THE LAW OF THE EULER SCHEME FOR STOCHASTIC DIFFERENTIAL EQUATIONS: II. CONVERGENCE RATE OF THE DENSITY

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Abstract

In the first part of this work [4] we have studied the approximation problem of $\mathbb{E}f(X_T)$ by $\mathbb{E}f(X_T^n)$, where (X_t) is the solution of a stochastic differential equation, (X_t^n) is defined by the Euler discretization scheme with step $\frac{T}{n}$, and $f(\cdot)$ is a given function, only supposed measurable and bounded; we have proven that the error can be expanded in terms of powers of $\frac{1}{n}$, under a nondegeneracy condition of Hörmander type for the infinitesimal generator of (X_t).

In this second part, we consider the density of the law of a small perturbation of X_T^n and we compare it to the density of the law of X_T: we prove that the difference between the densities can also be expanded in terms of $\frac{1}{n}$.

The results of this paper had been announced in special issues of journals devoted to the Proceedings of Conferences: see Bally, Protter and Talay [2] and Bally and Talay [3].

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1 Introduction

Let (X_t) be the process taking values in $\mathbb{R}^d$ solution to

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \ , \tag{1}$$

where (W_t) is a r-dimensional Brownian motion.

The problem of computing the expectation $\mathbb{E}f(X_t)$ on a time interval $[0,T]$ by a Monte Carlo algorithm appears in various applied problems; some of them are listed in [4]. The algorithm consists in approximating the unknown process (X_t) by an approximate process (X_t^n), where the parameter n governs the time discretization; that process can be simulated on a computer, and a simulation of a large number M of independent trajectories of X_t^n provides the following approximate value of $\mathbb{E}f(X_t)$:

$$\frac{1}{M}\sum_{i=1}^{M} f(X_t^n(\omega_i)) \ .$$

The resulting error of the algorithm depends on the choice of the approximate process and the two parameters M and n.

We consider the Euler scheme:

$$\left\{ \begin{array}{rcl} X_0^n & = & X_0 \ , \\ X_{(p+1)T/n}^n & = & X_{pT/n}^n + b(X_{pT/n}^n)\frac{T}{n} + \sigma(X_{pT/n}^n)(W_{(p+1)T/n} - W_{pT/n}) \ . \end{array} \right. \tag{2}$$

For $\frac{pT}{n} \leq t < \frac{(p+1)T}{n}$, X_t^n is defined by

$$X_t^n = X_{pT/n}^n + b(X_{pT/n}^n)\left(t - \frac{pT}{n}\right) + \sigma(X_{pT/n}^n)(W_t - W_{pT/n}) \ .$$

When $X_0 = x$ (resp. $X_0^n = x$) a.s., we write $X_t(x)$ (resp. $X_t^n(x)$).

The effects of n on the global error of the algorithm can be measured by the quantity

$$|\mathbb{E}f(X_T) - \mathbb{E}f(X_T^n)| \ . \tag{3}$$

This error can be expanded in terms of powers of $\frac{1}{n}$: see Talay and Tubaro [14] for smooth f's without any assumption on the infinitesimal generator of (X_t) and for the numerical interest of the result (i.e. the justification of Romberg extrapolations which exponentially accelerate the convergence rate with a linear increase of the numerical cost, which explains why we are not interested in more sophisticated schemes than the Euler scheme since one can obtain their accuracy with a weaker numerical cost: see Talay [13] for a discussion). Similar results hold when (X_t) is the solution of a Lévy driven stochastic differential equation, see Protter and Talay [12].

In Bally and Talay [4] the same expansion has been established for only measurable and bounded functions f's under a uniform nondegeneracy condition of Hörmander type on that generator (see below for a more precise formulation).

In this paper our objective is the following.

First, we prove that, when the infinitesimal generator of the process (X_t) is strongly elliptic, the density of the law of (X_T^n) and its derivatives have exponential bounds of the same type as the density of the law of (X_T) (with constants uniformly bounded with respect to the discretization step). It seems that this natural property cannot be proven by induction and that elementary techniques fail. This result will be very useful in the analysis of stochastic particle methods for nonlinear PDE's, where one often has to deal with quantities involving the behaviour at infinity of the distribution of the location of the particles (see Bernard, Talay and Tubaro [5] or Bossy and Talay [6] for examples of such a situation).

Second, we treat the case where the generator is not strongly elliptic. Observe that in this case the law of X_T^n may not have a density. Let $U_L \subset \mathbb{R}^d$ be the set of points for which Hörmander's condition involves Lie bracketts of length less or equal to L. For $x \in U_L$ the law of $X_T(x)$ has a density $p_T(x,\cdot)$ with respect to Lebesgue's measure. We approximate this density by the density $\tilde{p}_T^n(x,\cdot)$ of the law of of a small perturbation of $X_T^n(x)$. More precisely, for x and y in U_L, one has

$$p_T(x,y) - \tilde{p}_T^n(x,y) = -\frac{1}{n}\pi_T(x,y) + \frac{1}{n^2}R_T^n(x,y)$$

for some function $\pi_T(x,y)$ independent of n and some remainder term $R_T^n(x,y)$ which satisfy an exponential inequality of the type

$$|\pi_T(x,y)| + |R_T^n(x,y)| \leq \frac{K(T)}{T^q V_L(x)^{q'} V_L(y)^{q''}} \exp\left(-c\frac{\| x-y \|^2}{T}\right) .$$

(the functions $V_L(\cdot)$ and $K(\cdot)$ are defined below).

The above expansion and the exponential bounds for $\pi_T(x,y)$ and $R_T^n(x,y)$ give a local information on the approximation of the law of $X_T(x)$ by the law of $\tilde{X}_T^n(x)$: without any global nondegeneracy assumption but under the hypothesis that $x \in U_L$ and $A \subset \mathbb{R}^d$ is a Borel set whose boundary is a subset of U_L (neither A nor A^c is supposed included in U_L) we prove that

$$\mathbb{P}[X_T(x) \in A] - \mathbb{P}[\tilde{X}_T^n(x) \in A] = -\frac{1}{n}\pi_T(x,A) + \frac{1}{n^2}R_T^n(x,A)$$

where the functions $|\pi_T(\cdot,A)|$ and $|R_T^n(\cdot,A)|$ can be bounded exponentially from above.

During the very last days of the redaction of the present paper, the authors have received a paper by Kohatsu-Higa [9] which also deals with the approximation of $p_T(x,y)$ for $x \in U_L$. This density is approximated by

$$\mathbb{E}\phi_{n^{-\delta}}\left(X_T^n(x) + \frac{G}{n} - y\right)$$

where G is a standard Gaussian vector independent of W_1 and $\phi_\tau(\cdot)$ denotes a Gaussian density of mean 0 and of covariance matrix $\tau^2 Id(\mathbb{R}^d)$. Kohatsu-Higa shows that, for any $\delta \geq 1$, there exists a constant $C(\delta)$ such that

$$\sup_{y \in \mathbb{R}^d} |\mathbb{E}\phi_{n^{-\delta}}(X_T^n(x) - y) - p_T(x, y)| \leq \frac{C(\delta)}{n} \ .$$

It is clear that this estimate and our results are of different nature.

The organization of the paper is the following: in Section 2, we state and comment our main results; in Section 3, we prove these results, admitting technical estimates proven in Section 4; these estimates require a non trivial modification of a result due to Kusuoka and Stroock concerning the derivatives of $p_T(x, y)$: this work, interesting in itself, is done in Sections 5 and 6.

Notation.

In all the paper, given a smooth function $\phi(\cdot)$ and a multiindex α of the form

$$\alpha = (\alpha_1, \ldots, \alpha_k) \ , \ \alpha_i \in \{1, \ldots, d\}$$

the notation $\partial_\alpha^z \phi(t, z, \xi)$ means that the multiindex α concerns the derivation with respect to the coordinates of z, the variables t and ξ being fixed. When we write $\partial_\alpha \phi(t, z)$ it must be understood that we differentiate w.r.t. the space variable z only.

When $\gamma = (\gamma^{ij})$ is a matrix, $\hat{\gamma}$ denotes the determinant of γ, and γ_j denotes the $j - th$ column of γ.

When V is a vector, ∂V denotes the matrix $(\partial_i V^j)^{ij}$.

We will use the same notation $K(\cdot)$, q, c, μ, etc, for different functions and positive real numbers having the common property to be independent of T and of the approximation parameter n: typically, they will only depend on L^∞-norms of a finite number of partial derivatives of the coordinates of $b(\cdot)$ and $\sigma(\cdot)$ and on an integer L to be defined below.

As usual, we denote by $\mathbb{P}_z$ the law for which $X_0^n = X_0 = z$ a.s. and we denote the corresponding expectation by $\mathbb{E}_z$.

In all the paper, we reserve the letters x and y for elements of a set U_L defined below.

2 Main results

2.1 Density and local density.

Consider the stochastic differential equation (1).

In all the paper we suppose:

(H) The functions $b(\cdot)$ and $\sigma(\cdot)$ are $\mathcal{C}^\infty(\mathbb{R}^d)$ functions whose derivatives of any order are bounded.

Denote by $A_0, A_1, \ldots, A_r$ the vector fields defined by

$$\begin{aligned} A_0(\cdot) &= \sum_{i=1}^d b^i(\cdot)\partial_i \ , \\ A_j(\cdot) &= \sum_{i=1}^d \sigma^{ij}(\cdot)\partial_i \ , \quad j=1,\ldots,r \ . \end{aligned}$$

For multiindices $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{0, 1, \ldots r\}^k$, define the vector fields A_i^α $(1 \le i \le r)$ by induction: $A_i^\emptyset = A_i$ and, for $0 \le j \le r$, $A_i^{(\alpha,j)} := [A^j, A_i^\alpha]$.

For $L \ge 1$, define the quadratic forms

$$V_L(\xi,\eta) := \sum_{j=1}^r \sum_{|\alpha|\le L-1} < A_j^\alpha(\xi), \eta >^2$$

and set

$$V_L(\xi) = 1 \wedge \inf_{\|\eta\|=1} V_L(\xi,\eta) \ . \tag{4}$$

Denote by U_L the set

$$U_L := \{\xi; V_L(\xi) > 0\} \ . \tag{5}$$

Kusuoka and Stroock (Corollary 3.25 in [10]) have shown: for any integer $L \ge 1$ and any $x \in U_L$, the law of $X_T(x)$ has a smooth density $p_T(x,\cdot)$; besides, for any integers m, k, for any multiindices α and β such that $2m + |\alpha| + |\beta| \le k$, there exist an integer $M(k,L)$, a non decreasing function $K(\cdot)$ and real numbers c, q depending on $L, T, m, k, \alpha, \beta$ and on the bounds associated to the coefficients of the stochastic differential equation and their derivatives up to the order $M(k,L)$, such that the following inequality holds[1]:

$$\left|\frac{\partial^m}{\partial t^m}\partial_\alpha^x\partial_\beta^z p_t(x,z)\right| \le \frac{K(T)}{t^q V_L(x)^{q+2q/L}} \exp\left(-c\frac{\|x-z\|^2}{t}\right) \ , \ \forall 0<t\le T \ , \ \forall z \in \mathbb{R}^d \ , \ \forall x \in U_L \ . \tag{6}$$

A complementary result also holds whose proof is postponed to Section 5.

