10.1098/rspa.2003.1240

THE ROYAL SOCIETY

Convergence rate of the Sherman and Peskin branching stochastic particle method

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Received 27 February 2003; accepted 1 September 2003; published online 12 November 2003

In this paper, we analyse the convergence rate of a branching stochastic particle method proposed by Sherman & Peskin for the numerical resolution of one-dimensional convection-reaction–diffusion equations. We prove precise estimates in terms of the number of particles at time zero and the time-discretization step.

Keywords: stochastic particle methods; branching processes; convection-reaction–diffusion equations

1. Introduction and notation

In this paper, we study the Sherman & Peskin (1986) stochastic particle method to solve partial differential equations (PDEs) of the type

$$\left.\begin{aligned} &\frac{\partial V}{\partial t}(t,x) = b(x)\frac{\partial}{\partial x}V(t,x) + \tfrac{1}{2}\sigma(x)^2\frac{\partial^2}{\partial x^2}V(t,x) + f\circ V(t,x), \\ &V(0,x) = V_0(x). \end{aligned}\right\} \tag{1.1}$$

This family of PDEs include the Kolmogorov–Petrovskii–Piskunov (KPP) equation.

For equations of the type (1.1), the Puckett (1989) stochastic particle method is based on the simulation of independent particles with interacting weights. Bernard *et al.* (1994) have proven precise estimates on the global error of the method. In particular they have shown that, in spite of the fact that the particles are dependent, the global error is of order $\mathcal{O}(1/\sqrt{N}) + \mathcal{O}(\sqrt{\Delta t})$ for all N and Δt, where N stands for the number of simulated particles and Δt for the discretization time-step used to simulate the paths of the particles. The proof is based on an appropriate probabilistic interpretation of the gradient equation

$$\left.\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \tfrac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}(t,x) + [b(x)+\sigma(x)\sigma'(x)]\frac{\partial u}{\partial x}(t,x) + b'(x)u(t,x) \\ &\quad + f'\left(\int_{-\infty}^{x} u(t,y)\,\mathrm{d}y\right)u(t,x), \\ u(0,x) &= V_0'(x). \end{aligned}\right\} \tag{1.2}$$

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The Sherman–Peskin method is completely different from the Chorin–Puckett method, since it consists in simulating a measure-valued branching particle system with mean-field interaction. For a continuous-time version of the Sherman–Peskin system, Chauvin *et al.* (1991) have proven the propagation of chaos property and explicated the relationship between the limit law of the particle system and equations (1.1). In addition, Chauvin *et al*. have proven fluctuation results (see § 2).

Here our objective is twofold. Firstly, we aim to extend the Sherman–Peskin method, originally designed for the KPP equation (for which $\sigma(\cdot) \equiv 1$ and $b(\cdot) \equiv 0$), to the more general equations (1.1): as we will see, one has to be careful when defining the free motion of the particles. Secondly, we aim to provide accurate non-asymptotic estimates on the numerical approximation error of $V(t, x)$ by the empirical distribution function of the particles that are live at time t. We thus show that, as for the Chorin–Puckett method, the global error of the Sherman–Peskin method is of order $\mathcal{O}(1/\sqrt{N}) + \mathcal{O}(\sqrt{\Delta t})$ for all N and Δt, where N stands for the initial number of particles and Δt for the discretization time-step used to simulate the lifetimes of the particles and the free motions during the lifetimes.

The paper is organized as follows: in § 2 we present the interacting branching process (Z_t), which allows one to construct a probabilistic interpretation of (1.1); in § 3 we list the successive approximations of (Z_t) which are involved in the Sherman–Peskin algorithm, and we state our main result, that is, an estimate on the global error of the algorithm; in §§ 4–8 we prove estimates on the successive approximations listed in § 3.

The two main ingredients of the global error analysis are the decomposition of the algorithm in terms of a succession of approximations, and the technique used in § 8 consisting of introducing ‘independent intermediate trees’. For probabilistic methods for nonlinear PDEs, in order to get the optimal rate of convergence with respect to the number of particles, it is crucial to introduce in the error analysis appropriate independent objects close to the set of dependent particle weights or particle motions involved in the method under study (see Bernard *et al.* 1994; Bossy & Talay 1996, 1997). See also Talay (1996) for a review and Bossy (2004) for estimates which are optimal with respect to the number of particles and the time-discretization step.

The authors have taken a long time to write this paper, which is based on the PhD thesis by Régnier (1999), where complementary results and numerical experiments can be found. A summary of the main results has been published without proofs in Régnier & Talay (2001).

(*a*) *Notation*

Throughout the paper we use the following notation.

M_F denotes the set of the finite positive measures on $\mathbb{R}$.

$\mathcal{S} := \mathbb{D}([0, T], M_F)$ denotes the Skorokhod space of the M_F valued càdlàg (continue à droite, limite à gauche) functions endowed with the weak convergence topology.

For various sets S, $M^1(S)$ denotes the set of the probability measures defined on S.

$\mathcal{B}(\mathbb{R})$ denotes the set of Borel sets of $\mathbb{R}$.

For all $j \in \mathbb{N}$, $W^{2,-j}(\mathbb{R})$ is the dual space of $W^{2,j}(\mathbb{R})$, the Sobolev space of the functions j-times differentiable such that

$$\|\phi\|_{W^{2,j}(\mathbb{R})} = \left(\sum_{k=0}^{j} \int_{\mathbb{R}} \left| \frac{\partial^k \phi}{\partial x^k}(x) \right|^2 \mathrm{d}x \right)^{1/2} < +\infty.$$

For all positive integer-valued random variable $\mathcal{N}$, we set

$$\sum_{i=1}^{\mathcal{N}}(\cdots) = 0 \text{ on the event } [\mathcal{N}=0]. \tag{1.3}$$

We define the function $\mathcal{H}^x$ by

$$\mathcal{H}^x(\cdot) := H(x-\cdot), \tag{1.4}$$

where H denotes the Heaviside function: $H(z)=1$ if $z \geqslant 0$, $H(z)=0$ otherwise.

Finally, throughout the paper, C denotes a positive number which may vary from line to line but remains independent of the numerical parameters N and Δt (*a contrario*, it depends on all the data, particularly T and the L^∞ norms of derivatives of b, σ, f and V_0).

2. A measure-valued branching process

Hypothesis 2.1. *We are given*

(1) *a function f which belongs to $\mathcal{C}_b^2(\mathbb{R})$ and satisfies*

$$f(0)=f(1)=0; \quad f'(0)>0;$$
$$f'(x) \leqslant f'(0) \quad \textit{for all } x \in [0,1]; \quad f'(x)<0 \quad \textit{for all } x \in \mathbb{R}_+ \setminus [0,1];$$

(2) *functions b and σ, which belong to $\mathcal{C}_b^\infty(\mathbb{R})$, the function σ satisfying the condition that there exists a positive number C_0 such that $\sigma(x) \geqslant C_0 > 0$ for all x;*

(3) *a probability density V_0' such that*

$$\exists M>0, \quad \exists \eta>0, \quad \exists \lambda>0, \quad \forall |x|>M, \quad |V_0'(x)| \leqslant \eta \exp(-\lambda x^2).$$

We define the function $\alpha : \mathbb{R} \times M_F \to \mathbb{R}_+$ as

$$\alpha(x,\mu) := |f'(\langle \mu, \mathcal{H}^x \rangle)|, \tag{2.1}$$

where the function $\mathcal{H}^x$ is as in (1.4). We define the function $p_0 : \mathbb{R} \times M_F \to [0,1]$ as

$$p_0(x,\mu) := \mathbb{I}_{f'(\langle \mu, \mathcal{H}^x \rangle) \leqslant 0}, \tag{2.2}$$

and we set

$$p_2(x,\mu) := 1 - p_0(x,\mu). \tag{2.3}$$

In the following, the quantities $p_0(x,\mu)$ and $p_2(x,\mu)$, respectively, are the probabilities of destruction or creation of particles at point x.

Let $W=(W_t, t \geqslant 0)$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by (Y_t^y) the real-valued diffusion process starting from y at time $t=0$ solution of

$$Y_t = y + \int_0^t (\sigma(Y_s)\sigma'(Y_s) - b(Y_s))\,\mathrm{d}s + \int_0^t \sigma(Y_s)\,\mathrm{d}W_s. \tag{2.4}$$

Under hypothesis 2.1 it is well known that, for all positive bounded measure ν_0, there exists a unique flow of positive bounded measures ν_s $(s \geqslant 0)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle \nu_s, \phi\rangle = \langle \mathcal{L}^* \nu_s, \phi\rangle + \langle f' \circ V(s,\cdot)\nu_s, \phi\rangle, \quad \text{for all } \phi \in \mathcal{C}_K^\infty(\mathbb{R}),$$

where $V(s,\cdot)$ is the solution of (1.1) and $\mathcal{L}^*$ is the adjoint operator of the generator $\mathcal{L}$ of the process (Y_t^y), that is,

$$\mathcal{L}^* f(x) = \tfrac{1}{2}\sigma^2(x) f''(x) + (b(x) + \sigma(x)\sigma'(x)) f'(x) + b'(x) f(x). \tag{2.5}$$

Observe that (2.5) explains why we consider (2.4) to construct our probabilistic interpretation of (1.2).

