APPROXIMATION OF LYAPUNOV EXPONENTS OF NONLINEAR STOCHASTIC DIFFERENTIAL SYSTEMS *

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Abstract

We consider nonlinear stochastic differential systems defined either on a compact orientable manifold, or on $\mathbb{R}^d$; under our hypotheses, their Lyapunov exponents are deterministic. We propose an efficient algorithm of numerical computation of these exponents, and we give a theoretical estimate for the approximation error.

The method is based upon the discretization of the linearized stochastic flows of diffeomorphisms generated by the differential systems.

Results of numerical experiments are also presented.

APPROXIMATION DES EXPOSANTS DE LYAPUNOV DE SYSTEMES DIFFERENTIELS STOCHASTIQUES NON LINEAIRES

Résumé

Nous considérons des systèmes différentiels stochastiques non linéaires, définis soit sur une variété compacte orientable, soit sur $\mathbb{R}^d$; sous nos hypothèses, leurs exposants de Lyapounov sont déterministes. Nous en proposons un algorithme de calcul numérique efficace, dont nous donnons la vitesse de convergence.

La méthode est fondée sur la discrétisation des flots stochastiques de difféomorphismes linéarisés engendrés par les systèmes différentiels.

Nous présentons aussi des résultats de tests numériques.

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1 Introduction

The Lyapunov exponents of a stochastic dynamical system enable to study its stability. A survey of this important theory, for linear and nonlinear systems, may be found in [3], and in Arnold[1] (we will use the notations of this last reference).

From an applied point of view, most often it is necessary to numerically approximate the Lyapunov exponents. For the linear case, in Talay [21] an algorithm has been proposed to compute the upper one and its theoretical convergence rate is given; in the same paper, an industrial application (the stability of the motion of helicopter blades) is described.

In practice, most models are nonlinear, so it seems interesting to provide efficient numerical tools of studying their stability. This is the aim of this paper. The Lyapunov exponents are expressed in terms of the linearized flow of diffeomorphisms generated by the system; for a linear system, the linearized flow and the flow are equal, which simplifies the numerical analysis issues; some specific difficulties appear for fully nonlinear systems. The algorithm proposed here is based upon the time discretization of the linearized flow; we give a convergence rate of the method in terms of the discretization step; we approximate all the exponents (not only the top one) as well. The Lyapunov exponents can be expressed in terms of the invariant measure of a system on projective bundles (see Baxendale's formula (8) below or [5]). The procedure which we propose here avoids the numerical solving of stationary Fokker-Planck equations on these projective bundles, which may be numerically difficult or costly especially when the dimension of the state space is large.

We will limit ourselves to the case where the state space is $\mathbb{R}^d$ or a smooth compact orientable manifold. The reasons for this limitation are multiple: first, we need the existence of a stochastic flow of diffeomorphisms associated to the stochastic differential equation; second, the approximation algorithm of the solution of the system has a reasonable complexity in this framework (either one discretizes a system given in euclidian coordinates, or one just needs to use an exponential atlas satisfying a given property stated below); third, it is possible to write natural explicit conditions on the coefficients of the system ensuring the ergodicity of the solution; and, finally, we need to establish a technical result on the solution of a parabolic P.D.E. on the projective bundle of the state space, and our method could fail for non compact manifolds without supposing very stringent conditions on the coefficients of the system.

As we will see, the case of compact manifolds leads to particular developments; first, we propose an algorithm which takes into account the geometry of the state space, and is nevertheless realistic from a practical point of view; second, some details of the proof giving the convergence rate of the method are simpler than in the $\mathbb{R}^d$ case.

We have chosen not to separate the two situations in the presentation. Thus, we avoid a large amount of repetitions, and also it seems easier to distinguish where the compacity plays a role.

In Section 2, we state the hypotheses under which we will establish the convergence rate of our algorithm; in Sections 3 and 4, we present the discretization method of the linearized system, which will permit us to approximate the upper Lyapunov exponent; in Section 5, we prove that the Markov chain defined by the discretization admits a Lyapunov exponent; in Section 6, we prove that our method is a first order approximation; in Section 7, we give the proof of a technical Lemma used in Section 6; in Section 8, we explain how the method can be extended to get a first order approximation of any exponent of the spectrum; in Section 9, we present the results of numerical trials, for examples of systems on $\mathbb{R}^d$ and on the circle.

We will use results due to Arnold & San Martin, Baxendale and Caverhill for the continuous time processes (cf. the papers in [3]), and results due to Bougerol ([7] and [8]) for the discrete time approximating process.

Several statements below look like statements of a same nature in the references [22], [21], but cannot be obtained as Corollaries. Essentially, here one has to check that some properties of processes or P.D.E.'s which are true when the state space is the sphere of $\mathbb{R}^d$ or $\mathbb{R}^d$, remain true when the state space is the product of $\mathbb{R}^d$ and its projective space. We will use combinations of ideas of the two mentioned papers, but we think it would be difficult for a reader not familiar with them to be convinced that the claims of the present work are true without at least strong indications on what must be done in addition of the existing proofs: for the sake of clearness and completeness, we have chosen to make a rather detailed presentation.

2 Basic Hypotheses

2.1 Systems on $\mathbb{R}^d$

We consider a stochastic differential system on $\mathbb{R}^d$ defined by the equation

$$\begin{aligned} dx_t &= A(x_t)dt + \sum_{j=1}^{r} B_j(x_t) \circ dW_t^j \\ x_0 &= x \end{aligned} \tag{1}$$

with smooth vector fields A and B_j's; denoting by A' and B_j' the differential maps of A and B_j, the linearized system is defined on $\mathbb{R}^d \times \mathbb{R}^d$ by

$$\begin{aligned} dx_t &= A(x_t)dt + \sum_{j=1}^{r} B_j(x_t) \circ dW_t^j \\ dv_t &= A'(x_t)v_t dt + \sum_{j=1}^{r} B_j'(x_t)v_t \circ dW_t^j \\ x_0 &= x \ , \ v_0 \ = \ v \end{aligned} \tag{2}$$

We suppose:

(H1) *the vectors fields A and B_j ($j = 1 \ldots r$) are of class $\mathcal{C}^\infty$ and have bounded derivatives (for all order of derivation); the vector fields B_j ($j = 1...r$) are bounded.*

In order to ensure the existence of the Lyapunov exponents of the continuous time process and its discretized process, we need to formulate rather technical (but reasonable for the applications) hypotheses; we had to strengthen these hypotheses to get the rate of convergence of the approximate Lyapunov exponent in terms of the discretization step.

Let $S^{d-1} = \{x \in \mathbb{R}^d;\ |x| = 1\}$ be the unit sphere of $\mathbb{R}^d$, and $\mathbb{P}^{d-1}$ be the projective space of $\mathbb{R}^d$, i.e. the quotient of S^{d-1} with respect to the relation: $u \sim v$ iff $u = -v$. In the sequel, for any vector v in $\mathbb{R}^d$, $[v]$ will denote the equivalence class of $\frac{v}{|v|}$ in $\mathbb{P}^{d-1}$, and $\mathbb{P}\mathbb{R}^d$ will denote the space $\mathbb{R}^d \times \mathbb{P}^{d-1}$.

We know (cf Arnold and San Martin [2]) that the process $(Px_t(x,[v]))$ on $\mathbb{P}\mathbb{R}^d$ defined by $Px_t(x,[v]) = (x_t,[v_t])(x,[v])$ (i.e the process $(x_t,[v_t])$ with initial condition $(x,[v])$) is solution of the system:

$$\begin{aligned} dPx_t &= PA(Px_t)dt + \sum_{j=1}^{r} PB_j(Px_t) \circ dW_t^j \\ Px_0 &= (x,[v]) \end{aligned} \tag{3}$$

with $PC((x,[v])) = (Cx, C'(x)[v] - < C'(x)[v],[v] > [v])$ for $C = A$ or B_j.

We shall assume that the following hypothesis holds :

(H2) **(i)** *The differential operator $PA + \frac{1}{2}\sum_{j=1}^{r}(PB_j)^2$ is strongly elliptic.*

(ii) $\exists \beta > 0$, $\exists K$ *compact* $\in \mathbb{R}^d$ *such that:* $\forall x \in \mathbb{R}^d - K$, $< x, A(x) > \leq -\beta|x|^2$

2.2 Systems on compact manifolds

Let us consider a d-dimensional $\mathcal{C}^\infty$ compact manifold $\mathcal{M}$, and A, B_j ($j = 1,\ldots,r$) vector fields on $\mathcal{M}$. We also consider a r-dimensional standard Wiener process (W_t).

In this framework, we reformulate (H1) as follows:

(H1) *The vectors fields A and B_j ($j = 1 \ldots r$) are of class $\mathcal{C}^\infty$.*

We will deal with the stochastic differential system in the Stratonovich sense on $\mathcal{M}$:

$$dx_t = A(x_t)dt + \sum_{j=1}^{r} B_j(x_t) \circ dW_t^j \tag{4}$$

This system defines a stochastic flow of diffeomorphisms $(x_t(x))$ (cf. Ikeda & Watanabe [13] e.g.); if $Tx_t(x) : T_x\mathcal{M} \to T_{x_t(x)}\mathcal{M}$ is the linear part of x_t at x, and if the vector fields TA, TB_j are the linearizations of A, B_j, then the mapping Tx_t from $T\mathcal{M}$ to $T\mathcal{M}$ defined by $(x,v) \to (x_t(x), Tx_t(x)v)$ is a flow on the tangent bundle $T\mathcal{M}$, generated by the system

$$dTx_t = TA(Tx_t)dt + \sum_{j=1}^{r} TB_j(Tx_t) \circ dW_t^j \tag{5}$$

For $x \in \mathcal{M}$, let $\mathbb{P}_x\mathcal{M}$ be the projective fibre over x, and $\mathbb{P}\mathcal{M} = \cup_{x\in\mathcal{M}}\{x\} \times \mathbb{P}_x\mathcal{M}$ be the projective bundle over $\mathcal{M}$. For v in $T_x\mathcal{M}$, $[v]$ will denote the equivalence class of $\frac{v}{|v|}$ in $\mathbb{P}_x\mathcal{M}$.

