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A SYMMETRIZED EULER SCHEME FOR AN EFFICIENT APPROXIMATION OF REFLECTED DIFFUSIONS

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Abstract

In this article, we analyse the error induced by the Euler scheme combined with a symmetry procedure near the boundary for the simulation of diffusion processes with an oblique reflection on a smooth boundary. This procedure is easy to implement and, in addition, accurate: indeed, we prove that it yields a weak rate of convergence of order 1 with respect to the time-discretization step.

Keywords: Reflected diffusion; discretization scheme; weak approximation; Neumann boundary condition for parabolic PDE

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1. Introduction

1.1. Applications

In [2], several examples of models involving (possibly controlled) reflected diffusion processes can be found. The simulation of such diffusion processes, or in practice of approximate processes, allow the computation by Monte Carlo methods of statistics of constrained stochastic dynamics. In addition, the Feynman–Kac formulae in [2, Chapter 2, Section 4] justify the use of Monte Carlo methods, based upon the simulation of independent trajectories of reflected diffusion processes, to solve parabolic partial differential equations with general Neumann boundary conditions.

Let us now describe another example where the simulation of reflected diffusion processes is suitable. It concerns the identification of electrical and magnetic parameters at the surface of human brains. The data consist of some measurements obtained from sensors located on the patient's head. Faugeras *et al.* [6] (see also [3], [4], [12]) have developed a nice approach based on Maxwell equations. The identification of the parameters results is an inverse problem for these equations; the complex numerical procedure involves a kind of iterative gradient descent method which, at each step, requires the numerical resolution of a Poisson equation with conormal Neumann boundary conditions and a source term depending on numerical values obtained at the preceding step: the algorithm stops when the solution of the Poisson equations takes values at the locations of the sensors which are close enough to the experimental data.

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A deterministic numerical method for the Poisson equations computes the solutions at all the points of the domain of integration. Therefore, it seems worth elaborating a stochastic method allowing us to compute the solution at a few points only, namely the locations of the sensors. The construction and the error analysis of such a method lead to several technical difficulties. First, the elliptic operator under consideration is of the form

$$Lu(x) = \operatorname{div}(a(x)\nabla u(x)) \tag{1}$$

with conormal Neumann boundary conditions, where a is a discontinuous function; second, we have to approximate quantities of the form

$$\int_0^\infty \mathrm{E}_x\left[f(X_t) - \int f(\xi)\mu(\mathrm{d}\xi)\right]\mathrm{d}t,$$

where (X_t) is a Markov process with generator L and invariant measure μ. The boundary conditions of L imply that (X_t) needs to be reflected at the boundary (that is, the brain surface). Currently, whereas the approximation of diffusions in the whole Euclidian space, or of stopped diffusions, is well understood, many aspects of the approximation of reflected diffusion processes are still open questions (see our list of references and our comments below). Our objective here is to address the following aspect: the construction of a reflected Euler scheme which has a first-order convergence rate for the approximation of $\mathrm{E}_x(f(X_T))$ with a fixed horizon T, even in the case of a nonnormal reflection. We emphasize that we suppose here that the coefficients of the generator of the diffusion process under consideration are smooth, and that we do not study the long-time behaviour of our reflected Euler scheme. Thus, our results apply to the Monte Carlo methods for parabolic equations, but must be seen only as a first step to developing and analysing stochastic numerical methods for elliptic equations such as (1) with a discontinuous coefficient $a(x)$. Such questions are being investigated by M. Martinez (private communication, 2004).

1.2. Background results on discretization schemes

The numerical resolution by deterministic methods of second-order partial differential equations (PDEs) becomes inefficient in high dimensions. An alternative approach consists in developing Monte Carlo methods from the probabilistic representations of the solutions as expectations of functionals of diffusion processes $X = (X_t)_{t\geq 0}$. Usually, exact simulations of X are impossible and time-discretization procedures are needed.

A lot of attention has been paid to PDEs in the whole space. In that case, the process to simulate is the solution in the whole space of

$$X_t = x + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s,$$

where W is a standard multidimensional Brownian motion. Optimal convergence rates are now well established. For example, consider the Euler scheme with time-step $h = T/N$ ($t_i = ih$ being the discretization times of $[0, T]$):

$$X^N_{t_{i+1}} = X^N_{t_i} + b(X^N_{t_i})h + \sigma(X^N_{t_i})(W_{t_{i+1}} - W_{t_i}).$$

The weak error $\mathrm{E}(f(X_T)) - \mathrm{E}(f(X^N_T))$ can be expanded in terms of powers of h, given some regularity conditions on f (see [25]) or some nondegeneracy condition on the process X (i.e. hypo-ellipticity; see [1]).

If the PDE has a Dirichlet condition on the boundary ∂D of a domain D, then the diffusion process X needs to be killed or stopped when it hits ∂D. In that situation, if we naively kill or stop the Euler scheme, then the weak convergence rate is of order $\frac{1}{2}$. However, an efficient killing [8], [10] or stopping [19] procedure can be developed leading to a convergence rate of order 1.

For PDEs with Neumann boundary condition on ∂D, X needs to be a diffusion process with reflection on ∂D in some oblique direction γ, i.e. a solution of

$$X_t = x + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + \int_0^t \gamma(X_s)\,\mathrm{d}k_s,$$

where the so-called local time k_t is increasing only when X_t is on ∂D. In this article, we focus in the evaluation of quantities like $\mathrm{E}(f(X_T))$ for a fixed time T.

From the numerical point of view, using a regular mesh of the interval $[0, T]$ with time-step h, we can use an Euler scheme with projection, for which the weak error is of order $\frac{1}{2}$ as established in [5] for normal reflections $\gamma = n$ (see also [24], [20]). We can also use a penalty method: the convergence has been studied only in the $\mathcal{L}_p$ sense (see e.g. [18], [21], [11]). For a more complete presentation of these methods, see [10]. More recently, in [10] the second author combined Lépingle's procedure [14], [15] (which is exact when D is a half-space and the coefficients are constant) and a certain local half-space approximation to construct implementable procedures which are of order 1 under the condition that γ lies in the conormal direction: $\gamma(s)$ is parallel to $\sigma\sigma^\top(s)n(s)$ for any $s \in \partial D$, where $n(s)$ is the unit inward normal vector at s.

