Rate-based flow Control with delayed information

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1 Introduction

Rate-based flow control has been selected by the ATM forum in 1994 as the main approach for flow control in the Available Bit Rate service in ATM (Asynchronous Transfer Mode); see [1]. According to [1], the flow control should adapt its input rate according to feedback information on the available bandwidth and the congestion level in the network.

Many ad-hoc schemes for rate-based flow control have been proposed in recent years (see e.g. [3, 7, 8, 9, 10, 11] and references therein). Recently, two schemes were proposed in [2], based on two optimal-control approaches: the H^∞ approach, and the LQG (Linear-Quadratic-Gaussian). Several objectives were formulated: (1) to minimize the variation of the bottleneck queue size around some desired level, (2) to obtain good tracking between input and output rates, and (3) to minimize the jitter. The relative importance (weight) of each one of these objectives is given as data in the design of the controller. The optimal controllers that are obtained ware easy to implement: they turned to be linear in the (estimated) state. The information available to the controller was noisy measurements of the queue length and available bandwidth delayed by one unit, corresponding to a round trip delay.

In this paper we take the approach of [2] a step length further, and design controllers based on noisy information, delayed by several time units. This corresponds to a situation where round trip delays are large, and it is desirable to update the control several times, say $(\theta+1)$, during a single round-trip delay. Each update is computed based on the most recent available information, i.e. on information delayed by $(\theta+1)$ time units.

Due to the Certainty-Equivalence property of both H^∞ and LQG optimal control (see [5]), it turns out that the problem with delayed information does not require an increase in the dimension of the state space (as is generally the case in optimal control problems with delayed information, see e.g. [4, 11]). The optimal controller that we obtain maintains its simple structure, and is easy to implement. It is again linear in the (estimated) state.

2 The model

We consider a linearized, discrete time model [2] (for the justification and derivation of this model, see [3]). As explained above, a time unit here corresponds to $(\theta+1)$th of a round trip delay. Let q_n denote the queue length at a bottleneck link, and μ_n denote the effective service rate available for traffic of the given source in that link at the beginning of the nth time slot. Let x_n denote the source rate during the nth time slot. The queue length evolves according to

$$q_{n+1} = q_n + x_n - \mu_n. \tag{1}$$

Since several sources with varying transmission rates may share the same bottleneck link, the service rate μ_n available to the controlled source may change over time in an unpredictable way. The other sources may be represented as some interference, which we model as a stable ARMA process

$$\mu_n = \mu + \xi_n \qquad \xi_n = \sum_{i=1}^{d} \ell_i \xi_{n-i} + k\phi_n. \tag{2}$$

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We assume that ℓ_i, which will typically be obtained by some estimation/identification procedure, are such that (2) describes a stable system. The variable ϕ_n in (2) stands for disturbance, the nature of which will be described shortly. We assume that, at the end of step n, the source obtains noisy estimates of the values of $\mu_{n-\theta}$ and of $q_{n-\theta}$. We use $\hat{\mu}_n$ to denote a measurement for the bottleneck rate, and $\hat{q}_n$ to denote a measurement for the queue size. For our analysis, we assume that these measurements can be written as

$$\hat{\mu}_n = \mu_n + cv_n \qquad \hat{q}_n = q_n + aw_n \tag{3}$$

where v_n and w_n are disturbances whose nature will be described shortly, and a and c are some constants.

We define the shifted versions of the variables $x_n, q_n, \hat{\mu}_n, \hat{q}_n$:

$$\tilde{x}_n := x_n - \mu, \qquad \tilde{q}_n := q_n - Q, \qquad \tilde{\hat{\mu}}_n := \hat{\mu}_n - \mu, \qquad \tilde{\hat{q}}_n := \hat{q}_n - Q. \tag{4}$$

We further define the (scaled) variation in the input rate by bu_n, where u_n will be called the control, and b is some constant. In terms of the new variables, we now have the following dynamics:

$$\tilde{q}_{n+1} = \tilde{q}_n + \tilde{x}_n - \xi_n \qquad \xi_n = \sum_{i=1}^{d} \ell_i \xi_{n-i} + k\phi_n \qquad \tilde{x}_{n+1} = \tilde{x}_n + bu_n \tag{5}$$

and the observations

$$\tilde{\hat{\mu}}_n = \xi_n + cv_n, \qquad \tilde{\hat{q}}_n = \tilde{q}_n + aw_n. \tag{6}$$

