Continuum Equilibria for Routing in Dense Ad-hoc Networks

Eitan Altman
INRIA
2004 Route des Lucioles, B.P. 93,
06902 Sophia-Antipolis Cedex, France.
Email: Eitan.Altman@sophia.inria.fr

Alonso Silva
INRIA
2004 Route des Lucioles, B.P. 93,
06902 Sophia-Antipolis Cedex, France.
Email: Alonso.Silva@sophia.inria.fr

Pierre Bernhard
I3S, Université de Nice-Sophia Antipolis and CNRS
940 Route des Colles, B.P. 145,
06903 Sophia-Antipolis Cedex, France.
Email: Pierre.Bernhard@polytech.unice.fr

Merouane Debbah
SUPELEC
Plateau de Moulon, 3 rue Joliot-Curie,
91192 Gif-sur-Yvette Cedex, France.
Email: Merouane.Debbah@supelec.fr

***Abstract*— We consider massively dense ad-hoc networks and study their continuum limits as the node density increases and as the graph providing the available routes becomes a continuous area with location and congestion dependent costs. We study both the global optimal solution as well as the non-cooperative routing problem among a large population of users. Each user seeks a path from its source to its destination so as to minimize its individual cost. We seek for a (continuum version of the) Wardrop equilibrium. We first show how to derive meaningful cost models as a function of the scaling properties of the capacity of the network as a function of the density of nodes. We present various solution methodologies for the problem: (1) the viscosity solution of the Hamilton-Jacobi-Bellman equation, (2) a transformation into an equivalent global optimization problem that is obtained by identifying some potential related to the costs. We finally study the problem in which the routing decisions are taken by a finite number of competing service providers.**

I. Introduction

In the design and analysis of wireless networks, researchers frequently stumble on the scalability problem that can be summarized in the following sentence: "As the number of nodes in the network increases, problems become harder to solve" [20]. The sentence takes his meaning from several issues. Some examples are the following:

- In Routing: As the network size increases, routes consists of an increasing number of nodes, and so they are increasingly susceptible to node mobility and channel fading [19].
- In Transmission Scheduling: The determination of the maximum number of non-conflicting transmissions in a graph is a NP-complete problem [21].
- In Capacity of Wireless Networks: As the number of nodes increases, the determination of the precise capacity becomes an intractable problem.

Nevertheless when the system is sufficiently large, one may hope that a macroscopic view would provide a more useful abstraction of the network. The properties of the new macroscopic model would, however, consider microscopic considerations. Indeed we are going to sacrifice some details, but this macroscopic view will preserve enough information to allow a meaningful network optimization solution and the derivation of insightful results in a wide range of settings.

The physics-inspired paradigms used for the study of large ad-hoc networks go way beyond those related to statistical-mechanics in which macroscopic properties are derived from microscopic structure. Starting from the pioneering work by Jacquet in [17] in that area, a number of research groups have worked on massively dense ad-hoc networks using tools from geometrical optics [17][1] as well as electrostatics (see e.g. [13], [20], [25] and the survey [22] and references therein). We shall describe these in the next two sections.

The physical paradigms allow the authors to minimize various metrics related to the routing. In contrast, Hyytia and Virtamo propose in [15] an approach based on load balancing arguing that if shortest path (or cost minimization) arguments were used then some parts of the network would carry more traffic than others and may use more energy than others. This would result in a shorter life time of the network since some parts would be out of energy earlier than others and earlier than any part in a load balanced network.

The development of the original theory of routing in massively dense networks among the community of ad-hoc networks has emerged in a complete independent way of the existing theory of routing in massively dense networks which had been developed within the community of road traffic

[1]We note that this approach is restricted to costs that do not depend on the congestion.

engineers. Indeed, this approach had already been introduced on 1952 by Wardrop [27] and by Beckmann [4] and is still an active research area among that community, see [6], [7], [14], [16], [28] and references therein. We have thus chosen to devote part of this work to complement the physical-inspired methods that have been in use in the wireless networking community by some tools from road traffic engineering, and from the area of optimal control theory.

The structure of this paper is as follows. We begin by presenting models for costs that will later be used in various optimization models related to routing or to node assignment. We then present optimization problems related to massively dense ad-hoc network, and focus on limits obtainend from the use of directional antennas.