Definition 2.2. A probability π on the space $\mathcal{S}$ is said to be a solution of the nonlinear martingale problem (MP) if, for all functions $g \in \mathcal{C}_b^2(\mathbb{R}, (0,1))$, one has the fact that

$$\exp(\langle Z_t, \log g\rangle) \;-\; \exp(\langle Z_0, \log g\rangle) \;-\; \int_0^t \exp\langle Z_s, \log g\rangle \left\langle Z_s, \frac{\mathcal{H} + R(\cdot, \langle \pi, Z_s\rangle, g)}{g}\right\rangle \mathrm{d}s$$

is a $(\mathcal{F}_t)$-martingale, where (Z_t) and $(\mathcal{F}_t)$, respectively, denote the canonical process of $\mathcal{S}$ and the corresponding natural augmented filtration, and

$$R(x,\mu,g) := \alpha(x,\mu)(p_0(x,\mu) + p_2(x,\mu)g^2 - g).$$

Under the law π the process (Z_t) is called the *spatial branching process with mean-field interaction* (the reproduction measure depends on the state of the process) directed by the deterministic measure ν_t on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by

$$\forall B \in \mathcal{B}(\mathbb{R}), \quad \nu_t(B) = \langle \pi, Z_t\rangle(B) = \int \langle Z_t, \mathbb{I}_B\rangle \,\mathrm{d}\pi(Z).$$

The probability law π can be constructed as follows. At time zero, we are given initial discrete measures $Z_0^{N(i)}$; these measures are independent and identically distributed with common distribution $\pi_0^N \in M^1(M_F)$. Let $\mathcal{N}_t^N$ be the number of particles that are live at time t. Let Z_t^N be the branching process defined as

$$Z_t^N := \sum_{j=1}^{\mathcal{N}_t^N} \delta_{z_t^j},$$

where z_t^j denotes the position of the particle number j live at time t. We set

$$\mathcal{V}^N(t,x) := \frac{1}{N}\sum_{j=1}^{\mathcal{N}_t^N} H(x - z_t^j).$$

The dynamics of $\mathcal{N}_t^N$ and of the particles is as follows: between t and $t + \mathrm{d}t$, each particle moves independently of the other live particles, and the trajectory of each particle is a trajectory of the solution of (2.4). In addition, at all time t, the probability that one particle among the $\mathcal{N}_t^N$ live particles dies at time $t + \mathrm{d}t$ is

$$\mathrm{d}t \sum_{j=1}^{\mathcal{N}_t} |f' \circ \mathcal{V}^N(t, z_t^j)|,$$

and it is the particle located at z_t^J with probability

$$|f' \circ \mathcal{V}^N(t, z_t^J)| \biggl(\sum_{j=1}^{\mathcal{N}_t^N} |f' \circ \mathcal{V}^N(t, z_t^j)|\biggr)^{-1}.$$

It has no descendant if $f' \circ \mathcal{V}^N(t, z_t^J) \leqslant 0$ and gives birth to two particles otherwise (in the latter case we say that the dying particle 'branches').

One has the following strong law of large numbers and propagation of chaos results obtained by Chauvin *et al*. in the case where $\sigma(\cdot) \equiv 1$ and $b(\cdot) \equiv 0$, that is, when the differential operator of (1.1) reduces to the Laplace operator. It is straightforward to extend their results and their proofs to our more general situation†, so that we may state the following theorem.

Theorem 2.3 (Chauvin & Rouault 1990). *Suppose that hypothesis 2.1 holds. Suppose also that*

$$\int_{\mathbb{R}} (1 + |x|^p) \biggl| \frac{1}{N} Z_0^N - \mu_0 \biggr| (\mathrm{d}x) < \infty \quad \textit{for some integer } p > 0,$$

where μ_0 is the measure $V_0'(x)\,\mathrm{d}x$ and $|\cdots|$ stands for the total variation norm. The sequence of processes $((1/N)Z_\cdot^N)$ then converges weakly in $\mathbb{D}([0,T], W^{2,-3})$ to a deterministic process (μ_t). In addition, the flow μ_t is a solution in the sense of the distributions to (1.2), and therefore the distribution function of μ_t is solution of (1.1).

Theorem 2.4 (Chauvin *et al.* 1991). *Suppose that the hypothesis 2.1 holds. Suppose also that the initial measures $Z_0^{N(i)}$ $(i = 1, \dots, N)$ are i.i.d. with common distribution $\pi_0^N \in M^1(M_F)$, and $\mathrm{Law}(\pi_0^N) \Rightarrow \mathrm{Law}(\pi_0)$ in $M^1(M^1(M_F))$. The propagation of chaos then holds in the sense that $(\mathcal{L}\mathrm{aw}(\pi^N)) \Rightarrow \delta_\pi$ in $M^1(M^1(S))$, where π is the unique solution of the MP of the definition 2.2 with $\pi(Z_0 \in \cdot) = \pi_0(\cdot)$.*

In addition, the following fluctuation result holds: $\sqrt{N}((1/N)Z_\cdot^N - \mu_\cdot)$ converges weakly in $\mathbb{D}([0,T], W^{2,-5})$ to a generalized Ornstein–Uhlenbeck process.

Remark 2.5. We emphasize that the process which governs the free motions of the particles is the solution of (2.4). This is due to the fact that the empirical distribution of the particles approximates the gradient equation (1.2). Of course, this crucial observation was not needed by Sherman *et al*. to treat the KPP equation.

To prove the uniqueness of the limit of $((1/N)Z_\cdot^N)$ in theorem 2.3, Chauvin & Rouault make use of an integral equation which corresponds to the KPP equation and therefore involves Gaussian kernels (in the KPP case, (Y_t) reduces to the Brownian motion). In our extended situation one can use the result (see, for example, Rothe 1984) that assertions (I) and (II) are equivalent.

(I) The function $V(t, x)$ is the unique solution in $\mathcal{C}^{1,2}([0,T] \times \mathbb{R}, [0,1])$ of (1.1).

(II) The function $V(t, x)$ is in $L^\infty([0,T] \times \mathbb{R})$ and solves the integral equation

$$V(t, x) = T_t^0 V_0(x) + \int_0^t T_{t-s}^0 f(V(s, \cdot))(x)\,\mathrm{d}s, \tag{2.6}$$

† In Chauvin & Rouault (1990) the results are obtained under slightly weaker assumptions on V_0 and f than those we make here to obtain our convergence rate estimates.

where T_θ^0 is the transition operator of the process (X_t^x) starting from x at the time $t = 0$ solution of

$$X_t = x + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s)\,\mathrm{d}W_s. \tag{2.7}$$

3. Extension of Sherman & Peskin's method to general convection-reaction equations and main results

Our extension of Sherman & Peskin's method proceeds as follows.

(*a*) *Initialization of the particles*

At time zero, N particles with mass $1/N$ are located at points $V_0^{-1}(i/N)$, $i = 1, \dots, N-1$, and $V_0^{-1}(1-1/2N)$. These particles are destined to diffuse, branch and interact.

(*b*) *Offspring rule*

At each time-step $(k+1)\Delta t$ one looks at the particles that are live during the time-interval $[k\Delta t, (k+1)\Delta t)$, and one creates and deletes particles according to the following rule.

For $k \geqslant 1$ let $\bar{\mathcal{N}}_{k\Delta t}^N$ denote the number of particles that are live at time $k\Delta t$, and set

$$\bar{V}^N(k\Delta t, x) := \frac{1}{N} \sum_{j=1}^{\bar{\mathcal{N}}_{k\Delta t}^N} H(x - \bar{z}_{k\Delta t}^j),$$

where the $\{\bar{z}_{k\Delta t}^j\}$ are the locations of the simulated particles that are live at time $k\Delta t$. The particle located at $\{\bar{z}_{k\Delta t}^j\}$ dies with probability

$$\Delta t |f' \circ \bar{V}^N(k\Delta t, \bar{z}_{k\Delta t}^j)|.$$

It has no descendant if $f' \circ \bar{V}^N(k\Delta t, \bar{z}_{k\Delta t}^j) \leqslant 0$ and gives birth to two particles otherwise. That offspring procedure is achieved by the simulation of a family of independent random variables $\{\bar{\eta}_{(k+1)\Delta t}^j,\ j \geqslant 1,\ k \geqslant 0\}$ with uniform law on $[0,1]$.

(*c*) *Diffusion of the particles*

We discretize the time-interval $[0,T]$ by using a fixed step size $\Delta t = T/L$ for some integer L. In all the following we suppose that

$$0 < \Delta t < \min\left(1, \frac{1}{\max_{0 \leqslant x \leqslant 1} |f'(x)|}\right). \tag{3.1}$$

Between times $k\Delta t$ and $(k+1)\Delta t$, each particle which is live at time $k\Delta t$ moves independently of the other particles; its position at time $(k+1)\Delta t$ is

$$\begin{aligned}\bar{Y}_{(k+1)\Delta t} = \bar{Y}_{k\Delta t} &+ (\sigma(\bar{Y}_{k\Delta t})\sigma'(\bar{Y}_{k\Delta t}) - b(\bar{Y}_{k\Delta t}))\Delta t \\ &+ \sigma(\bar{Y}_{k\Delta t})(W_{(k+1)\Delta t} - W_{\Delta t}) \\ &+ \tfrac{1}{2}\sigma(\bar{Y}_{k\Delta t})\sigma'(\bar{Y}_{k\Delta t})((W_{(k+1)\Delta t} - W_{\Delta t})^2 - \Delta t).\end{aligned} \tag{3.2}$$

This Markov chain is the Milstein discretization scheme of (2.4). For a survey on discretization schemes of stochastic differential equations, see, for example, Talay (1996).