Hence, so far, the question of getting an easily implementable procedure providing a first-order convergence is still open for general oblique reflection problems. Our so-called *symmetrized Euler scheme* below solves this issue. Results in this article were presented at the conference on Monte Carlo and probabilistic methods for partial differential equations held at Monte Carlo, Monaco, in July 2000 and announced in [9]. This symmetrized scheme has been recently studied in [17], where the convergence is not analysed in detail.

1.3. Outline of the paper and notation

In Section 2, we set some preliminary geometry notation, state our assumptions and define the symmetrized Euler scheme. Then we state our main convergence result, Theorem 1. Section 3 is devoted to the proof of this result. In Section 4, a numerical example is considered which illustrates the efficiency of our algorithm.

We adopt the following usual convention on gradients: if $\psi : \mathbb{R}^{p_2} \to \mathbb{R}^{p_1}$ is a differentiable function, then its gradient

$$\nabla\psi(x) = (\partial_{x_1}\psi(x), \ldots, \partial_{x_{p_2}}\psi(x))$$

takes values in $\mathbb{R}^{p_1} \otimes \mathbb{R}^{p_2}$. In particular, the gradient of a linear function ψ is a *row* vector. Its Hessian matrix is denoted by H^ψ. Usually, the gradient is computed with respect to the space variables only. We denote the $d \times d$ identity matrix by $I_{\mathbb{R}^d \otimes \mathbb{R}^d}$, and the trace of a matrix A by $\mathrm{Tr}(A)$.

We use the generic notation $K(T)$ for all finite, nonnegative and nondecreasing functions: they are independent of x, the function f and the discretization step h, but they may depend on the coefficients b, σ, γ and on the domain D.

A quantity R is said to be equal to $O_{\exp}(h)$ if $|R| \leq K(T)\exp(-c/h)$ for some constants $K(T)$ and $c > 0$.

The conditional expectation $\mathrm{E}(Z \mid \mathcal{F}_{t_i})$ is denoted by $\mathrm{E}^{\mathcal{F}_{t_i}}(Z)$.

2. Assumptions and the main result

2.1. Assumptions

In the sequel, we consider a domain $D \subset \mathbb{R}^d$ with the following smoothness property.

Assumption 1. *The boundary ∂D is bounded and of class* C^5.

For $R > 0$, the set of points in the R-neighbourhood of ∂D is denoted by $V_{\partial D}(R) = \{x : d(x, \partial D) \leq R\}$. The vector field defining the reflection direction is uniformly nontangent to the boundary.

Assumption 2. *The unit vector field γ is of class C^4 and there exists a $\rho_0 > 0$ such that $\gamma(s) \cdot n(s) \geq \rho_0$ for all $s \in \partial D$.*

We recall some classical results concerning the distance to the boundary in the γ direction (see the appendix in [10]).

Proposition 1. *Let Assumptions 1 and 2 hold. There exists a constant $R > 0$ such that:*

(i) *For any $x \in V_{\partial D}(R)$, there exist a unique $s = \pi^{\gamma}_{\partial D}(x) \in \partial D$ and a unique $F^{\gamma}(x) \in \mathbb{R}$ such that*
$$x = \pi^{\gamma}_{\partial D}(x) + F^{\gamma}(x)\gamma(\pi^{\gamma}_{\partial D}(x)).$$

(ii) *The projection of x onto ∂D parallel to γ, that is, the function $x \mapsto \pi^{\gamma}_{\partial D}(x)$, is of class C^4 on $V_{\partial D}(R)$.*

(iii) *The algebraic distance of x to ∂D parallel to γ, that is, the function $x \mapsto F^{\gamma}(x)$, is of class C^4 on $V_{\partial D}(R)$. We have $F^{\gamma} > 0$ on $V_{\partial D}(R) \cap D$, $F^{\gamma} < 0$ on $V_{\partial D}(R) \cap \bar{D}^{\mathrm{c}}$ and $F^{\gamma} = 0$ on ∂D: we can extend F^{γ} to a $\mathrm{C}^4_{\mathrm{b}}(\mathbb{R}^d, \mathbb{R})$ function, with the conditions $F^{\gamma} > 0$ on D and $F^{\gamma} < 0$ on $\bar{D}^{\mathrm{c}}$.*

(iv) *The above extensions for F^{γ} and F^n can be performed in a way such that the functions F^{γ} and F^n are equivalent in the sense that*
$$\frac{1}{c_1}|F^n(x)| \leq |F^{\gamma}(x)| \leq c_1|F^n(x)| \quad \text{for all } x \in \mathbb{R}^d$$
for some constant $c_1 > 1$.

(v) *For $x \in \partial D$,*
$$\nabla F^{\gamma}(x) = \frac{n^{\top}}{n \cdot \gamma}(x).$$

We sometimes use the notation $n(x)$ or $\gamma(x)$ even if $x \notin \partial D$: for $x \in V_{\partial D}(R)$, we set $n(x) = n(\pi^{\gamma}_{\partial D}(x))$ and $\gamma(x) = \gamma(\pi^{\gamma}_{\partial D}(x))$ and, for $x \notin V_{\partial D}(R)$, arbitrary values are given.

The coefficients of the equation (2) are supposed to satisfy the following assumption.