II. Determining routing costs in dense ad-hoc networks

In optimizing a routing protocol in ad-hoc networks, or in decisions the optimization of placement of nodes, one of the starting points is the determination of the cost function. To determine it, we need a detailed specification of the network which includes the following:

- A model for the placement of nodes in the network.
- A forward rule that nodes will use to select the next hop of a packet.
- A model for the cost incurred in one hop, i.e. for transmitting a packet to an intermediate node.

Below we present several ways of choosing cost functions.

A. Costs derived from capacity scaling

Many models have been proposed in the literature that show how the transport capacity scales with the number of nodes n or with their density λ. Assume that we use a protocol that provides a transport capacity of the order of $f(\lambda)$ at some region in which the density of nodes is λ.

A typical cost (used at [25]) at a neighborhood of $\mathbf{x}$ is the density of nodes required there to carry a given flow. Assuming that a flow $\mathbf{T}(\mathbf{x})$[2] is assigned through a neighborhood of $\mathbf{x}$, the cost is taken to be

$$c(\mathbf{x}, \mathbf{T}(\mathbf{x})) = f^{-1}(\mathbf{T}(\mathbf{x})). \quad (1)$$

Examples for f:

- The capacity of a CSMA scheme with a fixed carrier sense range is $O\left(\frac{\sqrt{\lambda}}{\sqrt{\log \lambda}}\right)$
- Using a network theoretic approach based on multi-hop communication, in the paper of Gupta and Kumar [11] the authors show that the throughput of the system that can be transported by the network when the nodes are optimally located is $\Omega(\sqrt{\lambda})$ and when the nodes are randomly located this throughput becomes $\Omega(\frac{\sqrt{\lambda}}{\sqrt{\log \lambda}})$. Using percolation theory, the authors of [9] have shown that in the randomly located set the same $\Omega(\sqrt{\lambda})$ can be achieved.
- Baccelli, Blaszczyszyn and Mühlethaler introduce in [2] an access scheme, MSR (Multi-hop Spatial Reuse Aloha), reaching the Gupta and Kumar bound $O(\sqrt{\lambda})$ which does not require prior knowledge of the node density.
- A protocol introduced by Tse and Glosglauser [10] has a capacity that scales as $O(\lambda)$. However, it does not fall directly within the the class of massively dense ad-hoc networks and indeed, it relies on mobility and on relaying for handling disconnectivity.

We conclude that for the approach of Gupta and Kumar with either approaches the optimal location or the random location, as well as for the MSR protocol with a Poisson distribution of nodes, we obtain a quadratic cost of the form

$$c(\mathbf{T}(\mathbf{x})) = k|\mathbf{T}(\mathbf{x})|^2.$$

B. Congestion independent routing

A metric often used in the Internet for determining routing is the number of hops, which routing protocols try to minimize. The number of hops is proportional to the expected delay along the path in the context of ad-hoc networks, in case that the queueing delay is negligible with respect to the transmission delay over each hop. This criterion is insensitive to interference or congestion. We assume that the network is sufficiently dense so that the number of hops does not depend on the density of nodes. It depends only of the transmission range. We describe various cost criteria that can be formulated.

- If the range were constant then the cost density $c(\mathbf{x})$ is constant so that the cost of a path is its length in meters. The routing then follows a shortest path selection.
- Let's assume that the range $R(\mathbf{x})$ depends on local radio conditions at a point $\mathbf{x}$ (for example, it is influenced by weather conditions) but not on interference. The latter is justified when dedicated orthogonal channels (e.g. in time or frequency) can be allocated to traffic flows that would otherwise interfere with each other. Then determining the routing becomes a path cost minimization problem. We further assume, as in Gupta and Kumar, that the range is scaled to go to 0 as the total density λ of nodes grows to infinity. More precisely, let's consider a scaling of the range such that the following limit exists:

$$r(\mathbf{x}) := \lim_{\lambda\to\infty} \frac{R(\mathbf{x})}{f(\lambda)}$$

where $\lim_{\lambda\to\infty} f(\lambda) = 0$. Then in the dense limit, the density of nodes that participate in forwarding packets along a path is $1/r(\mathbf{x})$ and the path cost is the integral of this density along the path.

[2]We denote with bold font the vectors.

- The influence of varying radio conditions on the range can be eliminated using power control that can equalize the hop distance.

C. Costs related to energy consumption

In the absence of capacity constraints, the cost can represent energy consumption. In a general multi-hop ad-hoc network the hop distance can be optimized so as to minimize the energy consumption. Even within a single cell of 802.11 IEEE wireless LAN one can improve the energy consumption by using multiple hops, as it has been shown not to be efficient in terms of energy consumption to use a single hop [26].