Let $Px_t(x,[v])$ be the equivalence class of $\left(x_t(x), \frac{Tx_t(x)v}{|Tx_t(x)v|}\right)$ in $\mathbb{P}\mathcal{M}$; this process on $\mathbb{P}\mathcal{M}$ solves a stochastic differential system (cf Arnold & San Martin [2] or Carverhill [9]):

$$\begin{aligned} dPx_t &= PA(Px_t)dt + \sum_{j=1}^{r} PB_j(Px_t) \circ dW_t^j \\ Px_0 &= (x,[v]) \end{aligned} \tag{6}$$

In this framework, our hypothesis (H2) is:

(H2) *the differential operator* $PA + \frac{1}{2}\sum_{j=1}^{r}(PB_j)^2$ *is strongly elliptic.*

2.3 Existence of the Lyapunov exponent for the continuous time processes

A first consequence of the above hypotheses is the following proposition.

Proposition 2.1 *Under (H1) and (H2), the process* (Px_t) *on the tangent bundle* $\mathbb{P}\mathcal{M}$ *has a unique invariant probability law (henceafter denoted by* ν*), which has a strictly positive smooth density w.r.t.* $d\tau$*.*

Proof First, (Px_t) is a strong Feller process on a metric space; second, either $\mathcal{M}$ is a compact manifold, and then (Px_t) is a Feller process on the compact manifold $P\mathcal{M}$, or $\mathcal{M}$ is R^d and then there exists a strictly positive constant C such that: $\mathbb{E}|Px_t|^2 < C$ (this can be checked by using the Ito formula and (H2-ii)); in both cases, there exists at least one invariant probability measure (cf. Ethier & Kurtz [12] e.g.).

The conclusion of the Proposition comes from the fact that, under (H1) and (H2), for any deterministic initial condition, the law of (Px_t) $(t > 0)$ has a smooth and strictly positive density with respect to $d\tau$ (again, cf. Ikeda & Watanabe [13] e.g.).

This ensures that the process (Px_t) is an ergodic process.

□

In particular, of course this implies that the process (x_t) itself is ergodic.

As a result of the previous proposition, our hypotheses imply that there exists a real number λ such that, for any (x,v) in $T\mathcal{M}$:

$$\lambda = \lim_{t\to+\infty} \frac{1}{t} \log |Tx_t(x)v| \quad , a.s., \tag{7}$$

(cf Arnold & San Martin [2] for even weaker assumptions).

The number λ is called the top Lyapunov exponent of the system (2).

Remark 2.2 *Remember that ν denotes the unique invariant probability law of $(x_t, [v_t])$; Baxendale [5] gives the following expression for λ:*

$$\lambda = \int_{\mathbb{P}\mathcal{M}} \psi(\theta)\nu(d\theta) \tag{8}$$

where, ∇ (resp. $\mathcal{R}$) denoting the Riemannian covariant derivative (resp. the Riemannian curvature tensor) on $\mathcal{M}$, for x in $\mathcal{M}$ and v in $\mathbb{P}_x\mathcal{M}$,

$$\begin{aligned} \psi(x,v) &= < v, \nabla(A(x) + \frac{1}{2}\sum_{j=1}^{r} \nabla B_j(x)B_j(x))v > \qquad (9) \\ &\quad + \frac{1}{2}\sum_{j=1}^{r}\Big[|\nabla B_j(x)v|^2 - 2 < v, \nabla B_j(x)v >^2 + < \mathcal{R}(B_j(x),v)B_j(x), v >\Big] \end{aligned}$$

The numerical scheme is not based upon this representation of λ: first, most often the numerical cost corresponding to the evaluation of ψ is too high; second, the invariant measure ν is unknown and, as we already have stressed in the Introduction, may be difficult to approximate numerically. Therefore we prefer to derive the algorithm from the formula (7); nevertheless, the convergence rate analysis uses (8) (see Lemma 6.2).

3 Discretization of systems on $\mathbb{R}^d$

We are going to define a Markov chain $(\overline{x}_p^h, \overline{v}_p^h)$, which can be easily simulated on a computer, and approximates the solution of the system (2).

3.1 Discretization scheme

We begin by the following remark: under (H1), for any random variable U with a compact supported law $\mathbb{P}_U$, there exists $h_0(\mathbb{P}_U) > 0$ satisfying: for any $h \le h_0(\mathbb{P}_U)$, if

$$\tilde{A}(x) = A(x) + \frac{1}{2}\sum_{j=1}^{r} B_j'(x)B_j(x)$$

and if $(U^j, 1 \le j \le r)$ are independent copies of U, then:

$$||\tilde{A}'(x)h + \sqrt{h}\sum_{j=1}^{r} B_j'(x)U^j|| \le \frac{1}{2} \quad , \quad a.s.$$

Then we take an h in $\mathbb{R}_+^*$ and a family (U_{p+1}^j) of random variables which will be supposed to satisfy the following requirement:

(HU) **(i)** *the (U_{p+1}^j)'s are i.i.d., and the following conditions on the moments are fulfilled:* $\mathbb{E}[U_{p+1}^j] = \mathbb{E}[U_{p+1}^j]^3 = 0$, $\mathbb{E}[U_{p+1}^j]^2 = 1$;

(ii) *the common law $\mathbb{P}_U$ of the (U_{p+1}^j)'s has a continuous density w.r.t. the Lebesgue measure, whose support contains an open interval including 0 and is compact;*

(iii) *the step-size h is less than $h_0(\mathbb{P}_U)$.*

Let take a deterministic initial value (x_0, v_0) in $\mathbb{R}^d \times \mathbb{R}^d$ and set:

$$\begin{aligned}
\overline{x}_{p+1}^h &= \overline{x}_p^h + \tilde{A}(\overline{x}_p^h)h + \sum_{j=1}^{r} B_j(\overline{x}_p^h)U_{p+1}^j\sqrt{h} \\
\overline{M}_{p+1}^h &= Id + \tilde{A}'(\overline{x}_p^h)h + \sum_{j=1}^{r} B_j'(\overline{x}_p^h)U_{p+1}^j\sqrt{h} \\
\overline{v}_{p+1}^h &= \overline{M}_{p+1}^h \overline{v}_p^h
\end{aligned} \tag{10}$$

This system describes the passage from $(\overline{x}_p^h, \overline{v}_p^h)$ to the process $(\overline{x}_{p+1}^h, \overline{v}_{p+1}^h)$ in $\mathbb{R}^d \times \mathbb{R}^d$ obtained by discretizing the differential system (2) written in the Ito sense.

3.2 Remark

The condition (HU) does not allow the simulation of a gaussian law, whereas one could expect that $U_{p+1}^j\sqrt{h}$ would be $W_{(p+1)h}^j - W_{ph}^j$.

This is not a limitation, neither numerically (the computers prefer to simulate compact-supported laws !), nor theoretically: the rate of convergence of the algorithm would not be better if the exact gaussian law would be simulated (cf. also the approximation of the law of a diffusion process (Talay [20]).

3.3 About the choice of the scheme

Consider the stochastic differential equation (1) in $\mathbb{R}^d$, and the scheme

$$\overline{x}_{p+1}^h = \overline{x}_p^h + \tilde{A}(\overline{x}_p^h)h + \sum_{j=1}^{r} B_j(\overline{x}_p^h)(W_{(p+1)h}^j - W_{ph}^j) \; .$$

This is the usual Euler scheme for the system (1) written in the Itô sense. This scheme has a rather low convergence rate for the mean-square approximation on a finite time interval $[0, T]$. Indeed, suppose that the coefficients A and B_j are smooth enough; then, for any $T > 0$, there exists a constant $C_{Euler}(T)$ such that, for any h of the form $h = \frac{T}{M}$ for some integer M, one has

$$\sqrt{\mathbb{E}|x_T - \bar{x}_M^h|^2} \leq C_{Euler}(T)\sqrt{h}\ .$$

This has lead to the construction of various other schemes, among them the Milshtein scheme

$$\overline{x}_{p+1}^h = \overline{x}_p^h + \tilde{A}(\overline{x}_p^h)h + \sum_{j=1}^r B_j(\overline{x}_p^h)(W_{(p+1)h}^j - W_{ph}^j) + \sum_{i,j=1}^r \partial B_j(\overline{x}_p^h)B_i(\overline{x}_p^h)\int_{ph}^{(p+1)h}(W_s^i - W_{ph}^i)dW_s^j,$$

which, under appropriate assumptions (see Milshtein [16]), satisfies

$$\sqrt{\mathbb{E}|x_T - \bar{x}_M^h|^2} \leq C_{Milshtein}(T)h\ .$$

The constant $C_{Milshtein}(T)$ is not optimal for the family of the first-order schemes, so that Clark [11], Newton ([17], [18] and [19]), Castell & Gaines [10] have proposed asymptotically efficient schemes much more complex than the Milshtein scheme.

For the present problem (approximation of Lyapunov exponents), these schemes and the L^2-estimates are not satisfying, for the following reasons.

- First, the usual L^2-estimates are irrelevant: since in these works the process (x_t) is not supposed ergodic, the given constants $C_{Euler}(T)$ and $C_{Milshtein}(T)$ grow exponentially fast when T goes to infinity; non trivial refinements based upon the ergodicity of the exact process (x_t) are necessary to prove that all the multiplicative constants of h appearing in the error term can be bounded uniformly in T. Moreover, the Lyapunov exponents depend on the law of the process, not on its individual trajectories; for this type of problems, it usually appears that the Euler and the Milshtein scheme have the same order of convergence (see Talay: [20] for the approximation of $\mathbb{E}f(X_T)$, $f(\cdot)$ and T fixed, [21] for the approximation of the upper Lyapunov exponent of linear systems, [22] for the approximation of invariant measures of ergodic diffusions); consequently, one has no reason not to use the simplest scheme.