We define the following quadratic cost:

$$L = g\,\|\tilde{x} - \xi\|^2 + \|\tilde{q}\|^2 + \|u\|^2 \qquad \text{where } g > 0, \quad \|u\|^2 := \sum_{n=0}^{\infty} |u_n|^2, \quad \text{etc...} \tag{7}$$

This definition of the cost allows us to quantify optimization criteria that have been used in previous control models [3]:

The first term in (7) represents the quality with which the input rate tracks the available service rate, where g is a positive weighting term. A high rate of tracking is known to be desirable; see, e.g., the discussion in [7].

The second term is a penalty for deviating from a desirable queue length. Setting some desired level of queue reflects the fact that we do not wish to have a large queue, so as to avoid losses, and on the other hand, we do not wish to have a low level of queue, since if the queue is empty and the input rate is lower than the service rate then there is a waste of potential throughput. We have not included any additional weighting on this term, as any such positive weight can be absorbed into the other variables.

The last term stands for a penalty for high jitter, i.e. for high variability of (input) transmission rate, which is known to be undesirable; see the discussion in [7]. In many proposed telecommunication networks, the source will pay more for higher burstiness (variability) of its input rate, since highly variable input rate has typically a bad influence on other traffic. Again, as in the case of the second term, the weighting here has been normalized to 1.

Taking into account the delayed nature of the measurements, the controller has the form

$$u_n = f_n(\tilde{x}_m, \tilde{\hat{\mu}}_{m-\theta}, \tilde{\hat{q}}_{m-\theta}, u_{m-1}, m = 0, \ldots, n), \tag{8}$$

where the convention (throughout) is to take $\tilde{\hat{\mu}}_{m-\theta}$ and $\tilde{\hat{q}}_{m-\theta}$ to be identically *zero* for $m < \theta$. Here we could also have taken the dependence on $\tilde{x}_m$ to be with a delay of θ units, but since $\tilde{x}_m$ is generated by noise-free dynamics, this would not make any difference in the end result; that is, any performance that is achievable using $\tilde{x}_m$ could also be

We consider two types of models for the disturbances:

- <u>M1: The Gaussian model</u> Here v_n, w_n, ϕ_n are taken as independent identically distributed Gaussian random variables with zero mean and unit variance. It then follows that the measurements $\hat{\mu}_n$ and $\hat{q}_n$ are unbiased. The cost to be minimized in this case is the expected value of L, written as $E[L]$.

- M2: The H$^\infty$ approach In this case v_n, w_n and ϕ_n are taken as unknown (deterministic) disturbances. The cost to be minimized is the ℓ_2 gain from the disturbances to the cost L, i.e.,

$$\sup_{\{\phi_n, v_n, w_n\}_{n=0}^{\infty}} \frac{L}{\|\phi, v, w\|^2} \tag{9}$$

where

$$\|\phi, v, w\|^2 := \|\phi\|^2 + \|v\|^2 + \|w\|^2 = \sum_{n=0}^{\infty} |\phi_n|^2 + |v_n|^2 + |w_n|^2 . \tag{10}$$

Let us denote the square-root of the infimum of this cost over all admissible controllers $\{f_n\}$ by γ^*, i.e.

$$(\gamma^*)^2 = \inf_{\{f_n\}_{n=0}^{\infty}} \sup_{\{\phi_n, v_n, w_n\}_{n=0}^{\infty}} \frac{L(f; \phi, v, w)}{\|\phi, v, w\|^2}. \tag{11}$$

If there exists an actual minimizing control, say f^*, to the optimization problem above, then it can be shown [5] that, defining a soft-constrained cost function

$$L_\gamma(u; \phi, v, w) := L - \gamma^2 \|\phi, v, w\|^2$$

to be viewed as the kernel of a two player game, to be minimized by Player 1 (controlling u) and maximized by Player 2 (controlling (ϕ, v, w)), f^* has the property: $\sup_{\phi,v,w} L_{\gamma^*}(f^*; \phi, v, w) = \inf_f \sup_{\phi,v,w} L_{\gamma^*}(f; \phi, v, w)$. The quantity above is the upper value of the zero-sum dynamic game with kernel L_{γ^*}, which is in fact equal to zero. It can actually be shown that for any $\gamma \geq \gamma^*$, the upper value of the game with parametrized kernel L_γ is zero, and for $\gamma < \gamma^*$, its upper value is infinite. Hence, γ^* is the smallest positive scalar γ for which the zero-sum game with kernel L_γ has a finite upper value.