Alternatively, the cost can take into account the scaling of the nodes (as we had in Subsection II-A that is obtained when there are energy constraints. As an example, assuming random deployment of nodes, where each nodes has traffic to send to another randomly selected node, the capacity (in bits per Joule) has then the form $f(\lambda) = \Omega\left((\lambda/\log\lambda)^{(q-1)/2}\right)$ where q is the path-loss, see [18]. The cost is then obtained using (1).

III. Preliminary

In the work of Toumpis et al. ([13], [20], [22]–[25]) the authors address the problem of the optimal deployment of Wireless Sensor Networks by a parallel with Electrostatic.

Consider the continuous **information density function** $\rho(\mathbf{x})$, measured in bps/m^2, such that at locations $\mathbf{x}$ where $\rho(\mathbf{x}) > 0$ there is a distributed data source such that the rate with which information is created in an infinitesimal area of size $d\Phi$ centered at $\mathbf{x}$ is $\rho(\mathbf{x})d\Phi$. Similarly, at locations $\mathbf{x}$ where $\rho(\mathbf{x}) < 0$ there is a distributed data sink such that the rate with which information is absorbed by an infinitesimal area of size $d\Phi$, centered at point $\mathbf{x}$, is equal to $-\rho(\mathbf{x})d\Phi$.

The total rate at which sinks must absorb data is the same as the total rate which the data is created at the sources, i.e.

$$\int_{X_1\times X_2} \rho(\mathbf{x})\,\mathrm{d}S = 0.$$

Next we present the flow conservation [6], [25]. For information to be conserved over a surface Φ_0 of arbitrary shape on the $X_1 \times X_2$ plane it is necessary that the rate with which information is created in the area is equal to the rate with which information is leaving the area, i.e

$$\int_{\Phi_0} \rho(\mathbf{x})\,\mathrm{d}\mathbf{x} = \oint_{\partial\Phi_0} [\mathbf{T}(\ell)\cdot\mathbf{n}(\ell)]\,\mathrm{d}\ell$$

The integral on the left is the surface integral of $\rho(\mathbf{x})$ over Φ_0. The integral on the right is the path integral of the inner product $\mathbf{T}\cdot\mathbf{n}$ over the curve $\partial\Phi_0$. The vector $\mathbf{n}$ is the unit vector perpendicular to $\partial\Phi_0$ at the boundary point $\partial\Phi_0(s)$ and pointing outwards. The function $\mathbf{T}\cdot\mathbf{n}(s)$ measured in bps/m^2 is equal at the rate with which information is leaving the surface Φ_0 at the boundary point $\partial\Phi_0(s)$.

Dividing both terms by $|\Phi_0|$ and taking limit when $|\Phi_0|$ goes to 0, yields the following equivalent equation:

$$\nabla\cdot\mathbf{T} := \frac{\partial T_1(\mathbf{x})}{\partial x_1} + \frac{\partial T_2(\mathbf{x})}{\partial x_2} = \rho. \tag{2}$$

where "$\nabla\cdot$" is the divergence operator.

Extension to multi-class The work on massively dense ad-hoc networks, considered a single class of traffic. In the geometrical optics approach it corresponded to demand from a point **a** to a point **b**. In the electrostatic case it corresponded to a set of origins and a set of destinations where traffic from any origin point could go to any destination point. The analogy to positive and negative charges in electrostatics may limit the perspectives of multi-class problems where traffic from distinct origin sets has to be routed to distinct destination sets.

The model based on geometrical optics can directly be extended to include multiple classes as there are no elements in the model that suggest coupling between classes. This is due in particular to the fact that the cost density has been assumed to depend only on the density of the mobiles and not on the density of the flows.

In contrast, the cost in the model based on electrostatics is assumed to depend both on the location as well as on the local flow density. It thus models more complex interactions that would occur if we considered the case of ν traffic classes. Extending the relation (2) to the multi-class case, we have traffic conservation at each point in space to each traffic class as expressed in the following:

$$\nabla\cdot\mathbf{T}^j(\mathbf{x}) = \rho^j(\mathbf{x}), \quad \forall\mathbf{x}\in\Phi. \tag{3}$$

The function $\mathbf{T}^j$ is the flow distribution of class j and ρ^j corresponds to the distribution of the external sources and/or sinks.