- The preceding remark is particularly important for systems on a compact manifold, since the algorithm involves the expression in local coordinates of the coefficients of the linearized system (see the next section), whose successive derivatives may have so complex analytical expressions that the numerical efficiency may drastically be decreased.

- Besides, a statistical error is added to the discretization error, and this statistical error is of order $\frac{1}{\sqrt{N}}$ where N is the number of steps (see the discussion in [21]); thus to see the gain of accuracy due to a high order scheme, one may have to considerably lengthen the simulation time.

Anyhow, there exists a simple procedure to accelerate the convergence rate of the Euler scheme. In Talay & Tubaro [24] and in Bally & Talay [4], the structure of the Euler scheme error for the approximation of $\mathbb{E}f(X_T)$ and of invariant measures of ergodic diffusions is analyzed: the error can be expanded with respect to h; this result permits to construct efficient Romberg extrapolation procedures which accelerate the convergence rate; for example, under some appropriate assumptions, if μ denotes the invariant measure of (x_t) and $\bar{\mu}^h$ denotes the invariant mesure of the Markov chain $(\overline{x}_p^h)$, then, for any smooth enough function $f(\cdot)$, it holds that

$$\int_{\mathbb{R}^d} f(\theta)d\mu(\theta) - \int_{\mathbb{R}^d} f(\theta)d\bar{\mu}^h(\theta) = C_1 h + \mathcal{O}(h^2)\,,$$

so that the formula

$$\frac{1}{N}\sum_{p=1}^{2N} f(\bar{x}_p^{h/2}) - \frac{1}{N}\sum_{p=1}^{N} f(\overline{x}_p^h)$$

gives an approximation of $\int_{\mathbb{R}^d} f(\theta)d\mu(\theta)$ which, when N goes to infinity, is of order h^2: one has exponentially increased the accuracy by multiplying the computational effort by a factor 3 (this factor can even be reduced by a good implementation); we think that such an expansion also holds for our algorithm, but the proof would require additional developments. Numerical tests show that this extrapolation gives better results than the basic Euler scheme, whereas an extrapolation of rate h^3 seems useless because of the statistical error.

4 Discretization of systems on a compact manifold

For a system on a compact manifold, we can write a version of the previous algorithm. This version is easy to implement if one can define explicit coordinate maps satisfying a "goodness" property that we now define.

4.1 Preliminary

The construction of the approximate process requires the choice of local coordinates. Our error analysis needs that the chosen system of coordinates satisfies some properties (for example, the approximate process is needed to be a Feller process; see also the next proposition, e.g.), which lead to the following definition:

Definition 4.1 *An atlas* $\{(\phi_x, Dom(\phi_x)), x \in \mathcal{M}\}$ *of* $\mathcal{C}^\infty$ *charts will be called a* **good atlas** *if it satisfies the conditions:*

(a) *for all* $x \in \mathcal{M}$, $\phi_x(x) = 0$; *there exists a real number* $R > 0$ *such that, for all* $x \in \mathcal{M}$, $Val(\phi_x) \supset B(0, R)$ *and:*

(b) *for any sequence (x_n) in $\mathcal{M}$ converging to some x $(\phi_{x_n}^{-1}(y))$ converges to $(\phi_x^{-1}(y))$ uniformly in y in $B(0,R)$;*

(c) *for any smooth vector field V on $\mathcal{M}$, for any x in $\mathcal{M}$, if ν denotes the expression in local coordinates around x of V, then, for any non void multi-index I, the derivative $\partial_I \nu$ is bounded on $Val(\phi_x)$ by a constant depending only on I (not on x).*

A chart of a good atlas will be called a **good chart**.

In the last subsection of this section, we will give an example of a good atlas. The next proposition is an immediate application of this notion. For any vector-valued function γ, $\partial\gamma$ denotes the matrix $[\partial_k \gamma^i]_{i,k}$.

Proposition 4.2 *For any random variable U with a compact supported law $\mathbb{P}_U$, there exists $h_0(\mathbb{P}_U) > 0$ satisfying, for any $h \leq h_0(\mathbb{P}_U)$: $\forall x \in \mathcal{M}$, if α, β_j denote the expressions of the vector fields A, B_j in a good chart around x, and if*

$$\tilde{\alpha} := \alpha + \frac{1}{2}\sum_{j=1}^{r}(\partial\beta_j)\beta_j$$

then, $(U^j, 1 \leq j \leq r)$ being independent copies of U:

$$\tilde{\alpha}(0)h + \sqrt{h}\sum_{j=1}^{r}\beta_j(0)U^j \quad \in \quad B(0,R) \subset Val(\phi_x) \quad , \quad a.s.$$

and

$$||\partial\tilde{\alpha}(0)h + \sqrt{h}\sum_{j=1}^{r}\partial\beta_j(0)U^j|| \leq \frac{1}{2} \quad , \quad a.s. \tag{11}$$

4.2 Discretization scheme

The initial value is any deterministic pair (x_0, v_0) in $T\mathcal{M}$; let us describe the passage from $(\overline{x}_p^h, \overline{v}_p^h)$ to $(\overline{x}_{p+1}^h, \overline{v}_{p+1}^h)$: let $(\phi_p, Dom(\phi_p))$ be the good chart around $\overline{x}_p^h$, and let α_p (resp. $\beta_{j,p}$) be the representation of the vector field A (resp. B_j) in local coordinates; we set:

$$\tilde{\alpha_p} := \alpha_p + \frac{1}{2}\sum_{j=1}^{r}(\partial\beta_{j,p})\beta_{j,p}$$

The map ϕ_p induce a basis ϵ_p in $T_{\overline{x}_p^h}M$ given by: $\epsilon_p^j f = \partial_j(f \circ \phi_p^{-1})(0)$ for any smooth function f on $\mathcal{M}$ and $1 \leq j \leq d$. Let $\overline{V}_p^h$ denote the vector of $\mathbb{R}^d$ whose coordinates in the canonical basis are those of $\overline{v}_p^h$ in ϵ_p.

Then $\overline{x}^h_{p+1}$ is computed according to the formula:

$$\overline{x}^h_{p+1} = \phi_p^{-1}\left(\tilde{\alpha_p}(0)h + \sum_{j=1}^{r}\beta_{j,p}(0)U^j_{p+1}\sqrt{h}\right)$$

Let h and the random variables U^j_{p+1} be supposed to satisfy the requirement (HU) of the previous section, $h_0(\mathbb{P}_U)$ being now defined in the sense of the proposition (4.2); then $\overline{v}^h_{p+1}$ will be the vector of $T_{\overline{x}^h_{p+1}}\mathcal{M}$ whose coordinates in the basis ϵ_{p+1} are the coordinates in the canonical basis of $\mathbb{R}^d$ of

$$\overline{V}^h_{p+1} = \overline{M}^h_{p+1}\overline{V}^h_p$$

where

$$\overline{M}^h_{p+1} = Id + \partial\tilde{\alpha_p}(0)h + \sum_{j=1}^{r}\partial\beta_{j,p}(0)U^j_{p+1}\sqrt{h}.$$

4.3 Remark

In that case, the restriction to compact-supported laws in the hypothesis (HU) also ensures that for h small enough $\overline{V}^h_p \in B(0,R)$. For technical reasons appearing in the proof of the convergence, it is very important that it is obtained with the chart $(\phi_p, Dom(\phi_p))$, which is measurable w.r.t. the σ-field generated by $(\overline{U}^j_k\ ,\ 0 \le k \le p\ ,\ 1 \le j \le r)$.

4.4 An example of a good atlas

In the sequel, we will not use a particular good atlas, thus the aim of this subsection is just to check that the definition is nonvoid.

Below, we will give a Riemannian structure to $\mathcal{M}$; with this structure and its torsion-free connexion, the compact manifold $\mathcal{M}$ is complete (we can extend the geodesics up to infinity). Denote by Exp the mapping $(x,v) \to Exp_x v$ from $T\mathcal{M}$ to $\mathcal{M}$, where, for each $(x,v) \in T\mathcal{M}$, $Exp_x v$ is the point in $\mathcal{M}$ on the geodesic initiating at x in the direction v at time 1. For each $x \in \mathcal{M}$, Exp_x is a local diffeomorphism from $T_x\mathcal{M}$ to a neighborhood of x in $\mathcal{M}$; so using Exp_x^{-1}, choosing some basis in each $T_x\mathcal{M}$ and then identifying $T_x\mathcal{M}$ to $\mathbb{R}^d$, we can define, for each x in $\mathcal{M}$, a chart (that we will call "exponential") ϕ_x from a neighborhood $Dom(\phi_x)$ of x in $\mathcal{M}$ to $\mathbb{R}^d$. The Exponential Atlas $\mathcal{E}xp$ will be the set $\{(\phi_x, Dom(\phi_x)), x \in \mathcal{M}\}$ of exponential charts.

Let us recall known properties of $\mathcal{E}xp$ (cf. Kobayashi-Nomizu [14], section III, Bishop [6], chapter 8):

Properties of $\mathcal{E}xp$

(a) for each $(x, v) \in T\mathcal{M}$, the mapping $(x, v) \to Exp_x v$ from $T\mathcal{M}$ to $\mathcal{M}$ is C^∞;

(b) there exists a number $R > 0$ such that $Dom(\phi_x) \supset B(x, R)$ (the metric here is the Riemannian metric parallel to the connexion: see the lemma 8.2.3 in [6]);

(c) for each tensor K on $\mathcal{M}$, for each $x \in \mathcal{M}$ and for the canonical coordinate system $\{e_i, i = 1, \cdots, d\}$ associated with ϕ_x, the image of $\nabla_{e_i} K$ through $\mathcal{E}xp$ is $\frac{\partial \tilde{K}}{\partial x_i}$, where $\tilde{K}$ is the image through $\mathcal{E}xp$ of K (see the corollary III.8.5 in [14]).