Assumption 3. *The functions b and σ are $\mathrm{C}^4_{\mathrm{b}}(\bar{D}, \mathbb{R}^d)$ and $\mathrm{C}^4_{\mathrm{b}}(\bar{D}, \mathbb{R}^d \otimes \mathbb{R}^d)$ functions.*

Given a d-dimensional Brownian motion $(W_t)_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$ (satisfying the usual conditions), it is known (see [23], [16]) that under Assumptions 1, 2 and 3 there is an unique strong solution to

$$X_t = x + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s + \int_0^t \gamma(X_s)\,\mathrm{d}k_s, \tag{2}$$

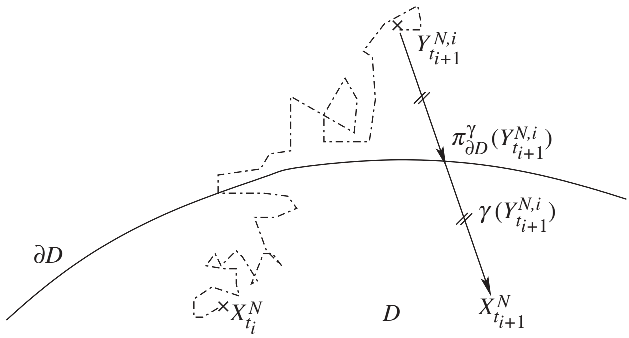

FIGURE 1: Description of the algorithm when $Y^{N,i}_{t_{i+1}}$ is outside D.

where k_t is a process increasing only when X_t is on ∂D: $k_t = \int_0^t \mathbf{1}_{\{X_s \in \partial D\}}\, \mathrm{d}k_s$. The initial value $x \in D$ is fixed in the sequel.

We also need the following nondegeneracy assumption.

Assumption 4. *The matrix $\sigma\sigma^\top$ is uniformly elliptic: for all $x \in \bar{D}$,*

$$\sigma\sigma^\top(x) \geq \sigma_0^2 I_{\mathbb{R}^d \otimes \mathbb{R}^d} \quad \textit{for some } \sigma_0 > 0.$$

2.2. The algorithm

We start with $X_0^N = x$ and assume that we have obtained $X_{t_i}^N \in \bar{D}$.

(a) For $t \in [t_i, t_{i+1}]$, we set

$$Y_t^{N,i} := X_{t_i}^N + b(X_{t_i}^N)(t - t_i) + \sigma(X_{t_i}^N)(W_t - W_{t_i}).$$

Observe that $Y^{N,i}_{t_{i+1}}$ is simulated by simply drawing d independent Gaussian variables.

(b) Then,

(i) if $Y^{N,i}_{t_{i+1}} \notin \bar{D}$ (i.e. if $F^\gamma(Y^{N,i}_{t_{i+1}}) < 0$), we set

$$X^N_{t_{i+1}} = \pi^\gamma_{\partial D}(Y^{N,i}_{t_{i+1}}) - F^\gamma(Y^{N,i}_{t_{i+1}})\gamma(Y^{N,i}_{t_{i+1}})$$

which coincides with the symmetric point of $Y^{N,i}_{t_{i+1}}$ with respect to $\pi^\gamma_{\partial D}(Y^{N,i}_{t_{i+1}})$ (see Figure 1);

(ii) if $Y^{N,i}_{t_{i+1}} \in \bar{D}$ (i.e. if $F^\gamma(Y^{N,i}_{t_{i+1}}) \geq 0$), we simply set

$$X^N_{t_{i+1}} = Y^{N,i}_{t_{i+1}}.$$

To sum up (i) and (ii), we have

$$X^N_{t_{i+1}} = Y^{N,i}_{t_{i+1}} + 2[F^\gamma(Y^{N,i}_{t_{i+1}})]^- \gamma(Y^{N,i}_{t_{i+1}}).$$

(c) It is possible that $Y^{N,i}_{t_{i+1}} \notin D \cup V_{\partial D}(R)$, that is, a huge increment has occurred: this event has a probability exponentially small with respect to h (see below) and, in that case, we suggest the simulation of $Y^{N,i}_{t_{i+1}}$ be restarted.

This way to proceed using a symmetry is actually very natural: indeed, in dimension 1, we know by Lévy's identity (see Section VI.2 of [22]) that the Brownian motion reflected on the positive axis has the same law as the absolute value of the standard Brownian motion. In more general situations, an analogous procedure is used by Freidlin [7] to prove the existence of a solution to (2).

2.3. Rate of convergence

We denote by L the infinitesimal generator of $(X_t)_{t\geq 0}$, that is,

$$Lu = \nabla u b + \tfrac{1}{2}\,\mathrm{Tr}(H^u a)$$

(with $a = \sigma\sigma^\top$).

We suppose that the class of test functions f used to describe the error's scheme, in relation to the partial differential equation (3) below, satisfies the following assumption.

Assumption 5. *The function f is of class $\mathrm{C}^5_{\mathrm{b}}(\bar{D}, \mathbb{R})$ and satisfies the compatibility condition on ∂D: for all $z \in \partial D$, $[\nabla f \gamma](z) = [\nabla(Lf)\gamma](z) = 0$.*

For $f \in \mathrm{C}^5_{\mathrm{b}}(\bar{D}, \mathbb{R})$, we set

$$\|f\|^{(5)} = \sum_{\alpha:|\alpha|\leq 5} \|\partial_x^\alpha f\|_\infty.$$

Our main result is the following.

Theorem 1. *Under Assumptions 1–5,*

$$|\,\mathrm{E}(f(X_T^N)) - \mathrm{E}(f(X_T))| \leq K(T)\|f\|^{(5)} h$$

for some constant $K(T)$ uniform in x and f.

The rest of the paper is devoted to the proof of this theorem.