Proposition 2.1 *Assume (H). Let L be such that U_L is non void. Then there exists a smooth function*

$$(t,z,y) \in (0,T] \times \mathbb{R}^d \times U_L \to q_t(z,y)$$

such that, for any measurable and bounded function $\phi(\cdot)$ with a compact support included in the set U_L, one has

$$\mathbb{E}_z\phi(X_t) = \int_{Supp(\phi)} \phi(\xi) q_t(z,\xi) d\xi \ . \tag{7}$$

[1] In the statement of Kusuoka and Stroock, the constants γ_j, $\mu_n(L)$ are equal to 0 under (H).

Let $L \geq 1$, m be arbitrary integers and let α, β be multiindices. There exist positive constants μ, c and there exists an increasing function $K(\cdot)$ such that, for any $0 < t \leq T$,

$$\forall y \in U_L \ , \ \forall z \in I\!\!R^d \ , \ \left| \frac{\partial^m}{\partial t^m} \partial^z_\alpha \partial^y_\beta q_t(z,y) \right| \leq \frac{K(T)}{(tV_L(y))^\mu} \exp\left(-c \frac{\| z - y \|^2}{t} \right) \ . \tag{8}$$

2.2 The perturbed scheme.

If the uniform ellipticity condition

(H1) $\exists \alpha > 0 \ , \ \|\sigma(t,x)\sigma^*(t,x)\| \geq \alpha \ , \ \forall (t,x)$

holds, then the law of $X^n_T(x)$ has a density $p^n_T(x,\cdot)$ w.r.t Lebesgue's measure. It may be false even if there exist $L > 0$ and $\alpha > 0$ such that $V_L(\xi) \geq \alpha > 0 \ , \ \forall \xi$. That leads us to consider a small perturbation of X^n_T whose law has a density.

In the whole paper, we refer to the following

Definition 2.2 *Let $\rho_0(\cdot)$ be a smooth and symmetric probability density function with a compact support in $(-1,1)$. For $\delta > 0$ and $\xi \in I\!\!R^d$ we define:*

$$\rho_\delta(\xi) := \prod_{i=1}^d \frac{\rho_0(\xi^i/\delta)}{\delta} \ . \tag{9}$$

We now define a new approximate value of $X_T(x)$, denoted by $\tilde{X}^n_T(x)$: let Z^n be a $I\!\!R^d$-valued random vector independent of $(W_t, 0 \leq t \leq T)$ whose components are i.i.d. and whose law is $\rho_{1/n}(\xi)d\xi$; we set:

$$\forall 0 \leq t < T \ , \ \tilde{X}^n_t(x) := X^n_t(x) \quad ; \quad \tilde{X}^n_T(x) = X^n_T(x) + Z^n \ . \tag{10}$$

We denote by $\tilde{p}^n_T(x,\cdot)$ the density of the law of $\tilde{X}^n_T(x)$ w.r.t. Lebesgue's measure.

2.3 Convergence rate for the density.

In [4] we have proven that, if $\inf_{z \in I\!\!R^d} V_L(z) > 0$ for some integer L and if the functions $b(\cdot)$ and $\sigma(\cdot)$ satisfy (H), then for any measurable and bounded function $f(\cdot)$,

$$I\!\!E_x f(X_T) - I\!\!E_x f(X^n_T) = -\frac{C_f(T,x)}{n} + \frac{Q_n(f,T,x)}{n^2} \ , \tag{11}$$

the terms $C_f(T,x)$ and $Q_n(f,T,x)$ having the following property: there exists an integer m, a non decreasing function $K(\cdot)$ depending on the coordinates of $\sigma(\cdot)$ and $b(\cdot)$ and on their derivatives up to the order m, and a positive real number q such that

$$|C_f(T,x)| + sup_n |Q_n(f,T,x)| \leq \|f\|_\infty \frac{K(T)}{T^q} \tag{12}$$

(in fact, the estimate given in [4] is slightly different: the simplified version (12) takes the boundedness of $b(\cdot)$ and $\sigma(\cdot)$ into account). To get an expansion for $p_T(x,y) - \tilde{p}_T^n(x,y)$, it is natural to fix y, to choose $f_\delta(\xi) = \rho_\delta(y-\xi)$ and to make δ tend to 0. But the above result is not sufficient since, when δ tends to 0, $(\| f_\delta \|_\infty)$ tends to infinity. Nevertheless, if $F_\delta(\cdot)$ is the distribution function of the measure $f_\delta(\xi)d\xi$, the sequence $(\| F_\delta \|_\infty)$ is constant: this gives the idea of proving inequalities of type (12) with $\| F \|_\infty$ instead of $\| f \|_\infty$ when $f(\cdot)$ has a compact support, $F(\cdot)$ being the distribution function of the measure $f(\xi)d\xi$.

Before stating our main results we need to introduce some definitions.

2.4 Definitions.

Definition 2.3 *Let $(a^{ij}(t,x))$ denote the matrix $\sigma(t,x)\sigma^*(t,x)$.*

Let $\phi(\cdot,\cdot)$ be a function of class $\mathcal{C}^4([0,T)\times I\!R^d)$. We define the differential operator $\mathcal{U}$ by

$$\begin{aligned}\mathcal{U}\phi(t,z) \;:=\; & \frac{1}{2}\sum_{i,j=1}^{d} b^i(z)b^j(z)\partial_{ij}\phi(t,z) + \frac{1}{2}\sum_{i,j,k=1}^{d} b^i(z)a^{jk}(z)\partial_{ijk}\phi(t,z) \\ & + \frac{1}{8}\sum_{i,j,k,l=1}^{d} a^{ij}(z)a^{kl}(z)\partial_{ijkl}\phi(t,z) + \frac{1}{2}\frac{\partial^2}{\partial t^2}\phi(t,z) \\ & + \sum_{i=1}^{d} b^i(\cdot)\frac{\partial}{\partial t}\partial_i\phi(t,z) + \frac{1}{2}\sum_{i,j=1}^{d} a^{ij}(\cdot)\frac{\partial}{\partial t}\partial_{ij}\phi(t,z)\ . \end{aligned} \tag{13}$$

Definition 2.4 *We set*

$$u(t,\xi) := I\!E_\xi f(X_{T-t}) = P_{T-t}f(\xi)\ ,\ \ 0 \le t \le T\ . \tag{14}$$

Proposition 2.1 shows that, if $f(\cdot)$ is a measurable and bounded function with a compact support included in a non void set U_L defined as in (5), the function $u(\cdot,\cdot)$ is of class $\mathcal{C}^\infty([0,T)\times I\!R^d)$. We set

$$\Psi(t,\cdot) := \mathcal{U}u(t,\cdot)\ . \tag{15}$$

Definition 2.5 *Let $L \in \mathbb{N} - \{0\}$ such that the set U_L is non void. For any measurable bounded function $f(\cdot)$ with a compact support included in U_L we set*

$$\nu_{f,L} = \inf_{\xi \in Supp(f)} V_L(\xi) > 0 \tag{16}$$

and we denote by $d_f(\xi)$ the distance of ξ to the support of $f(\cdot)$.

We also denote by $\mathcal{E}_{f,L}(\xi)$ any *function defined on U_L of the form*

$$\mathcal{E}_{f,L}(\xi) := \frac{K(T)}{T^Q \nu_{f,L}^q V_L(\xi)^{q'}} \exp\left(-c\frac{d_f(\xi)^2}{T}\right)\ , \tag{17}$$

for some strictly positive constants c, q, q', Q and some positive increasing function $K(\cdot)$.

2.5 Statements.

Theorem 2.6 *Assume (H). Let $L \in \mathbb{N} - \{0\}$ be such that U_L is non void and let $x \in U_L$. Let $f(\cdot)$ be a measurable and bounded function with a compact support included in U_L. Let $F(\cdot)$ denote the distribution function of the measure $f(\xi)d\xi$.*

Let $\Psi(\cdot,\cdot)$ be defined as in (15) and for x in U_L set

$$\pi_T^f(x) := \int_0^T \int_{\mathbb{R}^d} p_t(x,z)\Psi(t,z)dzdt \ . \tag{18}$$

The perturbed Euler scheme (10) satisfies: there exists some function $\mathcal{E}_{f,L}(\cdot)$ (for some strictly positive constants Q, c, q, q' and some increasing function $K(\cdot)$) and for each $n > \frac{2}{d_f(x)}$, there exists $R_T^{n,f}(x)$ such that

$$\mathbb{E}_x f(X_T) - \mathbb{E}_x f(\tilde{X}_T^n) = -\frac{1}{n}\pi_T^f(x) + \frac{1}{n^2}R_T^{n,f}(x) \tag{19}$$

and

$$|\pi_T^f(x)| + |R_T^{n,f}(x)| \leq \| F \|_\infty \mathcal{E}_{f,L}(x) \ . \tag{20}$$

The function $K(\cdot)$ entering in the definition of $\mathcal{E}_{f,L}(\cdot)$ depends on the $L^m(\mathbb{R}^d)$ norms (for some integer m) of a finite number of partial derivatives of the function $\rho_0(\cdot)$.

Under (H1), i.e. when the generator of (X_t) is strongly elliptic, (19)-(20) also hold for X^n instead of $\tilde{X}^n$ and any bounded measurable $f(\cdot)$ with a compact support:

$$\mathbb{E}_x f(X_T) - \mathbb{E}_x f(X_T^n) = -\frac{1}{n}\pi_T^f(x) + \frac{1}{n^2}R_T^{n,f}(x) \tag{21}$$

and

$$|\pi_T^f(x)| + |R_T^{n,f}(x)| \leq \| F \|_\infty \frac{K(T)}{T^Q} \exp\left(-c\frac{d_f(\xi)^2}{T}\right) \ . \tag{22}$$

Corollary 2.7 *Assume (H). Let $L \in \mathbb{N} - \{0\}$ be such that U_L is non void and let x and y be in U_L, so that*

$$V_L(x) \wedge V_L(y) > 0 \ . \tag{23}$$

Set

$$\pi_T(x,y) := \int_0^T \int_{\mathbb{R}^d} p_t(x,z)(\mathcal{U}q_{T-t}(\cdot,y))(z)dzdt \tag{24}$$

where the operator $\mathcal{U}$ is defined as in (13) and the function $q_{T-t}(\cdot,y)$ is defined as in (7).

There exists a non decreasing function $K(\cdot)$, there exists some strictly positive constants c, q, q', q'' and for each $n > \frac{2}{\|x-y\|}$, there exists a function $R_T^n(x,y)$ such that the perturbed Euler scheme satisfies

$$p_T(x,y) - \tilde{p}_T^n(x,y) = -\frac{1}{n}\pi_T(x,y) + \frac{1}{n^2}R_T^n(x,y) \tag{25}$$

with

$$|\pi_T(x,y)| + |R_T^n(x,y)| \leq \frac{K(T)}{T^q V_L(x)^{q'} V_L(y)^{q''}} \exp\left(-c\frac{\| x-y \|^2}{T}\right) \ . \tag{26}$$

The function $K(\cdot)$ depends on the $L^m(\mathbb{R}^d)$ norms (for some integer m) of a finite number of partial derivatives of the function $\rho_0(\cdot)$.