Using these properties of $\mathcal{E}xp$, we define a new atlas, again denoted by $\mathcal{E}xp$, for which, for each $x \in \mathcal{M}$, $Val(\phi_x)$ is equal to the ball $B(0, R)$ in $\mathbb{R}^d$. The charts of this atlas will still be called exponential charts.

Proposition 4.3 *Let V be a smooth vector field on $\mathcal{M}$, and, for each $x \in \mathcal{M}$, let ν the expression in exponential coordinates around x of the vector field V; then, for any non void multi-index I, the derivative $\partial_I \nu$ is bounded on $Val(\phi_x)$ by a constant depending only on I (not on x).*

Proof The vector field V being C^∞ on $\mathcal{M}$, is globally bounded. Any covariant derivative is also bounded. Then the values at 0 of the images through $\mathcal{E}xp$ of the covariant derivatives of V, which are the successive derivatives of ν (as a consequence of the property (c) of the Exponential atlas) are globally bounded by a constant depending only on the order of derivation. The usual Taylor formula in $Val(\phi_x)$ applied to ν permits to get the desired result. $\square$

5 Existence of the upper Lyapunov exponent for the discretized system

In all the sequel, if there is no precision, $\mathcal{M}$ will denote either $\mathbb{R}^d$ or a $\mathcal{C}^\infty$ compact manifold. In order to avoid repetitions, we will adopt the notations corresponding to the second case. If the reader is interested in the $\mathbb{R}^d$ case only, he should substitute $\mathbb{R}^d$ to each tangent space, and v_t to $Tx_t(x)v$.

Under (H1) and (H2) (cf. Ikeda & Watanabe [13], chapter 5, e.g.), we can define on $\mathbb{P}\mathcal{M}$ a Riemannian measure $d\tau$ associated to the differential operator

$$PA + \frac{1}{2}\sum_{j=1}^{r}(PB_j)^2.$$

From now on, we consider $\mathbb{P}\mathcal{M}$ equipped with this Riemannian structure.

Once for all, we give ourselves an atlas $\mathcal{A}_{\mathbb{P}\mathcal{M}}$ on the projective bundle. When $\mathcal{M}$ is compact, we just choose a good atlas (in the sense of Definition (4.1)) on the compact manifold $\mathbb{P}\mathcal{M}$. When $\mathcal{M} = \mathbb{R}^d$, we choose a good atlas on $\mathbb{P}^{d-1}$ considered as a compact manifold in $\mathbb{R}^D$ (for some D whose existence is implied by the Whitney Theorem); the set $\{(Id \times \phi_x, \mathbb{R}^d \times Dom(\phi_x)), x \in \mathbb{P}^{d-1}\}$ is a good atlas on $\mathbb{P}\mathbb{R}^d$, still denoted by $\mathcal{A}_{\mathbb{P}\mathcal{M}}$.

5.1 Ergodicity of the process $(P\overline{x}_p^h)$

Let $P\overline{x}_p^h$ be the equivalence class of $(\overline{x}_p^h, \frac{\overline{v}_p^h}{|\overline{v}_p^h|})$ in $\mathbb{P}\mathcal{M}$. We now adapt a proposition in [22].

Proposition 5.1 *Under (H1), (H2) and (HU), for all h small enough, the process $(P\overline{x}_p^h)$ is an ergodic process on $\mathbb{P}\mathcal{M}$. The unique invariant probability law $\overline{\nu}^h$ of the process $(P\overline{x}_p^h)$ has a support equal to $\mathbb{P}\mathcal{M}$.*

Proof When $\mathcal{M}$ is $\mathbb{R}^d$, we consider the measure $1_K(z)dz \otimes d\rho$, where dz is the Lebesgue measure on $\mathbb{R}^d$, $d\rho$ is the trace of $d\tau$ on $\mathbb{P}^{d-1}$, and K is the compact set of (H2-ii).

Denote by $\mathcal{O}(h)$ any random matrix or random vector (respectively real random variable) whose norm (respectively absolute value) can be upper bounded almost surely by $C \cdot h$, C deterministic and independent of h; denote also

$$\overline{s}_p^h := \frac{\overline{V}_p^h}{|\overline{V}_p^h|} \quad ;$$

using the compactness of the support of the U_p^j's (see (HU-ii)), for any h small enough one has:

$$\begin{aligned}
\bar{s}_{p+1}^h &= \frac{\bar{V}_{p+1}^h}{|\bar{V}_{p+1}^h|} = \frac{\bar{V}_p^h + \sqrt{h}\sum_{j=1}^r U_{p+1}^j B_j'(\overline{x}_p^h)\bar{V}_p^h + \mathcal{O}(h)\bar{V}_p^h}{|\bar{V}_p^h|\{1 + 2\sqrt{h}\sum_{j=1}^r U_{p+1}^j < B_j'(\overline{x}_p^h)\bar{s}_p^h, \bar{s}_p^h > +\mathcal{O}(h)\}^{1/2}} \\
&= (1 - \sqrt{h}\sum_{j=1}^r U_{p+1}^j < B_j'(\overline{x}_p^h)\bar{s}_p^h, \bar{s}_p^h > +\mathcal{O}(h)) \\
&\qquad (\overline{s}_p^h + \sqrt{h}\sum_{j=1}^r U_{p+1}^j B_j'(\overline{x}_p^h)\overline{s}_p^h + \mathcal{O}(h)) \quad ,
\end{aligned}$$

from which (see (3)) one gets

$$P\bar{x}_{p+1}^h = P\bar{x}_p^h + \sqrt{h}\sum_{j=1}^r PB_j(P\bar{x}_p^h)U_{p+1}^j + \mathcal{O}(h). \tag{12}$$

Using that expansion, we now show that $(P\overline{x}_p^h)$ reaches any open set of strictly positive $1_K(z)dz \otimes d\rho$ measure in finite time with a strictly positive probability. Indeed, we deduce from (H1) and (H2-ii) that, for any starting point x, the process $(\overline{x}_p^h)$ reaches the compact set K in finite time with a strictly positive probability; let $\tilde{x}$ be the reached point in K, and let K_0 be an arbitrary compact subset of $K \times \mathbb{P}^{d-1}$; as the law of the U_{p+1}^j's is absolutely continuous w.r.t. the Lebesgue measure, one deduces from (12) that under (H2-i), there exists h_0 independent of $\tilde{x}$ and K_0 such that, for any $h < h_0$, for any v, $(P\bar{x}_p^h)$ reaches K_0 from $(\tilde{x}, v)$ in a finite number of steps with a strictly positive probability.

As moreover, (H2-ii) implies that, for all small enough h, there exists ε strictly positive, such that for all x outside K:

$$\mathbb{E}|\overline{x}_1^h(x)|^2 \leq |x|^2 - \varepsilon,$$

a result of Tweedie [25] implies the ergodicity of the process $(P\overline{x}_p^h)$. From the proof it is clear that the unique invariant measure has a support equal to $\mathbb{P}\mathbb{R}^d$.

When $\mathcal{M}$ is compact, the definitions of $\overline{x}_{p+1}^h$ and of a good atlas insure that the process $(P\overline{x}_p^h)$ is a Feller process on a compact metric space, so there exists at least one invariant probability law; the uniqueness and the identification of the support come from the fact that, from any starting point on $\mathcal{M}$, the process can reach any compact set of $\mathbb{P}\mathcal{M}$ in finite time with a strictly positive probability (as above, this follows from (H2) and the construction of the scheme).

□

5.2 Existence of the Lyapunov exponent for the discrete time processes

In either the case $\mathbb{R}^d$ or the case of a compact manifold, the process $(\overline{x}_p^h, \overline{M}_p^h)$ in $\mathcal{M} \times Gl(\mathbb{R}^d)$ is a multiplicative Markov process in the sense of Bougerol [7]: indeed, if $\bar{R}_p^h$ is its transition operator, for any Borel sets E_1 in $\mathcal{M}$ and E_2 in $Gl(\mathbb{R}^d)$:

$$\bar{R}_p^h((x_0, M)\ ;\ E_1 \times E_2 M) = \bar{R}_p^h((x_0, Id)\ ;\ E_1 \times E_2) \quad , \quad \forall p \in \mathbb{N} \ , \ x_0 \in \mathcal{M} \ , \ M \in Gl(\mathbb{R}^d)$$

where: $E_2 M = \{NM \in Gl(\mathbb{R}^d)\ ;\ N \in E_2\}$.

The next statement gives important properties of this multiplicative process. We denote by $\mathbb{P}_x(A)$ the probability of A conditioned by $\bar{x}_0^h = x$ (note that, by construction, $\bar{M}_0^h = Id$); if ρ is a probability law on $\mathbb{R}^d$, we denote by $\mathbb{P}_\rho$ the probability defined by $\mathbb{P}_\rho(A) = \int_{\mathbb{R}^d} \mathbb{P}_x(A) d\rho(x)$ and $\mathbb{E}_\rho$ the corresponding expectation.