Instead of obtaining f^* defined above, we will in fact solve a parametrized class of controllers, $\{f^\gamma, \gamma > \gamma^*\}$, where f^γ is obtained from $\sup_{\phi,v,w} L_\gamma(f^\gamma; \phi, v, w) = \inf_f \sup_{\phi,v,w} L_\gamma(f; \phi, v, w)$. The controller f^γ will clearly have the property that it ensures a performance level γ^2 for the index adopted for M2, i.e. the attenuation is bounded by

$$\mathcal{A}(f^\gamma; \phi, v, w) := \frac{\{L(f^\gamma; \phi, v, w)\}^{1/2}}{\|\phi, v, w\|} \leq \gamma \quad \text{for all } \phi, v, w. \tag{12}$$

It will turn out that the limit $\lim_{\gamma \to \infty} f^\gamma =: f^\infty$ is a well-defined controller, and solves uniquely the control problem with the Gaussian model, M1.

3 Solution to problem M2 with additional perturbation

The theory for H$^\infty$ control problems in descrete time with delayed imperfect information has been established in Başar and Bernhard [5], Chap. 6. The control problem we study does not fall precisely into that setting, and to bring it to the standard setting, the perfect state measurements (corresponding to the component $\tilde{x}_n$) has to be perturbed by small noise. The limit of the solution thus obtained, as the perturbation noise vanishes, will provide the optimal control for the original problem, see [2]. We first write (5)-(5) in standard state variable form. Introduce the d-dimensional vector

$$\eta_n := (\xi_{n-d+1}, \ldots, \xi_n)' \tag{13}$$

in terms of which (5) can be re-written as a first-order difference equation:

$$\eta_{n+1} = C\eta_n + k \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \phi_n \qquad \text{where} \qquad C := \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \ldots & \\ \ell_d & \ell_{d-1} & \ell_{d-2} & & \ldots & \ell_1 \end{pmatrix} \tag{14}$$

Further introducing the $(d+2)$-dimensional vector $z_n := (\tilde{x}_n, \tilde{q}_n, \eta_n')'$ we can write (5)-(5) in the compact form: $z_{n+1} = Az_n + D\phi_n + Bu_n$, where

$$A := \left(\begin{array}{cc|cccc} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & -1 \\ \hline & & & & & \\ & 0 & & & \mathrm{C} & \\ & & & & & \end{array} \right) \qquad D := \begin{pmatrix} 0 \\ . \\ . \\ . \\ 0 \\ k \end{pmatrix} \qquad B := \begin{pmatrix} b \\ 0 \\ . \\ . \\ . \\ 0 \end{pmatrix} \tag{15}$$

Now, the stability of the ARMA process (2) is equivalent to the stability of the matrix C in (14), which is captured in the following assumption required to hold throughout the remaining analysis:

- **<u>A1.</u>** All eigenvalues of C are in the unit circle in the complex plane.

Under A1, we have the following fact, which will be useful in the statement of our main result:
<u>F1:</u> Under A1, the pair (A,B) is <u>stabilizable</u>.

As far as the measurements available to the controller go, we have two disturbance corrupted state measurements, given by (6), and one perfect measurement of the state, $\tilde{x}_n$. This is a "mixed" structure, that leads to a *singular* measurement equation, which is not allowed in standard theory. To circumvent this difficulty, we will first perturb $\tilde{x}_n$ with a small independent disturbance, to arrive at the measurement equation $\hat{\tilde{x}}_n = \tilde{x}_n + \epsilon\tilde{\tilde{v}}_n$, $0 < \epsilon << 1$, and assume that now $\hat{\tilde{x}}_n$ is available for control purposes, instead of $\tilde{x}_n$. In the above, $\tilde{\tilde{v}}_n$ is the auxiliary disturbance introduced, and ϵ is a small positive quantity. This modification is only for mathematical convenience, which will in the end be made to vanish asymptotically by letting $\epsilon \to 0$, so as to recover the original information structure.