Theorem 3.1. *Suppose that the hypothesis 2.1 and the condition (3.1) hold. Let Φ be a positive function in $L^1(\mathbb{R}) \bigcap L^\infty(\mathbb{R})$. Set*

$$\|g\|_{L^{1,\Phi}(\mathbb{R})} := \int_{\mathbb{R}} |g(x)|\Phi(x)\,\mathrm{d}x$$

for all Lebesgue integrable functions g. One then obtains the fact that there exists $C > 0$ such that

$$\max_{0 \leqslant k\Delta t \leqslant T} \mathbb{E}\|V(k\Delta t, \cdot) - \bar{V}^N(k\Delta t, \cdot)\|_{L^{1,\Phi}(\mathbb{R})} \leqslant \frac{C}{\sqrt{N}} + C\sqrt{\Delta t}.$$

Remark 3.2.

(i) The use of the norm $\|\cdot\|_{L^{1,\Phi}(\mathbb{R})}$ comes from the fact that the random function $\bar{V}^N(t,x)$ does not tend to unity when x tends to $+\infty$ (whereas $V(t,x)$ does); consequently, almost surely the function $V(t,\cdot) - \bar{V}^N(t,\cdot)$ is not integrable at infinity with respect to Lebesgue's measure.

(ii) All our results below also hold, e.g. when the function V_0' has a compact support and is continuous on its support, or when μ_0 is a Dirac measure (see remark 4.2).

We now give a sketch of the lengthy and technical proof before going into details. We decompose the global error $\mathbb{E}\|V(k\Delta t,\cdot) - \bar{V}^N(k\Delta t,\cdot)\|_{L^{1,\Phi}(\mathbb{R})}$ into several terms, each one corresponding to one of the local approximations to which the algorithm proceeds.

The first ingredient of the algorithm consists in approximating the initial condition $V(0,\cdot)$ by

$$V^N(0,x) := \frac{1}{N}\sum_{i=1}^{N} H(x - z_0^i), \tag{3.3}$$

where

$$z_0^i = \begin{cases} V_0^{-1}\left(\dfrac{i}{N}\right), & i = 1,\dots,N-1, \\[2ex] V_0^{-1}\left(1 - \dfrac{1}{2N}\right), & i = N. \end{cases}$$

Let $V^N(t,\cdot)$ be the solution of (1.1) with initial condition $V^N(0,x)$. In § 4 we prove

$$\max_{0 \leqslant t \leqslant T} \|V(t,\cdot) - V^N(t,\cdot)\|_{L^1(\mathbb{R})} \leqslant C\frac{\sqrt{\log(N)}}{N}. \tag{3.4}$$

The second ingredient consists in discretizing the lifetimes. To estimate its effect on the global error, we consider a branching process $(\tilde{Z}_t^N)$ similar to the process (Z_t^N) of § 2, except that the lifetimes take the discrete values $k\Delta t$ only. In § 5, denoting by

$\tilde{V}^N(t,\cdot)$ the mean value of the distribution function of the particles which are live at time t, we show that

$$\max_{0\leqslant k\Delta t\leqslant T} \mathbb{E}\|V^N(k\Delta t,\cdot) - \tilde{V}^N(k\Delta t,\cdot)\|_{L^{1,\Phi}(\mathbb{R})} \leqslant C\sqrt{\Delta t}. \tag{3.5}$$

The third ingredient consists in governing the independent free motions of the particles by the Milstein discretization scheme (3.2) (instead of the solution of (2.4) itself). We denote by $\breve{V}^N(t,\cdot)$ the corresponding mean value of the distribution function at time t. In § 6 we show that

$$\max_{0\leqslant k\Delta t\leqslant T} \|\tilde{V}^N(k\Delta t,\cdot) - \breve{V}^N(k\Delta t,\cdot)\|_{L^{1,\Phi}(\mathbb{R})} \leqslant C\sqrt{\Delta t}. \tag{3.6}$$

The fourth ingredient consists in approximating $\breve{V}^N(k\Delta t,\cdot)$ by the empirical distribution

$$\widehat{\breve{V}}^N(k\Delta t,\cdot)$$

of the particles which are live at time $k\Delta t$. For this statistical error we prove in § 7 that

$$\max_{0\leqslant k\Delta t\leqslant T} \mathbb{E}\|\breve{V}^N(k\Delta t,\cdot) - \widehat{\breve{V}}^N(k\Delta t,\cdot)\|_{L^{1,\Phi}(\mathbb{R})} \leqslant \frac{C}{\sqrt{N}}. \tag{3.7}$$

The last ingredient consists in replacing the independent branching processes by the simulated branching processes which are in mean-field interaction. In § 8 we prove that

$$\max_{0\leqslant k\Delta t\leqslant T} \mathbb{E}\|\widehat{\breve{V}}^N(k\Delta t,\cdot) - \bar{V}^N(k\Delta t,\cdot)\|_{L^{1,\Phi}(\mathbb{R})} \leqslant \frac{C}{\sqrt{N}} + C\Delta t. \tag{3.8}$$

The conclusion of theorem 3.1 results from collecting the inequalities (3.4)–(3.8). The rest of the paper is devoted to the proof of these inequalities. To start with, we readily get the following estimates on the total number of live particles in the various trees we consider the following lemmas.

Lemma 3.3. *Let*

$$\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}, \qquad \breve{\mathcal{N}}^{N(i)}_{k\Delta t} \qquad \bar{\mathcal{N}}^{N(i)}_{k\Delta t},$$

respectively, denote the number of particles of the trees

$$\tilde{Z}^{N(i)}_{k\Delta t}, \qquad \breve{Z}^{N(i)}_{k\Delta t}, \qquad \bar{Z}^{N(i)}_{k\Delta t},$$

issued from z_0^i, which are live at time $k\Delta t$. For all integer m there exists $C > 0$ (depending on m but not on N) such that

$$\mathbb{E}[\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}]^m \leqslant \exp(Ck\Delta t), \tag{3.9}$$

$$\mathbb{E}[\breve{\mathcal{N}}^{N(i)}_{k\Delta t}]^m \leqslant \exp(Ck\Delta t), \tag{3.10}$$

$$\mathbb{E}[\bar{\mathcal{N}}^{N(i)}_{k\Delta t}]^m \leqslant \exp(Ck\Delta t) \tag{3.11}$$

for all $i = 1,\dots,N$.

Proof. Since similar arguments allow us to prove the three preceding inequalities, we only prove (3.9). We obviously can choose C large enough to have

$$\tilde{\mathcal{N}}^{N(i)}_{(k+1)\Delta t} \leqslant \tilde{\mathcal{N}}^{N(i)}_{k\Delta t} + \sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}} \mathbb{I}_{C\Delta t \geqslant \tilde{\eta}^{j}_{(k+1)\Delta t}}. \tag{3.12}$$

Use

$$\left(\mathcal{N} + \sum_{j=1}^{\mathcal{N}} \mathbb{I}_{\omega_j}\right)^m \leqslant \mathcal{N}^m + 2^m \mathcal{N}^{m-1} \sum_{j=1}^{\mathcal{N}} \mathbb{I}_{\omega_j} \quad \text{for all integers } m \text{ and } \mathcal{N} \geqslant 1,$$

which readily follows from

$$\left(\sum_{j=1}^{\mathcal{N}} \mathbb{I}_{\omega_j}\right)^p \leqslant \mathcal{N}^{p-1} \sum_{j=1}^{\mathcal{N}} \mathbb{I}_{\omega_j}$$

in view of Hölder's inequality. This results in

$$\mathbb{E}[\tilde{\mathcal{N}}^{N(i)}_{(k+1)\Delta t}]^m \leqslant (1 + 2^m C\Delta t)\mathbb{E}[\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}]^m.$$

The conclusion follows from $\tilde{\mathcal{N}}^{N(i)}_0 = 1$. ∎

4. The error due to the discretization of the initial condition $V(0, x)$

The aim of this section is to prove the following proposition.

Proposition 4.1. *Let $V^N(t, \cdot)$ be the solution of (1.1) with initial condition $V^N(0, \cdot)$ defined as in (3.3). There exists a positive number C such that*

$$\max_{0 \leqslant t \leqslant T} \|V(t, \cdot) - V^N(t, \cdot)\|_{L^1(\mathbb{R})} \leqslant C\|V(0, \cdot) - V^N(0, \cdot)\|_{L^1(\mathbb{R})} \tag{4.1}$$

and

$$\|V(0, \cdot) - V^N(0, \cdot)\|_{L^1(\mathbb{R})} \leqslant \frac{\epsilon(N)}{N}, \tag{4.2}$$

where $\epsilon(N) := C\sqrt{\log(N)}$.

Proof. Let L denote the generator of the solution (X_t^x) of (2.7), and set $w(t, x) := V(t, x) - V^N(t, x)$. That function satisfies

$$\begin{aligned} \frac{\partial w}{\partial t}(t, x) &= Lw(t, x) + w(t, x)\, \frac{f \circ V(t, x) - f \circ V^N(t, x)}{V(t, x) - V^N(t, x)}, \\ w(0, x) &= V(0, x) - V^N(0, x). \end{aligned}$$

As the function f is Lipschitz, the Feynman–Kac formula implies

$$|w(t, x)| \leqslant C\mathbb{E}|w_0(X_t^x)|.$$

Thus the inequality (4.1) results from the following inequality, which is easy to obtain (see Bernard *et al.* 1994, lemma 2.2, p. 559),

$$\int_{\mathbb{R}} p_t^X(x, y)\, \mathrm{d}x \leqslant C, \tag{4.3}$$

where $p_t^X(x,y)$ denotes the density of X_t^x whose existence is ensured by the hypothesis 2.1 (2) (see, for example, Friedman 1975). To get (4.2) one can then proceed as in the proof of lemma 2.4 in Bossy & Talay (1997). ■

Remark 4.2. When V_0 is the Heaviside function, then obviously $\epsilon(N)=0$. When the function V_0' has a compact support and is continuous on its support, $\epsilon(N)$ can be chosen as a constant function (see Bossy & Talay 1997).