Proposition 5.2 *Under the hypotheses (H1), (H2), (HU), for any h small enough:*

(i) the process $(\overline{x}_p^h)$ is ergodic;

(ii) let $\overline{\mu}^h$ be its unique invariant probability law; then

$$\sup_{p\leq \frac{1}{h}} \mathbb{E}_{\overline{\mu}^h}(Log||\overline{M}_p^h|| + Log||(\overline{M}_p^h)^{-1}||) < \infty$$

(iii) the system $(\overline{x}_p^h, \overline{M}_p^h)$ is irreducible, in the sense that there does not exist a measurable family $\{V^h(x), x \in \mathcal{M}\}$ of proper subspaces of $\mathbb{R}^d$ such that, for any p:

$$\bar{M}_p^h \dots \bar{M}_1^h V^h(x_0) = V^h(\overline{x}_p^h) \quad , \quad \mathbb{P}_{\bar{\mu}^h} - a.s.$$

Proof We showed in the Proposition 5.1 that the process $(P\overline{x}_p^h)$ is an ergodic process on $\mathbb{P}\mathcal{M}$, therefore $(\overline{x}_p^h)$ is an ergodic process.

The statement (ii) is satisfied whenever h is smaller than $h_0(\mathbb{P}_U)$.

Let us know prove that, for any h small enough, the irreductibility holds. If the system were not irreducible, the ergodic theorem would imply:

$$1 = \frac{1}{N}\sum_{p=0}^{N} \mathbb{1}_{[\bar{v}_p^h \in V^h(\bar{x}_p^h)]} = \int_{\mathbb{P}\mathcal{M}} \mathbb{1}_{[v\in V^h(x)]} d\overline{\nu}^h(x,v).$$

As the support of $\overline{\nu}^h$ is $\mathbb{P}\mathcal{M}$, this contradicts the fact that $\{V^h(x)\}$ is a family of proper subspaces. □

A Corollary of Proposition 5.2 (cf. Bougerol [8], Section 2) is:

Theorem 5.3 *Under the hypotheses (H1), (H2), (HU), for any h small enough, there exists $\overline{\lambda}^h \in \mathbb{R}$ such that, for any x_0 in $\mathcal{M}$, for any v_0 in $T_{x_0}\mathcal{M} - \{0\}$:*

$$\lim_{N\to\infty} \frac{1}{Nh} \log|\overline{v}_N^h(v_0)| = \overline{\lambda}^h \quad , \quad a.s.$$

Remark 5.4 *The limit of $\{\frac{1}{Nh}log|\overline{v}_N^h(v_0)|, N \to \infty\}$ holds also in L^1 because the sequence $\{\frac{1}{N}log|\overline{v}_N^h(v_0)|, N \in \mathbb{N}\}$ is uniformly integrable (cf. Proposition 4.1 in Talay [21]).*

The number $\overline{\lambda}^h$ is called the top Lyapunov exponent of the Markov process $(\overline{x}_p^h, \overline{v}_p^h)$.

Numerically, we will compute, for N large enough, the quantity:

$$\frac{1}{Nh} \log|\overline{v}_N^h(v_0)| \tag{13}$$

6 Approximation error

The theoretical approximation error is given by the

Theorem 6.1 *Under the hypotheses (H1), (H2), (HU):* $|\lambda - \overline{\lambda}^h| = \mathcal{O}(h)$.

Now, let us again denote:

$$\overline{s}_p^h := \frac{\overline{V}_p^h}{|\overline{V}_p^h|}$$

Using the Remark (5.4), and the definition of $\overline{V}_p^h$, we can write:

$$\overline{\lambda}^h = \lim_{N\to\infty} \frac{1}{Nh} \sum_{p=1}^{N} \mathbb{E} \log |\overline{M}_{p+1}^h \overline{s}_p^h|$$

The proof of Theorem (6.1) will be a succession of lemmas.

Lemma 6.2

$$\mathbb{E} \log |\overline{M}_{p+1}^h \overline{s}_p^h| = \mathbb{E}\psi(P\overline{x}_p^h)h + \overline{r}_p^h h^2 \tag{14}$$

with: ψ *defined in (9),* $|\overline{r}_p^h| \leq C$ *(for some constant* C *independent from* p, h*).*

Proof For any Stochastic Differential Equation in $\mathbb{R}^d$ whose coefficients are smooth with bounded derivatives of any order:

$$dy_t = a(y_t)dt + \sum_{j=1}^{r} b_j(y_t) \circ dW_t^j$$

and for any smooth bounded function f with bounded derivatives on the set of values taken by $\overline{y}_p^h$, one can check that the Euler scheme

$$\overline{y}_{p+1}^h = \overline{y}_p^h + (a(\overline{y}_p^h) + \frac{1}{2}\sum_{j=1}^{r} \partial b_j(\overline{y}_p^h) b_j(\overline{y}_p^h))h + \sum_{j=1}^{r} b_j(\overline{y}_p^h) U_{p+1}^j \sqrt{h}$$

satisfies:

$$\mathbb{E}f(\overline{y}_{p+1}^h) = \mathbb{E}f(\overline{y}_p^h) + \mathbb{E}\mathcal{L}f(\overline{y}_p^h)h + \overline{z}_{p+1}^h h^2$$

where $\mathcal{L}$ is the infinitesimal generator of (y_t), and $|\overline{z}_{p+1}^h| \leq C$ (C independent of p, h).

Besides, we know (Baxendale [5]) that, for any initial condition (x_0, v_0) of the system (5), if (Px_t) is the solution of (6), then:

$$\frac{d}{dt}\mathbb{E} \log |Tx_t(x_0)v_0| = \mathbb{E}\psi(Px_t(x_0, [v_0]))$$

In other words, if we define $\phi(x, v) = \log|v|$, then $\psi \circ \pi = \mathcal{L}\phi$, where $\mathcal{L}$ is the infinitesimal generator of the process (Tx_t) and π is the application from $T\mathcal{M}$ to $\mathbb{P}\mathcal{M}$ defined by: $\pi(x, v) = (x, [v])$.

Now consider the function ψ defined by (9). In the case $\mathcal{M} = \mathbb{R}^d$, the covariant derivative is just the ordinary derivative, and $\mathcal{R} = 0$. Thus, under (H1), ψ is a $\mathcal{C}^\infty$ bounded function. If $\mathcal{M}$ is a compact manifold, applying Proposition (4.3), we remark that, for each multi-index I, there exists a positive constant C_I such that, for any chart $(\phi, Dom(\phi))$ of the atlas $\mathcal{A}_{\mathbb{P}\mathcal{M}}$ on $\mathbb{P}\mathcal{M}$, the partial derivative $\partial_I\psi$ of the function ψ expressed in local coordinates satisfies:

$$|\partial_I(\psi \circ \phi^{-1})(y)| < C_I \quad , \quad \forall y \in Val(\phi) .$$

Therefore, to get (14) in the $\mathbb{R}^d$ case, we simply choose $y_t = (x_t, v_t)$ and $f(x, v) = \log|v|$, remarking that the inequality (11) ensures that the process $(\overline{M}^h_{p+1}\overline{s}^h_p)$ lives in a compact manifold which does not include 0, on which f and its derivatives are smooth. In the compact case, we express $(x_t(x), Tx_t(x)v)$ and $\log|v|$ in the local coordinates given by the atlas $\mathcal{A}_{\mathbb{P}\mathcal{M}}$ on $\mathbb{P}\mathcal{M}$, and we remark that $\overline{v}^h_p = \overline{V}^h_p$.

□

Iterating (14), one gets: $\mathbb{E}\log|\overline{M}^h_N\overline{s}^h_{N-1}| = \sum_{p=1}^{N-1}\mathbb{E}\psi(P\overline{x}^h_p)h + \sum_{p=1}^{N-1}\overline{r}^h_p h^2$.

Let us divide each term of the previous equality by Nh and make N tend to infinity.

Remember that $\overline{\nu}^h$ denotes the unique invariant probability law of the process $(P\overline{x}^h_p)$R; then the limit

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{p=1}^{N}\mathbb{E}\psi(P\overline{x}^h_p)$$

exists, and is equal to $\int_{\mathbb{P}\mathcal{M}}\psi(\theta)\overline{\nu}^h(d\theta)$.

Therefore, the conclusion of our Theorem will be implied by the following Lemma, whose hypothesis on f is weaker than necessary here (bounded would be sufficient, but the boundedness brings no simplification in the proof):

Lemma 6.3 *Let f be a smooth function on $\mathbb{P}\mathcal{M}$; when $\mathcal{M} = \mathbb{R}^d$, we suppose that for φ being f or any of its partial derivatives of any order with respect to the x-coordinates:*

$$\exists C > 0 \quad , \quad \exists n \in \mathbb{N} \quad , \quad \forall\theta = (x, [v]) \in \mathbb{P}\mathbb{R}^d \quad : \quad |\varphi(\theta)| \leq C(1 + |x|^n)$$

Then:

$$|\int_{\mathbb{P}\mathcal{M}} f(\theta)\nu(d\theta) - \int_{\mathbb{P}\mathcal{M}} f(\theta)\overline{\nu}^h(d\theta)| = \mathcal{O}(h)$$

Proof Let L be the infinitesimal generator of the process (Px_t) solution of (6) on $\mathbb{P}\mathcal{M}$, and, for a given smooth function f on $\mathbb{P}\mathcal{M}$, and $\theta \in \mathbb{P}\mathcal{M}$, let $u(t, \theta) := \mathbb{E}_\theta f(Px_t)$.