Under (H1), (25)-(26) also hold for all (x,y) and for $p_T^n(x,y)$ instead of $\tilde{p}_T^n(x,y)$:

$$\forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \ p_T(x,y) - p_T^n(x,y) = -\frac{1}{n}\pi_T(x,y) + \frac{1}{n^2} R_T^n(x,y) \tag{27}$$

and

$$|\pi_T(x,y)| + |R_T^n(x,y)| \leq \frac{K(T)}{T^q} \exp\left(-c\frac{\| x-y \|^2}{T}\right) \ . \tag{28}$$

Theorem 2.6 and Corollary 2.7 cannot be seen as extensions of (11) which holds for the Euler scheme itself and for unbounded coefficients $b(\cdot)$, $\sigma(\cdot)$. Nevertheless, the expansion (25) can be used to get a result similar to (11) when $V_L(\cdot)$ is bounded below by a strictly positive constant uniform. Even weaker assumptions are admissible as shown by the following proposition.

Proposition 2.8 *Assume (H). L et A be a Borel set such that ∂A (the boundary of A) is included into a non void set U_L for some integer $L \geq 1$ (neither A nor A^c is supposed included in U_L). Let $x \in U_L$.*

Set

$$\pi_T(x,A) := \int_0^T \int_{\mathbb{R}^d} p_t(x,z) \mathcal{U}(P_{T-t} \ \mathbb{1}_A)(z) dz \ .$$

Then

$$P_x[X_T \in A] - P_x\left[\tilde{X}_T^n \in A\right] = -\frac{\pi_T(x,A)}{n} + \frac{R_T^n(x,A)}{n^2} \ . \tag{29}$$

Besides,

$$|\pi_T(x,A)| + |R_T^n(x,A)| \leq \frac{K(T)}{T^Q} \mathcal{E} \ \mathbb{1}_{A,L}(x) \ . \tag{30}$$

Proof. Since U_L is an open set and ∂A is included in U_L one can find a smooth function $\zeta(\cdot)$ such that $\zeta(\cdot) = \mathbb{1}_A(\cdot)$ on U_L^c. As $\mathbb{1}_A = \zeta + (\mathbb{1}_A - \zeta)$, the result follows from Theorem 2.6 applied to $f(\cdot) = \mathbb{1}_A - \zeta(\cdot)$ (since the support of $f(\cdot)$ is included in U_L by construction) and from Theorem 1 of Talay and Tubaro [14] applied to the smooth function $\zeta(\cdot)$ (the proof in [14] must be combined with the classical inequality (42) below to get the exponential in (30)). ∎

3 Proofs of Theorem 2.6 and Corollary 2.7

For the sake of simplicity, in this section we use several technical estimates whose proofs are deferred in the next sections. Besides, we do not treat the restrictive case where (H1) holds, for which the arguments below can be used with some simplifications.

3.1 Proof of Theorem 2.6.

We recall the lemma 4.4 of [4]. The function $\Psi(\cdot,\cdot)$ being defined as in (15) there holds

$$\mathbb{E}_x f(X_T^n) - \mathbb{E}_x f(X_T) = \frac{T^2}{n^2} \sum_{k=0}^{n-2} \mathbb{E}_x \Psi\left(\frac{kT}{n}, X_{kT/n}^n\right) + \sum_{k=0}^{n-1} r_k^n(x) \ , \tag{31}$$

where

$$r_{n-1}^n(x) := \mathbb{E}_x f(X_T^n) - \mathbb{E}_x (P_{T/n} f)(X_{T-T/n}^n) \ ,$$

and for $k < n-1$, $r_k^n(x)$ can be explicited under a sum of terms, each of them being of the form

$$\begin{aligned}\mathbb{E}_x \Big[\varphi_\alpha^\natural(X_{kT/n}^n) \int_{kT/n}^{(k+1)T/n} \int_{kT/n}^{s_1} \int_{kT/n}^{s_2} \varphi_\alpha^\sharp(X_{s_3}^n) \partial_\alpha u(s_3, X_{s_3}^n) ds_3 ds_2 ds_1 \\ + \varphi_\alpha^\dagger(X_{kT/n}) \int_{kT/n}^{(k+1)T/n} \int_{kT/n}^{s_1} \int_{kT/n}^{s_2} \varphi_\alpha^\flat(X_{s_3}) \partial_\alpha u(s_3, X_{s_3}) ds_3 ds_2 ds_1 \Big]\end{aligned} \tag{32}$$

where $|\alpha| \leq 6$ and the $\varphi_\alpha^\natural$'s, $\varphi_\alpha^\sharp$'s, $\varphi_\alpha^\dagger$'s, $\varphi_\alpha^\flat$'s are products of functions which are partial derivatives up to the order 3 of the a^{ij}'s and b^i's.

Thus,

$$\begin{aligned}\mathbb{E}_x f(\tilde{X}_T^n) - \mathbb{E}_x f(X_T) &= \frac{T}{n} \int_0^T \mathbb{E}_x \Psi(s, X_s) ds \\ &\quad + \frac{T^2}{n^2} \sum_{k=0}^{n-2} \mathbb{E}_x \Psi\left(\frac{kT}{n}, X_{kT/n}\right) - \frac{T}{n} \int_0^T \mathbb{E}_x \Psi(s, X_s) ds \\ &\quad + \frac{T^2}{n^2} \sum_{k=0}^{n-2} \mathbb{E}_x \left(\Psi\left(\frac{kT}{n}, X_{kT/n}^n\right) - \Psi\left(\frac{kT}{n}, X_{kT/n}\right)\right) \\ &\quad + \sum_{k=0}^{n-2} r_k^n(x) + \mathbb{E}_x f(\tilde{X}_T^n) - \mathbb{E}_x (P_{T/n} f)(X_{T-T/n}^n) \\ &=: \frac{T}{n} \int_0^T \mathbb{E}_x \Psi(s, X_s) ds + A^n + B^n + \sum_{k=0}^{n-2} r_k^n(x) + C^n \ .\end{aligned} \tag{33}$$

We observe that the estimate (38) below ensures that $\int_0^T \mathbb{E}_x |\Psi(s, X_s)| ds$ is finite.

First consider A^n: using Itô's formula and the estimate (38), we get

$$\left| \frac{T}{n} \sum_{k \leq n-1} \mathbb{E}_x \Psi\left(\frac{kT}{n}, X_{kT/n}\right) - \int_0^T \mathbb{E}_x \Psi\left(s, X_s\right) ds \right| \leq \|F\|_\infty \frac{\mathcal{E}_{f,L}(x)}{n^2} . \tag{34}$$

Now we treat B^n. For $1 \leq k \leq n-2$, one applies the expansion (33), substituting the function

$$f_{n,k}(\cdot) := \Psi\left(\frac{kT}{n}, \cdot\right)$$

to $f(\cdot)$.

Set $u_{n,k}(t,x) := P_{kT/n-t} f_{n,k}(\cdot)$ and denote by $\Psi_{n,k}(t,\cdot)$ the function defined in (15) with $u_{n,k}(t,\cdot)$ instead of $u(t,\cdot)$ and kT/n instead of T. Thus, for some functions $g_\lambda(\cdot) \in \mathcal{C}_b^\infty(\mathbb{R}^d)$ one has that, for $t \leq \frac{kT}{n}$,

$$\Psi_{n,k}(t,\cdot) = \sum_\lambda g_\lambda(\cdot) \partial_\lambda \left[P_{kT/n-t} \Psi\left(\frac{kT}{n}, \cdot\right)\right] .$$

There holds:

$$\mathbb{E}_x \Psi\left(\frac{kT}{n}, X^n_{kT/n}\right) - \mathbb{E}_x \Psi\left(\frac{kT}{n}, X_{kT/n}\right) = \frac{T^2}{n^2} \sum_{j=0}^{k-2} \mathbb{E}_x \Psi_{n,k}\left(\frac{jT}{n}, X^n_{jT/n}\right) + \sum_{j=0}^{k-1} r_j^{n,k}(x) ,$$

where the $r_j^{n,k}(x)$'s are sums of terms of type (32) with $u_{n,k}$ instead of u.

We use the inequalities (36) and (37) of the next section to upper bound the right side; we get:

$$|B^n| \leq \frac{T^2}{n^2} \sum_{k=0}^{n-2} \left| \mathbb{E}_x \Psi\left(\frac{kT}{n}, X^n_{kT/n}\right) - \mathbb{E}_x \Psi\left(\frac{kT}{n}, X_{kT/n}\right)\right| \leq \|F\|_\infty \frac{\mathcal{E}_{f,L}(x)}{n^2} . \tag{35}$$

We proceed similarly to upper bound $\left| \sum_{k=0}^{n-2} r_k^n(x) \right|$ and we apply (40) below to upper bound C^n. That ends the proof. ∎

3.2 Proof of Corollary 2.7.

Fix $y \in U_L$ and choose δ small enough to ensure that the support of the function

$$\xi \to \rho_\delta(\xi - y) = \prod_{i=1}^d \frac{1}{\delta} \rho_0\left(\frac{\xi^i - y^i}{\delta}\right)$$

is included in the set U_L (as $V_L(\cdot)$ is continuous the set U_L is an open set).

Apply the estimate (19) with $f(\cdot)=\rho_\delta(\cdot - y)$. Then $F(\cdot)$ is the cumulative distribution function of the measure $\rho_\delta(\xi - y)d\xi$; we denote by $\pi_{T,\delta,y}(\cdot)$ and $R^n_{T,\delta,y}(\cdot)$ the functions appearing in the right side.

There holds:

$$\mathbb{E}_x \rho_\delta(\tilde{X}^n_T - y) = \mathbb{E}_x \rho_\delta(X_T - y) + \frac{1}{n}\pi_{T,\delta,y}(x) - \frac{1}{n^2} R^n_{T,\delta,y}(x) \ .$$

From Proposition 2.1 it is easy to check that $\pi_{T,\delta,y}(x)$ tends to $\pi_T(x,y)$ when δ tends to 0. Besides, as $\| F \|_\infty \leq 1$ it comes

$$\lim_{\delta\to 0}\sup \left(|\pi_{T,\delta,y}(x)| + |R^n_{T,\delta,y}(x)| \right) \leq \frac{K(T)}{T^q V_L(x)^{q'} V_L(y)^{q''}} \exp\left(-c\frac{\| x-y \|^2}{T} \right) \ .$$

That ends the proof. ∎

4 Upper bounds uniform w.r.t. n

In this section we prove some technical propositions which have permitted us to upper bound the remaining terms of the expansion (33) (cf. the inequalities (35) and (34)).

4.1 Statements.

The following proposition was used to treat the term B^n of (33).