To introduce the modified measurement equation, let $y_n := (\hat{\tilde{x}}_n, \hat{\tilde{q}}_n, \hat{\tilde{\mu}}_n)'$. Then $y_n = Hz_n + E_\epsilon \tilde{v}_n$ where

$$H := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \qquad E_\epsilon := \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & c \end{pmatrix} \qquad \tilde{v}_n := \begin{pmatrix} \tilde{\tilde{v}}_n \\ w_n \\ v_n \end{pmatrix} \tag{16}$$

Since $E_\epsilon E_\epsilon' > 0$ for all $\epsilon > 0$, this is a nonsingular measurement equation. Compatible with this modification, we also modify the state equation to write it as $z_{n+1} = Az_n + Bu_n + D_\epsilon\zeta_n$, where

$$D_\epsilon := \begin{pmatrix} 0 & 0 & \dots & 0 & k \\ \epsilon & 0 & \dots & 0 & 0 \end{pmatrix}' \qquad \zeta_n := (\tilde{\phi}_n, \phi_n)' \tag{17}$$

with $\tilde{\phi}_n$ being the new disturbance introduced.

To complete the description, we have to write the cost function (7) in terms of these new variables $L = \sum_{n=0}^{\infty} \left\{ |z_n|_N^2 + |u_n|^2 \right\}$, where

$$N := \left(\begin{array}{cc|cccccc} g & 0 & 0 & . & . & . & 0 & -g \\ 0 & 1 & 0 & . & . & . & 0 & 0 \\ \hline 0 & 0 & & & & & & 0 \\ . & . & & & & & & . \\ . & . & & & 0 & & & . \\ . & . & & & & & & 0 \\ -g & 0 & 0 & . & . & . & 0 & g \end{array} \right) \tag{18}$$

and the H^∞ criterion (to be minimized) now becomes $\sup_{\tilde{v},\zeta} \left(L/\|\tilde{v},\zeta\|^2 \right)$, the square-root of whose infimum over all admissible controllers we now denote by $\gamma^*(\epsilon)$. It turns out that $\gamma^*(\epsilon)$ is actually continuous at $\epsilon = 0$, and the quantity $\gamma^*(0)$ is equal to the γ^* defined earlier (for the unmodified problem).

Three additional properties, related to the system matrices, are now worth noting, which we state below as facts: **<u>F2:</u>** The pair (A, H) is observable. **<u>F3:</u>** The pair (A, N) is observable. **<u>F4:</u>** For each $\epsilon > 0$, the pair (A, D_ϵ) is controllable.

Proposition 1 *Let $\epsilon > 0$ be fixed, and assumption A1 hold. Then, $\gamma^*(\epsilon)$ defined above is finite, and for all $\gamma > \gamma^*(\epsilon)$: 1) There exists a minimal positive-definite solution, M_ϵ, to*

$$M = N + A'M\Lambda_\epsilon^{-1}A \tag{19}$$

such that

$$\gamma^2 I - D_\epsilon' M_\epsilon D_\epsilon > 0, \tag{20}$$

where

$$\Lambda_\epsilon := I + (BB' - \gamma^{-2}D_\epsilon D_\epsilon')M. \tag{21}$$

2) There exists a minimal positive-definite solution, Σ_ϵ, to

$$\Sigma = D_\epsilon D_\epsilon' + A\Sigma R_\epsilon^{-1}A' \tag{22}$$

such that

$$\gamma^2 I - N^{1/2}\Sigma_\epsilon N^{1/2} > 0, \tag{23}$$

where

$$R_\epsilon := I + \left(H'(E_\epsilon E_\epsilon')^{-1}H - \gamma^{-2}N\right)\Sigma. \tag{24}$$