It is well known that

$$\begin{aligned} \frac{d}{dt}u(t,\theta) &= Lu(t,\theta) \\ u(0,\theta) &= f(\theta) \end{aligned} \tag{15}$$

Let us suppose that we have proved (this will be done in the next Section):

Lemma 6.4 *Under (H1), (H2), if $\mathcal{M}$ is compact:*

(i) there exist strictly positive constants Γ and γ such that

$$\forall\theta \in \mathbb{P}\mathcal{M} \quad : \quad |u(t,\theta) - \int_{\mathbb{P}\mathcal{M}} f(s)d\nu(s)| \leq \Gamma \exp(-\gamma t) \tag{16}$$

(ii) for any multi-index I, there exist strictly positive constants Γ_I and γ_I such that, for any $\theta = (x,[v])$ in $\mathbb{P}\mathcal{M}$, any z in $Val(\phi_\theta)$ (where $(\phi_\theta, Dom(\phi_\theta))$ is a good chart around θ in the sense defined at the beginning of Section (5)), the spatial derivative $\partial_I u(t,\phi_\theta^{-1}(z))$ satisfies:

$$|\partial_I u(t,\phi_\theta^{-1}(z))| \leq \Gamma_I \exp(-\gamma_I t) \tag{17}$$

When $\mathcal{M}$ is $\mathbb{R}^d$, the inequality (16) becomes the following, where n_I is an integer:

$$\forall\theta \in \mathbb{P}\mathbb{R}^d \quad : \quad |u(t,\theta) - \int_{\mathbb{P}\mathcal{M}} f(s)d\nu(s)| \leq \Gamma \exp(-\gamma t)(1+|x|^{n_I}) \tag{18}$$

and the inequality (17) becomes:

$$|\partial_I u(t,\phi_\theta^{-1}(z))| \leq \Gamma_I \exp(-\gamma_I t)(1+|x|^{n_I}) \tag{19}$$

Then tedious computations (where the remark in Section (4.3) concerning the measurability of the charts around $\overline{x}_p^h$, and the Proposition (4.3) play a role) show that our construction implies:

$$\forall k \quad , \quad \forall p \quad , \quad \mathbb{E}u(kh, P\overline{x}_{p+1}^h) = \mathbb{E}u(kh, P\overline{x}_p^h) + \mathbb{E}Lu(kh, P\overline{x}_p^h)h + \overline{r}_{k,p+1}^h h^2$$

where $|\overline{r}_{k,p+1}^h| \leq C_1 e^{-C_2 kh}$ (for some strictly positive constants C_1, C_2 independent from p,k,h). Then we proceed as in the proof of Lemma 4.4 of Talay [21].

□

7 Proof of Lemma (6.4)

In Talay [21] (Lemma 4.3), a similar result has been obtained for processes on the sphere S^{d-1}; the construction of the approximating process was different, but the proof can be adapted without difficulty for the case where $\mathcal{M}$ is compact, just by choosing the atlas $\mathcal{A}_{\mathbb{P}\mathcal{M}}$ on the compact manifold $\mathbb{P}\mathcal{M}$.

For the case $\mathbb{R}^d$, the adaptation is more intricate.

7.1 Preliminary inequalities

Let us state a result concerning the partial derivatives of u. We recall that we work with the atlas $\mathcal{A}_{\mathbb{P}\mathcal{M}}$ on $\mathbb{P}\mathbb{R}^d$ (in the sense of the Section (5)).

Lemma 7.1 *For any t and any multi-index I, there exist strictly positive constants $C_I(t)$ and n_I (n_I independent of t) such that, for any $\theta = (x, [v])$ in $\mathbb{P}\mathbb{R}^d$, any (x, y) in $Val(\phi_\theta)$, the spatial derivative $\partial_I u(t, \phi_\theta^{-1}(x, y))$ satisfies:*

$$|\partial_I u(t, \phi_\theta^{-1}(x, y))| \leq C_I(t)(1 + |x|^{n_I}) \tag{20}$$

Proof Having imbedded $\mathbb{P}^{d-1}$ in $\mathbb{R}^D$ (for some D), we can extend the vector fields PA and PB_j ($j = 1, \ldots, r$) to $\mathcal{C}^\infty$ vector fields on $\mathbb{R}^d \times \mathbb{R}^D$ with supports of the form $\mathbb{R}^d \times \mathcal{K}$, $\mathcal{K}$ compact. Let us denote by (Z_t) the solution of the corresponding stochastic differential system. It is well known (cf. Kunita [15] e.g.) that the stochastic flow $(Z_t(z))$ associated to this system satisfies: for any integer i ($i = 1, \ldots, d$), there exists a random variable L_t^i, having moments of all orders, such that

$$|\partial_i Z_t(z)| \leq L_t^i (1 + |z|^2)$$

We also extend the function f to a $\mathcal{C}^\infty$ function g on $\mathbb{R}^d \times \mathbb{R}^D$, with a support of the form $\mathbb{R}^d \times \mathcal{K}_1$, $\mathcal{K}_1$ compact. Let us define: $\zeta(t, z) = \mathbb{E} g(Z_t(z))$. We then have, for any multi-index I and some constants n_I, $C_I(t)$:

$$|\partial_I \zeta(t, z)| \leq C_I(t)(1 + |z|^{n_I})$$

The restriction of $\zeta(t, z)$ to the cartesian product of $\mathbb{R}^d$ and the imbedding of $\mathbb{P}^{d-1}$ is $u(t, \theta)$; expressing θ in the local coordinates given by the atlas $\mathcal{A}_{\mathbb{P}\mathcal{M}}$, we then deduce (20).

□

Remarks

- In the sequel, we will not go on working with the process (Z_t), because the associated differential operator is not strongly elliptic. Thus we cannot avoid working on $\mathbb{P}\mathbb{R}^d$.
- For any integer s, let us now define:

$$\pi_s(x) = \frac{1}{(1 + |x|^2)^s}$$

  The previous Lemma shows that, for any integer $n \geq 0$, there exists an integer s_n such that, for any multi-index I of length $l(I)$ smaller than n, for any $t \geq 0$:

$$|\partial_I u(t, \phi_\theta^{-1}(x, y))| \pi_{s_n}(x) \in L^2(\mathbb{R}^d \times Val(\phi_\theta)) \tag{21}$$

  This will be often implicitly used in the sequel to justify the existence of integrals w.r.t. $d\nu$ (we recall that ν denotes the unique invariant probability law of $(x_t, [v_t])$), or measures of type $\pi_s(x)dx$.

Without loss of generality, thereafter we will assume:

$$\int_{\mathbb{P}\mathbb{R}^d} f(s)d\nu(s) = 0 \tag{22}$$

(if it is not the case, we change f in $\tilde{f} = f - \int_{\mathbb{P}\mathbb{R}^d} f(s)d\nu(s)$).

Now, we will adapt for $u(t,\theta)$ the method used in [22] to establish some exponential decay results on the function $(t,x) \to \mathbb{E}g(x_t(x))$ and its derivatives.

We begin by an easy Proposition, proved in [22].

Proposition 7.2 *(i) For any integer n:*

$$\exists C_n > 0 \ , \ \exists \gamma_n > 0 \ : \ \mathbb{E}|x_t(x)|^n \leq C_n(1 + |x|^n \exp(-\gamma_n t)) \quad , \quad \forall t \ , \ \forall x \tag{23}$$

(ii) The unique invariant probability measure of (x_t), μ, has a smooth density $p(x)$ and finite moments of any order.

This implies that any function f satisfying the requirements of the Lemma (6.3) belongs to $L^2(\mathbb{P}\mathcal{M}; d\nu)$.

The plan of the sequel of the proof is the following (in all the inequalities which follow, the constants must be understood strictly positive):

- we show:
$$\forall t > 0 \quad , \quad \int_{\mathbb{P}\mathbb{R}^d} |u(t,\theta)|^2 d\nu(\theta) \leq C \exp(-\kappa t)$$
- this inequality permits to show that, for some family of differential operators L_k defined below, we have:
$$\int_{\mathbb{P}\mathbb{R}^d} |L_{k_1} \dots L_{k_q} u(t,\theta)|^2 d\nu(\theta) \leq C_q \exp(-\kappa_q t)$$
- then, $d\rho([v])$ denoting the marginal distribution of $d\tau(x,[v])$ on $\mathbb{P}^{d-1}$, we use these estimates to prove that, for s large enough:
$$\forall t > 0 \quad , \quad \int |u(t,x,[v])|^2 \pi_s(x) dx \otimes d\rho([v]) \leq C \exp(-\lambda t)$$
- the previous inequality will permit to get:
$$\int_{\mathbb{P}\mathbb{R}^d} L\left(|L_{k_1} \dots L_{k_q} u(t,x,[v])|^2\right) \pi_s(x) dx \otimes d\rho([v]) \leq C'_q \exp(-\lambda_q t)$$
- finally, we get:
$$\int_{\mathbb{P}\mathbb{R}^d} |L_{k_1} \dots L_{k_q} u(t,x,[v])|^2 \pi_s(x) dx \otimes d\rho([v]) \leq D_q \exp(-\gamma_q t)$$

  Expressing this inequality in local coordinates, we use the Sobolev imbedding theorem to conclude.

7.2 Upper bounds in $L^2(\mathbb{P}\mathbb{R}^d; d\nu(\theta))$

We now state a result, which is an immediate extension of the Lemma 6.1 of [22] (one just needs to remark that the hypothesis (H2-i) implies that ν, the unique invariant probability law of $(x_t, [v_t])$, has a strictly positive smooth density).