Let $\mathbf{T}(\mathbf{x})$ be the total flow vector at point $\mathbf{x}\in\Phi$. A generic multi-class optimization problem would then be: minimize Z over the flow distributions $\{T_i^j\}$

$$Z = \int_\Phi g(\mathbf{x},\mathbf{T}(\mathbf{x}))\,\mathrm{d}x_1\,\mathrm{d}x_2 \quad \text{subject to} \tag{4a}$$

$$\nabla\cdot\mathbf{T}^j(\mathbf{x}) = \rho^j(\mathbf{x}),\ j=1,...,\nu \quad \forall\mathbf{x}\in\Phi. \tag{4b}$$

IV. Directional Antennas

For energy efficiency, it is assumed that each terminal is equipped with one or with two directional antennas, allowing transmission at each hop to be directed either from North to South or from West to East. Thus $T_1^j \geq 0,\ T_2^j \geq 0,\ j = 1,...,\nu$. In the dense limit, a curved path can be viewed as a limit of a path with many such hops as the hop distance tends to zero. The approach that we follow is based on the work of Dafermos [6] where the author gave a similar approach on the road traffic equilibrium context. In our paper we give a mathematical development and formulate the problem in the multi-class framework.

Some assumptions on the cost:

- **Individual cost:** We allow the cost for a horizontal (West-East) transmission from a point $\mathbf{x}$ to be different than the cost for a vertical transmission (North-South). It is assumed that a packet located at the point $\mathbf{x}$ and traveling in the direction of the axis x_i incurs a **transportation cost** g_i and such transportation cost depends upon the traffic flow. We thus allow for a vector valued cost $\mathbf{g} = \mathbf{g}(\mathbf{x}, \mathbf{T}(\mathbf{x}))$.
- The local cost corresponding to the global optimization problem is given by $g(\mathbf{x}, \mathbf{T}(\mathbf{x})) = \mathbf{g}(\mathbf{x}, \mathbf{T}(\mathbf{x})) \cdot \mathbf{T}(\mathbf{x})$ if it is perceived as the sum of costs of individuals.
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- The global cost will be the integral of the local cost density.
- The local cost $g(\mathbf{x}, \mathbf{T}(\mathbf{x}))$ is assumed to be non-negative, convex increasing in each of the components of $\mathbf{T}$ (T_1 and T_2 in our 2-dimensional case).

The **boundary conditions** will be determined by the options that travelers have in selecting their origins and/or destinations. Examples of the boundary conditions are:

- Assignment problem: users of the network have predetermined origins and destinations and are free to choose their travel paths.
- Combined distribution and assignment: users of the network have predetermined origins and are free to choose their destinations (within a certain destination region) as well as their paths.
- Combined generation, distributions and assignment: users are free to choose their origins, their destinations, as well as their travel paths.

The problem formulation is again to minimize Z as defined in (4).

Kuhn-Tucker conditions. Define the Lagrangian as

$$L^{\zeta}(\mathbf{x}, \mathbf{T}) := \int_{\Phi} \ell^{\zeta}(\mathbf{x}, \mathbf{T})\, \mathrm{d}\mathbf{x} \quad \text{where}$$

$$\ell^{\zeta}(\mathbf{x}, \mathbf{T}) = g(\mathbf{x}, \mathbf{T}(\mathbf{x})) - \sum_{j=1}^{\nu} \zeta^j(\mathbf{x}) \left[\nabla \cdot \mathbf{T}^j(\mathbf{x}) - \rho^j(\mathbf{x})\right]$$

where $\zeta^j(\mathbf{x}) \in H^1(\Phi)$ are Lagrange multipliers. The criterion is convex, and the constraint (3) affine. Therefore the Kuhn-Tucker theorem holds, stating that the lagrangian is minimumm at the optimum.

A variation $\delta\mathbf{T}(\cdot)$ will be admissibe if $\mathbf{T}(\mathbf{x}) + \delta\mathbf{T}(\mathbf{x}) \geq 0$ for all $\mathbf{x}$, hence in particular, $\forall \mathbf{x} : T_i^j(\mathbf{x}) = 0$, $\delta T_i^j(\mathbf{x}) \geq 0$.

