational difficulties can arise in Step 2 and Step 3, if the functions $r_j(\varepsilon)$ and $y_l(\varepsilon)$ are identically zero, namely, if $r_j(\varepsilon)=0, y_l(\varepsilon)=0$ for any ε. In this case we are forced to calculate all terms in corresponding expansions up to the $(m+1)$-st term. Of course, we are interested in identifying such identically zero elements by an efficient method.

One simple heuristic solution for the above problem is proposed here. Note that if $r_j(\varepsilon) = 0$ or $y_l(\varepsilon)=0$ for $\varepsilon \in (0,\bar{\varepsilon}]$ then these equalities hold for any $\varepsilon \in \mathbb{R}$. This fact follows from the observation that $r_j(\varepsilon)$ and $y_l(\varepsilon)$ are rational functions of ε and every rational function has either no zero or isolated zeros or it is identically zero. Therefore we can detect elements that are identically zero not only in the neighbourhood of $\varepsilon = 0$ but also at any point $\varepsilon \in \mathbb{R}$.

For instance, choose an ε^* such that the basis matrix $A_B(\varepsilon^*)$ is well-conditioned. Then calculate $r_j(\varepsilon^*)$ and $y_l(\varepsilon^*)$ (now we can use directly formulas $r_N(\varepsilon^*) = \lambda(\varepsilon^*)A_N(\varepsilon^*) - c_N(\varepsilon^*)$ and $y(\varepsilon^*) = A_B^{-1}(\varepsilon^*)a_k(\varepsilon^*)$ instead of expansions). If $r_j(\varepsilon^*) \neq 0$ and $y_l(\varepsilon^*) \neq 0$, then the functions are certainly not identically zero. If we obtain some zero elements, then we should add to ε^* arbitrary small variation and check whether this is a local zero or an identical zero. Of course, ε^* and its small variation (if necessary) are chosen according to the features of the specific problem.

4 Generalizations

In this section we show how the asymptotic simplex method can be modified to solve problems more general than LP (4), (5) under Assumptions 1 and 2.

We first point out that our method can be readily generalized to an LP with polynomial perturbations. Namely, suppose that the coefficients $A(\varepsilon), b(\varepsilon)$ and $c(\varepsilon)$ in the perturbed LP are polynomials of ε. In particular, a basis matrix has the form

$$A_B(\varepsilon) = A_B^{(0)} + \varepsilon A_B^{(1)} + ... + \varepsilon^p A_B^{(p)}. \tag{21}$$

Clearly, the lexicographical entry and exit rules will need only some "technical" changes. Note that in the case of polynomial perturbations one needs at worst to check $(m+1)p$ terms of Laurent expansions for the entry rule and $2mp$ terms for the exit rule. The main difficulties that we face now are in the inversion of the polynomial matrix (21). It turns out that the methods of Lamond [19, 20] and Huang [14] for calculating the Laurent series

$$A_B^{-1}(\varepsilon) = \frac{1}{\varepsilon^s}U^{(-s)} + \frac{1}{\varepsilon^{s-1}}U^{(-s+1)} + ... \tag{22}$$

cannot be applied (at least not directly) to the polynomial perturbation problem. This is because their methods are heavily dependent on the linearity of the perturbation. Note that the iterative formulae (11),(12) that we use in our analysis were also derived for the case of linear perturbations. However, they can be generalized for the case of polynomial perturbations. The next generalized recursive formula is due to Korolyuk and Turbin [18]

$$U^{(k+1)} = -\sum_{i=0}^{p-1}\left(\sum_{j=0}^{p-1-i} U^{(-j)}A^{(j+i+1)}\right)U^{(k-i)}, \quad k = 0, 1, ... \quad . \tag{23}$$

Thus, only some singular and the first regular terms of Laurent expansion (22) have to be obtained or updated. The other terms, if needed, can be computed in an efficient way by the recursive formula (23). On the first iteration of the asymptotic simplex method one may use the algorithm from [1, 13, 18] to get the initial Laurent expansion. For the next iterations one can use the generalized version of the updating algorithm that we introduced in the previous section.

Note that Assumption 1 guarantees that an a-optimal solution can be expanded in Taylor series. We have introduced this assumption in order to restrict ourselves to the most common and interesting case where the a-optimal solution differs from the optimal solution of the unperturbed problem but both solutions are finite. In this case there exists a computationally stable updating formula (20). Of course, one can consider a perturbed LP without this restriction. Then one will need to deal with the basic solutions in the form of general Laurent series with singular terms. Again, the asymptotic algorithm for that case would not be much different from the one presented in Section 3.

Appendix

Here we present several auxiliary theoretical results that help to develop the asymptotic simplex method.

The next elegant property of the Laurent expansion for the singularly perturbed inverse matrix was originally established by Langenhop [21]. Later Schweitzer [25] greatly simplified the proof. Here we give yet another modification of Schweitzer's proof.