Proposition 4.1 *Let x, $L \in \mathbb{N} - \{0\}$, U_L and the function $f(\cdot)$ be as in Theorem 2.6.*

Let $g(\cdot), g_\lambda(\cdot)$ be smooth functions in $\mathcal{C}^\infty_b(\mathbb{R}^d, \mathbb{R})$. Let λ be a multiindex. Then there exist strictly positive constants c, q, q', Q, a positive increasing function $K(\cdot)$ and the corresponding function $\mathcal{E}_{f,L}(\cdot)$ such that

$$|\mathbb{E}_x[g(X^n_t)\partial_\lambda u(t, X^n_t)]| \leq \| F \|_\infty \mathcal{E}_{f,L}(x) \ , \ \forall 0 \leq t \leq T - \frac{T}{n} \tag{36}$$

and more generally, for any θ in $[t, T-\frac{T}{n}]$, for any function $\Psi_\theta(t,\cdot)$ of the form

$$\Psi_\theta(t,\cdot) = g_\lambda(\cdot)\partial_\lambda[P_{\theta - t}\Psi(\theta,\cdot)] \ ,$$

one has

$$|\mathbb{E}_x[\Psi_\theta(t, X^n_t)]| \leq \| F \|_\infty \mathcal{E}_{f,L}(x) \ , \ \forall 0 \leq t \leq \theta \leq T - \frac{T}{n} \ . \tag{37}$$

Similar inequalities hold for the processes (X_t) instead of (X_t^n); in that case, one may take $0 \le t \le \theta \le T$:

$$|\mathbb{E}_x[g(X_t)\partial_\lambda u(t,X_t)]| \le \| F \|_\infty \mathcal{E}_{f,L}(x) \ , \ \forall 0 \le t \le T \tag{38}$$

and

$$|\mathbb{E}_x[\Psi_\theta(t,X_t)]| \le \| F \|_\infty \mathcal{E}_{f,L}(x) \ , \ \forall 0 \le t \le \theta \le T \ . \tag{39}$$

The next proposition was used to treat the term C^n of (33).

Proposition 4.2 *Assume the hypotheses of Proposition 4.1. Then*

$$|\mathbb{E}_x f(\tilde{X}_T^n) - \mathbb{E}_x(P_{T/n}f)(X_{T-T/n}^n)| \le \| F \|_\infty \mathcal{E}_{f,L}(x) \ . \tag{40}$$

Before proving the two above propositions, we need to prove the two following technical lemmas, easy to obtain. The second one is interesting by itself.

4.2 Preliminary lemmas.

Lemma 4.3 *Let $x \in \mathbb{R}^d$ and $\Lambda \subset \mathbb{R}^d$ a closed set. The distance of x to Λ is denoted by $d(x,\Lambda)$. Let $c > 0$. For some strictly positive constants C_0, C_1 and C_2 uniform w.r.t. n, T and $\theta \in (0,T]$, one has*

$$\mathbb{E}_x \exp\left(-c\frac{d(X_\theta^n,\Lambda)^2}{\theta}\right) \le C_0 \exp\left(C_1\theta - C_2\frac{d(x,\Lambda)^2}{\theta}\right) \ , \ \forall 0 < \theta \le T \ . \tag{41}$$

Proof. If $d(x,\Lambda) = 0$, the right side of (41) is larger than 1 for $C_0 > 1$, thus the inequality is true. If $d(x,\Lambda) > 0$, one splits the left side in two parts corresponding to the events $A := [\| X_\theta^n - x \| < \frac{1}{2}d(x,\Lambda)]$ and A^c. On A one has that $d(X_\theta^n,\Lambda) \ge \frac{1}{2}d(x,\Lambda)$, which gives

$$\mathbb{E}_x\left[\mathbb{1}_A \exp\left(-c\frac{d(X_\theta^n,\Lambda)^2}{\theta}\right)\right] \le C_0 \exp\left(-C_2\frac{d(x,\Lambda)^2}{\theta}\right) \ .$$

On the other hand,

$$\mathbb{E}_x\left[\mathbb{1}_{A^c} \exp\left(-c\frac{d(X_\theta^n,\Lambda)^2}{\theta}\right)\right] \le \mathbb{P}\left[\| X_\theta^n(x) - x \| \ge \frac{1}{2}d(x,\Lambda)\right] \ .$$

Since $b(\cdot)$ and $\sigma(\cdot)$ are bounded functions, we can use a standard inequality for certain continuous Brownian semimartingales (obtained from a Girsanov transformation combined with Bernstein's exponential inequality for continuous martingales):

$$\mathbb{P}\left[\| X_\theta^n(x) - x \| \ge \frac{1}{2}d(x,\Lambda)\right] \le C_0 \exp\left(C_1\theta - C_2\frac{d(x,\Lambda)^2}{\theta}\right) \ . \tag{42}$$

■

Lemma 4.4 *Under the hypotheses of Theorem 2.6, there exist some strictly positive constants Q, q, c and μ such that*

$$\forall T > t \geq 0,\ \forall z \in \mathbb{R}^d,\ |\partial_\alpha^z u(t,z)| \leq \ \parallel F \parallel_\infty \frac{K(T)}{(T-t)^Q \nu_{f,L}^\mu} \exp\left(-c\frac{d_f(z)^2}{T-t}\right) . \tag{43}$$

Proof. If $d_f(z) = 0$, the inequality is a consequence of (6) since $Supp(f) \subset U_L$. We now fix $z \in \mathbb{R}^d$ such that $d_f(z) > 0$.

We first observe that since $V_L(\cdot)$ is a continuous function, one constructs a smooth function $\zeta_{z,f,L} : \mathbb{R}^d \to [0,1]$ such that

(a) if $d_f(z) \leq \sqrt{T-t}$ then $\zeta_{z,f,L}(\bar{y}) = 1$ for all $\bar{y}$;

(b) if $d_f(z) > \sqrt{T-t}$ then:

- **(b1)** $\zeta_{z,f,L}(\bar{y}) = 0$ *if* $V_L(\bar{y}) \leq \frac{\nu_{f,L}}{2}$ *or* $d_f(\bar{y}) \geq \frac{d_f(z)}{2}$,
- **(b2)** $\zeta_{z,f,L}(\bar{y}) = 1$ *if* $\bar{y} \in Supp(f)$,
- **(b3)** $\int_{\mathbb{R}^d} \zeta_{z,f,L}(\bar{y}) d\bar{y} = 1$,
- **(b4)** for any multiindex γ with $1 \leq |\gamma| \leq d$, $|\partial_\gamma^{\bar{y}} \zeta_{f,L}(\bar{y})| \leq \frac{C_\gamma}{d_f(z)^q} \leq \frac{C_\gamma}{(T-t)^{q/2}}$, where C_γ is uniform with respect to $d_f(z)$ and $\nu_{f,L}$ and where q is positive and depends on the dimension d only.

Since the support of $f(\cdot)$ is included in $\{\xi; \zeta_{z,f,L}(\xi) = 1\}$, one has (see Proposition 2.1 for the definition of the smooth function $q_{T-t}(\cdot,\cdot)$):

$$\begin{aligned}
\partial_\alpha^z u(t,z) &= \int_{Supp(f)} \partial_\alpha^z q_{T-t}(z,\bar{y}) f(\bar{y}) \zeta_{z,f,L}(\bar{y}) d\bar{y} \\
&= (-1)^d \int \partial_{\bar{y}_1 \ldots \bar{y}_d} \{\partial_\alpha^z q_{T-t}(z,\bar{y}) \zeta_{z,f,L}(\bar{y})\} F(\bar{y}) d\bar{y} .
\end{aligned}$$

We now use the inequality (8) proven below. For some constants c and μ, for some increasing function $K(\cdot)$, it comes

$$\begin{aligned}
&\forall T > t \geq 0 \ ,\ \forall z \in \mathbb{R}^d, \\
&|\partial_\alpha^z u(t,z)| \leq \ \parallel F \parallel_\infty \frac{K(T)}{(T-t)^\mu} \sum_{0 \leq |\gamma| \leq d} \int \exp\left(-c\frac{\parallel z - \bar{y} \parallel^2}{T-t}\right) \frac{|\partial_\gamma \zeta_{z,f,L}(\bar{y})|}{V_L(\bar{y})^\mu} d\bar{y} .
\end{aligned}$$

From the definition of $\zeta_{z,f,L}(\cdot)$, one has

$$|\partial_\gamma \zeta_{z,f,L}(\bar{y})| > 0 \Rightarrow \parallel z - \bar{y} \parallel \geq \frac{d_f(z)}{2} \ \textit{and} \ V_L(\bar{y}) \geq \frac{\nu_{f,L}}{2} .$$

Using the above conditions (a) or (b3) and (b4), we deduce (changing the definition of $K(\cdot)$ from line to line and remembering that $\nu_{f,L} \leq 1$):

$$\begin{aligned} |\partial_\alpha^z u(t,z)| &\leq \quad \| F \|_\infty \frac{K(T)}{\nu_{f,L}^\mu (T-t)^\mu} \exp\left(-c\frac{d_f(z)^2}{T-t}\right) \\ &\qquad \left\{ \exp(c) \exp\left(-c\frac{d_f(z)^2}{T-t}\right) \ 1\!\!1_{d_f(z)\leq\sqrt{T-t}} + \frac{1}{d_f(z)^q} \ 1\!\!1_{d_f(z)>\sqrt{T-t}} \right\} \\ &\leq \| F \|_\infty \frac{K(T)}{(T-t)^Q \nu_{f,L}^\mu} \exp\left(-c\frac{d_f(z)^2}{T-t}\right) \ . \end{aligned}$$

■

4.3 Some recalls on Malliavin calculus.

Now we are in the position to prove Proposition 4.1. The proof uses some material of [4] and some wellknown results which for convenience we first recall in this subsection. We refer to Nualart [11] for an exposition of Malliavin calculus and the notation we use here concerning the stochastic calculus of variations.

For $G := (G^1, \ldots, G^m) \in (I\!\!D^\infty)^m$, we denote by γ_G its Malliavin covariance matrix, i.e. the $m \times m$-matrix defined by

$$\gamma_G^{ij} :=< DG^i, DG^j >_{L^2(0,T)} \ .$$

Definition 4.5 *We say that the random vector G satisfies the nondegeneracy assumption if its Malliavin covariance matrix is a.s. invertible and $\hat{\Gamma}_G$ the determinant of the inverse matrix $\Gamma_G := \gamma_G^{-1}$ satisfies*

$$\hat{\Gamma}_G \in \bigcap_{p\geq 1} L^p(\Omega) \ . \tag{44}$$

Our analysis deeply uses the following integration by parts formula (see the section V-9 in Ikeda-Wanabe [8] for the proof of (46) and see [4] for the proof of (47)):

Proposition 4.6 *Let $G_0 \in (I\!\!D^\infty)^m$ satisfy the nondegeneracy condition of Definition 4.5 and let G in $I\!\!D^\infty$.*

Let $\{H_\beta\}$ be the family of random variables depending on multiindices β of length strictly larger than 1 and with coordinates $\beta_j \in \{1, \ldots, m\}$, recursively defined in the following way:

$$\begin{aligned} H_i(G_0, G) &= H_{(i)}(G_0, G) \\ &:= -\sum_{j=1}^m \left\{ G < D\Gamma_{G_0}^{ij}, DG_0^j >_{L^2(0,T)} + \Gamma_{G_0}^{ij} < DG, DG_0^j >_{L^2(0,T)} \right. \end{aligned}$$

$$
\begin{aligned}
& \qquad +\Gamma^{ij}_{G_0} \cdot G \cdot LG^j_0 \Big\} \ , \\
H_\beta(G_0, G) \ &= \ H_{(\beta_1,\dots,\beta_k)}(G_0, G) \\
&:= \ H_{\beta_k}(G_0, H_{(\beta_1,\dots,\beta_{k-1})}(G_0, G)) \ . \qquad (45)
\end{aligned}
$$

Let $\phi(\cdot)$ be a smooth function with polynomial growth.