5. The lifetimes discretization error

Remember that the step size $\Delta t<1$ is of the form T/L for some integer L. We define a branching process, the lifetimes of which are of the form $k\Delta t$. We denote the number of particles which are live at time $k\Delta t$ by $\tilde{\mathcal{N}}_{k\Delta t}^N$ and we set

$$\tilde{Z}_{k\Delta t}^N := \sum_{j=1}^{\tilde{\mathcal{N}}_{k\Delta t}^N} \delta_{\tilde{z}_{k\Delta t}^j},$$

where $\tilde{z}_{k\Delta t}^j$ denotes the location of the particle number j at time $k\Delta t$. We set

$$\tilde{V}^N(k\Delta t,x) := \frac{1}{N}\mathbb{E}\sum_{j=1}^{\tilde{\mathcal{N}}_{k\Delta t}^N} H(x-\tilde{z}_{k\Delta t}^j).$$

During their lives the particles move independently of the others according to the description in § 2. In addition, at all times $k\Delta t$, the particle located at $\tilde{z}_{k\Delta t}^j$ dies with probability

$$\Delta t\ |f'\circ\tilde{V}^N(k\Delta t,\tilde{z}_{k\Delta t}^j)|.$$

It then has no descendant if $f'\circ\tilde{V}^N(k\Delta t,\tilde{z}_{k\Delta t}^j)\leqslant 0$ and gives birth to two particles otherwise. We aim to prove the following proposition.

Proposition 5.1. *There exists a positive real number C such that*

$$\max_{0\leqslant k\Delta t\leqslant T}\|V^N(k\Delta t,\cdot)-\tilde{V}^N(k\Delta t,\cdot)\|_{L^{1,\Phi}(\mathbb{R})}\leqslant C\sqrt{\Delta t}. \tag{5.1}$$

We start with a series of lemmas.

(a) *Preliminaries*

Lemma 5.2. *For all Δt satisfying (3.1) and all $x\in\mathbb{R}$ one has*

$$V^N((k+1)\Delta t,x)=\mathbb{E}V^N(k\Delta t,X_{\Delta t}^x)+\Delta t\mathbb{E}f\circ V^N(k\Delta t,X_{\Delta t}^x)+R_{k\Delta t}(x), \tag{5.2}$$

with

$$\|R_{k\Delta t}(\cdot)\|_{L^1(\mathbb{R})}\leqslant C\Delta t^{3/2},\quad \textit{for all } \Delta t\leqslant k\Delta t\leqslant T. \tag{5.3}$$

Proof. See Bernard *et al.* (1994, theorem 6.1) for the technical calculation. ■

Lemma 5.3. *Let $(\tilde{\mathcal{F}}_{k\Delta t}^N)$ denote the filtration generated by $(\tilde{Z}_{k\Delta t}^N)$. Fix an arbitrary time*

$$\Delta t\leqslant k\Delta t\leqslant T$$

and an arbitrary real number x. Let (X^x_t) be a solution of (2.7) starting at x at time zero and independent of $\tilde{\mathcal{F}}^N_{k\Delta t}$. There holds

$$\tilde{V}^N((k+1)\Delta t, x) = \mathbb{E}\tilde{V}^N(k\Delta t, X^x_{\Delta t}) + \Delta t \mathbb{E} f \circ \tilde{V}^N(k\Delta t, X^x_{\Delta t}) + \tilde{R}^N_{k\Delta t}(x), \quad (5.4)$$

with

$$\max_{0\leqslant k\Delta t\leqslant T} \|\tilde{R}^N_{k\Delta t}(\cdot)\|_{L^{1,\varPhi}(\mathbb{R})} \leqslant C\Delta t^{3/2}. \quad (5.5)$$

Proof. Because of the combinatory of the mechanism of creation and deletion of the particles, it appears convenient to consider the trees $\tilde{Z}^{N(i)}_{k\Delta t}$, where the index i means that the tree is issued from z^i_0. We denote by $\tilde{z}^{(i)j}_{k\Delta t}$ the locations of the particles of the tree $\tilde{Z}^{N(i)}_{k\Delta t}$, and we set

$$\tilde{V}^{N(i)}(k\Delta t, x) := \frac{1}{N}\mathbb{E}\sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}} H(x - \tilde{z}^{(i)j}_{k\Delta t}).$$

We then set

$$\tilde{Y}^{(i)j}_{\Delta t, k\Delta t} := \xi^Y_{0,\Delta t}(y)|_{y=\tilde{z}^{(i)j}_{k\Delta t}},$$

where $\xi^Y_{\theta,t}(y)$ stands for the continuous version of the stochastic flow defined by (2.4) governed by a Brownian motion independent of $\tilde{\mathcal{F}}^N_{k\Delta t}$. The desired expansion results from equalities (5.6)–(5.8) and inequality (5.9), where the expectation $\mathbb{E}^Y$ is computed under the law of $\xi^Y_{0,\Delta t}$:

$$\begin{aligned}\tilde{V}^{N(i)}&((k+1)\Delta t, x)\\ &= \frac{1}{N}\mathbb{E}\sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}} \mathbb{E}^Y H(x - \tilde{Y}^{(i)j}_{\Delta t, k\Delta t})\\ &\quad + \frac{\Delta t}{N}\mathbb{E}\sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}} f' \circ \tilde{V}^N(k\Delta t, \tilde{z}^{(i)j}_{k\Delta t})\mathbb{E}^Y H(x - \tilde{Y}^{(i)j}_{\Delta t, k\Delta t}) + \tilde{\mathcal{R}}^{N(i)}_{(k+1)\Delta t}(x), \quad (5.6)\end{aligned}$$

$$\frac{1}{N}\mathbb{E}\sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}} \mathbb{E}^Y H(x - \tilde{Y}^{(i)j}_{\Delta t, k\Delta t}) = \mathbb{E}\tilde{V}^{N(i)}(k\Delta t, X^x_{\Delta t}), \quad (5.7)$$

$$\sum_{i=1}^{N}\mathbb{E}\sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}} f' \circ \tilde{V}^N(k\Delta t, \tilde{z}^{(i)j}_{k\Delta t})\mathbb{E}^Y H(x - \tilde{Y}^{(i)j}_{\Delta t, k\Delta t}) = \mathbb{E} f \circ \tilde{V}^N(k\Delta t, X^x_{\Delta t}), \quad (5.8)$$

$$\max_{0\leqslant k\Delta t\leqslant T} \|\tilde{R}^{N(i)}_{k\Delta t}(\cdot)\|_{L^{1,\varPhi}(\mathbb{R})} \leqslant C\Delta t^2. \quad (5.9)$$

To prove equation (5.6) we consider the event $\tilde{\mathcal{O}}^{(i)j}_{(k+1)\Delta t}$: 'the particle located at $\tilde{z}^{(i)j}_{k\Delta t}$, and that particle only, dies at time $k\Delta t$', that is,

$$\begin{aligned}\tilde{\mathcal{O}}^{(i)j}_{(k+1)\Delta t} = [\tilde{\mathcal{N}}^{N(i)}_{k\Delta t} \geqslant 1] \cap [\Delta t|f' \circ \tilde{V}^N(k\Delta t, \tilde{z}^{(i)j}_{k\Delta t})| \geqslant \tilde{\eta}^{(i)j}_{(k+1)\Delta t}]&\\ \bigcap_{\theta\neq j}[\Delta t|f' \circ \tilde{V}^N(k\Delta t, \tilde{z}^{(i)\theta}_{k\Delta t})| \leqslant \tilde{\eta}^{(i)\theta}_{(k+1)\Delta t}].& \quad (5.10)\end{aligned}$$

We postpone for a while the proof of the equality

$$\mathbb{P}^{\tilde{\mathcal{F}}^N_{k\Delta t}}\tilde{\mathcal{O}}^{(i)j}_{(k+1)\Delta t} = \Delta t|f' \circ \tilde{V}^N(k\Delta t, \tilde{z}^{(i)j}_{k\Delta t})|\mathbb{I}_{\tilde{\mathcal{N}}^N_{k\Delta t}\geqslant 1} + r^{(i)j}_{(k+1)\Delta t}, \tag{5.11}$$

where the reminder satisfies

$$\mathbb{E}\sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}} |r^{(i)j}_{(k+1)\Delta t}| \leqslant C\Delta t^2. \tag{5.12}$$

Let $\tilde{\mathcal{Q}}^{N(i)}_{(k+1)\Delta t}$ denote the event 'at least two particles of $\tilde{Z}^{N(i)}_{k\Delta t}$ die'. We then have that

$$\begin{aligned}\sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{(k+1)\Delta t}} H(x - \tilde{z}^{(i)j}_{(k+1)\Delta t}) &= \sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}} H(x - \tilde{Y}^{(i)j}_{\Delta t, k\Delta t}) \\ &\quad + \sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}} H(x - \tilde{z}^{(i)j}_{(k+1)\Delta t})\mathbb{I}_{\tilde{\mathcal{O}}^{(i)j}_{(k+1)\Delta t}}\mathbb{I}_{f' \circ \tilde{V}^N(k\Delta t, \tilde{z}^{(i)j}_{k\Delta t})>0} \\ &\quad - \sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}} H(x - \tilde{z}^{(i)j}_{k\Delta t})\mathbb{I}_{\tilde{\mathcal{O}}^{(i)j}_{(k+1)\Delta t}}\mathbb{I}_{f' \circ \tilde{V}^N(k\Delta t, \tilde{z}^{(i)j}_{k\Delta t})\leqslant 0} \\ &\quad + \sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{(k+1)\Delta t}} H(x - \tilde{z}^{(i)j}_{(k+1)\Delta t})\mathbb{I}_{\tilde{\mathcal{Q}}^{N(i)}_{(k+1)\Delta t}}.\end{aligned}$$

Owing to (5.11), (5.12) and obvious independence arguments, we get (5.6) with

$$\tilde{\mathcal{R}}^{N(i)}_{(k+1)\Delta t}(x) := \frac{1}{N}\sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{(k+1)\Delta t}} H(x - \tilde{z}^{(i)j}_{(k+1)\Delta t})\mathbb{I}_{\tilde{\mathcal{Q}}^{N(i)}_{(k+1)\Delta t}}.$$

Lemma 5.4 below provides (5.9).