3. Proofs

We follow the usual trick consisting of decomposing the error into a sum of local errors using an appropriate PDE. For this, we consider a smooth solution of the following PDE (see [13, Theorem 5.3, p. 320]), with Neumann boundary condition:

$$\begin{aligned} (\partial_t u + Lu)(t,x) &= 0 && \text{for } (t,x) \in [0,T]\times \bar{D},\\ \nabla u(t,x)\gamma(x) &= 0 && \text{for } (t,x) \in [0,T]\times \partial D,\\ u(T,x) &= f(x) && \text{for } x \in D. \end{aligned} \tag{3}$$

Under the assumptions of Theorem 1, the solution u is at least of class $\mathrm{C}^{2,4}([0,T]\times\bar{D})$ with uniformly bounded derivatives (the compatibility condition Assumption 5 is crucial for this): namely, for $2p + |\alpha| \leq 4$,

$$|\partial_t^p \partial_x^\alpha u(t,x)| \leq K(T)\|f\|^{(5)} \quad \text{for all } (t,x)| \in [0,T]\times \bar{D} \tag{4}$$

(see [13, Theorem 5.3, p. 320]). Then we can easily show that $u(t,x) = \mathrm{E}[f(X_{T-t}) \mid X_0 = x]$. We extend u to a $\mathrm{C}^{2,4}([0,T]\times\mathbb{R}^d)$ function (see [13]) which still satisfies the estimates (4).

We introduce a continuous-time version of the symmetrized Euler scheme by setting $X_t^N = Y_t^{N,i} + 2[F^\gamma(Y_t^{N,i})]^- \gamma(Y_t^{N,i})$ for $t \in [t_i, t_{i+1}[$. Define

$$\tau = \inf\{t \geq 0 : Y_t^{N,i} \notin D \cup V_{\partial D}(R) \text{ with } t_i \leq t \leq t_{i+1}\}.$$

On the event $\{\tau > T\}$, $(X_t^N)_{0 \leq t \leq T}$ lives in D. In addition,

$$\mathrm{P}[\tau \leq T] = O_{\exp}(h), \tag{5}$$

which is a straightforward consequence of the following standard lemma.

Lemma 1. *Consider an Itô process with uniformly bounded coefficients:* $\mathrm{d}U_t = b_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t$. *There exist some constants $c > 0$ and $K(T)$ (depending on $p \geq 1$) such that, for any stopping times S and S' (with $0 \leq S \leq S' \leq \delta \leq T$) and any $\eta \geq 0$,*

$$\mathrm{P}\Big[\sup_{t \in [S, S']} \|U_t - U_S\| \geq \eta\Big] \leq K(T) \exp\Big(-c\frac{\eta^2}{\delta}\Big), \tag{6}$$

$$\mathrm{E}\Big[\sup_{t \in [S, S']} \|U_t - U_S\|^p\Big] \leq K(T) \delta^{p/2}. \tag{7}$$

The first estimate is based on Bernstein's inequality for martingales (see e.g. Lemma 4.1 of [8]), and the second follows from the Burkholder–Davis–Gundy inequalities.

Now, set

$$\begin{aligned}\mathcal{E}_i &:= \mathrm{E}(u(t_{i+1} \wedge \tau, X^N_{t_{i+1}\wedge\tau}) - u(t_i \wedge \tau, X^N_{t_i \wedge \tau}))\\ &= \mathrm{E}(\mathbf{1}_{\{t_i < \tau\}}\, \mathrm{E}^{\mathcal{F}_{t_i}}[u(t_{i+1} \wedge \tau, X^N_{t_{i+1}\wedge\tau}) - u(t_i, X^N_{t_i})]).\end{aligned}$$

In view of (3) and (5), the weak error can be decomposed as follows:

$$\begin{aligned}\mathrm{E}(f(X_T^N)) - \mathrm{E}(f(X_T)) &= \mathrm{E}\big(u(T, X_T^N) - u(T \wedge \tau, X^N_{T\wedge\tau}) + u(T \wedge \tau, X^N_{T\wedge\tau}) - u(0, X_0^N)\big)\\ &= \|f\|_\infty O_{\exp}(h) + \sum_{i=0}^{N-1} \mathcal{E}_i.\end{aligned}$$

We then need the following two crucial results, which we prove later.

Lemma 2. *Under Assumptions 1–4, for all $c > 0$,*

$$h\,\mathrm{E}\bigg(\sum_{i=0}^{N-1} \mathbf{1}_{\{t_i < \tau\}} \exp\bigg(-c\frac{d^2(X^N_{t_i}, \partial D)}{h}\bigg)\bigg) \leq K(T)\sqrt{h}.$$

Lemma 3. *Under Assumptions 1–5, for all $x \in \partial D$,*

$$\mathcal{C}^u(x) := \bigg(-\nabla u \nabla\gamma a \frac{n}{n \cdot \gamma} + \gamma^\top H^u \gamma \frac{n^\top a n}{(n \cdot \gamma)^2} - \frac{n^\top a H^u \gamma}{n \cdot \gamma}\bigg)(x) = 0.$$

In view of Tanaka's formula [22], $(X_t^N)_{0 \leq t \leq T}$, defined as in the step (b) of the algorithm, is a continuous semimartingale for $t \in [t_i, t_{i+1}[$. Easy computations lead to

$$\begin{aligned}\mathrm{d}X_t^N = {} & \mathrm{d}Y_t^{N,i} + \gamma(Y_t^{N,i})\,\mathrm{d}L_t^0(F^\gamma(Y_\cdot^{N,i}))\\ & + [F^\gamma(Y_t^{N,i})]^- \{2\nabla\gamma(Y_t^{N,i})\,\mathrm{d}Y_t^{N,i} + \mathrm{Tr}[H^\gamma(Y_t^{N,i}) a(X^N_{t_i})]\,\mathrm{d}t\}\\ & - \mathbf{1}_{\{Y_t^{N,i} \notin D\}} \big\{2\nabla\gamma(Y_t^{N,i}) a(X^N_{t_i})[\nabla F(Y_t^{N,i})]^\top\,\mathrm{d}t + 2\gamma(Y_t^{N,i})\nabla F^\gamma(Y_t^{N,i})\,\mathrm{d}Y_t^{N,i}\\ & \qquad + \gamma(Y_t^{N,i})\,\mathrm{Tr}[H^{F^\gamma}(Y_t^{N,i}) a(X^N_{t_i})]\,\mathrm{d}t\big\}\end{aligned} \tag{8}$$