Lemma 7.3 *Under the hypotheses of Theorem (6.1) and (22), there exist strictly positive constants C and κ such that*

$$\forall t > 0 \quad , \quad \int_{\mathbb{P}\mathbb{R}^d} |u(t,\theta)|^2 d\nu(\theta) \leq C \exp(-\kappa t) \tag{24}$$

Let us define some differential operators on $\mathbb{P}\mathbb{R}^d$ by:

$$L_j = PB_j$$

Now, remembering that L is the differential operator associated to $(x_t, [v_t])$, we are equipped to prove:

Lemma 7.4 *Under (H1), (H2) and (22), there exist strictly positive constants C_1 and κ_1 such that*

$$\sum_{k=1}^{r} \int_{\mathbb{P}\mathbb{R}^d} L_k |u(t,\theta)|^2 d\nu(\theta) \leq C_1 \exp(-\kappa_1 t) \tag{25}$$

Proof We remark:

$$\frac{d}{dt}|u(t,\theta)|^2 - L|u(t,\theta)|^2 = -\sum_{k=1}^{r} (L_k u(t,\theta))^2$$

Let us choose $0 < \delta < \kappa$. Multiplying the previous equality by $e^{\delta t}$, integrating with respect to $d\nu$, we get, using $L^*\nu = 0$:

$$e^{\delta t}\frac{d}{dt}\int_{\mathbb{P}\mathbb{R}^d} |u(t,\theta)|^2 d\nu(\theta) + e^{\delta t}\sum_{k=1}^{r}\int_{\mathbb{P}\mathbb{R}^d} |L_k u(t,\theta)|^2 d\nu(\theta) \leq 0$$

Now, let us choose an arbitrarily large time T and integrate from 0 to T the previous inequality; we obtain:

$$\begin{aligned} e^{\delta T}\int_{\mathbb{P}\mathbb{R}^d} |u(T,\theta)|^2 d\nu(\theta) &+ \int_0^T e^{\delta t}\left(\sum_{k=1}^{r}\int_{\mathbb{P}\mathbb{R}^d} |L_k u(t,\theta)|^2 d\nu(\theta)\right) dt \\ &\leq \int_{\mathbb{P}\mathbb{R}^d} |f(\theta)|^2 d\nu(\theta) + \delta\int_0^T e^{\delta t}\left(\int_{\mathbb{P}\mathbb{R}^d} |u(t,\theta)|^2 d\nu(\theta)\right) dt \end{aligned}$$

Thus, using (24):

$$\sum_{k=1}^{r}\int_0^{+\infty} e^{\delta t}\left(\int_{\mathbb{P}\mathbb{R}^d} |L_k u(t,\theta)|^2 d\nu(\theta)\right) dt < +\infty$$

Then, we remark that there exist strictly positive constants C_2, C_3 and C_4 such that

$$\begin{aligned}\frac{d}{dt}\sum_{k=1}^{r}|L_k u(t,\theta)|^2 - L\sum_{k=1}^{r}|L_k u(t,\theta)|^2 \\ \leq -C_2\sum_{k,l=1}^{r}|L_k L_l u(t,\theta)|^2 + (C_3|x|+C_4)\sum_{k=1}^{r}|L_k u(t,\theta)|^2\end{aligned}$$

Let us choose $0 < \kappa_1 < \delta$. Multiplying the previous equality by $e^{\kappa_1 \xi}$, integrating with respect to $d\nu(\theta)$, and then with respect to ξ from 0 to t, we obtain, for some strictly positive constant C_1:

$$\sum_{k=1}^{r}\int_{\mathbb{P}\mathbb{R}^d} |L_k u(t,\theta)|^2 d\nu(\theta) \leq C_1 e^{-\kappa_1 t}$$

That ends the proof.

□

Corollary 7.5 *Under (H1), (H2) and (22), for any integer q, there exist strictly positive constants C_q and κ_q such that, for any $k_1, \dots k_q$ in $\{1, \dots r\}$, we have:*

$$\int_{\mathbb{P}\mathbb{R}^d} |L_{k_1}\dots L_{k_q} u(t,\theta)|^2 d\nu(\theta) \leq C_q \exp(-\kappa_q t) \tag{26}$$

Proof One can proceed by a recurrence over q, and performing the same kind of integrations as before, starting from the inequality (where C_J and C_q are some strictly positive constants):

$$\begin{aligned}\frac{d}{dt}|L_{k_1}\dots L_{k_q}u(t,\theta)|^2 - L|L_{k_1}\dots L_{k_q}u(t,\theta)|^2 \\ \leq -C_q\sum_{j=1}^{r}L_j|L_{k_1}\dots L_{k_q}u(t,\theta)|^2 + \sum_{m=1}^{q}\sum_{J=(j_1\dots j_m)} C_J(|x|+1)|L_{j_1}\dots L_{j_m}u(t,\theta)|^2\end{aligned}$$

□

We recall that for any integer s, we have defined:

$$\pi_s(x) = \frac{1}{(1+|x|^2)^s}$$

and that $d\rho([v])$ denotes the trace of $d\tau(x,[v])$ on $\mathbb{P}^{d-1}$.

7.3 Upper bounds in $L^2(\mathbb{P}\mathbb{R}^d; \pi_s(x)dx \otimes d\rho([v]))$

Lemma 7.6 *Under the hypotheses (H1), (H2) and (22), there exist strictly positive constants C and λ such that*

$$\forall t > 0 \quad , \quad \int_{\mathbb{P}\mathbb{R}^d} |u(t,x,[v])|^2 \pi_s(x) dx \otimes d\rho([v]) \leq C \exp(-\lambda t) \tag{27}$$

Proof As, for any multi-index J, we have:

$$\partial_J \pi_s(x) = \psi_J(x) \pi_s(x) \quad , \quad \psi_J(x) \ \textit{bounded function} \tag{28}$$

we remark that there exist an integer s and functions $\phi_1(x)$ and $\phi_2(x)$ such that

- $\phi_1(x)$ is a bounded function independent of s;
- $\phi_2(x)$ is a function depending on s, but tending to 0 when $|x| \longrightarrow +\infty$;
- the following inequality holds:

$$\begin{aligned}
&\int_{\mathbb{P}\mathbb{R}^d} u(t,x,[v]) Lu(t,x,[v]) \pi_s(x) dx \otimes d\rho([v]) \\
&\quad \leq \int_{\mathbb{P}\mathbb{R}^d} \left(\phi_1(x,[v]) + \phi_2(x,[v]) + s \frac{< A(x), x >}{1+|x|^2} \right) |u(t,x,[v])|^2 \pi_s(x) dx \otimes d\rho([v]) \\
&\qquad - C \sum_{k=1}^{r} \int |L_k u(t,x,[v])|^2 \pi_s(x) dx \otimes d\rho([v])
\end{aligned}$$

After having possibly increased the value of s, we can choose a ball $B = B(0, R_0)$ such that

$$\forall x \in \mathbb{R}^d - B \quad , \quad \forall v \quad , \quad \phi_1(x,[v]) + \phi_2(x,[v]) + s \frac{< A(x), x >}{1+|x|^2} < -1 \tag{29}$$

We separate the previous integral over $\mathbb{P}\mathbb{R}^d$, in an integral over B plus an integral over the complementary set of B. The estimation (24) permits to get an upper bound of the form $C \exp(-\lambda t)$ for the integration on B, since ν has a smooth and strictly positive density.

An easy computation then shows that, for some strictly positive constant C:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{P}\mathbb{R}^d} |u(t,x,[v])|^2 \pi_s(x) dx \otimes d\rho([v]) \leq - \int |u(t,x[v])|^2 \pi_s(x) dx \otimes d\rho([v]) + C \exp(-\lambda t)$$

That ends the proof.

$\square$

Now, we remark that there exist an integer s and functions $\varphi_1(x)$ and $\varphi_2(x)$ such that

- $\varphi_1(x)$ is a bounded function independent of s;
- $\varphi_2(x)$ is a function depending on s, but tending to 0 when $|x| \longrightarrow +\infty$;
- the following equality holds:

$$\begin{aligned}&\int_{\mathbb{PR}^d} L|u(t,x,[v])|^2\pi_s(x)dx \otimes d\rho([v])\\ &\quad= \int_{\mathbb{PR}^d} |u(t,x,[v])|^2 L^*\pi_s(x)dx \otimes d\rho([v])\\ &\quad= \int_{\mathbb{PR}^d} \left(\varphi_1(x) + \varphi_2(x) + 2s\frac{< A(x), x >}{1+|x|^2}\right)|u(t,x,[v])|^2\pi_s(x)dx \otimes d\rho([v])\end{aligned}$$

Therefore, similar arguments as those used for the previous Lemma and (27) show that there exist an integer s and strictly positive constant C_1, λ_1 satisfying:

$$\int_{\mathbb{PR}^d} L|u(t,x,[v])|^2\pi_s(x)dx \otimes d\rho([v]) \leq C_1 \exp(-\lambda_1 t) \tag{30}$$

As well, using also arguments employed in the previous Section, we can prove the existence of strictly positive constants λ_q and C_q such that, for any $k_1, \dots k_q$ in $\{1, \dots r\}$, we have:

$$\int_{\mathbb{PR}^d} L\left(|L_{k_1}\dots L_{k_q}u(t,x,[v])|^2\right)\pi_s(x)dx \otimes d\rho([v]) \leq C_q \exp(-\lambda_q t) \tag{31}$$

7.4 End of the Proof of Lemma (6.4)

We remark again:

$$\frac{d}{dt}|u(t,\theta)|^2 - L|u(t,\theta)|^2 = -\sum_{k=1}^{r}(L_k u(t,\theta))^2$$

Let us choose $0 < \delta < \lambda_0$. Multiplying the previous equality by $e^{\delta t}$, integrating with respect to $\pi_s(x)dx \otimes d\rho([v])$, we get, using (30):

$$\begin{aligned}&e^{\delta t}\frac{d}{dt}\int_{\mathbb{PR}^d} |u(t,x,[v])|^2\pi_s(x)dx \otimes d\rho([v]) + e^{\delta t}\sum_{k=1}^{r}\int_{\mathbb{PR}^d}(L_k u(t,x,[v]))^2\pi_s(x)dx \otimes d\rho([v])\\ &\quad\leq C_0 e^{-(\lambda-\delta)t}\end{aligned}$$