3)

$$\Sigma_\epsilon^{1/2}M_\epsilon\Sigma_\epsilon^{1/2} < \gamma^2 I. \tag{25}$$

4) An H^∞-controller that ensures a bound $\gamma^2 > (\gamma^*(\epsilon))^2$ is

$$u_{n,\epsilon,\theta}^{\gamma} = f_{\epsilon,\theta}^{\gamma}(\hat{z}_{n|n-\theta}) = -B'M_\epsilon\Lambda_\epsilon^{-1}A\left(I - \gamma^{-2}\tilde{\Sigma}_{\epsilon,\theta}M_\epsilon\right)^{-1}\hat{z}_{n|n-\theta}, \qquad n = 0, 1, \ldots \tag{26}$$

where $\hat{z}_{n|n-\theta} = \xi_n^{(n)}$, with $\xi_n^{(n)}$ being the last step of the iteration $\{\xi_k^{(n)}\}_{k=n-\theta+1}^{n}$ generated by

$$\xi_{k+1}^{(n)} = A\left(I - \gamma^{-2}\tilde{\Sigma}_{\epsilon,\theta}N\right)^{-1}\xi_k^{(n)} + Bu_k\,; \quad \xi_{n-\theta+1}^{(n)} = \begin{cases} \check{z}_{n-\theta+1} & n \geq \theta \\ 0 & else \end{cases}, \tag{27}$$

and $\{\check{z}_n\}_{n=0}^{\infty}$ is as generated by

$$\check{z}_{n+1} = A\check{z}_n + Bu_n + A\Sigma_\epsilon R_\epsilon^{-1}\left[\gamma^{-2}N\check{z}_n + H'(E_\epsilon E_\epsilon')^{-1}(y_n - H\check{z}_n)\right]. \tag{28}$$

The matrix $\tilde{\Sigma}_{\epsilon,\theta}$ is the θ-th term generated by the recursion

$$\tilde{\Sigma}_{\epsilon,k+1} = A\tilde{\Sigma}_{\epsilon,k}\left(I - \gamma^{-2}N\tilde{\Sigma}_{\epsilon,k}\right)^{-1}A' + D_\epsilon D_\epsilon'\,; \quad \tilde{\Sigma}_{\epsilon,1} = \Sigma_\epsilon\,, \tag{29}$$

satisfying the two spectral radius conditions

$$N^{1/2}\tilde{\Sigma}_{\epsilon,k}N^{1/2} < \gamma^2 I\,, \quad k = 1, \ldots, \theta \tag{30}$$

$$M_\epsilon^{1/2}\tilde{\Sigma}_{\epsilon,k}M_\epsilon^{1/2} < \gamma^2 I\,, \quad k = 1, \ldots, \theta + 1. \tag{31}$$

The initializing matrix Σ_ϵ in (29) is a minimal positive-definite solution of (22), satisfying (23).

If any one of the three properties (1)-(3) above or (30)-(31) do not hold, then $\gamma < \gamma^*(\epsilon)$.

Proof. Follows directly from Thm 6.6 (p. 276) of [5]. Even though this Theorem states the solution of only the finite-horizon case, its extension to the infinite-horizon case (which is dealt with here) follows by mimicking the arguments used in the non-delay case; see Theorem 5.5, p. 226 of [5]. $\diamond$

Remark 1 The results of Proposition 1 are valid not only for all finite γ's, larger than $\gamma^*(\epsilon)$, but also as $\gamma \to \infty$. In this limiting case, which captures the solution to ϵ-perturbed problem under the Gaussian model (M1) of the disturbance [6], the constraints (20), (23) and (24) become irrelevant, and all other expressions admit "well-defined" forms, obtained by simply setting $\gamma^{-2} = 0$. $\diamond$

4 The limiting solution as $\epsilon \downarrow 0$

We next study the limit of the solution given in Proposition 1 as $\epsilon \downarrow 0$. For the limit to be well defined, it will be sufficient to show that (19)-(25) are well-defined in the limit. The first three depend continuously on $\epsilon \geq 0$, and hence in the limit (19) will be replaced by

$$M = N + A' M \Lambda_0^{-1} A \tag{32}$$

where

$$\Lambda_0 := I + \mathrm{diag}(b^2, 0, \ldots, 0, -\gamma^2 k^2) M \tag{33}$$

and (32) will have to be solved under the scalar condition (as counterpart of (20)):

$$k^2 (0, \ldots, 0, 1) M (0, \ldots, 0, 1)' < \gamma^2 \tag{34}$$

which guarantees invertibility of (33). Again, for each $\gamma > \gamma^*(0)$, (32) admits a unique minimal positive-definite solution, which also satisfies (34). The limiting analysis of (22)-(24) is more complicated, since it involves the inverse of a matrix that becomes singular at $\epsilon = 0$ (which is $E_\epsilon E_\epsilon'$). However, this isolated singularity does not lead to any singularity for the solution of (22) as to be shown below. We will in fact show that in the limit as $\epsilon \downarrow 0$, the solution of (22) will be in the form