Then, for any multiindex α,

$$
I\!E\left[(\partial_\alpha \phi)(G_0)G\right] = I\!E[\phi(G_0)H_\alpha(G_0, G)] \ . \qquad (46)
$$

Besides, for any $p > 1$ and any multiindex β, there exist a constant $C(p, \beta) > 0$ and integers $k(p, \beta)$, $m(p, \beta)$, $m'(p, \beta)$, $N(p, \beta)$, $N'(p, \beta)$, such that, for any measurable set $A \subset \Omega$ and any G_0, G as above, one has

$$
I\!E[|H_\beta(G_0, G)|^p \ 1\!\!1_{\mathrm{A}}]^{\frac{1}{p}} \le C(p, \beta) \ \|\hat{\Gamma}_{G_0} \ 1\!\!1_{\mathrm{A}}\|_{k(p,\beta)} \ \|G\|_{N(p,\beta),m(p,\beta)} \ \|G_0\|_{N'(p,\beta),m'(p,\beta)} \ . \quad (47)
$$

In this paper we need the following "local" version of the preceding, where G_0 satisfies the nondegeneracy condition of Definition 4.5 only locally:

Proposition 4.7 *Let $G_0 \in (I\!D^\infty)^m$ and let G in $I\!D^\infty$.*

Suppose that for some multiindex α, γ_{G_0} is invertible a.s. on the set

$$
[G \neq 0] \bigcup_{|\beta| \le |\alpha|} [D^\beta G \neq 0]
$$

and that, for any $p \ge 1$,

$$
I\!E\left[\left|\hat{\Gamma}_{G_0} \ 1\!\!1_{[G\neq 0]\bigcup_{|\beta|\le|\alpha|}[D^\beta G \neq 0]}\right|^p\right] < \infty \ . \qquad (48)
$$

Let $\{H_\beta\}$ be defined as above.

Let $\phi(\cdot)$ be a smooth function with polynomial growth.

Then

$$
I\!E[(\partial_\alpha \phi)(G_0)G] = I\!E[\phi(G_0)H_\alpha(G_0, G) \ 1\!\!1_{[G\neq 0]\bigcup_{|\beta|\le|\alpha|}[D^\beta G \neq 0]}] \ . \qquad (49)
$$

Besides, if $A \subset [G \neq 0] \bigcup_{|\beta|\le|\alpha|}[D^\beta G \neq 0]$ then the inequality (47) holds for β such that $|\beta| \le |\alpha|$.

Proof. Consider a Gaussian standard variable $\tilde{G}$ independent of (G_0, G) and define $G_\varepsilon := G_0 + \varepsilon \tilde{G}$. Since

$$
\hat{\Gamma}_{G_\varepsilon} \le \frac{1}{\varepsilon^{2d}} \ ,
$$

we may apply (46):

$$\begin{aligned} \mathbb{E}[(\partial_\alpha\phi)(G_\varepsilon)G] &= \mathbb{E}[\phi(G_\varepsilon)H_\alpha(G_\varepsilon,G)] \\ &= \mathbb{E}\left[\phi(G_\varepsilon)H_\alpha(G_\varepsilon,G)]\ \mathbb{1}_{[G\neq 0]\bigcup_{|\beta|\leq|\alpha|}[D^\beta G\neq 0]}\right] . \end{aligned}$$

Observe that

$$\phi(G_\varepsilon)H_\alpha(G_\varepsilon,G)\ \mathbb{1}_{[G\neq 0]\bigcup_{|\beta|\leq|\alpha|}[D^\beta G\neq 0]]} \to \phi(G_0)H_\alpha(G_0,G)\ \ \mathbb{1}_{[G\neq 0]\bigcup_{|\beta|\leq|\alpha|}[D^\beta G\neq 0]}\ a.s.$$

Besides, for all $p>1$,

$$\mathbb{E}\left|\hat{\Gamma}_{G_\varepsilon}\ \mathbb{1}_{[G>0]\bigcup_{|\beta|\leq|\alpha|}[D^\beta G\neq 0]}\right|^p \leq \mathbb{E}\left|\hat{\Gamma}_{G_0}\ \mathbb{1}_{[G>0]\bigcup_{|\beta|\leq|\alpha|}[D^\beta G\neq 0]}\right|^p .$$

Thus, the hypothesis (48) and a uniform integrability argument lead to the conclusion. ■

4.4 Proof of Proposition 4.1.

We only prove (36): the arguments to add in order to get (37) follow the same guidelines as those we used in the proof of Lemma 4.2 in [4]. We also limit ourselves to the process (X_t^n): for (X_t), the proof below can be simplified, in the sense that a localization procedure is unnecessary (which explains that the result holds for $T-\frac{T}{n}\leq t\leq T$ also).

We also suppose $d_f(x)>0$: the case $d_f(x)=0$ only differs by some simplifications in the proof.

We successively will consider the cases where t is "small" (less than $\frac{T}{2}$) and "large" (between $\frac{T}{2}$ and $T-\frac{T}{n}$).

4.4.1 Small t: t in $[0,\frac{T}{2}]$.

Since $g(\cdot)$ is a bounded function, in view of (43) it comes

$$|\mathbb{E}_x[g(X_t^n)\partial_\alpha u(t,X_t^n)]| \leq \ \| F \|_\infty \frac{K(T)}{\nu^\mu_{f,L}(T-t)^Q}\sqrt{1+\mathbb{E}_x d_f(X_t^n)^{2q}}\mathbb{E}_x\left[\exp\left(-c\frac{d_f(X_t^n)^2}{T-t}\right)\right] .$$

Use the fact that $T\geq T-t\geq\frac{T}{2}$. Then the inequality (41) permits to conclude.

4.4.2 Large t: t in $[\frac{T}{2}, T - \frac{T}{n}]$.

Let $\phi \in \mathcal{C}_b^\infty(\mathbb{R})$ such that $\phi(x) = 1$ for $|x| \leq \frac{1}{4}$, $\phi(x) = 0$ for $|x| \geq \frac{1}{2}$ and $0 < \phi(x) < 1$ for $|x| \in (\frac{1}{4}, \frac{1}{2})$.

Let γ_t (resp. γ_t^n) be the Malliavin covariance matrix of X_t (resp. X_t^n), and let $\hat{\gamma}_t$ (resp. $\hat{\gamma}_t^n$) be their determinants. Define

$$r_t^n := \frac{(\hat{\gamma}_t^n - \hat{\gamma}_t)}{\hat{\gamma}_t} .$$

Consider

$$\begin{aligned} \mathbb{E}_x[g(X_t^n)\partial_\alpha u(t, X_t^n)] &= \mathbb{E}_x[(1-\phi(r_t^n))g(X_t^n)\partial_\alpha u(t, X_t^n)] \\ &\quad + \mathbb{E}_x[\phi(r_t^n)g(X_t^n)\partial_\alpha u(t, X_t^n)] \\ &=: A + B . \end{aligned} \tag{50}$$

Using the inequalities (41) and (43) and then the fact that $\frac{2}{T} \leq \frac{1}{T-t} \leq \frac{n}{T}$ since $\frac{T}{2} \leq T - \frac{T}{n}$, one gets

$$\begin{aligned} |A| &\leq \parallel F \parallel_\infty \frac{K(T)}{\nu_{f,L}^\mu (T-t)^\mu} \exp\left(-c\frac{d_f(x)^2}{T-t}\right) \left[\mathbb{E}_x|1-\phi(r_t^n)|^2\right]^{\frac{1}{2}} \\ &\leq n^\mu \parallel F \parallel_\infty \mathcal{E}_{f,L}(T,x) \left[\mathbb{E}_x|1-\phi(r_t^n)|^2\right]^{\frac{1}{2}} . \end{aligned}$$

Slightly modifying the proof of Lemma 5.1 in [4] to take the boundedness of $b(\cdot)$ and $\sigma(\cdot)$ into account, we can prove that for any $p > 1$, there exists an increasing function $K(\cdot)$ depending on p such that

$$[\mathbb{E}_x|1-\phi(r_t^n)|^2]^{\frac{1}{2}} \leq K(T)\ n^{-\frac{p}{4}} ,\ \forall 0 < t \leq T . \tag{51}$$

It remains to choose $p = 4\mu$ to get the expected upper bound for A:

$$|A| \leq \parallel F \parallel_\infty \mathcal{E}_{f,L}(T,x) . \tag{52}$$

Let us now treat B which is the really interesting term.

Consider X_t^n as an element of $(\mathbb{D}^\infty)^d$. Apply the "local" Malliavin integration by parts formula (49). Setting

$$H_\alpha^n(t) := H_\alpha(X_t^n, g(X_t^n)\phi(r_t^n)) ,$$

it comes

$$\begin{aligned} B &= \mathbb{E}_x[u(t, X_t^n)H_\alpha^n(t)] , \\ &= \mathbb{E}_x[(P_{T-t}f)(X_t^n)H_\alpha^n(t)] . \end{aligned}$$

Consider a process $(\tilde{X}_t)$ which is a weak solution of (1) independent of (X_t); denote by $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ the probability space on which $(\tilde{X}_t)$ is defined, and $\tilde{\mathbb{E}}$ the expectation under $\tilde{\mathbb{P}}$. It comes:

$$B = \mathbb{E}_x \left[H_\alpha^n(t) \tilde{\mathbb{E}}_z f(\tilde{X}_{T-t}) \mathcal{B}_{z=X_t^n} \right] .$$

Now, choose a $\mathcal{C}^\infty(\mathbb{R}^d)$ function with compact support $\zeta_{x,f,L}(\cdot)$ such that [2]:

(a) if $d_f(x) \leq \sqrt{T}$ then $\zeta_{z,f,L}(\xi) = 1$ for all ξ;

(b) if $d_f(x) > \sqrt{T}$ then:

- **(b1)** $\zeta_{z,f,L}(\xi) = 0 \ if \ d_f(\xi) \geq \frac{d_f(x)}{2}$,
- **(b2)** $\zeta_{z,f,L}(\xi) = 1 \ if \ \xi \in Supp(f)$,
- **(b3)** for any multiindex γ $|\partial_\gamma^{\bar{y}} \zeta_{f,L}(\xi)| \leq \frac{C_\gamma}{d_f(x)^q} \leq \frac{C_\gamma}{(T)^{q/2}}$, where C_γ is uniform with respect to $d_f(x)$ and where q is positive and depends on the dimension d and on γ only.

The role of this localization function is to keep the memory of the support of $f(\cdot)$ in the Malliavin integration by parts procedure, in order to make the exponential term of $\mathcal{E}_{f,L}(x)$ appear.