Now, let us choose an arbitrarily large time T and integrate from 0 to T the previous inequality; we obtain, for some strictly positive constants C_1 and s large enough:

$$\begin{aligned}&e^{\delta T}\int_{\mathbb{PR}^d} |u(T,x,[v])|^2\pi_s(x)dx \otimes d\rho([v]) +\\ &+\int_0^T e^{\delta t}\left(\sum_{k=1}^{r}\int_{\mathbb{PR}^d}|(L_k u(t,x,[v])|^2\pi_s(x)dx \otimes d\rho([v])\right)dt\\ &\qquad\leq \int_{\mathbb{PR}^d}|f(x,[v])|^2\pi_s(x)dx \otimes d\rho([v])\\ &\qquad+\delta\int_0^T e^{\delta t}\left(\int_{\mathbb{PR}^d}|u(t,x,[v])|^2\pi_s(x)dx \otimes d\rho([v])\right)dt + C_1\end{aligned}$$

Thus:

$$\sum_{k=1}^{r}\int_0^{+\infty} e^{\delta t}\left(\int_{\mathbb{PR}^d} |L_k u(t,x,[v])|^2 \pi_s(x)dx \otimes d\rho([v])\right) dt < +\infty$$

If necessary, we increase the value of s, in order to obtain that, for any constants D_1 and D_2:

$$\sum_{k=1}^{r}\int_0^{+\infty} e^{\delta t}\left(\int_{\mathbb{PR}^d} (D_1|x|+D_2)|L_k u(t,x,[v])|^2 \pi_s(x)dx \otimes d\rho([v])\right) dt < +\infty \tag{32}$$

Then, we remark that there exist strictly positive constants C_2, C_3 and C_4 such that

$$\begin{aligned}&\frac{d}{dt}\Big(\sum_{k=1}^{r}|L_k u(t,x,[v])|^2\Big) - L\Big(\sum_{k=1}^{r}|L_k u(t,x,[v])|^2\Big)\\ &\quad \le -C_2\sum_{k,l=1}^{r}|L_k L_l u(t,x,[v])|^2 + (C_3|x|+C_4)\sum_{k=1}^{r}|L_k u(t,x,[v])|^2\end{aligned}$$

Let us choose $0 < \delta_1 < \delta$. Multiplying the previous equality by $e^{\delta_1 \xi}$, integrating with respect to $\pi_s(x)dx \otimes d\rho([v])$, and then with respect to ξ from 0 to t, using (31) and (32), we obtain for some strictly positive constant C_5:

$$\sum_{k=1}^{r}\int_{\mathbb{PR}^d} |L_k u(t,x,[v])|^2 \pi_s(x)dx \otimes d\rho([v]) \le C_5 e^{-\delta_1 t}$$

From the previous inequality and (H2-i), we deduce the existence of strictly positive constants C, γ, independent on $\theta = (x,[v])$, such that

$$\int_{Val(\phi_\theta)} |\nabla u(t,\phi_\theta^{-1}(z,y))|^2 \pi_s(z)dzdy \le Ce^{-\gamma t}$$

By the same procedure, one may show that, for any sequence $J = (k_1,\ldots,k_q)$ of integers in $1,\ldots,r$, there exist strictly positive constants C_J and γ_J such that

$$\int_{\mathbb{PR}^d} |L_{k_1}\ldots L_{k_q} u(t,x,[v])|^2 \pi_s(x)dx \otimes d\rho([v]) \le C_J \exp(-\gamma_J t)$$

We deduce that for any multi-index J (refering to derivatives w.r.t. the coordinates (x,y)), there exist strictly positive constants C_J and γ_J, independent of θ, such that

$$\int_{Val(\phi_\theta)} |\partial_J u(t,\phi_\theta^{-1}(z,y))|^2 \pi_s(z)dzdy \le C_J \exp(-\gamma_J t)$$

We already have remarked that for any multi-index J:

$$\partial_J \pi_s(z) = \psi_J(z)\pi_s(z) \quad , \quad \psi_J(z) \ \textit{bounded}$$

Therefore we get that, for any multi-index I, any integer M, there exist an integer s and strictly positive constants C and λ such that

$$\forall m \leq M \ , \ \forall t > 0 \ \ , \quad \int_{Val(\phi_\theta)} |D^m(u(t,x,y)\pi_s(x))|^2 dxdy \leq C \exp(-\lambda t)$$

so that we deduce (20) as a consequence of the Sobolev imbedding Theorem (in (4.3), we have constrained $Val(\phi_\theta)$ to be a convex domain).

8 Approximation of the spectrum

The approximation of all the Lyapunov exponents may be performed in the following way.

For x in $\mathcal{M}$, and any integer k ($1 \leq k \leq d$), let $G_k(T_x\mathcal{M})$ be the Grassmann manifold of all k-dimensional subspaces of the tangent space $T_x\mathcal{M}$, and $G_k(T\mathcal{M})$ be the Grassmann bundle $\cup_{x\in\mathcal{M}} G_k(T_x\mathcal{M})$.

For any K in $G_k(T_x\mathcal{M})$, let $K_t = Tx_t(x)K \subset T_{x_t(x)}\mathcal{M}$, and $J_t(K) = |det(Tx_t(x)|_K)|$.

Let us suppose that the above hypothesis (H1) holds and that the infinitesimal generators of the processes (K_t) are strongly elliptic.

In this context (and under even weaker hypotheses: Baxendale [5]), for any $1 \leq k \leq d$, there exist real numbers $\lambda_1, \ldots, \lambda_k$ and a smooth function ψ_k on $G_k(T\mathcal{M})$ such that, for any x in $\mathcal{M}$ and K in $G_k(T_x\mathcal{M})$, if ν now denotes the unique invariant probability law of (K_t):

$$a.s. \lim_{t\to+\infty} \frac{1}{t} \log J_t(K) = \lambda_1 + \ldots + \lambda_k = \int_{G_k(T\mathcal{M})} \psi_k(\theta)\nu(d\theta)$$

To approximate the exponents, we first choose $k = 1$ and approximate $\lambda_1 = \lambda$ as described in the previous Sections; then we choose an orthonormal basis (v_0^1, v_0^2) of a sub-space K of $T_{x_0}\mathcal{M}$ for initial value of $\overline{v}_p^h$, and we compute

$$\frac{1}{N} \log |det(\overline{v}_N^{h,1}, \overline{v}_N^{h,2})| \tag{33}$$

Going on up to d, we successively get approximations of all the sums $\lambda_1 + \ldots + \lambda_k$. This provides the approximate values of the exponents. With the same technique as above, we may prove that the approximation error on each of these sums is of order h, and therefore the error on each λ_k is also of order h.

9 Numerical tests

In this section, we present numerical results corresponding to two simple examples, our objective being to show the typical behaviour of the approximation in terms of the integration time and of the true value of the Lyapunov exponent.

Note also that, even in the simple second situation, if a more complex scheme than the Euler scheme is chosen, the complexity of the expressions of several derivatives of the coefficients in the local coordinates corresponding to the proposed good atlas, drastically decreases the efficiency of the algorithm.

The Fortran programs corresponding to the two examples below, have been generated by Presto, a software of automatic generation of programs for the simulation of S.D.E.'s [23].

9.1 System in $\mathbb{R}^d$

We choose $d = r = 1$, and consider the following system, in the Ito sense:

$$dx_t = (-ax_t + F(x_t))dt + G(x_t)dW_t.$$

Linearizing it, it is easy to see that, for $a > 0$ and F, G with continuous bounded derivatives, if $G(x) > G_0 > 0$ for all x, then the upper Lyapunov exponent of the system exists and is given by:

$$\lambda = -a + \int_{\mathbb{R}} (F'(x) - \frac{1}{2} G'^2(x))p(x)dx$$

where $p(x)$ is the density of the unique invariant probability law of (x_t).

We then choose: $F(x) = \arctan(x)$ and $G(x) = \sqrt{1+x^2}$.

Solving the stationary Fokker-Planck equation, we get the explicit expression of the unnormalized stationary density. For each value of a, we can compute numerically the normalization constant (just integrating over $\mathbb{R}$ the unnormalized density). This permits us to compute the "true" value of λ.

For example, for $a = 2$, we find $\lambda = -1.3385$. The figure 1 below shows the time evolution of $\overline{\lambda}_p^h$ (in the x-axis: ph), for $h = 0.01$.

9.2 System on the circle

Let us consider the stochastic differential system in the Stratonovich sense on $\mathcal{M} = \mathcal{S}^1$:

$$\begin{aligned} d\varphi_t &= \sin(\varphi_t) \circ dW_t^1 + \cos(\varphi_t) \circ dW_t^2 \\ \varphi_0 &= \varphi^* \end{aligned} \tag{34}$$

where $W_t = (W_t^1, W_t^2)$ is a standard vectorial Brownian Motion. This equation describes a Brownian Motion on the circle and the upper Lyapunov exponent is $\lambda = -\frac{1}{2}$ (cf. Arnold [1]).

Figure 1: Variations of the Lyapunov exponent in terms of *ph*

Considering S^1 imbedded in $\mathbb{R}^2$, let us denote by $x_t := (\cos\varphi_t, \sin\varphi_t)$ the image of the process in $\mathbb{R}^2$. We choose a good atlas $\mathcal{A}$ defined by $\{(\phi_x, Dom\ (\phi_x)), x \in S^1\}$, where ϕ_x is the stereographic projection of pole $(-x)$, and $Dom(\phi_x)$ is such that $Val(\phi_x) =]-2, 2[$ for all x in S^1.

Below (figure 2), we show the time evolution of $\overline{\lambda}_p^h$ (in the x-axis: *ph*), for $h = 0.0001$.

Figure 2: Variations of the Lyapunov exponent in terms of *ph*

We observe that the discretization step needed to get a good approximation of λ is much smaller than in the previous example. This is due to the fact that the system is less stable, since here the Lyapunov exponent is near 0.

10 Conclusion

We have proposed an algorithm of approximation of the Lyapunov exponents of nonlinear stochastic differential systems, and given a theoretical estimate of its convergence rate.

From a numerical point of view, as in the linear case [21], the pertinent choice of the number N of steps in the formula (33) may present important difficulties. Further studies in that direction are necessary.

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