We now proceed to the proof of (5.7). It is well known that the stochastic flow defined by (2.4) has a version which is strictly increasing, and that the inverse flow $\xi^{Y,-1}_{\theta,t}(y)$ satisfies

$$\xi^{Y,-1}_{\theta,t}(y) = y + \int_\theta^t b(\xi^{Y,-1}_{s,t}(y))\,\mathrm{d}s - \int_\theta^t \sigma(\xi^{Y,-1}_{s,t}(y))\,\hat{\mathrm{d}}W_s \quad \text{for all } \theta < t, \tag{5.13}$$

where $\hat{\mathrm{d}}W_s$ denotes the 'backward stochastic integral' (see Kunita 1984). Therefore, conditionally to $\tilde{\mathcal{F}}^N_{k\Delta t}$, $H(x - \tilde{Y}^{(i)j}_{\Delta t,k\Delta t})$ has the same law as $H(X^x_{\Delta t} - \tilde{z}^{(i)j}_{k\Delta t})$, which provides (5.7).

To prove (5.8) we observe that, for all $k \geqslant 1$, the function $\tilde{V}^N(k\Delta t, \cdot)$ is differentiable (this results from the fact that, in view of hypothesis 2.1 (2), the law of Y^y_t has a continuous density for all $t > 0$ and all $y \in \mathbb{R}$). In view of Fubini's theorem†,

† The justification of the use of Fubini's theorem results from (3.9).

one has

$$\mathbb{E}\sum_{j=1}^{\tilde{\mathcal{N}}^N_{k\Delta t}} f'\circ\tilde{V}^N(k\Delta t,\tilde{z}^j_{k\Delta t})\mathbb{E}^Y H(x-\tilde{Y}^j_{\Delta t,k\Delta t})$$

$$= \int \mathbb{E}\int_{-\infty}^{\xi} f'\circ\tilde{V}^N(k\Delta t,z)\tilde{Z}^N_{k\Delta t}(\mathrm{d}z)\,\mathrm{d}\mathbb{P}_{X^x_{\Delta t}}(\xi)$$

$$=: \int \rho(\xi)\,\mathrm{d}\mathbb{P}_{X^x_{\Delta t}}(\xi).$$

Now, using the integrations by parts formula for Stieltjes integrals, we get

$$\begin{aligned}\rho(\xi) &= \mathbb{E}f'\circ\tilde{V}^N(k\Delta t,\xi)\tilde{Z}^N_{k\Delta t}(-\infty,\xi)\\ &\qquad -\mathbb{E}\int_{-\infty}^{\xi}\tilde{Z}^N_{k\Delta t}(-\infty,z)f''\circ\tilde{V}^N(k\Delta t,z)\frac{\partial}{\partial z}\tilde{V}^N(k\Delta t,z)\,\mathrm{d}z\\ &= f'\circ\tilde{V}^N(k\Delta t,\xi)\tilde{V}^N(k\Delta t,\xi)\\ &\qquad -\int_{-\infty}^{\xi}\tilde{V}^N(k\Delta t,z)f''\circ\tilde{V}^N(k\Delta t,z)\frac{\partial}{\partial z}\tilde{V}^N(k\Delta t,z)\,\mathrm{d}z\\ &= f\circ\tilde{V}^N(k\Delta t,\xi),\end{aligned}$$

so that (5.8) is proved. ■

In the preceding proof we used the following result.

Lemma 5.4. *We have*

$$\mathbb{P}^{\tilde{\mathcal{F}}^N_{k\Delta t}}\tilde{\mathcal{O}}^{(i)j}_{(k+1)\Delta t} = \Delta t|f'\circ\tilde{V}^N(k\Delta t,\tilde{z}^{(i)j}_{k\Delta t})|\mathbb{I}_{\tilde{\mathcal{N}}^N_{k\Delta t}\geqslant 1} + r^{(i)j}_{(k+1)\Delta t}, \tag{5.14}$$

where the reminder satisfies

$$\mathbb{E}\sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}}|r^{(i)j}_{(k+1)\Delta t}| \leqslant C\Delta t^2. \tag{5.15}$$

There also holds

$$\mathbb{P}^{\tilde{\mathcal{F}}^N_{k\Delta t}}\tilde{\mathcal{Q}}^{N(i)}_{(k+1)\Delta t} \leqslant C\Delta t^2. \tag{5.16}$$

Proof. Using (5.10) and independence arguments we get

$$\begin{aligned}\mathbb{P}^{\tilde{\mathcal{F}}^N_{k\Delta t}}\tilde{\mathcal{O}}^{(i)j}_{(k+1)\Delta t} &= \mathbb{P}^{\tilde{\mathcal{F}}^N_{k\Delta t}}[\Delta t|f'\circ\tilde{V}^{N(i)}(k\Delta t,\tilde{z}^{(i)j}_{k\Delta t})| \geqslant \tilde{\eta}^{(i)j}_{(k+1)\Delta t}]\\ &\qquad\times\prod_{\theta\neq j}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}}\mathbb{P}^{\tilde{\mathcal{F}}^N_{k\Delta t}}[\Delta t|f'\circ\tilde{V}^{N(i)}(k\Delta t,\tilde{z}^{(i)\theta}_{k\Delta t})| \leqslant \tilde{\eta}^{(i)\theta}_{(k+1)\Delta t}]\mathbb{I}_{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}\geqslant 2}\\ &= \Delta t|f'\circ\tilde{V}^{N(i)}(k\Delta t,\tilde{z}^{(i)j}_{k\Delta t})|\prod_{\substack{\theta=1,\\ \theta\neq j}}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}}(1-\Delta t|f'\circ\tilde{V}^{N(i)}(k\Delta t,\tilde{z}^{(i)\theta}_{k\Delta t})|).\end{aligned}$$

We now set

$$r^{(i)j}_{(k+1)\Delta t} = \Delta t|f' \circ \tilde{V}^{N(i)}(k\Delta t, \tilde{z}^{(i)j}_{k\Delta t})|\Bigg(\prod_{\theta\neq j}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}} (1 - \Delta t|f' \circ \tilde{V}^{N(i)}(k\Delta t, \tilde{z}^{(i)\theta}_{k\Delta t})|) - 1\Bigg).$$

We have the following rough estimate,

$$1 - \prod_{j=1}^{\mathcal{N}}(1 - a_j) \leqslant -\sum_{j=1}^{\mathcal{N}} \log(1 - a_j) \leqslant \sum_{j=1}^{\mathcal{N}} \frac{a_j}{1 - a_j} \leqslant \frac{\mathcal{N}\ \max(a_j)}{1 - \max(a_j)}, \tag{5.17}$$

for all integer $\mathcal{N}$ and $\{a_j, 0 < a_j < 1\}$. In view of (3.1) one thus has

$$\mathbb{E} \sum_{j=1}^{\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}} |r^{(i)j}_{(k+1)\Delta t}| \leqslant C\mathbb{E}[\tilde{\mathcal{N}}^{N(i)}_{k\Delta t}]^2\, \Delta t^2.$$

It now remains to use the lemma 3.3. The expansion (5.14) follows.

One can prove (5.16) with similar arguments and the following rough estimate deduced from the inequalities in (5.17):

$$\begin{aligned} 1 - \prod_{j=1}^{\mathcal{N}}(1 - a_j) - \sum_{j=1}^{\mathcal{N}} a_j \prod_{\substack{k=1,\\ k\neq j}}^{\mathcal{N}}(1 - a_k) &\leqslant \sum_{j=1}^{\mathcal{N}} \frac{a_j}{1 - a_j} - \sum_{j=1}^{\mathcal{N}} \frac{a_j}{1 - a_j} \prod_{k=1}^{\mathcal{N}}(1 - a_k) \\ &\leqslant \frac{\mathcal{N}^2 \max(a_j)^2}{(1 - \max(a_j))^2}. \end{aligned} \tag{5.18}$$

We omit the details. ■

We are now in a position to proceed to the proof of proposition 5.1.

(b) *Proof of proposition 5.1*

Let $\varPhi_k$ be the density of $X_{k\Delta t}$ when the distribution function of X_0 is $\varPhi$. Set $w_{k\Delta t}(x) := V^N(k\Delta t, x) - \tilde{V}^N(k\Delta t, x)$. In view of (5.2) and (5.4), as f is Lipschitz we have

$$\begin{aligned} \|w_{(k+1)\Delta t}(\cdot)\|_{L^{1,\varPhi}(\mathbb{R})} &\leqslant (1 + C\Delta t)\|\mathbb{E}w_{k\Delta t}(\cdot)\|_{L^{1,\varPhi_1}(\mathbb{R})} \\ &\quad + \|R_{k\Delta t}(\cdot)\|_{L^{1,\varPhi}(\mathbb{R})} + \|\tilde{R}_{k\Delta t}(\cdot)\|_{L^{1,\varPhi}(\mathbb{R})}. \end{aligned} \tag{5.19}$$

We then use (5.3) and (5.5) and proceed by iteration up to $k = 0$. The desired result follows.