$$\Sigma_0 = \begin{pmatrix} 0 & . & . & . & 0 \\ . & & & & \\ . & & P & & \\ . & & & & \\ 0 & & & & \end{pmatrix} \tag{35}$$

where $P > 0$ is of dimension $(d+1) \times (d+1)$. To see this, let us first rewrite ΣR_ϵ^{-1} as

$$\Sigma R_\epsilon^{-1} = \Sigma^{1/2} \left[I + \Sigma^{1/2} H' (E_\epsilon E_\epsilon')^{-1} H \Sigma^{1/2} - \gamma^{-2} \Sigma^{1/2} N \Sigma^{1/2} \right]^{-1} \Sigma^{1/2}$$

where $\Sigma^{1/2}$ denotes the unique nonnegative definite square root of Σ. Let the $(d+1) \times (d+1)$-dimensional lower block of N be denoted by $\tilde{N}$, which is

$$\tilde{N} = \mathrm{diag}(1, 0, \ldots, 0, g) \tag{36}$$

and suppose that Σ is given by (35), where P is arbitrary. Then, straightforward manipulations lead to

$$\Sigma^{1/2} N \Sigma^{1/2} = \text{block-diag}\left(0,\ P^{1/2} \tilde{N} P^{1/2}\right),$$

$$\Sigma^{1/2} H' (E_\epsilon E_\epsilon')^{-1} H \Sigma^{1/2} = \text{block-diag}\left(0,\ P^{1/2} \tilde{H}' (\tilde{E} \tilde{E}')^{-1} \tilde{H} P^{1/2}\right)$$

where

$$\tilde{H} := \underbrace{\begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}}_{2 \times (d+1)} \qquad \tilde{E} := \underbrace{\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}}_{2 \times 2} \tag{37}$$

Hence,

$$\begin{aligned} \Sigma R_\epsilon^{-1} &= \Sigma^{1/2} \left[\text{block-diag}\left(1,\ I + P^{1/2} \left(\tilde{H}' (\tilde{E} \tilde{E}')^{-1} \tilde{H} - \gamma^{-2} \tilde{N} \right) P^{1/2} \right) \right]^{-1} \Sigma^{1/2} \\ &= \text{block-diag}\left(0,\ P^{1/2} \left[I + P^{1/2} \left(\tilde{H}' (\tilde{E} \tilde{E}')^{-1} \tilde{H} - \gamma^{-2} \tilde{N} \right) P^{1/2} \right]^{-1} P^{1/2} \right) \\ &= \text{block-diag}(0,\ \tilde{S}) \end{aligned} \tag{38}$$

which is independent of ϵ. Pre- and post-multiplying this expression by A and A', we obtain

$$A \Sigma R_\epsilon^{-1} A' = \text{block-diag}\left(0,\ \tilde{A} P^{1/2} \left[I + P^{1/2} \left(\tilde{H}' (\tilde{E} \tilde{E}')^{-1} \tilde{H} - \gamma^{-2} \tilde{N} \right) P^{1/2} \right]^{-1} P^{1/2} \tilde{A}' \right)$$

where $\tilde{A}$ is the $(d+1)\times(d+1)$ dimensional lower block of A, i.e.

$$\tilde{A} := \left(\begin{array}{c|ccccc} 1 & 0 & . & . & 0 & \text{-1} \\ \hline 0 & & & & & \\ . & & & & & \\ . & & & \text{C} & & \\ . & & & & & \\ 0 & & & & & \end{array}\right) \tag{39}$$

Furthermore, since $\lim_{\epsilon\to 0} D_\epsilon D_\epsilon' = \text{diag}(0,\ldots,0,k^2)$, it readily follows that the structure (35) is consistent with (22). Introducing a $(d+1)$-dimensional vector $\tilde{D}$,

$$\tilde{D} := (0,\ldots,0,k)', \tag{40}$$

the equation to be satisfied by P can now be written as

$$P = \tilde{D}\tilde{D}' + \tilde{A}P\left[I + \left(\tilde{H}'(\tilde{E}\tilde{E}')^{-1}\tilde{H} - \gamma^{-2}\tilde{N}\right)P\right]^{-1}\tilde{A}', \tag{41}$$

and (23) becomes equivalent to

$$\gamma^2 I - \tilde{N}^{1/2}P\tilde{N}^{1/2} > 0. \tag{42}$$

Finally, (25) simplifies to

$$P^{1/2}\tilde{M}P^{1/2} < \gamma^2 I \tag{43}$$

where $\tilde{M}$ is the $(d+1)\times(d+1)$-dimensional lower block of M.