Euler's inequality reads:

$$\forall \delta\mathbf{T} \text{ admissible}, \quad \mathrm{D}L^{\zeta} \cdot \delta\mathbf{T} \geq 0,$$

therefore here

$$\int_{\Phi} \sum_j \langle \nabla_{\mathbf{T}^j} g(\mathbf{x}, \mathbf{T}(\mathbf{x})), \delta\mathbf{T}^j(\mathbf{x})\rangle \, \mathrm{d}\mathbf{x} - \int_{\Phi} \sum_j \zeta^j(\mathbf{x}) \nabla \cdot \delta\mathbf{T}^j(\mathbf{x}) \, \mathrm{d}\mathbf{x} \geq 0.$$

Integrating by parts with Green's formula, this is equivalent to

$$\int_{\Phi} \sum_j \left[\langle \nabla_{\mathbf{T}^j} g, \delta\mathbf{T}^j\rangle + \langle \nabla_{\mathbf{x}} \zeta^j, \delta\mathbf{T}^j\rangle\right] \mathrm{d}\mathbf{x} - \int_{\partial\Phi} \sum_j \zeta^j \langle \delta\mathbf{T}^j, \mathbf{n}\rangle \, \mathrm{d}\ell \geq 0.$$

We may choose all the $\delta\mathbf{T}^k = 0$ except $\delta\mathbf{T}^j$, and choose that one in $(H_0^1(\Phi))^2$, *i.e.* such that the boundary integral be zero. This is always feasible and admissible. Then the last term above vanishes, and it is a classical fact that the inequality implies for $i = 1, 2$:

$$\frac{\partial g(\mathbf{x}, \mathbf{T})}{\partial T_i^j} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} = 0 \quad \text{if } T_i^j(\mathbf{x}) > 0 \tag{5a}$$

$$\frac{\partial g(\mathbf{x}, \mathbf{T})}{\partial T_i^j} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} \geq 0 \quad \text{if } T_i^j(\mathbf{x}) = 0. \tag{5b}$$

Placing this back in Euler's inequality, and using a $\delta\mathbf{T}^j$ non zero on the boundary, it follows that necessarily [3] $\zeta^j(\mathbf{x}) = 0$ at any $\mathbf{x}$ of the boundary $\partial\Phi$ where $T(\mathbf{x}) > 0$. This provides the boundary condition to recover ζ^j from the condition (3).

Consider the following special cases that we shall need later. We assume a single traffic class, but this could easily be extended to several. Let

$$g(\mathbf{x}, \mathbf{T}(\mathbf{x})) = \sum_{i=1,2} g_i(\mathbf{x}, \mathbf{T}(\mathbf{x})) T_i(\mathbf{x}).$$

1) Monomial cost per packet:

$$g_i(\mathbf{x}, \mathbf{T}(\mathbf{x})) = c_i(\mathbf{x}) \left(T_i(\mathbf{x})\right)^{\beta} \tag{6}$$

for some $\beta > 1$. Then (5a)-(5b) simplify to

$$(\beta+1) c_i(\mathbf{x}) \left(T_i(\mathbf{x})\right)^{\beta} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} = 0 \quad \text{if } T_i^j(\mathbf{x}) > 0, \tag{7a}$$

$$(\beta+1) c_i(\mathbf{x}) \left(T_i(\mathbf{x})\right)^{\beta} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} \geq 0 \quad \text{if } T_i^j(\mathbf{x}) = 0. \tag{7b}$$

2) Affine cost per packet:

$$g_i(\mathbf{x}, \mathbf{T}(\mathbf{x})) = \frac{1}{2} k_i(\mathbf{x}) T_i(\mathbf{x}) + h_i(\mathbf{x}). \tag{8}$$

[3]This is a complementary slackness condition on the boundary.

Then (5a)-(5b) simplify to

$$k_i(\mathbf{x})T_i(\mathbf{x}) + h_i(\mathbf{x}) + \frac{\partial \zeta(\mathbf{x})}{\partial x_i} = 0$$
$$\text{if } T_i(\mathbf{x}) > 0$$
$$k_i(\mathbf{x})T_i + h_i(\mathbf{x}) + \frac{\partial \zeta(\mathbf{x})}{\partial x_i} \geq 0$$
$$\text{if } T_i(\mathbf{x}) = 0.$$

Assume that the $k_i(\cdot)$ are everywhere positive and bounded away from 0. For simplicity, let $a_i = 1/k_i$, and b be the vector with coordinates $b_i = h_i/k_i$, all assumed to be square integrable. Assume that there exists a solution where $T(\mathbf{x}) > 0$ for all $\mathbf{x}$. Then

$$T_i(\mathbf{x}) = -\left(a_i(\mathbf{x}) \frac{\partial \zeta(\mathbf{x})}{\partial x_i} + b_i(\mathbf{x}) \right).$$