6. The path-discretization error

The process $(\tilde{Z}^N_{k\Delta t})$ has continuous paths between times $k\Delta t$ and $(k+1)\Delta t$. We now consider its time-discrete approximation $(\breve{Z}^N_{k\Delta t})$. We define the $\breve{z}^j_{k\Delta t}$ and $\breve{\mathcal{N}}^N_{k\Delta t}$ in an obvious way. The free motion of the particles during their lifetimes is described by the Milstein scheme for equation (2.4) (see (3.2)). We set

$$\breve{V}^N(k\Delta t, x) := \frac{1}{N}\mathbb{E}\sum_{j=1}^{\breve{\mathcal{N}}_{k\Delta t}} H(x - \breve{z}^j_{k\Delta t}). \tag{6.1}$$

At time $(k+1)\Delta t$ the particle located at $\breve{z}^j_{k\Delta t}$ dies with probability

$$\Delta t|f' \circ \breve{V}^N(k\Delta t, \breve{z}^j_{k\Delta t})|.$$

It then has no descendant if $f' \circ \breve{V}^N(k\Delta t, \breve{z}^j_{k\Delta t}) \leqslant 0$ and gives birth to two particles otherwise.

Proposition 6.1. *There exists $C > 0$ such that*

$$\max_{0\leqslant k\Delta t\leqslant T} \|\tilde{V}^N(k\Delta t, \cdot) - \breve{V}^N(k\Delta t, \cdot)\|_{L^{1,\Phi}(\mathbb{R})} \leqslant C\sqrt{\Delta t}. \tag{6.2}$$

The proof of the preceding proposition proceeds as the proof of proposition 5.1, once we have proven lemma 6.2.

Lemma 6.2. *For all $k\Delta t$ in $[0, T - \Delta t]$ and all $x \in \mathbb{R}$, one has*

$$\breve{V}^N((k+1)\Delta t, x) = \mathbb{E}\breve{V}^N(k\Delta t, X^x_{\Delta t}) + \mathbb{E}f \circ \breve{V}^N(k\Delta t, X^x_{\Delta t}) + \breve{\mathcal{R}}_{(k+1)\Delta t}(x), \tag{6.3}$$

with

$$\max_{0\leqslant k\Delta t\leqslant T} \|\breve{\mathcal{R}}_{(k+1)\Delta t}(\cdot)\|_{L^{1,\Phi}(\mathbb{R})} \leqslant C\Delta t^{3/2}. \tag{6.4}$$

Proof. We denote by $(\breve{Y}^y_{k\Delta t})$ the Milstein scheme for (2.4) with initial condition y at time zero, and set

$$\breve{Y}^j_{\Delta t,k\Delta t} := \breve{Y}^y_{\Delta t}|_{y=\breve{z}^j_{k\Delta t}}.$$

Proceeding as in § 5 we get

$$\begin{aligned}\breve{V}^N((k+1)\Delta t, x) = \frac{1}{N}\mathbb{E}&\sum_{j=1}^{\breve{\mathcal{N}}^N_{k\Delta t}} \mathbb{E}^{\breve{Y}} H(x - \breve{Y}^j_{\Delta t,k\Delta t}) \\ &+ \frac{\Delta t}{N}\mathbb{E}\sum_{j=1}^{\breve{\mathcal{N}}^N_{k\Delta t}} f' \circ \breve{V}^N(k\Delta t, \breve{z}^j_{k\Delta t})\mathbb{E}^{\breve{Y}} H(x - \breve{Y}^j_{\Delta t,k\Delta t}) \\ &+ \breve{\mathcal{R}}_{(k+1)\Delta t}(x).\end{aligned}$$

It then remains to prove that

$$\left\|\frac{1}{N}\mathbb{E}\sum_{j=1}^{\breve{\mathcal{N}}^N_{k\Delta t}} \mathbb{E}^{\breve{Y}} H(x - \breve{Y}^j_{\Delta t,k\Delta t}) - \mathbb{E}\breve{V}^N(k\Delta t, X^x_{\Delta t})\right\|_{L^{1,\Phi}(\mathbb{R})} \leqslant C(\Delta t)^{3/2}, \tag{6.5}$$

and

$$\left\| \frac{1}{N}\mathbb{E}\sum_{j=1}^{\breve{\mathcal{N}}_{k\Delta t}^N} f'\circ \breve{V}^N(k\Delta t, \breve{z}_{k\Delta t}^j)\mathbb{E}^{\breve{Y}}H(x-\breve{Y}_{\Delta t,k\Delta t}^j) \right.$$

$$\left. -\mathbb{E}f\circ \breve{V}^N(k\Delta t, X_{\Delta t}^x)\right\|_{L^{1,\Phi}(\mathbb{R})} \leqslant C(\Delta t)^{3/2}. \quad (6.6)$$

To prove (6.5) we observe that the left-hand side is bounded from above by

$$\frac{1}{N}\mathbb{E}\sum_{j=1}^{\breve{\mathcal{N}}_{k\Delta t}^N} \|\mathbb{E}^{\breve{Y}}H(x-\breve{Y}_{\Delta t,k\Delta t}^j) - \mathbb{E}^{Y}H(x-\xi_{0,\Delta t}^{Y}\circ \breve{z}_{k\Delta t}^j)\|_{L^1(\mathbb{R})},$$

from which, as $\int |H(x-a)-H(x-b)|\,\mathrm{d}x = |a-b|$, we get the new bound from above

$$\frac{1}{N}\mathbb{E}\sum_{j=1}^{\breve{\mathcal{N}}_{k\Delta t}^N} \mathbb{E}|\breve{Y}_{\Delta t}^x - Y_{\Delta t}^x|_{x=\breve{z}_{k\Delta t}^j}.$$

One then uses the following property of the Milstein scheme:

$$\exists C>0, \quad \mathbb{E}|\breve{Y}_{\Delta t}^x - Y_{\Delta t}^x| \leqslant C\Delta t^{3/2} \quad \text{for all } x\in\mathbb{R} \quad \text{and} \quad 0\leqslant \Delta t<1. \quad (6.7)$$

Lemma 3.3 allows then us to obtain (6.5).

To prove (6.6) we use (6.7) and a calculation similar to that made to prove (5.8). ■

Remark 6.3. The choice of the Milstein scheme rather than the Euler scheme is due to the estimate (6.7). It is likely that one could adapt the technique developed by Bossy (2004) and then get a global discretization error of order Δt for the Euler scheme, but we have chosen not to do this because of the complexity of the corresponding calculation.

7. The statistical error

We now estimate the effect of substituting

$$\widehat{\breve{V}}^N(k\Delta t, x) := \frac{1}{N}\sum_{j=1}^{\breve{\mathcal{N}}_{k\Delta t}^N} H(x-\breve{z}_{k\Delta t}^j) \quad (7.1)$$

with $\breve{V}^N$.

Proposition 7.1. *There exists $C>0$ such that*

$$\max_{0\leqslant k\Delta t\leqslant T}\mathbb{E}\|\breve{V}^N(k\Delta t,\cdot) - \widehat{\breve{V}}^N(k\Delta t,\cdot)\|_{L^{1,\Phi}(\mathbb{R})} \leqslant \frac{C}{\sqrt{N}}. \quad (7.2)$$

Proof. Set

$$\varepsilon_{k\Delta t} := \mathbb{E}\|\breve{V}^N(k\Delta t,\cdot) - \widehat{\breve{V}}^N(k\Delta t,\cdot)\|_{L^{1,\Phi}(\mathbb{R})},$$

and denote by $\breve{Z}^{N(i)}_{k\Delta t}$ the tree issued from z_0^i. With obvious notation we have

$$\langle \breve{Z}^{N(i)}_{k\Delta t}, \mathcal{H}^x \rangle = \sum_{j=1}^{\breve{\mathcal{N}}^{N(i)}_{k\Delta t}} H(x - \breve{z}^{(i)j}_{k\Delta t}),$$

and therefore

$$\varepsilon_{k\Delta t} = \frac{1}{N} \int_{\mathbb{R}} \mathbb{E}\biggl| \sum_{i=1}^{N} (\mathbb{E}\langle \breve{Z}^{N(i)}_{k\Delta t}, \mathcal{H}^x \rangle - \langle \breve{Z}^{N(i)}_{k\Delta t}, \mathcal{H}^x \rangle) \biggr| \Phi(x)\, \mathrm{d}x.$$

By independence one has

$$\mathbb{E}\biggl| \sum_{i=1}^{N} (\mathbb{E}\langle \breve{Z}^{N(i)}_{k\Delta t}, \mathcal{H}^x \rangle - \langle \breve{Z}^{N(i)}_{k\Delta t}, \mathcal{H}^x \rangle) \biggr|^2 = \sum_{i=1}^{N} \mathrm{Var}\langle \breve{Z}^{N(i)}_{k\Delta T}, \mathcal{H}^x \rangle.$$

In view of lemma 3.3 one has

$$\mathrm{Var}\langle \breve{Z}^{N(i)}_{k\Delta t}, \mathcal{H}^x \rangle \leqslant \mathbb{E}(\breve{\mathcal{N}}^{N(i)}_{k\Delta t})^2 \leqslant C,$$

so that, since Φ belongs to $L^1(\mathbb{R})$, one gets

$$\varepsilon_{k\Delta t} \leqslant \frac{C}{\sqrt{N}}.$$

■

8. The interaction error

The process $(\breve{Z}_{k\Delta t})$ cannot be simulated, since the distribution function $\mathbb{E}\breve{V}^N(\cdot,\cdot)$ is unknown. We thus finally consider the branching process $(\bar{Z}^N_{k\Delta t})$, the offspring law of which depends on the empirical measure of the particles. We define the $\bar{z}^j_{k\Delta t}$ and $\bar{\mathcal{N}}^N_{k\Delta t}$ in an obvious way and we set

$$\bar{V}^N(k\Delta t, x) := \frac{1}{N} \sum_{j=1}^{\bar{\mathcal{N}}^N_{k\Delta t}} H(x - \bar{z}^j_{k\Delta t}).$$

At time $(k+1)\Delta t$ the particle located at $\bar{z}^j_{k\Delta t}$ dies with probability

$$\Delta t | f' \circ \bar{V}^N(k\Delta t, \bar{z}^j_{k\Delta t})|.$$

It then has no descendant if $f' \circ \bar{V}^N(k\Delta t, \bar{z}^j_{k\Delta t}) \leqslant 0$ and gives birth to two particles otherwise.

