Approximation of quantiles of components of diffusion processes

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Abstract

In this paper we study the convergence rate of the numerical approximation of the quantiles of the marginal laws of (X_t), where (X_t) is a diffusion process, when one uses a Monte Carlo method combined with the Euler discretization scheme. Our convergence rate estimates are obtained under two sets of hypotheses: either (X_t) is uniformly hypoelliptic (in the sense of Condition (UH) below), or the inverse of the Malliavin covariance of the marginal law under consideration satisfies the condition (M) below.

In order to deduce the required numerical parameters from our error estimates in view of a prescribed accuracy, one needs to get an as accurate as possible lower bound estimate for the density of the marginal law under consideration. This usually is a very hard task. Nevertheless, in our section 3 of this paper, we treat a case coming from a financial application.

Keywords: Stochastic Differential Equations; Euler method; Monte Carlo methods, simulation.

1 Introduction

We recently encountered several applications leading to the following approximation problem: given a d-dimensional diffusion process $(X_t(x))$, one needs to approximate quantiles of the random variable $X_T^d(x)$ for some prescribed time T. For example, in some Random Mechanics models the motion of a mechanical system submitted to random forces is described by the dynamics of the position P_t and the velocity V_t of the centre of gravity. The process

$(X_t) := (P_t, V_t)$ satisfies a stochastic differential equation of the type

$$\begin{cases} dP_t & = V_t \; dt, \\ dV_t & = A(P_t, V_t) \; dt + \sigma(X_t, V_t) \; dW_t. \end{cases} \tag{1}$$

Commonly admitted safety and reliability factors are the quantiles of certain components of the position or the velocity. As well, in Finance, the Value-at-Risk of a financial position is defined as one quantile of the possible large losses induced by the position at the end of a given period. Consider a self–financing portfolio with $d-1$ financial assets. The asset prices are denoted by $(X_t^j,\ 1 \leq j \leq d-1)$. Suppose that the investing strategies of the portfolio are functions of the asset prices only, so that the portfolio value X_t^d satisfies an equality of the type

$$X_t^d = \sum_{j=1}^{d-1} \pi^j(s, X_s) X_s^j$$

for all t. Since the portfolio is self-financing, one has

$$\begin{cases} X_t^j & = x^j + \int_0^t \sigma_0^j(s, X_s) X_s^j \; ds \\ & \quad + \sum_{i=1}^r \int_0^t \sigma_i^j(s, X_s) \; X_s^j \; dW_s^i, \quad j = 1, \ldots, d-1, \\ X_t^d & = x^d + \sum_{j=1}^{d-1} \int_0^t \pi^j(s, X_s) \; dX_s^j. \end{cases} \tag{2}$$

The VaR of confidence level δ of this portfolio at a given period of time T is

$$\rho(x, \delta) := \inf\{\rho \in \mathbb{R};\ \mathbb{P}\left[X_T^d \leq \rho\right] = \delta\}.$$

To approximate safety factors or Value-at-Risk factors, the numerical resolution of the Fokker–Planck equation may be impossible because d typically is larger than 4. One thus uses a Monte Carlo method. As one cannot simulate exact independent trajectories of the solution (X_t) since that solution is not known exactly, one has to use a discretization scheme. The Euler scheme has the weakest possible complexity and, combined with extrapolation techniques, allows one to get good accuracies (see below). We therefore aim to estimate the approximation error on quantiles of the Monte Carlo method based on the simulation of the Euler scheme.

We now emphasize a difficulty that we had to overcome. In the above mechanical and financial examples, the infinitesimal generator of the diffusion process (X_t) is degenerate: on one hand, no noise appears in the dynamics of the position P_t; on the other hand, the noises appearing in the dynamics of (X_t^d) are those of the $d-1$ first coordinates of (X_t). Because of that strong lack of ellipticity we had to use Malliavin calculus techniques in the error analysis (see, e.g., Nualart [14, 13] for introductions to Malliavin calculus).