since $\{F^\gamma(Y_t^{N,i}) \le 0\} = \{Y_t^{N,i} \notin D\}$; here we have denoted by $\mathrm{Tr}[H^\gamma(Y_t^{N,i})a(X_{t_i}^N)]$ the vector with jth row equal to $\mathrm{Tr}[H^{\gamma_j}(Y_t^{N,i})a(X_{t_i}^N)]$. Thus, Itô's formula yields that

$$\mathrm{E}^{\mathcal{F}_{t_i}}(u(t_{i+1}\wedge\tau, X^N_{t_{i+1}\wedge\tau}) - u(t_i, X^N_{t_i})) = A_i^1 + A_i^2,$$

where

$$A_i^1 := \mathrm{E}^{\mathcal{F}_{t_i}}\biggl(\int_{t_i}^{t_{i+1}\wedge\tau}[\partial_t u(t, X_t^N)\,\mathrm{d}t + \nabla u(t, X_t^N)\,\mathrm{d}Y_t^{N,i} + \tfrac{1}{2}\mathrm{Tr}(H^u(t, X_t^N)a(X_{t_i}^N))\,\mathrm{d}t]\biggr),$$
$$A_i^2 := \mathrm{E}^{\mathcal{F}_{t_i}}\biggl(\int_{t_i}^{t_{i+1}\wedge\tau}\bigl[\nabla u(t, X_t^N)(\mathrm{d}X_t^N - \mathrm{d}Y_t^{N,i}) + \tfrac{1}{2}\mathrm{Tr}(H^u(t, X_t^N)(\mathrm{d}\langle X_\cdot^N, X_\cdot^N\rangle_t - a(X_{t_i}^N)\,\mathrm{d}t))\bigr]\biggr).$$

The term A_i^1 should not be a surprise for a reader familiar with the approximation of diffusions in the whole space (remember that $\mathrm{d}\langle Y_\cdot^{N,i}, Y_\cdot^{N,i}\rangle_t = a(X_{t_i}^N)\,\mathrm{d}t$); it is actually related to the approximation of $(X_t)_{t\ge 0}$ inside the domain. The term A_i^2 comes from the approximation near the boundary.

Case 1. (*The term* A_i^1.) Using (5), Itô's formula and simplifications coming from $\partial_t u + Lu = 0$ inside $[0, T] \times \bar{D}$, we easily find that

$$A_i^1 = \mathrm{E}^{\mathcal{F}_{t_i}}\biggl(\int_{t_i}^{t_{i+1}}\mathrm{d}t\int_{t_i}^{t\wedge\tau}[\mathcal{B}_s^{u,1}\,\mathrm{d}s + \mathbf{1}_{\{Y_s^{N,i}\notin D\}}\,\mathcal{B}_s^{u,2}\,\mathrm{d}s + \mathcal{B}_s^{u,3}\,\mathrm{d}L_s^0(F^\gamma(Y_\cdot^{N,i}))]\biggr) + \|f\|^{(5)}O_{\exp}(h),$$

where the processes $(\mathcal{B}_s^{u,1})_s$, $(\mathcal{B}_s^{u,2})_s$, $(\mathcal{B}_s^{u,3})_s$ are continuous, adapted and uniformly bounded by $K(T)\|f\|^{(5)}$ since they can be expressed as a sum of products of spatial derivatives of u (up to the order 4) and of coefficients b and σ and their derivatives, each of them being evaluated at point $(s, X_{t_i}^N)$ or (s, X_s^N). Hence, from Tanaka's formula,

$$|A_i^1| \le K(T)\|f\|^{(5)}[h^2 + h\,\mathrm{E}^{\mathcal{F}_{t_i}}[L^0_{t_{i+1}\wedge\tau}(F^\gamma(Y_\cdot^{N,i})) - L^0_{t_i}(F^\gamma(Y_\cdot^{N,i}))] + O_{\exp}(h)]$$
$$\le K(T)\|f\|^{(5)}(h^2 + h\,\mathrm{E}^{\mathcal{F}_{t_i}}[|F^\gamma(Y^{N,i}_{t_{i+1}\wedge\tau})| - |F^\gamma(Y^{N,i}_{t_i})|] + O_{\exp}(h))$$

and, thus,

$$\biggl|\mathrm{E}\biggl(\sum_{i=0}^{N-1}\mathbf{1}_{\{t_i<\tau\}}A_i^1\biggr)\biggr| \le K(T)\|f\|^{(5)}h\biggl(1 + \mathrm{E}\biggl(\sum_{i=0}^{N-1}|F^\gamma(Y^{N,i}_{t_{i+1}\wedge\tau})| - |F^\gamma(Y^{N,i}_{t_i\wedge\tau})|\biggr)\biggr).$$

On $\{\tau \le T\}$, the above sum is $O_{\exp}(h)$. On $\{\tau > T\}$, $|F^\gamma(Y^{N,i}_{t_{i+1}})| = |F^\gamma(Y^{N,i+1}_{t_i})|$ because of the symmetry procedure, thus the sum is telescoping: this proves that

$$\biggl|\mathrm{E}\biggl(\sum_{i=0}^{N-1}\mathbf{1}_{\{t_i<\tau\}}A_i^1\biggr)\biggr| \le K(T)\|f\|^{(5)}h.$$

Case 2. (*The term* A_i^2.) When we substitute (8) into the expression for A_i^2, the integral with respect to the local time vanishes because of the Neumann condition (3), while the other contributions can be gathered into a sum A_i^{21} involving the terms with a factor of $[F^\gamma(Y_t^{N,i})]^-$ and a sum A_i^{22} involving the terms with a factor of $\mathbf{1}_{\{Y_t^{N,i}\notin D\}}$.