Proposition 8.1. *There exists $C > 0$ such that*

$$\max_{0 \leqslant k\Delta t \leqslant T} \mathbb{E} \| \hat{\bar{V}}^N(k\Delta t, \cdot) - \bar{V}^N(k\Delta t, \cdot) \|_{L^{1,\Phi}(\mathbb{R})} \leqslant C\Delta t + \frac{C}{\sqrt{N}}. \tag{8.1}$$

To prove proposition 8.1 we need the following expansion.

Lemma 8.2. *Set*

$$\varepsilon_{k\Delta t} := \mathbb{E}\bigg\|\frac{1}{N}\sum_{j=1}^{\breve{\mathcal{N}}^N_{k\Delta t}} H(\cdot - \breve{z}^j_{k\Delta t}) - \frac{1}{N}\sum_{j=1}^{\bar{\mathcal{N}}^N_{k\Delta t}} H(\cdot - \bar{z}^j_{k\Delta t})\bigg\|_{L^{1,\Phi}(\mathbb{R})}, \tag{8.2}$$

and denote the particles of the tree $\breve{Z}^{N(i)}_{k\Delta t,(k-\ell)\Delta t}$ *issued from* z^i_0 *by* $\breve{z}^{(i)j}_{k\Delta t,(k-\ell)\Delta t}$. *There exists* $C>0$ *such that*

$$\varepsilon_{k\Delta t} \leqslant \frac{C\Delta t}{N}\sum_{i=1}^{N}\sum_{\ell=1}^{k-1}\mathbb{E}\sum_{j=1}^{\breve{\mathcal{N}}^{N(i)}_{(k-\ell)\Delta t}} |\breve{V}^N((k-\ell)\Delta t, \breve{z}^{(i)j}_{(k-\ell)\Delta t}) - \bar{V}^N((k-\ell)\Delta t, \breve{z}^{(i)j}_{(k-\ell)\Delta t})| + C\Delta t \tag{8.3}$$

for all $0 \leqslant k\Delta t \leqslant T$.

For the right-hand side of (8.3) we have the following estimate.

Lemma 8.3. *There exists* $C>0$ *such that*

$$\begin{aligned}\mathbb{E}\sum_{j=1}^{\breve{\mathcal{N}}^{N(i)}_{k\Delta t}} &|\breve{V}^N(k\Delta t, \breve{z}^{(i)j}_{k\Delta t}) - \bar{V}^N(k\Delta t, \breve{z}^{(i)j}_{k\Delta t})| \\ &\leqslant C\Delta t\sum_{\ell=1}^{k-1}\mathbb{E}\sum_{j=1}^{\breve{\mathcal{N}}^{N(i)}_{(k-\ell)\Delta t}} |\breve{V}^N((k-\ell)\Delta t, \breve{z}^{(i)j}_{(k-\ell)\Delta t}) - \bar{V}^N((k-\ell)\Delta t, \breve{z}^{(i)j}_{(k-\ell)\Delta t})| \\ &\quad + C\Delta t + \frac{C}{\sqrt{N}}\end{aligned} \tag{8.4}$$

for all $0 \leqslant k\Delta t \leqslant T$.

Suppose for a while that the lemmas 8.2 and 8.3 are proven. Set

$$\eta(k\Delta t) := \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}\sum_{j=1}^{\breve{\mathcal{N}}^{N(i)}_{k\Delta t}} |\breve{V}^N(k\Delta t, \breve{z}^{(i)j}_{k\Delta t}) - \bar{V}^N(k\Delta t, \breve{z}^{(i)j}_{k\Delta t})|.$$

In view of (8.4) one has

$$\eta(k\Delta t) \leqslant C\Delta t\sum_{\ell=1}^{k-1}\eta(\ell\Delta t) + C\Delta t + \frac{C}{\sqrt{N}}. \tag{8.5}$$

We deduce that

$$\sum_{\ell=1}^{k+1}\eta(\ell\Delta t) - \sum_{\ell=1}^{k}\eta(\ell\Delta t) \leqslant C\Delta t\sum_{\ell=1}^{k}\eta(\ell\Delta t) + C\Delta t + \frac{C}{\sqrt{N}}.$$

An easy induction shows that

$$\sum_{\ell=1}^{k}\eta(\ell\Delta t) \leqslant C\Delta t + \frac{C}{\sqrt{N}}$$

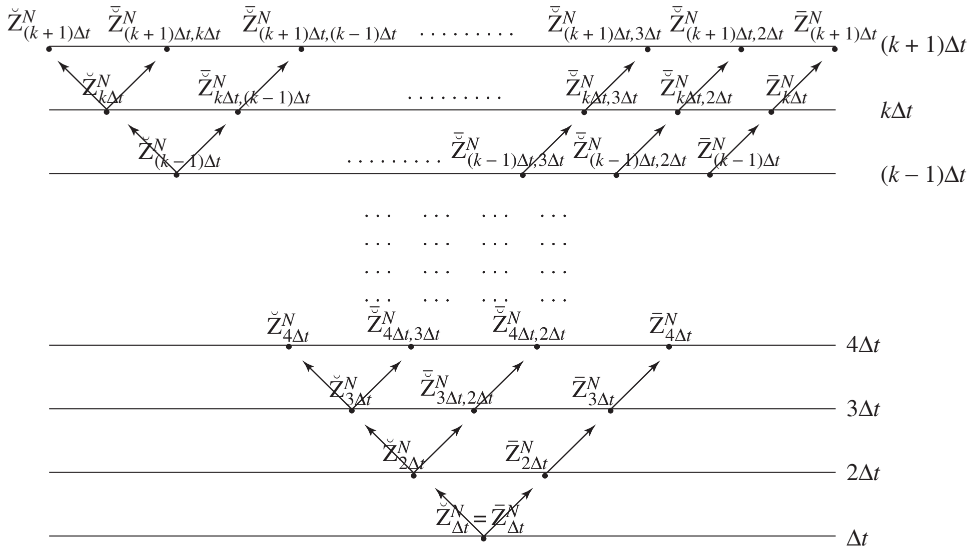

Figure 1. Construction of the intermediate trees.

for all $k\Delta t \leqslant T$, from which $\eta(k\Delta t) \leqslant C\Delta t + C/\sqrt{N}$ in view of (8.5). The desired inequality (8.1) then follows from the lemma 8.2.

It now remains to prove lemmas 8.2 and 8.3. The calculation is tricky because of interaction between the particles induced by the randomness of the offspring rule which governs the process $(\bar{Z}_t^N)$. This leads us to introduce a family of new trees whose representation is as follows. In figure 1, $\nwarrow$ means that the offspring rule is governed by the deterministic function $\breve{V}^N$, whereas $\nearrow$ means that the offspring rule is governed by the random function $\bar{V}^N$. For two trees whose names are on the same horizontal line and are neighbours, the particles having the same ancestor die, branch and diffuse by means of the same random trials.

(*a*) *Proof of lemma 8.2*

Denote by $\bar{\breve{z}}^{(i)j}_{k\Delta t,(k-\ell)\Delta t}$ the particles of the tree $\bar{\breve{Z}}^{N(i)}_{k\Delta t,(k-\ell)\Delta t}$ issued from $\breve{Z}^{N(i)}_{(k-\ell)\Delta t}$, and set

$$\bar{\breve{Q}}^{N(i)}_{k,\ell}(x) := \left| \sum_{j=1}^{\bar{\breve{\mathcal{N}}}^{N(i)}_{k\Delta t,(k-\ell+1)\Delta t}} H(x - \bar{\breve{z}}^{(i)j}_{k\Delta t,(k-\ell+1)\Delta t}) - \sum_{j=1}^{\bar{\breve{\mathcal{N}}}^{N(i)}_{k\Delta t,(k-\ell)\Delta t}} H(x - \bar{\breve{z}}^{(i)j}_{k\Delta t,(k-\ell)\Delta t}) \right|. \tag{8.6}$$

We obviously have

$$\varepsilon_{k\Delta t} \leqslant \frac{1}{N} \sum_{i=1}^{N} \sum_{\ell=1}^{k-1} \mathbb{E} \| \bar{\breve{Q}}^{N(i)}_{k,\ell}(\cdot) \|_{L^{1,\Phi}(\mathbb{R})}. \tag{8.7}$$