Theorem 2 (Langenhop,Schweitzer) *Consider a linearly perturbed matrix $A(\varepsilon) = A_0 + \varepsilon A_1 \in \mathbb{R}^{m\times m}$ and suppose that $detA(\varepsilon) \neq 0, 0 < \varepsilon < \overline{\varepsilon}$ and $det(A_0) = 0$. Then the inverse operator $U(\varepsilon) = A^{(-1)}(\varepsilon)$ is expanded as a Laurent series*

$$U(\varepsilon) = \frac{1}{\varepsilon^s}U^{(-s)} + ... + \frac{1}{\varepsilon}U^{(-1)} + U^{(0)} + \varepsilon U^{(1)} + ..., \quad s \leq m,$$

The regular part $U^R(\varepsilon) = U^{(0)} + \varepsilon U^{(1)} + ...$ and the singular part $U^S(\varepsilon) = \frac{1}{\varepsilon^s}U^{(-s)} + ... + \frac{1}{\varepsilon}U^{(-1)}$ of this series are represented in the close analytic forms

$$U^R(\varepsilon) = U^{(0)}(I + \varepsilon A_1 U^{(0)})^{-1} = (I + \varepsilon U^{(0)} A_1)^{-1} U^{(0)} \tag{24}$$

$$U^S(\varepsilon) = \varepsilon^{-1}U^{(-1)}(I + \varepsilon^{-1}A_0 U^{(-1)})^{-1} = (I + \varepsilon^{-1}U^{(-1)}A_0)^{-1}\varepsilon^{-1}U^{(-1)} \tag{25}$$

Moreover if the coefficients $U^{(0)}, U^{(-1)}$ are given, then the other coefficients of the Laurent expansion are calculated by simple iterative formulae

$$U^{(k)} = (-U^{(0)}A_1)^{k-1}U^{(0)} = U^{(0)}(-A_1 U^{(0)})^{k-1}, \quad k = 0, 1, ... \tag{26}$$

$$U^{(-k)} = (-U^{(-1)}A_0)^{k-1}U^{(-1)} = U^{(-1)}(-A_0 U^{(-1)})^{k-1}, \quad k = 1, ..., s \tag{27}$$

PROOF: It is easy to check that the matrix $U(\varepsilon)$ satisfies the following resolvent-type identity

$$U(\varepsilon_2) - U(\varepsilon_1) = (\varepsilon_1 - \varepsilon_2)U(\varepsilon_2)A_1 U(\varepsilon_1)$$

Suppose ε_2 is fixed and take regular parts of the above identity with respect to ε_1

$$U(\varepsilon_2) - U^R(\varepsilon_1) = (\varepsilon_1 - \varepsilon_2)U(\varepsilon_2)A_1 U^R(\varepsilon_1) + U(\varepsilon_2)A_1 U^{(-1)}, \tag{28}$$

and let $\varepsilon_1 = \varepsilon_2$,

$$U(\varepsilon_2) - U^R(\varepsilon_2) = U(\varepsilon_2)A_1 U^{(-1)},$$

and again take regular parts (with respect to ε_2)

$$0 = U^R(\varepsilon_2)A_1 U^{(-1)} \tag{29}$$

Now in (28) take regular parts with respect to ε_2 (with ε_1 fixed).

$$U^R(\varepsilon_2) - U^R(\varepsilon_1) = (\varepsilon_1 - \varepsilon_2)U^R(\varepsilon_2)A_1 U^R(\varepsilon_1) - U^{(-1)}A_1 U^R(\varepsilon_1) + U(\varepsilon_2)A_1 U^{(-1)}$$

The last summand in the above equation is zero due to (29). Let $\varepsilon_2 = \varepsilon_1$,

$$0 = 0 - U^{(-1)}A_1 U^R(\varepsilon_1).$$

So, we have obtained

$$U^R(\varepsilon_2) - U^R(\varepsilon_1) = (\varepsilon_1 - \varepsilon_2)U^R(\varepsilon_2)A_1 U^R(\varepsilon_1). \tag{30}$$

By setting $\varepsilon_2 = \varepsilon, \varepsilon_1 = 0$, we get

$$U^R(\varepsilon) - U^{(0)} = -\varepsilon U^R(\varepsilon) A_1 U^{(0)}$$

$$U^R(\varepsilon)[I + \varepsilon A_1 U^{(0)}] = U^{(0)}$$

or $U^R(\varepsilon) = U^{(0)}[I + \varepsilon A_1 U^{(0)}]^{-1}$. To obtain the other formula $U^R(\varepsilon) = [I + \varepsilon U^{(0)} A_1]^{-1} U^{(0)}$ just set in (30) $\varepsilon_2 = 0, \varepsilon_1 = \varepsilon$.

The similar considerations are applied to obtain the resolvent-type identity for the singular part

$$U^S(\varepsilon_2) - U^S(\varepsilon_1) = (\varepsilon_1 - \varepsilon_2) U^S(\varepsilon_2) A_1 U^S(\varepsilon_1)$$

and then, via equating coefficients of $\varepsilon_{1,2}^{-1}$ in this identity, we get the formulae (25).

Finally, expanding the analytic formulas (24),(25) in the Neumann power series, we immediately obtain (26),(27).