(i) *The term* A_i^{21}. We have

$$A_i^{21} = \mathrm{E}^{\mathcal{F}_{t_i}}\left(\int_{t_i}^{t_{i+1}\wedge\tau} [F^\gamma(Y_t^{N,i})]^- \mathcal{B}_t^{u,4} dt\right),$$

where $(\mathcal{B}_t^{u,4})_t$ has the same properties as $(\mathcal{B}_t^{u,j})_t$ $(j \le 3)$. The Cauchy–Schwarz inequality combined with the estimates (7) and $[F^\gamma(Y_{t_i}^{N,i})]^- = 0$, and (6) with $\eta = d(Y_{t_i}^{N,i}, \partial D) = d(X_{t_i}^N, \partial D)$ implies that

$$\begin{aligned}|A_i^{21}| &\le K(T)\|f\|^{(5)} \int_{t_i}^{t_{i+1}} \sqrt{\mathrm{E}^{\mathcal{F}_{t_i}}([[F^\gamma(Y_t^{N,i})]^- - [F^\gamma(Y_{t_i}^{N,i})]^-]^2)}\sqrt{\mathrm{P}^{\mathcal{F}_{t_i}}[Y_t^{N,i} \notin D]}\,\mathrm{d}t \\ &\le K(T)\|f\|^{(5)} h^{3/2} \exp\left(-\frac{c}{2}\frac{d^2(X_{t_i}^N, \partial D)}{h}\right)\end{aligned}$$

for some $c > 0$ and, thus, we obtain that

$$\left|\mathrm{E}\left(\sum_{i=0}^{N-1} \mathbf{1}_{\{t_i<\tau\}} A_i^{21}\right)\right| \le K(T)\|f\|^{(5)} h$$

using Lemma 2.

(ii) *The term* A_i^{22}. This term is equal to

$$A_i^{22} = \mathrm{E}^{\mathcal{F}_{t_i}}\left(\int_{t_i}^{t_{i+1}\wedge\tau} \mathbf{1}_{\{Y_t^{N,i}\notin D\}} \mathcal{B}^{u,5}(t, Y_{t_i}^{N,i}, X_t^N, Y_t^{N,i})\,\mathrm{d}t\right)$$

with

$$\begin{aligned}&\mathcal{B}^{u,5}(t, x_i, x, y) \\ &\quad = \nabla u(t,x)\{-2\nabla\gamma(y)a(x_i)[\nabla F^\gamma(y)]^\top - 2\gamma(y)\nabla F^\gamma(y)b(x_i) - \gamma(y)\,\mathrm{Tr}[H^{F^\gamma}(y)a(x_i)]\} \\ &\qquad + \tfrac{1}{2}\,\mathrm{Tr}\{H^u(t,x)(-4\gamma(y)\nabla F^\gamma(y)a(x_i) + 4\gamma(y)\nabla F^\gamma(y)a(x_i)[\nabla F^\gamma(y)]^\top\gamma^\top(y))\}.\end{aligned}$$

Now, we notice that the function $\mathcal{B}^{u,5}$ vanishes when $x_i = x = y \in \partial D$ and $t < T$. Indeed, in view of the Neumann condition in (3), the second and third terms with a factor of ∇u vanish. In addition, $\nabla F^\gamma = n^\top / n \cdot \gamma$ on ∂D (see Proposition 1(v)). Thus, for all $z \in \partial D$,

$$\begin{aligned}&\mathcal{B}^{u,5}(t, z, z, z) \\ &\quad = -2\nabla u(t,z)\nabla\gamma(z)a(z)\frac{n(z)}{n(z)\cdot\gamma(z)} \\ &\qquad + 2\,\mathrm{Tr}\left\{H^u(t,z)\left[-\gamma(z)\frac{n^\top(z)}{n(z)\cdot\gamma(z)}a(z) + \gamma(z)\frac{n^\top(z)}{n(z)\cdot\gamma(z)}a(z)\frac{n(z)}{n(z)\cdot\gamma(z)}\gamma^\top(z)\right]\right\}.\end{aligned}$$

From easy linear algebra ($\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, etc.), it follows that $\mathcal{B}^{u,5}(t,z,z,z) = 2\mathcal{C}^u(z) = 0$ in view of Lemma 3.

Now, set $\tau_i = \inf\{t \ge t_i : Y_t^{N,i} \notin D\}$: on $\{Y_t^{N,i} \notin D\}$, we have $\tau_i \le t$, $Y_{\tau_i}^{N,i} \in \partial D$ and $\mathcal{B}^{u,5}(t, Y_{\tau_i}^{N,i}, Y_{\tau_i}^{N,i}, Y_{\tau_i}^{N,i}) = 0$. Since $\mathcal{B}^{u,5}(t,\cdot)$ is continuously differentiable with first derivatives bounded by $K(T)\|f\|^{(5)}$, we can easily deduce that

$$\begin{aligned}
&|A_i^{22}| \\
&\le K(T)\|f\|^{(5)}\,\mathrm{E}^{\mathcal{F}_{t_i}}\biggl(\int_{t_i}^{t_{i+1}\wedge\tau} \mathbf{1}_{\{Y_t^{N,i}\notin D\}}(|Y_{t_i}^{N,i} - Y_{\tau_i}^{N,i}| + |X_t^N - Y_{\tau_i}^{N,i}| + |Y_t^{N,i} - Y_{\tau_i}^{N,i}|)\,\mathrm{d}t\biggr) \\
&\le K(T)\|f\|^{(5)}\,\mathrm{E}^{\mathcal{F}_{t_i}}\biggl(\int_{t_i}^{t_{i+1}\wedge\tau} \mathbf{1}_{\{Y_t^{N,i}\notin D\}}(|Y_{t_i}^{N,i} - Y_{\tau_i}^{N,i}| + [F^\gamma(Y_t^{N,i})]^- + |Y_t^{N,i} - Y_{\tau_i}^{N,i}|)\,\mathrm{d}t\biggr)
\end{aligned}$$

for a constant $K(T)$ changing from line to line. We now apply arguments already used for A_i^{21}: thus,

$$|A_i^{22}| \le K(T)\|f\|^{(5)} h^{3/2} \exp\biggl(-\frac{c}{2}\frac{d^2(X_{t_i}^N, \partial D)}{h}\biggr),$$

and then $|\,\mathrm{E}(\sum_{i=0}^{N-1}\mathbf{1}_{\{t_i<\tau\}}\,A_i^{22})| \le K(T)\|f\|^{(5)}h$.