Then,

$$\begin{aligned} B &= \mathbb{E}_x \left[H_\alpha^n(t) \tilde{\mathbb{E}}_z \left[f(\tilde{X}_{T-t}) \zeta_{x,f,L}(\tilde{X}_{T-t}) \right] \mathcal{B}_{z=X_t^n} \right] \\ &= \mathbb{E}_x \left[H_\alpha^n(t) \tilde{\mathbb{E}}_z \left[(\partial_{x_1,\dots,x_d} F)(\tilde{X}_{T-t}) \zeta_{x,f,L}(\tilde{X}_{T-t}) \right] \mathcal{B}_{z=X_t^n} \right] . \end{aligned}$$

We now apply the proposition 5.2 in [4]: let $\tilde{X}_\theta(\cdot, \tilde{\omega})$ denote a version of class $\mathcal{C}^\infty$ of the stochastic flow $z \to \tilde{X}_\theta(z, \tilde{\omega})$; let $\tilde{Y}_\theta(\cdot, \tilde{\omega})$ denote its Jacobian matrix and $\tilde{Z}_\theta(\cdot, \tilde{\omega})$ the inverse matrix of $\tilde{Y}_\theta(\cdot, \tilde{\omega})$; there exists processes $(\tilde{Q}_\theta^\lambda)$ such that

$$(\partial_{x_1,\dots,x_d} F)(\tilde{X}_\theta(z)) = \sum_{|\lambda| \leq d} \tilde{Q}_\theta^\lambda(z) \partial_\lambda \{ \partial_{x_1,\dots,x_d} F \circ \tilde{X}_\theta(\cdot, \tilde{\omega}) \}(z) \ a.s. \ , \tag{53}$$

and $\tilde{Q}_\theta^\lambda(z)$ is a polynomial function of the coordinates of $\tilde{Z}_\theta(z, \tilde{\omega})$. Thus, choosing $z = X_t^n$ and $\theta = T - t$, one gets

$$B = \sum_{|\lambda| \leq d} \tilde{\mathbb{E}} \mathbb{E}_x \left[H_\alpha^n(t) \partial_\lambda^z \{ F \circ \tilde{X}_{T-t}(z, \tilde{\omega}) \} \tilde{Q}_{T-t}^\lambda(z) \zeta_{x,f,L}(\tilde{X}_{T-t}(z, \tilde{\omega})) \mathcal{B}_{z=X_t^n} \right] .$$

As in Section 5.1.2 of [4] we observe that $H_\alpha^n(t)$ is a sum of terms, each one being a product which includes a partial derivative of $\phi(\cdot)$ evaluated at point r_t^n; we thus may

[2] At the beginning of the proof we have supposed that $d_f(x) > 0$.

again apply the integration by parts formula (49) with $G_0 = X_t^n$; we obtain for some processes $(\tilde{H}^n_{\alpha,\lambda}(t))$,

$$B = \sum_\lambda \tilde{\mathbb{E}}\mathbb{E}_x \left[\tilde{H}^n_{\alpha,\lambda}(t) F \circ \tilde{X}_{T-t}(z,\tilde{\omega}) \mathcal{B}_{z=X_t^n} \right] \ ,$$

from which

$$|B| \leq \| F \|_\infty \sum_\lambda \tilde{\mathbb{E}}\mathbb{E}_x \left| \tilde{H}^n_{\alpha,\lambda}(t) \right| \ .$$

Observe that $\tilde{H}^n_{\alpha,\lambda}(t)$ is a sum of terms, each one being a product which includes a partial derivative of $\zeta_{x,f,L}(\cdot)$ evaluated at a point $\tilde{X}_{T-t}(z)\mathcal{B}_{z=X_t^n}$ and of a partial derivative of $\phi(\cdot)$ evaluated at point r_t^n. Thus,

$$\tilde{H}^n_{\alpha,\lambda}(t) = \tilde{H}^n_{\alpha,\lambda}(t) \ \mathbb{1}_{[0,1/2]}(|r_t^n|) \ \mathbb{1}_{Supp(\zeta_{x,f,L})} \left(\tilde{X}_{T-t}(z,\tilde{\omega}) \mathcal{B}_{z=X_t^n(\omega)} \right) \ .$$

On $|r_t^n| \leq \frac{1}{2}$ one has $\frac{3}{2}\hat{\gamma}_t \geq \hat{\gamma}_t^n \geq \frac{1}{2}\hat{\gamma}_t$ and therefore

$$\tilde{H}^n_{\alpha,\lambda}(t) = \tilde{H}^n_{\alpha,\lambda}(t) \ \mathbb{1}_{[\frac{3}{2}\hat{\gamma}_t \geq \hat{\gamma}_t^n \geq \frac{1}{2}\hat{\gamma}_t]} \mathbb{1}_{Supp(\zeta_{x,f,L})} \left(\tilde{X}_{T-t}(z,\tilde{\omega}) \mathcal{B}_{z=X_t^n(\omega)} \right) \ .$$

We fix $\tilde{\omega}$. The inequality (47) (remember Proposition 4.7) leads to the following inequality, where the Sobolev norms are computed w.r.t. $\mathbb{P}$ on Ω:

$$\begin{aligned} \mathbb{E}_x \left| \tilde{H}^n_{\alpha,\lambda}(t,\tilde{\omega}) \right| \ \leq \ & C \left\| \Gamma_t^n(x) \ \mathbb{1}_{[\hat{\gamma}_t^n(x) \geq \frac{1}{2}\hat{\gamma}_t(x)]} \mathbb{1}_{Supp(\zeta_{x,f,L})} \left(\tilde{X}_{T-t}(z,\tilde{\omega}) \mathcal{B}_{z=X_t^n(x)} \right) \right\|_k \\ & \|X_t^n(x)\|_{N,m} \ \left\| H^n_{\alpha,\lambda}(t) \tilde{Q}^\lambda_{T-t}(z) \zeta_{x,f,L}(\tilde{X}_{T-t}(z,\tilde{\omega})) \mathcal{B}_{z=X_t^n(x)} \right\|_{N',m'} \end{aligned} \tag{54}$$

for some integers k, N, m, N', m'. We are now going to treat each term of the right side.

First, it is clear that

$$\left\| \Gamma_t^n(x) \ \mathbb{1}_{[\hat{\gamma}_t^n(x) \geq \frac{1}{2}\hat{\gamma}_t(x)]} \right\|_{2k} \leq K(T) \ .$$

Second, let us check that

$$\tilde{\mathbb{E}} \left\| \ \mathbb{1}_{Supp(\zeta_{x,f,L})} \left(\tilde{X}_{T-t}(z,\tilde{\omega}) \mathcal{B}_{z=X_t^n(x)} \right) \right\|_{2k} \leq K(T) \exp\left(-c\frac{d_f(x)^2}{T} \right) \ .$$

Indeed,

$$\begin{aligned} \tilde{\mathbb{E}} \left[\ \mathbb{1}_{Supp(\zeta_{x,f,L})} \left(\tilde{X}_{T-t}(z,\tilde{\omega}) \mathcal{B}_{z=X_t^n(x)} \right) \right] \ \leq \ & \mathbb{P} \otimes \tilde{\mathbb{P}} \left[\| \tilde{X}_{T-t}(z) \mathcal{B}_{z=X_t^n(x)} - x \| \geq \frac{d_f(x)}{2} \right] \\ \leq \ & \mathbb{P} \otimes \tilde{\mathbb{P}} \left[\| \tilde{X}_{T-t}(z) \mathcal{B}_{z=X_t^n(x)} - X_t^n(x) \| \geq \frac{d_f(x)}{4} \right] \\ & + \mathbb{P} \left[\| X_t^n(x) - x \| \geq \frac{d_f(x)}{4} \right] \ . \end{aligned}$$

Using (42) again, we get

$$\tilde{E}\left[\ 1\!\!1_{Supp(\zeta_{x,f,L})}\left(\tilde{X}_{T-t}(z,\tilde{\omega})\mathcal{B}_{z=X_t^n(x)}\right)\right] \leq K(T)\left[\exp\left(-c\frac{d_f(x)^2}{T}\right)+\exp\left(-c\frac{d_f(x)^2}{T-t}\right)\right] \ .$$

We now use the fact that $T \geq t \geq \frac{T}{2}$.

Next, proceeding as in the proof of Lemma 5.1 in [4] and using the additional hypothesis that $b(\cdot)$ and $\sigma(\cdot)$ are bounded, we get

$$\sup_{n\geq 1}\|X_t^n(x)\|_{N,m} < K(t) \ .$$

Obviously, from the above condition (b3) of the definition of $\zeta_{x,f,L}(\cdot)$ there holds (see the detailed arguments in the proof of Lemma 4.4):

$$\tilde{E}\left\|H^n_{\alpha,\lambda}(t)\tilde{Q}^{\lambda}_{T-t}(z)\zeta_{x,f,L}(\tilde{X}_{T-t}(z,\tilde{\omega}))\mathcal{B}_{z=X_t^n(x)}\right\|_{N',m'} \leq \frac{K(T)}{T^q}\exp\left(-c\frac{d_f(x)^2}{T}\right) \ .$$

Combining all the preceding remarks, we have got that

$$|B| \leq \| F \|_\infty \frac{K(T)}{T^q}\exp\left(-c\frac{d_f(x)^2}{T}\right) \ .$$

In conclusion, the preceding estimate, (50) and (52) prove that the inequality (36) holds for $\frac{T}{2} \leq t \leq T - \frac{T}{n}$. ∎

4.5 Proof of Proposition 4.2.

With $\phi(\cdot)$ and r_t^n defined as in the preceding proof, consider

$$\begin{aligned}
A^* &:= E_x\left[f(\tilde{X}_T^n)(1-\phi(r^n_{T-T/n}))\right] \ , \\
B^* &:= E_x\left[(u(T,\tilde{X}_T^n) - u(T-T/n, X^n_{T-T/n}+Z^n))\phi(r^n_{T-T/n})\right] \ , \\
C^* &:= E_x\left[(u(T-T/n, X^n_{T-T/n}+Z^n) - u(T-T/n, X^n_{T-T/n}))\phi(r^n_{T-T/n})\right] \ , \\
D^* &:= E_x\left[u(T-T/n, X^n_{T-T/n})(1-\phi(r^n_{T-T/n}))\right] \ .
\end{aligned}$$

Clearly, it is sufficient to prove that $|A^*|$, $|B^*|$, $|C^*|$, $|D^*|$ can be bounded from above by the right side of (40).

Let us start with A^*. We again consider the smooth function $\zeta_{x,f,L}(\cdot)$ of the preceding proof. We observe that, for a certain set Λ of multiindices of length smaller than d,

$$\begin{aligned}
A^* &= E_x\left[(1-\phi(r^n_{T-T/n}))\int f(X_T^n+z)\zeta_{x,f,L}(X_T^n+z)\rho_{1/n}(z)dz\right] \\
&= E_x\left[(1-\phi(r^n_{T-T/n}))\sum_{\lambda\in\Lambda}\int F(X_T^n+z)\partial_\lambda(\rho_{1/n}(z)\zeta_{x,f,L}(X_T^n+z))dz\right] \ .
\end{aligned}$$

Remembering that we have supposed $n > \frac{2}{d_f(x)}$, we deduce that, for some constant C linearly depending on the $L^2(I\!R^d)$-norm of the $\partial_\lambda \rho_0$'s, for some $q > 0$ and $Q > 0$ there holds

$$|A^*| \leq \frac{C}{n^q T^Q} \parallel F \parallel_\infty \sqrt{I\!E_x(1-\phi(r^n_{T-T/n}))^2} \sqrt{I\!P_x\left[d_f(X^n_T) \leq \frac{d_f(x)}{2} - \frac{1}{n}\right]} \ .$$

Observe that

$$\left[d_f(X^n_T) \leq \frac{d_f(x)}{2} - \frac{1}{n}\right] \subset \left[\|X^n_T - x\| \geq \frac{d_f(x)}{2}\right] \ .$$

We then conclude by applying the inequalities (42) and (51):

$$|A^*| \leq \frac{K(T)}{T^Q} \parallel F \parallel_\infty \exp\left(-c\frac{d_f(x)^2}{T}\right) \ .$$

Next, we observe that

$$B^* - \frac{T^2}{n^2} I\!E_x\left[\Psi(T - T/n, X^n_{T-T/n} + Z^n)\phi(r^n_{T-T/n})\right]$$

is a sum of terms of the type (32). We then apply the arguments used in the subsection 4.4.2, especially the integration by parts formula (49) with $G_0 = X^n_{T-T/n}(x)$. Of course, we also use the fact that Z^n is independent of the process (W_t).