□

Lemma 1 (Lamond,Huang) *Suppose $c(\varepsilon) = a(\varepsilon)/b(\varepsilon)$ is a rational function with the degrees of the polynomials $a(\varepsilon)$ and $b(\varepsilon)$ being p and q, respectively. Then the function $c(\varepsilon)$ can be expanded as a Laurent series*

$$c(\varepsilon) = \frac{1}{\varepsilon^s} c^{(-s)} + \frac{1}{\varepsilon^{s-1}} c^{(-s+1)} + \ldots$$

in some punctured neighbourhood of zero with the order of pole s that is at most q. Moreover, if $c^{(-s)} = c^{(-s+1)} = \ldots = c^{(p)} = 0$, then $c(\varepsilon) \equiv 0$.

Proof: See papers of Lamond [19, 20] and Huang [14].

□

Lemma 2 *Let Assumption 1 hold. Then, any basic feasible solution of the perturbed program (4), (5) is expanded as a Taylor series*

$$x_B(\varepsilon) = x_B^{(0)} + \varepsilon x_B^{(1)} + \ldots$$

for $\varepsilon \in (0, \overline{\varepsilon}]$, where $\overline{\varepsilon} > 0$ is sufficiently small.

Proof: Recall that any basic feasible solution is given by the formula

$$x_B(\varepsilon) = A_B^{-1}(\varepsilon) b(\varepsilon). \tag{31}$$

According to Theorem 2, the inverse basis matrix $A_B^{-1}(\varepsilon)$ possesses, in general, a Laurent series expansion. Thus one can see from the formula (31), that $x_B(\varepsilon)$ possesses a Laurent series expansion in some punctured neighbourhood of $\varepsilon = 0$ as well.

Now we shall show that the Laurent series for $x_B(\varepsilon)$ does not have a singular part. Suppose this is not the case and some basic feasible solution has a Laurent series with a non-trivial singular part. The latter implies that there exists a sequence $\{\varepsilon_k\}_{k=0}^{\infty}$ such that $\varepsilon_k \to 0$ and $||x_B(\varepsilon_k)|| \to \infty$ as $k \to \infty$.

Next we define the following auxiliary sequence

$$y_k := \frac{x_B(\varepsilon_k)}{||x_B(\varepsilon_k)||}.$$

Note that $||y_k|| = 1, \ k = 1, 2, \ldots$. According to the Bolzano-Weierstrass principle on the bounded sequences in the finite dimensional spaces, there exists a convergent subsequence $\{y_{k_l}\}$ with the limit $\overline{y}$ as $k_l \to \infty$.

From the definition of the basic feasible solution one gets

$$A(\varepsilon_{k_l})y_{k_l} = \frac{b(\varepsilon_{k_l})}{||x(\varepsilon_{k_l}||},$$

for each k_l. Now let k_l go to the infinity to obtain

$$A^{(0)}\overline{y} = 0 \quad \text{and} \quad \overline{y} \geq 0.$$

Take any feasible solution of the unperturbed problem x_f such that

$$A^{(0)}x_f = b^{(0)}, \quad x_f \geq 0.$$

It is easy to see that $x_f + \lambda\overline{y}$ is also feasible solution for any $\lambda \geq 0$. The latter means that the original feasible region M_0 is unbounded, which is a contradiction to Assumption 1.

Thus, every basic feasible solution of the perturbed program can be expanded as a Taylor series.

□

Remark 6 *We would like to note that the first term $x_B^{(0)}$ of the Taylor expansion for $x_B(\varepsilon)$ might not be a* basic *feasible solution for the original program. This may occur in the case of singular perturbations.*

Lemma 3 (Markushevich) *Let analytic functions $a(\varepsilon)$ and $b(\varepsilon)$ be represented by the power series $a(\varepsilon) = \varepsilon^t a^{(t)} + \varepsilon^{t+1}a^{(t+1)} + ..., a^{(t)} \neq 0$ and $b(\varepsilon) = \varepsilon^q b^{(q)} + \varepsilon^{q+1}b^{(q+1)} + ..., b^{(q)} \neq 0$ respectively. Then the division of these analytic functions can be also expressed as a power series (for sufficiently small ε)*

$$c(\varepsilon) = \frac{a(\varepsilon)}{b(\varepsilon)} = \varepsilon^{t-q}c^{(0)} + \varepsilon^{t-q+1}c^{(1)} + ..., \quad c^{(0)} \neq 0, \tag{32}$$

whose coefficients are calculated by the recurrent formula

$$c^{(k)} = [a^{(t+k)} - \sum_{i=0}^{k-1} b^{(q+k-i)}c^{(i)}]/b^{(q)}, \quad k = 0, 1, 2, ... \tag{33}$$

Proof: It is well known [23] that the division of two analytic functions with poles with finite orders is also an analytic function with a pole of finite order. Then $c(\varepsilon)$ can be expanded as a Laurent series near $\varepsilon = 0$ that converges in a punctured disc of the origin with non-zero radius. Moreover, the coefficients of expansion (32) are unique. The recursive formula (33) is easily derived from the system of equations obtained by equating the terms with the same power of ε in the identity $b(\varepsilon)c(\varepsilon) = a(\varepsilon)$.

□

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