The proof of Theorem 1 is complete.

Proof of Lemma 2. Should $X_{t_i}^N$ have a density with respect to the Lebesgue measure on D and should this density be uniformly bounded in N near ∂D, then we could easily conclude that $\mathcal{A}_i := \mathrm{E}(\mathbf{1}_{\{t_i<\tau\}}\exp(-cd^2(X_{t_i}^N, \partial D)/h)) \le K(T)\sqrt{h}$. But the desired property on the density of $X_{t_i}^N$ seems difficult to prove, even by using Malliavin calculus tools (because of the $[F^\gamma]^-$ terms).

The idea of our proof is to use the occupation times formula. By (iii) and (iv) of Proposition 1, if $d(x, \partial D) \le R$, then $d(x, \partial D) = |F^n(x)| \ge |F^\gamma(x)|/c_1$, and thus

$$\mathcal{A}_{i+1} \le \mathrm{E}\biggl(\mathbf{1}_{\{t_{i+1}<\tau\}}\exp\biggl(-c\frac{[F^\gamma(X_{t_{i+1}}^N)]^2}{c_1^2 h}\biggr)\biggr) + O_{\exp}(h).$$

Set $c' = c/2c_1^2 > 0$ and $g(x) = \exp(-2c'x^2/h)$: it is easy to check that $|g(x)| + \sqrt{h}|g'(x)| + h|g''(x)| \le K(T)\exp(-c'x^2/h)$. Hence, for $t \in [t_i, t_{i+1}]$, Itô's formula combined with the decomposition (8) and the estimate (5) yields that

$$\begin{aligned}
&\mathrm{E}\biggl(\mathbf{1}_{\{t_{i+1}<\tau\}}\exp\biggl(-2c'\frac{[F^\gamma(X_{t_{i+1}}^N)]^2}{h}\biggr)\biggr) \\
&\quad\le K(T)\biggl[\mathrm{E}\biggl(\mathbf{1}_{t<\tau}\exp\biggl(-c'\frac{[F^\gamma(X_t^N)]^2}{h}\biggr)\biggr) \\
&\qquad\qquad + \frac{1}{h}\int_t^{t_{i+1}}\mathrm{E}\biggl(\mathbf{1}_{s<\tau}\exp\biggl(-c'\frac{[F^\gamma(X_s^N)]^2}{h}\biggr)\biggr)\,\mathrm{d}s\biggr] + O_{\exp}(h);
\end{aligned}$$

notice that the local time involved in (8) provides no contribution to the preceding computation because $g'(0) = 0$. Integrate this inequality with respect to t over $[t_i, t_{i+1}]$ to get

$$h\mathcal{A}_{i+1} \le K(T)\int_{t_i}^{t_{i+1}}\mathrm{E}\biggl(\mathbf{1}_{\{s<\tau\}}\exp\biggl(-c'\frac{[F^\gamma(X_s^N)]^2}{h}\biggr)\biggr)\,\mathrm{d}s + O_{\exp}(h).$$

Observe that, when $|F^\gamma(y)| \le R$,

$$\mathrm{d}\langle F^\gamma(X_\cdot^N), F^\gamma(X_\cdot^N)\rangle_s = \nabla F^\gamma(X_s^N)a(X_{t_i}^N)[F^\gamma(X_s^N)]^\top\,\mathrm{d}s \ge \frac{\sigma_0^2}{4}\,\mathrm{d}s$$

using Assumption 4 and $|\nabla F^\gamma(y)| \geq \frac{1}{2}$ when $|F^\gamma(y)| \leq R$ (we can assume this last property by decreasing R in Proposition 1 if necessary). It readily follows from the occupation times formula that

$$h\mathcal{A}_{i+1} \leq K(T)\int_{-R}^{R} \mathrm{d}y \exp\biggl(-c'\frac{y^2}{h}\biggr) \mathrm{E}(L^y_{t_{i+1}\wedge\tau}(F^\gamma(X^N_\cdot)) - L^y_{t_i\wedge\tau}(F^\gamma(X^N_\cdot))) + O_{\exp}(h).$$

Now,

$$\begin{aligned}
&\tfrac{1}{2}\,\mathrm{E}(L^y_{t_{i+1}\wedge\tau}(F^\gamma(X^N_\cdot)) - L^y_{t_i\wedge\tau}(F^\gamma(X^N_\cdot)))\\
&\quad = \mathrm{E}((F^\gamma(X^N_{t_{i+1}\wedge\tau}) - y)^+ - (F^\gamma(X^N_{t_i\wedge\tau}) - y)^+ - \int_{t_i\wedge\tau}^{t_{i+1}\wedge\tau} \mathbf{1}_{\{F^\gamma(X^N_s)\geq y\}}\,\mathrm{d}(F^\gamma(X^N_s)))\\
&\quad \leq \mathrm{E}((F^\gamma(X^N_{t_{i+1}\wedge\tau}) - y)^+ - (F^\gamma(X^N_{t_i\wedge\tau}) - y)^+) + K(T)h\\
&\qquad - \mathrm{E}\biggl(\int_{t_i\wedge\tau}^{t_{i+1}\wedge\tau} \mathbf{1}_{\{F^\gamma(X^N_s)\geq y\}}\,\nabla F^\gamma(X^N_s)\gamma(Y^{N,i}_s)\,\mathrm{d}L^0_s(F^\gamma(Y^{N,i}_\cdot))\biggr).
\end{aligned}$$

We have used (8) to get the last inequality. The above integral with respect to the local time is nonnegative since $\nabla F^\gamma \gamma = 1$ on ∂D.