For the term C^* we directly apply (49). For the term D^*, we use the same arguments as for A^*. ∎

5 An exponential bound for the local density q

The objective of this section is to prove Proposition 2.1. For convenience we recall it.

Proposition 5.1 *Assume (H). Let L be such that U_L is non void.*

(i) *Then there exists a smooth function*

$$(t, z, \xi) \in (0, T] \times I\!R^d \times U_L \to q_t(z, \xi)$$

such that, for any measurable and bounded function $\phi(\cdot)$ with a compact support included in U_L, one has

$$I\!E_z \phi(X_t) = \int_{Supp(\phi)} \phi(\xi) q_t(z, \xi) d\xi \ , \ \forall z \in I\!R^d \ . \qquad (55)$$

(ii) *Let $L \geq 1$, m be arbitrary integers and let α, β be multiindices. There exist positive constants μ, c and there exists an increasing function $K(\cdot)$ such that, for any $0 < t \leq T$,*

$$\forall y \in U_L \ , \ \forall z \in \mathbb{R}^d \ , \quad \left| \frac{\partial^m}{\partial t^m} \partial^z_\alpha \partial^y_\beta q_t(z,y) \right| \leq \frac{K(T)}{(tV_L(y))^\mu} \exp\left(-c \frac{\| z - y \|^2}{t} \right) \ . \quad (56)$$

Inequalities of this type are classical when the infinitesimal generator of (X_t) is strongly elliptic: for example, see Friedman [7]. We have not found (8) in the literature under our hypotheses. Observe that nevertheless it is a variation of (6): the roles of z and y are permuted.

To prove the result, we first note that the Fokker-Planck equation permits to only consider the case of spatial derivatives: from now on, we set $m = 0$.

Theorem 2.1 in Bally and Pardoux [1] provides a "localized" version of Malliavin's absolute continuity criterion and permits to construct a smooth function $q_t(z, \cdot)$ for each $(t, z) \in (0, T] \times \mathbb{R}^d$. Here we prove the differentiability with respect to all the variables.

Proof. *(i)* Consider the sequence of open sets

$$U^\epsilon_L := \{\xi \ ; \ V_L(\xi) > \epsilon\} \bigcap B\left(0, \frac{1}{\epsilon}\right) \ .$$

Let $\zeta^\epsilon(\cdot)$ be a smooth function with a compact support in U_L and such that $\zeta^\epsilon(\xi) = 1$ in U^ϵ_L. Consider the finite measure

$$d\mathbb{P}_{X_t(z)}(\xi)\zeta^\epsilon(\xi)dz \ .$$

Let $\psi(z, \xi)$ be a smooth function with a compact support in $\mathbb{R}^d \times \mathbb{R}^d$ and let α, β be multiindices. One has:

$$\begin{aligned}
&\left| \int \left(\partial^z_\alpha \partial^\xi_\beta \psi\right)(z,\xi)\zeta^\epsilon(\xi)d\mathbb{P}_{X_t(z)(\xi)}dz \right| \ = \\
&\qquad = \left| \mathbb{E} \int \left[\left(\partial^z_\alpha \partial^\xi_\beta \psi\right)(z, X_t(z))\zeta^\epsilon(X_t(z)) \right] dz \right| \\
&\qquad = \left| \mathbb{E} \int \left[\left(\partial^\xi_\beta \psi\right)(z, X_t(z))\partial^z_\alpha \left\{\zeta^\epsilon(X_t(z))\right\} \right] dz \right| \\
&\qquad = \left| \int \mathbb{E} \left[\left(\partial^\xi_\beta \psi\right)(z, X_t(z)) \sum_{|\gamma| \leq |\alpha|} \left(\partial_\gamma \zeta^\epsilon\right)(X_t(z))Q_\gamma(t,z) \right] \right|
\end{aligned}$$

for some polynomial functions $Q_\gamma(t, z)$ of the derivatives of the flow of $X^t(z)$. We now apply Proposition 4.7: for some positive constant C independent of $\psi(\cdot)$ it holds that

$$\left| \int \left(\partial^z_\alpha \partial^\xi_\beta \psi\right)(z,\xi)\zeta^\epsilon(\xi)d\mathbb{P}_{X_t(z)}(\xi)dz \right| \leq C\|\psi\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \ .$$

Thus, the measure $\zeta^\epsilon(\xi) d\mathbb{P}_{X_t(z)}(\xi) dz$ has a smooth density $q_t^\epsilon(z,\xi)$ with respect to Lebesgue's measure. Therefore, for any $\epsilon > 0$ and $t > 0$, for any smooth function $\psi^\epsilon(\cdot)$ with a compact support in U_L^ϵ, one has

$$\mathbb{E}_z \psi^\epsilon(X_t) = \int q_t^\epsilon(z,\xi)\psi^\epsilon(\xi) d\xi \ .$$

For $y \in U_L$ set $\epsilon(y) := \sup\{\epsilon; y \in U_L^\epsilon\}$. Now for $(z,y) \in \mathbb{R}^d \times U_L$ define

$$q_t(z,y) := q_t^{\epsilon(y)}(z,y) \ .$$

By construction this function is a smooth function and satisfies (55).

(ii) We now turn our attention to (56). Let $y \in U_L$.

Define $\rho_\delta(\cdot)$ as before and set $t > 0$. We consider the case $m = |\alpha| = 0$ and $|\beta| > 0$: one can treat the other cases with analoguous arguments (for $m > 0$ one must use the Fokker-Planck equation in addition).

As $q_t(\cdot,\cdot)$ is a smooth function,

$$\partial_\beta^y q_t(z,y) = \lim_{\delta \to 0} \int \rho_\delta(\xi) \partial_\beta^y q_t(z, y+\xi) d\xi \ .$$

Thus, integrating by parts we get

$$|\partial_\beta q_t(z,y)| = |\lim_{\delta \to 0} \mathbb{E}[(\partial_\beta^y \rho_{\delta,y,z})(X_t(z) - y)]| \ ,$$

where

$$\rho_{\delta,y,z}(\xi) := \prod_{i=1}^d \frac{1}{\delta} \left\{ \rho_0\left(\frac{\xi^i - y}{\delta}\right) \ \mathbb{1}_{[y^i - z^i > 0]} + \left(\rho_0\left(\frac{\xi^i - y}{\delta}\right) - 1\right) \ \mathbb{1}_{[y^i - z^i \leq 0]} \right\} \ .$$

Define

$$\begin{aligned} A_L(y) &= \left\{ \xi \ ; \ V_L(\xi) > \frac{1}{4} V_L(y) \right\} \ , \\ B_L(y) &= \left\{ \xi \ ; \ V_L(\xi) > \frac{1}{2} V_L(y) \right\} \ . \end{aligned}$$

Since the application $V_L(\cdot)$ is continuous, the closure of $B_L(y)$ is included in $A_L(y)$ and one may find a function $\zeta_{y,L}(\cdot)$ in $\mathcal{C}^\infty(\mathbb{R}^d)$ taking its value in $[0,1]$ such that

(a) $\zeta_{y,L}(z) = 0 \ if \ z \in A_L^c(y)$ or $d(z, B_L) \geq V_L(y)$,

(b) $\zeta_{y,L}(z) = 1 \ if \ z \in B_L(y)$,

(c) for any multiindex λ,

$$|\partial_\lambda^z \zeta_{y,L}(z)| \leq \frac{C_\lambda}{V_L(y)^{q_\lambda}}$$

where C_λ and q_λ are positive and uniform w.r.t $y \in U_L$ and L.

Observe that, for all δ small enough, for all $\xi \in I\!\!R^d$,

$$\partial^y_\beta \rho_{\delta,y,z}(\xi - y) = \partial_\beta \rho_{\delta,y,z}(\xi - y)\zeta_{y,L}(\xi) \ .$$

Consequently,

$$I\!\!E_z[(\partial_\beta \rho_{\delta,y,z})(X_t - y)] = I\!\!E_z[(\partial_\beta \rho_{\delta,y,z})(X_t - y)\zeta_{y,L}(X_t)] \ .$$

We again apply Proposition 4.7: the Malliavin integration by parts formula implies that the right side is equal to

$$I\!\!E_z[\Phi_{\delta,y,z}(X_t - y) H_\beta(X_t, \zeta_{y,L}(X_t))] \ ,$$

with

$$\Phi_{\delta,y,z}(\xi) := \int_0^{\xi_1} \dots \int_0^{\xi_d} \rho_{\delta,y,z}(\xi_1, \dots, \xi_d) d\xi_1 \dots d\xi_d \ .$$

Thus, making δ tend to 0, we get

$$\begin{aligned} \partial^y_\beta q_t(z,y) \ = \ I\!\!E_z \Bigg[H_\beta(X_t, \zeta_{y,L}(X_t)) \prod_{i=1}^d \Big\{ \ & 1\!\!1_{[0,+\infty)}(X^i_t - y^i) \ 1\!\!1_{[y^i - z^i > 0]} \\ & - 1\!\!1_{(-\infty,0]}(X^i_t - y^i) \ 1\!\!1_{[y^i - z^i \leq 0]} \Big\} \Bigg] \ . \end{aligned}$$

$H_\beta(X_t, \zeta_{y,L}(X_t))$ is a sum of terms, each one containing $\zeta_{y,L}(\cdot)$ or a derivative of $\zeta_{y,L}(\cdot)$, functions which vanish on the set $A_L(y)^c$, so that

$$H_\beta(X_t, \zeta_{y,L}(X_t)) = H_\beta(X_t, \zeta_{y,L}(X_t)) \ 1\!\!1_{A_L(y)}(X_t)$$

whence

$$\begin{aligned} |\partial^y_\beta q_t(z,y)| \ \leq \ & \sqrt{I\!\!E_z \left[H_\beta(X_t, \zeta_{y,L}(X_t))^2 \ 1\!\!1_{A_L(y)}(X_t) \right]} \\ & \sqrt{I\!\!E_z \prod_{i=1}^d \Big[\ 1\!\!1_{[0,+\infty)}(X^i_t - y^i) \ 1\!\!1_{[y^i - z^i > 0]} + \ 1\!\!1_{(-\infty,0]}(X^i_t - y^i) \ 1\!\!1_{[y^i - z^i \leq 0]} \Big]} \ . \end{aligned}$$

Using the inequality (42) and standard computations (see [10] or [4]) one gets

$$|\partial^y_\beta q_t(z,y)| \leq K(T) \parallel \ 1\!\!1_{A_L(y)}(X_t(z)) \hat{\Gamma}_t(z) \parallel_k \sum_{\lambda \in \Lambda} \|\partial_\lambda \zeta_{y,L}(X_t(z))\|_{N,m} \exp\left(-c \frac{\parallel z - y \parallel^2}{t} \right)$$

for some integers k, N and m, and a finite set Λ of multiindices.