We observe that, by construction, the particles which belong to one of the trees

$$\bar{\breve{Z}}^{N(i)}_{k\Delta t,(k-\ell+1)\Delta t} \quad \text{and} \quad \bar{\breve{Z}}^{N(i)}_{k\Delta t,(k-\ell)\Delta t},$$

but do not belong to the other, are descendants of particles which have appeared in one of the trees

$$\bar{\breve{Z}}^{N(i)}_{(k-\ell+1)\Delta t,(k-\ell)\Delta t} \quad \text{and} \quad \breve{Z}^{N(i)}_{(k-\ell+1)\Delta t},$$

and not in the other one. This leads us to introduce the events $\bar{\breve{\omega}}^{N(i)}_{(k-\ell+1)\Delta t}(m)$ defined as

$$\{\text{Exactly } m \text{ particles of the trees } \bar{\breve{Z}}^{N(i)}_{(k-\ell+1)\Delta t,(k-\ell)\Delta t} \text{ and } \breve{Z}^{N(i)}_{(k-\ell+1)\Delta t} \text{ belong to one tree and not the other one}\} \cap \{\breve{\mathcal{N}}^{N(i)}_{(k-\ell)\Delta t} \geqslant m\}.$$

Observe that, on the event $\bar{\breve{\omega}}^{N(i)}_{(k-\ell+1)\Delta t}(0)$, whose probability may be close to unity (see (8.9)), one has $\breve{Q}^{N(i)}_{k,\ell}(\cdot) \equiv 0$, since the free motions of the particles of the two trees are identical. We now 'localize' by proving that, for $m \geqslant 2$, the contribution of the event $\bar{\breve{\omega}}^{N(i)}_{(k-\ell+1)\Delta t}(m)$ is small, that is,

$$\exists C > 0, \quad \mathbb{E}\bigg\{\bar{\breve{Q}}^{N(i)}_{k,\ell}(x) \sum_{m=2}^{\breve{\mathcal{N}}^{N(i)}_{(k-\ell)\Delta t}} \mathbb{I}_{\bar{\breve{\omega}}^{N(i)}_{(k-\ell+1)\Delta t}}(m)\bigg\} \leqslant C\Delta t^2 \quad \text{for all } x \in \mathbb{R}. \tag{8.8}$$

To this end, set

$$\begin{aligned}
\alpha^{(i)j}_{(k-\ell)\Delta t} &:= \min(|f' \circ \breve{V}^N((k-\ell)\Delta t, \breve{z}^{(i)j}_{(k-\ell)\Delta t})|, |f' \circ \bar{V}^N((k-\ell)\Delta t, \breve{z}^{(i)j}_{(k-\ell)\Delta t})|),\\
\beta^{(i)j}_{(k-\ell)\Delta t} &:= \max(|f' \circ \breve{V}^N((k-\ell)\Delta t, \breve{z}^{(i)j}_{(k-\ell)\Delta t})|, |f' \circ \bar{V}^N((k-\ell)\Delta t, \breve{z}^{(i)j}_{(k-\ell)\Delta t})|),\\
\delta^{(i)j}_{(k-\ell)\Delta t} &:= \beta^{(i)j}_{(k-\ell)\Delta t} - \alpha^{(i)j}_{(k-\ell)\Delta t}.
\end{aligned}$$

On $\bar{\breve{\omega}}^{N(i)}_{(k-\ell+1)\Delta t}(0)$, either no particle dies at time $(k-\ell)\Delta t$, or any dying particle dies in the two trees simultaneously (that is, all the jth uniform trials have taken values outside the respective intervals $(\Delta t\alpha^{(i)j}_{(k-\ell)\Delta t}, \Delta t\beta^{(i)j}_{(k-\ell)\Delta t})$). We obviously have

$$\mathbb{P}^{\breve{\mathcal{F}}^N_{(k-\ell)\Delta t}} \bar{\breve{\omega}}^{N(i)}_{(k-\ell+1)\Delta t}(0) = \prod_{j=1}^{\breve{\mathcal{N}}^{N(i)}_{(k-\ell)\Delta t}} (1 - \Delta t\delta^{(i)j}_{(k-\ell)\Delta t}). \tag{8.9}$$

Similarly,

$$\mathbb{P}^{\breve{\mathcal{F}}^N_{(k-\ell)\Delta t}} \bar{\breve{\omega}}^{N(i)}_{(k-\ell+1)\Delta t}(1) = \Delta t \sum_{j=1}^{\breve{\mathcal{N}}^{N(i)}_{(k-\ell)\Delta t}} \delta^{(i)j}_{(k-\ell)\Delta t} \prod_{\substack{\theta=1,\\ \theta\neq j}}^{\breve{\mathcal{N}}^{N(i)}_{(k-\ell)\Delta t}} (1 - \Delta t\delta^{(i)\theta}_{(k-\ell)\Delta t}). \tag{8.10}$$

We thus get (8.8) by using

$$\bigcup_{m\geqslant 2} \bar{\breve{\omega}}^{N(i)}_{(k-\ell+1)\Delta t}(m) = \Omega - \bar{\breve{\omega}}^{N(i)}_{(k-\ell+1)\Delta t}(0) - \bar{\breve{\omega}}^{N(i)}_{(k-\ell+1)\Delta t}(1)$$

and (5.18), and by proceeding as in the proof of lemma 3.3, to get

$$\mathbb{E}^{\breve{\mathcal{F}}^N_{(k-\ell+1)\Delta t}} \bar{\breve{Q}}^{N(i)}_{k,\ell}(x) \leqslant C(\bar{\breve{\mathcal{N}}}^{N(i)}_{(k-\ell)\Delta t,(k-\ell+1)\Delta t} + \breve{\mathcal{N}}^{N(i)}_{(k-\ell+1)\Delta t}) \leqslant 4C\breve{\mathcal{N}}^{N(i)}_{(k-\ell)\Delta t},$$

and

$$\mathbb{E}\breve{\mathcal{N}}^{N(i)}_{(k-\ell)\Delta t} \leqslant C.$$

Let us now restrict ourselves to the event $\bar{\breve{\omega}}^{N(i)}_{(k-\ell+1)\Delta t}(1)$ and consider the particle belonging to one of the trees and not to the other one; conditionally to $\breve{\mathcal{F}}^N_{(k-\ell)\Delta t}$, the moments of the number of its descendants at time $k\Delta t$ obviously satisfy inequalities of the type (3.10). By conditioning with respect to $\breve{\mathcal{F}}^N_{(k-\ell)\Delta t}$, and then using (8.10) and proceeding as in the proof of (5.14), we finally obtain

$$\mathbb{E}\bar{\breve{Q}}^{N(i)}_{k,\ell}(x) \leqslant C\Delta t \mathbb{E} \sum_{j=1}^{\breve{\mathcal{N}}^{N(i)}_{(k-\ell)\Delta t}} |\breve{V}^N((k-\ell)\Delta t, \breve{z}^{(i)j}_{(k-\ell)\Delta t}) - \bar{V}^N((k-\ell)\Delta t, \breve{z}^{(i)j}_{(k-\ell)\Delta t})| + C(\Delta t)^2 \quad (8.11)$$

for some positive number C which is independent of x. In view of (8.7) the proof of the lemma 8.2 is thus completed.

(b) *Proof of lemma 8.3*

Consider the left-hand side of (8.4) and insert $\hat{\breve{V}}^N(k\Delta t, \breve{z}^{(i)j}_{k\Delta t})$ into the sum.

The expression

$$\mathbb{E} \sum_{j=1}^{\breve{\mathcal{N}}^{N(i)}_{k\Delta t}} |\breve{V}^N(k\Delta t, \breve{z}^{(i)j}_{k\Delta t}) - \hat{\breve{V}}^N(k\Delta t, \breve{z}^{(i)j}_{k\Delta t})|$$

can be bound from above by

$$\frac{1}{N}\mathbb{E} \sum_{j=1}^{\breve{\mathcal{N}}^{N(i)}_{k\Delta t}} \{|(\mathbb{E}\langle \breve{Z}^{N(i)}_{k\Delta t}, \mathcal{H}^x\rangle - \langle \breve{Z}^{N(i)}_{k\Delta t}, \mathcal{H}^x\rangle)|_{x=\breve{z}^{(i)j}_{k\Delta t}}\} + \frac{1}{N}\mathbb{E} \sum_{j=1}^{\breve{\mathcal{N}}^{N(i)}_{k\Delta t}} \left\{ \left| \sum_{\substack{i'=1,\\ i'\neq i}}^{N} (\mathbb{E}\langle \breve{Z}^{N(i')}_{k\Delta t}, \mathcal{H}^x\rangle - \langle \breve{Z}^{N(i')}_{k\Delta t}, \mathcal{H}^x\rangle) \right|_{x=\breve{z}^{(i)j}_{k\Delta t}} \right\}.$$

The first term can be bounded from above by C/N in view of lemma 3.3. The second term can be bounded from above by $C/\sqrt{N}$ by using the same arguments as in the proof of (7.2) and, again, the lemma 3.3.

We finally have to consider

$$\mathbb{E} \sum_{j=1}^{\breve{\mathcal{N}}^{N(i)}_{k\Delta t}} |\hat{\breve{V}}^N(k\Delta t, \breve{z}^{(i)j}_{k\Delta t}) - \bar{V}^N(k\Delta t, \breve{z}^{(i)j}_{k\Delta t})|.$$

Proceeding as in the proof of lemma 3.3 we easily get

$$\mathbb{E}\bar{\breve{Q}}^{N(i)}_{k,\ell}(\breve{z}^{(i)j}_{k\Delta t})\mathbb{I}_{\bar{\breve{\omega}}^m_{(k-\ell+1)\Delta t}(1)} \leqslant C\mathbb{P}[\bar{\breve{\omega}}^m_{(k-\ell+1)\Delta t}(1)].$$

Similar arguments to those of the proof of lemma 8.2 then allow us to conclude. We omit the details.

While this paper was in press, Axel Grorud, from the Université de Provence, suddenly died. Axel had been a member of the Omega research group at INRIA since its creation. He helped the first author a lot while studying for his PhD, and he was more than a good friend to the second author. We dedicate this paper to his memory, in remembrance of the happy moments we spent doing mathematics together.

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