Hence, (19)-(21) are well-defined at $\epsilon = 0$, and (22)-(25) admit well-defined limits in the space of dimension $(d+1)$ instead of $(d+2)$.

Next, we deternie $\tilde{\Sigma}_{\epsilon,k}$. Compatible with that of $\Sigma_0 = \lim_{\epsilon\downarrow 0}\Sigma_\epsilon$, it is not difficult to show that $\tilde{\Sigma}_{0,k} =$ block-diag $\left(0,\ \tilde{P}_k\right)$ where the $(d+1)$-dimensional square matrix $\tilde{P}_k$ is generated by the recursion

$$\tilde{P}_{k+1} = \tilde{A}\tilde{P}_k\left(I - \gamma^{-2}\tilde{N}\tilde{P}_k\right)^{-1}\tilde{A}' + \tilde{D}\tilde{D}'; \quad \tilde{P}_1 = P, \tag{44}$$

where P is a minimal positive-definite solution of (41), satisfying (42), and $\tilde{A}$, $\tilde{D}$ are as defined by (39) and (40), respectively. In view of this simplification, the two spectral radius conditions (30) and (31) become, respectively

$$\tilde{N}^{1/2}\tilde{P}_k\tilde{N}^{1/2} < \gamma^2 I, \quad k = 1,\ldots,\theta \tag{45}$$

$$\tilde{M}^{1/2}\tilde{P}_k\tilde{M}^{1/2} < \gamma^2 I, \quad k = 1,\ldots,\theta+1, \tag{46}$$

where $\tilde{N}$ was defined by (36) and $\tilde{M}$ in conjunction with (43). To complete the limiting analysis, we still have to find the limiting form of the controller as $\epsilon\downarrow 0$. Denote $\hat{z}_{n|n-\theta} = \left(\hat{\alpha}_{n|n-\theta},\ \hat{\beta}'_{n|n-\theta}\right)'$; $\quad \xi_k^{(n)} = \left(\tilde{\alpha}_k^{(n)},\ \tilde{\beta}_k^{(n)'}\right)'$, where $\hat{\alpha}_{n|n-\theta}$ is of dimension 1, and $\hat{\beta}'_{n|n-\theta}$ of dimension $d+1$. Then, some manipulations yield:

$$\begin{array}{l} \tilde{\alpha}_{k+1}^{(n)} = \tilde{\alpha}_k^{(n)} + bu_k; \quad \tilde{\alpha}_{n-\theta+1}^{(n)} = \left\{\begin{array}{cc} \tilde{x}_{n-\theta+1} & n\geq\theta \\ 0 & \text{else} \end{array}\right. \\ \Rightarrow \quad \tilde{\alpha}_k^{(n)} = \tilde{x}_k, \quad k = 0,1,\ldots\ \forall\, n\geq 0 \qquad \Rightarrow \qquad \hat{\alpha}_{n|n-\theta} = \tilde{x}_n \end{array} \tag{47}$$

and

$$\begin{array}{c} \hat{\beta}_{n|n-\theta} = \tilde{\beta}_n^{(n)} \\ \tilde{\beta}_{k+1}^{(n)} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}}_{(d+1)\times(d+2)} A\left(I - \gamma^{-2}\tilde{\Sigma}_{0,\theta}N\right)^{-1}\begin{pmatrix} \tilde{x}_k \\ \tilde{\beta}_k^{(n)} \end{pmatrix}; \\ \tilde{\beta}_{n-\theta+1}^{(n)} = \left\{\begin{array}{cc} \breve{\beta}_{n-\theta+1} & n\geq\theta \\ 0 & \text{else} \end{array}\right. \end{array} \tag{48}$$

where $\breve{\beta}_n,\ n \geq 1$ is generated by

$$\breve{\beta}_{n+1} = \tilde{A}\breve{\beta}_n + \tilde{A}\tilde{S}\left[\gamma^{-2}\tilde{N}\breve{\beta}_n + \tilde{H}'(\tilde{E}\tilde{E}')^{-1}(\tilde{y}_n - \tilde{H}\breve{\beta}_n)\right], \qquad \tilde{y}_n := (\hat{\tilde{q}}_n, \hat{\tilde{\mu}}_n)' \tag{49}$$