As a consequence, from (3) and the above remark, we get that $\zeta(\cdot)$ is to be found as the solution in $H_0^1(\Phi)$ of the elliptic equation (an equality in $H^{-1}(\Phi)$)

$$\sum_i \frac{\partial}{\partial x_i} \left(a_i(\mathbf{x}) \frac{\partial \zeta}{\partial x_i} \right) + \nabla \cdot b(\mathbf{x}) + \rho(\mathbf{x}) = 0\,.$$

This is a well behaved Dirichlet problem, known to have a unique solution in $H_0^1(\Phi)$, furthermore easy to compute numerically.

V. User optimization and Wardrop equilibrium

We expand on the shortest path approach for optimization that has already appeared using geometrical optics tools [17]. We present general optimization frameworks for handling shortest path problems and more generally, minimum cost paths. We go beyond the approach of geometrical optics by allowing the cost to depend on congestion. Shortest path costs can be a system objective as we shall motivate below. But it can also be the result of decentralized decision making by many "infinitesimally small" players where a player may represent a single packet (or a single session) in a context where there is a huge population of packets (or of sessions). The result of such a decentralized decision making can be expected to satisfy the following properties which define the so called, user (or Wardrop) equilibrium:

"*Under equilibrium conditions traffic arranges itself in congested networks such that all used routes between OD pair have equal and minimum costs while all unused routes have greater or equal costs*" [27].

Related work. Both the framework of global optimization as well as the one of minimum cost path had been studied extensively in the context of road traffic engineering. The use of a continuum network approach was already introduced on 1952 by Wardrop [27] and by Beckmann [4]. For more recent papers in this area, see e.g. [6], [7], [14], [16], [28] and references therein. We formulate it below and obtain some of its properties.

Motivation. One popular objective in some routing protocols in ad-hoc networks is to assign routes for packets in a way that each packet follows a minimal cost path (given the others' paths choices) [12]. This has the advantage of equalizing source-destination delays of packets that belong to the same class, which allows one to minimize the amount of packets that come out of sequence. (This is desirable since in data transfers, out of order packets are misinterpreted to be lost which results not only in retransmissions but also in drop of systems throughput.

Traffic assignment that satisfies the above definition is known in the context of road traffic as Wardrop equilibrium [27].

A. Directional antennas and congestion-dependent costs

Congestion independent cost

We consider the model of Subsection IV. We assume that the local cost depends on the direction of the flow but not on its size. The cost is $c_1(\mathbf{x})$ for a flow that is locally horizontal and is $c_2(\mathbf{x})$ for a flow that is locally vertical. We first assume that c_1 and c_2 do not depend on $\mathbf{T}$. The cost incurred by a user traveling along a path p is given by the line integral

$$\mathbf{c}_p = \int_p \mathbf{c} \cdot \mathbf{dx}.$$

Let $V^j(\mathbf{x})$ be the minimum cost to go from a point $\mathbf{x}$ to a set B^j, $j = 1, ..., \nu$. We shall assume here that $B^j \subset \partial\Phi$ is part of the boundary of Φ and that the south-east corner of the rectangle is included in all B^j's. The fact that the destinations are on the boundary can represent a situation of access points situated at the boundary of a region in which sensors are deployed. Then

$$V^j(\mathbf{x}) = \min \Big(c_1(\mathbf{x})\,\mathrm{d}x_1 + V^j(x_1 + \mathrm{d}x_1, x_2), \\ c_2(\mathbf{x})\,\mathrm{d}x_2 + V^j(x_1, x_2 + \mathrm{d}x_2) \Big) \qquad (10)$$

This can be written as

$$0 = \min \left(c_1(\mathbf{x}) + \frac{\partial V^j(\mathbf{x})}{\partial x_1}, c_2(\mathbf{x}) + \frac{\partial V^j(\mathbf{x})}{\partial x_2} \right), \qquad (11)$$
$$\forall \mathbf{x} \in B^j\,,\; V^j(\mathbf{x}) = 0\,.$$

If V^j is differentiable then, under suitable conditions, it is the unique solution (11). In the case that V^j is not everywhere differentiable then, under suitable conditions, it is the unique viscosity solution of (11) see [3], [8]. There are many numerical approaches for solving the HJB equation. One can discretize the HJB equation and obtain a discrete dynamic programming for which efficient solution methods exist. If one repeats this for various discretization steps, then we know that the solution of the discrete problem converges to the viscosity solution of the original problem (under suitable conditions) as the step size converges to zero [3].