We now use the condition (c) of the definition of $\zeta_{y,L}(\cdot)$. To conclude, it remains to apply the inequality (58) below with $A = A_L(y)$ using the fact that $V_L(A) = V_L(A_L(y)) \geq \frac{1}{4} V_L(y) > 0$. We then get (56). $\blacksquare$

6 An L^p-estimate for the inverse Malliavin covariance matrix

The aim of this section is to prove the following

Proposition 6.1 *Assume (H).*

Fix two arbitrary integers p and L in $\mathbb{N}-\{0\}$. Then there exists a positive constant μ and an increasing function $K_{L,p}(\cdot)$ both depending on p and L which satisfy the following: for any Borel set A, if

$$V_L(A) := \inf_{\xi\in A} V_L(\xi) > 0 \tag{57}$$

then, for all $0 < t \le T$, for all $z \in \mathbb{R}^d$,

$$\| \ \mathbb{1}_A(X_t(z))\hat{\gamma}_t(z)^{-1} \ \|_p \le \frac{K_{L,p}(t)}{(tV_L(A))^\mu} \ . \tag{58}$$

Proof. First we show that it is sufficient to prove that, for all $p \ge 1$ there exist strictly positive constants μ, Q, Q' and an increasing function $K_{L,p}(\cdot)$ such that, for all $z \in \mathbb{R}^d$, for all $T \ge t > 0$, for all $0 < \varepsilon < \min(t^{-Q}, t^{Q'})$, for all Borel set A for which (57) holds,

$$\mathbb{P}_z[\ \mathbb{1}_A(X_t)|\hat{\gamma}_t| \le \varepsilon\ ,\ X_t\in A] \le \frac{K_{L,p}(t)}{(tV_L(A))^\mu}\varepsilon^{p+2} \ . \tag{59}$$

Indeed, we would then have

$$\begin{aligned}
\mathbb{E}_z\Big[\ \mathbb{1}_A(X_t)|\hat{\gamma}_t^{-1}|^p\Big] &= \sum_{k=0}^{\infty}\mathbb{E}_z\Big[\ \mathbb{1}_A(X_t)|\hat{\gamma}_t^{-1}|^p\ \mathbb{1}_{[k\le\ \mathbb{1}_A(X_t)|\hat{\gamma}_t^{-1}|\le k+1]}\ \mathbb{1}_{[X_t\in A]}\Big] \\
&\le 1+\sum_{k=1}^{\infty}(k+1)^p\mathbb{P}_z\Big(\Big[\ \mathbb{1}_A(X_t)|\hat{\gamma}_t|\le\frac{1}{k}\Big]\bigcap[X_t\in A]\Big) \\
&\le \frac{K_{L,p}(t)}{t^{Q''}}+\frac{K_{L,p}(t)}{t^\mu V_L(A)^\mu}\sum_{k\ge1}^{\infty}\frac{(k+1)^p}{k^{p+2}} \ .
\end{aligned}$$

Thus, as by definition $0 < V_L(A) \le 1$, we would have obtained (58) (with a new function $K_{L,p}(\cdot)$ and a possibly new constant μ).

Thus, we are going to prove (59).

We start with some localizations.

Under (H), the function $V_L(\cdot)$ is uniformly Lipschitz. Thus, there exists a constant δ_L depending only on L, $b(\cdot)$ and $\sigma(\cdot)$ such that

$$d(z,A) \le \delta_L V_L(A) \Rightarrow V_L(z) \ge \frac{V_L(A)}{2} \tag{60}$$

Set

$$\gamma := \frac{2}{2L+1} \tag{61}$$

and

$$B_\varepsilon := [\| X_t - X_{t-\varepsilon^\gamma t} \| \leq \delta_L V_L(A)] \ . \tag{62}$$

A Girsanov transformation and Tschebycheff's inequality show that under (H), for any $q \geq 1$ and for some increasing function $K_{L,q}(\cdot)$ uniform w.r.t. A,

$$\mathbb{P}\left(\Omega - B_\varepsilon\right) \leq \frac{K_{L,q}(t)\varepsilon^{\gamma q}}{V_L(A)^{2q}} \ . \tag{63}$$

We introduce another set of localization. Again denote by $Y_t(z)$ the matrix $\partial^z X_t(z)$ and denote by $Z_t(z)$ the inverse matrix of $Y_t(z)$. Set

$$C_\varepsilon := \left[\inf_{\|\xi\|=1} \| Z^*_{t-\varepsilon^\gamma t}\xi \| \geq 1 \right] \ . \tag{64}$$

An easy algebraic computation shows that

$$\inf_{\|\xi\|=1} \| Z^*_{t-\varepsilon^\gamma t}\xi \| = \frac{1}{\sup_{\|\xi\|=1} \| Y_{t-\varepsilon^\gamma t}\xi \|} \ ,$$

so that for any $q \geq 1$ and for some increasing function $K_q(\cdot)$ only depending on $b(\cdot)$, $\sigma(\cdot)$ and q,

$$\mathbb{P}(\Omega - C_\varepsilon) \leq \mathbb{E}\left[\sup_{\|\xi\|=1} \|Y_{t-\varepsilon^\gamma t}\xi\| \right]^{2q} \leq K_q(t)\varepsilon^{\gamma q} \ . \tag{65}$$

Thus, (63), (65) and $0 < V_L(A) < 1$ reduce the proof of (59) to the proof of

$$\mathbb{P}_z \left[\, \mathbb{1}_A(X_t)|\hat{\gamma}_t| \leq \varepsilon \ ; \ [X_t \in A] \bigcap C_\varepsilon \bigcap B_\varepsilon \right] \leq \frac{K_{L,p}(t)}{t^\mu V_L(A)^\mu}\varepsilon^{p+2} \ . \tag{66}$$

Instead of keeping γ_t, we will use a new matrix τ_t which we now define. Set

$$\tau_t := \sum_{i=1}^d \int_0^t (Z_s\sigma^i(X_s))(Z_s\sigma^i(X_s))^* ds \ .$$

Observe that, by the variation of constants formula, one has

$$\gamma_t = Y_t\tau_t Y^*_t \ .$$

Since, for all $q \geq 1$, there exists an increasing function $K_q(\cdot)$ such that

$$\| \, (\hat{Y}_t)^{-1} \, \|_q < K_q(t) \ ,$$

it is sufficient to prove (66) with τ_t instead of γ_t.

For any symmetric nonnegative definite $d \times d$ matrix M, one has:

$$det(M)^{1/d} \geq \inf_{\|\xi\|=1} < M\xi, \xi > \quad .$$

Therefore (observe that $0 < \varepsilon \leq 1$ by definition),

$$\begin{aligned} \hat{\tau}_t^{1/d} &\geq \sum_{i=1}^{d} \inf_{\|\xi\|=1} \int_0^t \xi^*(Z_s\sigma^i(X_s))(Z_s\sigma^i(X_s))^*\xi ds \\ &\geq \sum_{i=1}^{d} \inf_{\|\xi\|=1} \int_{t-\varepsilon^\gamma t}^t \xi^*(Z_s\sigma^i(X_s))(Z_s\sigma^i(X_s))^*\xi ds \ . \end{aligned}$$

Observe that, for $t - \varepsilon^\gamma t \leq s \leq t$ the inverse matrix of $Y_s(x)$, $Z_s(x)$, satifies:

$$Z_s(x) = Z_{t-\varepsilon^\gamma t}(x) Z_{s-(t-\varepsilon^\gamma t)}(X_{t-\varepsilon^\gamma t}(x)) \ ,$$

so that

$$\hat{\tau}_t^{1/d} \geq \sum_{i=1}^{d} \inf_{\|\xi\|=1} \int_{t-\varepsilon^\gamma t}^t < \xi^* Z_{t-\varepsilon^\gamma t}, Z_{s-(t-\varepsilon^\gamma t)}(y)\sigma^i(X_{s-(t-\varepsilon^\gamma t)}(y)) >^2 \Big|_{y=X_{t-\varepsilon^\gamma t}} ds \ .$$

On C_ϵ, for $\|\xi\| = 1$ one has $\|\xi^* Z_{t-\varepsilon^\gamma t}\| \geq 1$. Thus, on C_ϵ,

$$\begin{aligned} \hat{\tau}_t^{1/d} &\geq \sum_{i=1}^{d} \inf_{\|\xi\|\geq 1} \int_{t-\varepsilon^\gamma t}^t < \xi, Z_{s-(t-\varepsilon^\gamma t)}(y)\sigma^i(X_{s-(t-\varepsilon^\gamma t)}(y)) >^2 \Big|_{y=X_{t-\varepsilon^\gamma t}} ds \\ &= \sum_{i=1}^{d} \inf_{\|\xi\|=1} \int_0^{\varepsilon^\gamma t} < \xi, Z_s(y)\sigma^i(X_s(y)) >^2 \ \Big|_{y=X_{t-\varepsilon^\gamma t}} ds \\ &=: G_{t,\varepsilon}^{1/d} \ . \end{aligned}$$

Therefore, a sufficient condition for (66) is

$$I\!P_z \left[G_{t,\varepsilon} \leq \varepsilon \ ; \ [X_t \in A] \bigcap B_\varepsilon \right] \leq \frac{K_{L,p}(t)}{t^\mu V_L(A)^\mu} \varepsilon^{p+2} \ . \tag{67}$$

Let us now turn our attention to the localization in B_ε. On $B_\varepsilon \bigcap [X_t \in A]$, one has

$$V_L(X_{t-\varepsilon^\gamma t}) \geq \frac{V_L(A)}{2} \ .$$

Thus, a new sufficient condition for (66) is

$$\begin{aligned} \int I\!P_y \left[\sum_{i=1}^{d} \inf_{\|\xi\|=1} \int_0^{\varepsilon^\gamma t} < \xi, Z_s\sigma^i(X_s) >^2 ds \leq \varepsilon \right] \ \mathbb{1}_{[V_L(y) \geq \frac{V_L(A)}{2}]} dI\!P_{X_{t-\varepsilon^\gamma t}(z)}(y) \\ \leq \frac{K_{L,p}(t)}{t^\mu V_L(A)^\mu} \varepsilon^{p+2} \ . \end{aligned}$$

We finally observe that the preceding inequality follows from the estimate (68) in the theorem below (with $\tau = t^{L+1}\varepsilon^{1/(2L+1)}$, $\lambda = t^{L(L+1)}\varepsilon^{-(L+1)/(2L+1)}$ and $\varepsilon \leq \min(t^{L(L+1)}, t^{-(L+1)(2L+1)})$), γ being defined as in (61), which ends the proof. ∎

We have just referred to the following theorem (Theorem 2.17 in [10]):

Theorem 6.2 (Kusuoka-Stroock) *Assume (H). There exists a positive constant C depending only on $b(\cdot)$ and $\sigma(\cdot)$ and for any $L \geq 1$ there exist positive constants μ_L, C_L such that, for any $0 < \tau < 1$, for any $\lambda \geq 1$,*

$$\mathbb{P}_y\left[\sum_{i=1}^{d} \inf_{\|\xi\|=1} \int_0^{\tau/\lambda^{1/(L+1)}} < \xi, Z_s\sigma^i(X_s) >^2 \, ds \leq \frac{\tau^L}{\lambda}\right] \leq C_L \exp\left(-CV_L(y)^{(L+2)\mu_L}\lambda^{\mu_L}\right) . \tag{68}$$

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