Therefore, $\sum_{i=0}^{N-1} \mathrm{E}(L^y_{t_{i+1}\wedge\tau}(F^\gamma(Y^{N,i}_\cdot)) - L^y_{t_i\wedge\tau}(F^\gamma(Y^{N,i}_\cdot))) \leq K(T)$ uniformly in $|y| \leq R$ since the sum is telescoping. We can then conclude that $h\sum_{i=0}^{N-1}\mathcal{A}_{i+1} \leq K(T)\sqrt{h}$. The proof of Lemma 2 is complete.

Proof of Lemma 3. In the following, $t < T$ is fixed and we omit it. Since the function $\nabla u\gamma$ vanishes on ∂D, $n[\nabla(\nabla u\gamma)n] = [\nabla(\nabla u\gamma)]^\top$. Taking into account the fact that $\nabla(\nabla u\gamma) = \gamma^\top H^u + \nabla u\nabla\gamma$, we derive the following identity on the boundary:

$$H^u\gamma = n(\gamma^\top H^u n) + n(\nabla u\nabla\gamma n) - (\nabla u\nabla\gamma)^\top.$$

We thus have

$$\begin{aligned}
\mathcal{C}^u(x) &= -\nabla u\nabla\gamma a\frac{n}{n\cdot\gamma} + \frac{(n^\top an)}{(n\cdot\gamma)^2}\gamma^\top[n(\gamma^\top H^u n) + n(\nabla u\nabla\gamma n) - (\nabla u\nabla\gamma)^\top]\\
&\quad - \frac{1}{n\cdot\gamma}n^\top a[n(\gamma^\top H^u n) + n(\nabla u\nabla\gamma n) - (\nabla u\nabla\gamma)^\top]\\
&= -\frac{(n^\top an)}{(n\cdot\gamma)^2}(\nabla u\nabla\gamma\gamma)^\top.
\end{aligned}$$

The right-hand side is equal to 0 on ∂D since $\nabla\gamma\gamma = 0$ on ∂D: indeed, it follows from the facts that $\gamma(x + \lambda\gamma(x)) = \gamma(x)$ for $x \in \partial D$ and $|\lambda| \leq R$ (see Proposition 1). We are finished.

4. A numerical example

Let X be a three-dimensional Brownian motion normally reflected at the boundary of the unit ball $D = S_3(0, 1)$. We are interested in the evaluation of $\mathrm{E}(\|X_1\|^2)$: to the best of the authors' knowledge, the exact value is unknown. To make the experiment more interesting, we compare the scheme of this paper with two other ones: the usual *projected Euler scheme* (see [5]) with order of convergence equal to $\frac{1}{2}$, and the *reflected Euler scheme* on local half-space approximations (see [10]) with order of convergence equal to 1 in this example. We also

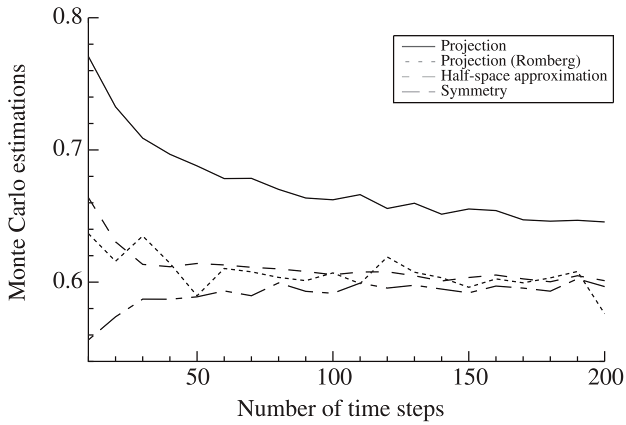

FIGURE 2: Comparison of the weak error for four schemes.

TABLE 1: Computational time for each scheme when $N = 50$.

	Projected scheme	Projected scheme with Romberg extrapolation	Symmetrized scheme	Reflected scheme in local half-space approximation
CPU time	0.92 s	1.37 s	0.92 s	1.52 s

consider the *Romberg extrapolation* (see [25]) with the projected scheme, assuming that an expansion of the error at order $\frac{1}{2}$ is available: it gives

$$\mathrm{E}\bigg(\frac{\sqrt{2}f(X_T^N) - f(X_T^{N/2})}{\sqrt{2}-1}\bigg) = \mathrm{E}(f(X_T)) + o(\sqrt{h}).$$

The number of simulated paths is $M = 10\,000$: it provides a width of the 95%-confidence interval essentially equal to 0.03 for each scheme (except for the Romberg extrapolation for which it is larger, that is 0.035).

The efficiency of each procedure is illustrated by Figure 2. It turns out that procedures with symmetry, half-space approximation and Romberg extrapolation all behave very well. However, the computational time is much smaller for the method presented here because of the simplicity of the symmetry (in fact, as simple as the projection method): see Table 1.

5. Conclusion

We have proved that an Euler scheme with a symmetry procedure yields an accurate approximation of obliquely reflected diffusions at a fixed time. We give three open issues that we have not been able to handle:

1. How to get an expansion of the error with respect to h? It seems that sharper estimates on the law of X^N near the ∂D are needed.
2. How to adapt the current analysis to the stationary problem where $\int_0^\infty \mathrm{E}_x(\phi(X_t) - \int \phi(\xi)\mu(\mathrm{d}\xi))\,\mathrm{d}t$ needs to be evaluated? The use of new ergodic type estimates seems to be crucial.

3. While with other approximation methods ([5], [10]) it is possible to simulate the local time on ∂D (and hence to evaluate expectations of more complex functionals of type $\mathrm{E}(\int_0^T g(X_t)\,\mathrm{d}k_t)$), it is not known how to adapt our algorithm to approximate these quantities in a satisfactory and accurate way (with a first-order convergence).

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