Congestion dependent cost

We now add to c_1 the dependence on T_1 and to c_2 the dependence on T_2, as in section IV. Let $V^j(\mathbf{x})$ be the minimum cost to go from a point $\mathbf{x}$ to B^j at equilibrium. The equation (10) still holds but this time with c_i that depends on T_i^j, $i = 1, 2$. and on the total flows T_i, $i = 1, 2$. Thus (11) becomes

$$0 = \min_{i=1,2} \left(c_i(\mathbf{x}, T_i) + \frac{\partial V^j(\mathbf{x})}{\partial x_i} \right), \quad \forall \mathbf{x} \in B^j, V^j(\mathbf{x}) = 0. \tag{12}$$

We note that if $T_i^j(\mathbf{x}) > 0$ then by the definition of the equilibrium, i attains the minimum at (12). Hence (12) implies the following relations for each traffic class j, and for $i = 1, 2$:

$$c_i(\mathbf{x}, T_i) + \frac{\partial V^j}{\partial x_i} = 0 \quad \text{if} \quad T_i^j > 0, \tag{13a}$$

$$c_i(\mathbf{x}, T_i) + \frac{\partial V^j}{\partial x_i} \geq 0 \quad \text{if} \quad T_i^j = 0. \tag{13b}$$

Beckmann transformation

As Beckmann et al. did in [5] for discrete networks, we transform the minimum cost problem into an equivalent global minimization one. We shall restrict here to the single class case. To that end we note that (13a)-(13b) has exactly the same form as the Kuhn-Tucker conditions (5a)-(5b) except that $c_i(\mathbf{x}, T_i)$ in the former are replaced by $\partial g(\mathbf{x}, \mathbf{T})/\partial T_i(\mathbf{x})$ in the latter. We conclude that if there exists a scalar function (potential) $\psi(\mathbf{x}, \mathbf{T})$ such that for both $i = 1, 2$:

$$c_i(\mathbf{x}, T_i) = \frac{\partial \psi(\mathbf{x}, \mathbf{T})}{\partial T_i(\mathbf{x})}$$

then the user equilibrium flow is the one obtained from the global optimization problem where we use $\psi(\mathbf{x}, \mathbf{T})$ as local cost. Hence, the Wardrop equilibrium is obtained as the solution of

$$\min_{T(\mathbf{x}),\ x \in \Phi} \int_\Phi \psi(\mathbf{x}, \mathbf{T}) d\mathbf{x} \quad \text{subject to}$$
$$\nabla \cdot \mathbf{T}(\mathbf{x}) = \rho(\mathbf{x}), \quad \forall \mathbf{x} \in \Phi.$$

The potential ψ is given by

$$\psi(\mathbf{x}, \mathbf{T}) = \sum_{i=1,2} \int_0^{T_i(\mathbf{x})} c_i(\mathbf{x}, s)\, \mathrm{d}s$$

In the special case where costs are given as a power of the flow as defined in eq (6), we observe that equations (13a)-(13b) coincide with equations (7a)-(7b) (up-to a multiplicative constant of the cost). We conclude that for such costs, the user equilibrium and the global optimization solution coincide.

VI. Competitive routing

We describe in this Section a competitive framework for routing decisions taken by a finite number N of competing decision makers that represent service providers. We restrict to the framework of directional antennas.

There are n classes. T^j is the flow distribution of class j and ρ^j corresponds to the distribution of the external sources and/or sinks.

Assumption A1: The local cost $g^j(\mathbf{x}, \mathbf{T}^j, \mathbf{T})$ corresponding to player j may depend on

- The total horizontal and the total vertical flow and not directly on the amount of flow of each class.
- The total horizontal and vertical flow of that same player,
- The location.