As compared with the result in [2], in the delayed measurement case we do not have a separation between $\hat{\alpha}$ and $\hat{\beta}$ as clean as in the case with $\theta = 0$. (The dynamics for $\tilde{\beta}^{(n)}$ above, and thereby that of $\hat{\beta}_{n|n-\theta}$ use also $\tilde{x}$ as an input, whereas (49) did not.) In view of these limiting expressions, the limiting expression for the controller (26), as $\epsilon \downarrow 0$, is

$$u^{\gamma}_{n,\theta} = f^{\gamma}_{\theta}(x_n, \hat{\beta}_{n|n-\theta}) = -B'M_0\Lambda_0^{-1}A\left(I - \gamma^{-2}\tilde{\Sigma}_{0,\theta}M_0\right)^{-1}\begin{pmatrix} \tilde{x}_n \\ \hat{\beta}_{n|n-\theta} \end{pmatrix}, \qquad n = 0, 1, \ldots \tag{50}$$

This now leads to the following Theorem:

Theorem 1 *Consider the original control problem formulated in Section 2, under the H^{∞} model* (M2) *of the disturbance. Let Assumption A1 hold. Then γ^* defined by (11) is finite, and for all $\gamma > \gamma^*$: 1) There exists a minimal positive-definite solution, M_0, to (32) such that (34) holds.*
2) There exists a minimal positive-definite solution, P, to (41), such that (42) holds.
3) Conditions (43), (45) and (46) hold, and the matrix sequence generated by (44) is well defined and nonnegative definite.
4) An H^{∞}-controller that ensures the bound γ^2 in (9) under θ-delayed measurements is given by (50).
For a given $\gamma > 0$, if any one of the conditions above does not hold, then $\gamma < \gamma^*$. ◇

We immediately have the following corollary to Theorem 1, obtained by letting $\gamma \to \infty$, which solves the original problem under the Gaussian interpretation (M1) of the disturbances (see Remark 1).

Corollary 1 *Under A1, the optimal control problem with the disturbance model M1 and θ-delayed measurements, admits the unique solution:*

$$u^{\infty}_{n,\theta} = f^{\infty}_{\theta}(x_n, \hat{\beta}_{n|n-\theta}) = -B'\overline{M}(I + BB'\overline{M})^{-1}A\begin{pmatrix} \tilde{x}_n \\ \hat{\beta}_{n|n-\theta} \end{pmatrix}, \qquad n = 0, 1, \ldots \tag{51}$$

where $\overline{M}$ is the unique positive-definite solution of $\overline{M} = N + A'\overline{M}(I + BB'\overline{M})^{-1}A$ in the class of nonnegative-definite matrices, and

$$\hat{\beta}_{n|n-\theta} = \tilde{\beta}^{(n)}_n; \qquad \tilde{\beta}^{(n)}_{k+1} = \tilde{A}\tilde{\beta}^{(n)}_k,\ \tilde{\beta}^{(n)}_{n-\theta+1} = \begin{cases} \breve{\beta}_{n-\theta+1} & n \geq \theta \\ 0 & else \end{cases} \tag{52}$$

$$\breve{\beta}_{n+1} = \tilde{A}\breve{\beta}_n + \tilde{A}\overline{S}\tilde{H}'(\tilde{E}\tilde{E}')^{-1}(\tilde{y}_n - \tilde{H}\breve{\beta}_n) \tag{53}$$

where $\overline{S} := \overline{P}\left[I + \tilde{H}'(\tilde{E}\tilde{E}')^{-1}\tilde{H}\overline{P}\right]^{-1}$, and $\overline{P}$ is the unique positive-definite solution of $\overline{P} = \tilde{D}\tilde{D}' + \tilde{A}\overline{S}\tilde{A}'$ in the class of nonnegative definite matrices. ◇

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