Player j (controlling the routing of class j) minimizes the total cost Z^j for its traffic, where

$$Z^j = \int_\Phi g^j(\mathbf{x}, \mathbf{T})\, \mathrm{d}\mathbf{x},$$

subject to

$$\nabla \cdot \mathbf{T}^j(\mathbf{x}) = \rho^j(\mathbf{x}), \qquad \forall \mathbf{x} \in \Phi.$$

Kuhn-Tucker conditions. Define the Lagrangian for player j as

$$L^{\zeta,j}(\mathbf{x}, \mathbf{T}) := \int_\Phi \ell^{\zeta,j}(\mathbf{x}, \mathbf{T})\, \mathrm{d}\mathbf{x} \quad \text{where}$$
$$\ell^{\zeta,j}(\mathbf{x}, \mathbf{T}) = g^j(\mathbf{x}, \mathbf{T}(\mathbf{x})) + \zeta^j(\mathbf{x}) \left[\nabla \cdot \mathbf{T}^j(\mathbf{x}) - \rho^j(\mathbf{x}) \right] \tag{14}$$

The Kuhn-Tucker (KT) conditions corresponding to this problem are

$$\frac{\partial \ell^{\zeta,j}(\mathbf{x}, \mathbf{T})}{\partial T_i^j(\mathbf{x})} = 0 \quad \text{if } T_i^j(\mathbf{x}) > 0,$$

$$\frac{\partial \ell^{\zeta,j}(\mathbf{x}, \mathbf{T})}{\partial T_i^j(\mathbf{x})} \geq 0 \quad \text{if } T_i^j(\mathbf{x}) = 0$$

for $i = 1, 2$. We thus obtain for $i = 1, 2$:

$$\frac{\partial g^j(\mathbf{x}, \mathbf{T})}{\partial T_i^j(\mathbf{x})} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} = 0 \quad \text{if } T_i^j(\mathbf{x}) > 0, \tag{15a}$$

$$\frac{\partial g^j(\mathbf{x}, \mathbf{T})}{\partial T_i^j(\mathbf{x})} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} \geq 0 \quad \text{if } T_i^j(\mathbf{x}) = 0. \tag{15b}$$

Next we assume that the per packet local *cost density* is linear in the congestion:

$$g^j(\mathbf{x}, \mathbf{T}(\mathbf{x})) = \sum_{i=1,2} \left(c_i(\mathbf{x}) T_i(\mathbf{x}) + d_i(\mathbf{x}) \right) T_i^j(\mathbf{x})$$

Then

$$\frac{\partial g^j(\mathbf{x}, \mathbf{T})}{\partial T_i(\mathbf{x})} = c_i(\mathbf{x})(T_i(\mathbf{x}) + T_i^j(\mathbf{x})) + d_i(\mathbf{x}).$$

Eq. (15a)-(15b) simplify to

$$c_i(\mathbf{x})(T_i(\mathbf{x}) + T_i^j(\mathbf{x})) + d_i(\mathbf{x}) + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} = 0 \quad \text{if } T_i^j(\mathbf{x}) > 0, \tag{16a}$$

$$c_i(\mathbf{x})(T_i(\mathbf{x}) + T_i^j(\mathbf{x})) + d_i(\mathbf{x}) + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} \geq 0 \quad \text{if } T_i^j(\mathbf{x}) = 0. \tag{16b}$$

Assume now that at equilibrium, there is positive density of flow $\mathbf{T}^j(\mathbf{x})$ over the whole plane Φ for every player j. Then (16a) holds for all j. Summing over j we obtain for $i = 1, 2$:

$$(n+1)T_i(\mathbf{x})c_i(\mathbf{x}) + nd_i(\mathbf{x}) + \frac{\partial \zeta(\mathbf{x})}{\partial x_i} \geq 0$$

where $\zeta = \sum_{j=1}^n \zeta^j$. We thus obtained the KT conditions for the globally optimal problem in which $\rho = \sum_j \rho^j$ and in which the local cost is given by

$$g(\mathbf{x}, \mathbf{T}(\mathbf{x})) = \tag{17}$$

$$\sum_{i=1,2} \left(\frac{n+1}{2} c_i(\mathbf{x}) T_i(\mathbf{x}) + n d_i(\mathbf{x}) \right) T_i(\mathbf{x}) \tag{18}$$

Hence as in the Wardrop case, it is possible to transform the game into an equivalent globally optimal problem with a single decision maker.

VII. Conclusions

Routing in adhoc networks have received much attention in the massively dense limit. The main tools to describe the limits had been electrostatics and geometric optics. We exploit another approach for the problem that has its roots in road traffic theory, and present both quantitative as well as qualitative results for various optimization frameworks. For lack of space we have omitted numerical results that can be found in [1].

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