Let us briefly present and comment on the main result of this paper. Let $(X_t(x))$ be a d-dimensional smooth version of the stochastic flow solution to

$$X_t(x) = x + \int_0^t A_0(s, X_s(x))ds + \sum_{i=1}^r \int_0^t A_i(s, X_s(x))dW_s^i, \tag{3}$$

where (W_s) is a r-dimensional Brownian motion, and the functions $A_0, A_1, \ldots, A_r$ are smooth with bounded derivatives. The Euler scheme is defined as follows:

$$\begin{aligned} X^n_{(p+1)T/n}(x) = X^n_{pT/n}(x) + A_0(pT/n, X^n_{pT/n}(x))\frac{T}{n} \\ + \sum_{i=1}^r A_i(pT/n, X^n_{pT/n}(x))(W^i_{(p+1)T/n} - W^i_{pT/n}) \end{aligned}$$

for $p = 0, \ldots, n-1$. Talay & Tubaro [17] have shown that the discretization error

$$\mathbb{E}\, f(X_T(x)) - \mathbb{E}\, f(X_T^n(x)) \tag{4}$$

can be expanded in terms of powers of $\frac{1}{n}$ when f is a smooth function. The result requires some smoothness hypotheses on the functions A_i only. In addition, formulae for the coefficients of the expansion can be derived. The existence of the expansion implies that one can improve the convergence rate of the Euler scheme by linear combinations of the results obtained with different step-sizes (Romberg–Richardson extrapolation technique). When the infinitesimal generator of (X_t) is uniformly hypoelliptic in the sense of Condition (UH) in Section 2, Bally & Talay [3, 4] proved that the expansion holds true even when f is only supposed measurable and bounded, and also when f is a δ-function. See also Kohatsu–Higa [8], Kohatsu–Higa and Pettersson [9] for related results. For similar estimates when the stochastic differential equation is driven by a Lévy process, see Protter & Talay [16].

The above mentioned results do not provide error estimates on the approximation of quantiles of the marginal laws of $X_T(x)$. Our objective in this article is to show that, under a suitable condition on the Malliavin covariance of the d-*th* component $X_T^d(x)$ (see Condition (M) in Subsection 2.2), the time discretization error on the quantile of level δ, $\rho(x, \delta)$, of the law of $X_T^d(x)$ is bounded from above by

$$C(T)\ \frac{1 + \|x\|^Q}{\overline{p}_T^d(\rho(x,\delta))}\ \frac{1}{n},$$

where the positive numbers $C(T)$ and Q do not depend on n and x and, denoting by $p_T^d(x,y)$ the density of $X_T^d(x)$, we have set

$$\overline{p}_T^d(\rho(x,\delta)) := \inf_{y\in(\rho(x,\delta)-1,\rho(x,\delta)+1)} p_T^d(x,y).$$

Combined with a classical estimate on the statistical error on quantiles of the Monte Carlo method with N simulations (see, e.g., Cramer [6, p.367]), we then get that the empirical quantile of N independent simulations of the Euler scheme leads to a global error of order

$$\mathcal{O}\left(\frac{1}{\overline{p}_T^d(\rho(x,\delta))n}\right) + \mathcal{O}\left(\frac{1}{p_T^{n,d}(x,\rho(x,\delta))\sqrt{N}}\right), \tag{5}$$

where $p_T^{n,d}(x,\xi)$ denotes the density at time T of the d-*th* component of the Euler scheme.

In our framework, the Malliavin covariance matrix of $X_T(x)$ may be degenerate since our condition (M) concerns the Malliavin covariance of $X_T^d(x)$ only. Of course, our study would be of limited interest if the condition (M) were seldom satisfied, or difficult to check. The examples in our paper [18] show that it does not seem to be the case. We point out that our proof involves a convergence rate result on the marginal laws of the Euler scheme which is new (see Theorem 2.6).

We finally emphasize an ultimate difficulty. In view of (5), in practice one has to seek a precise estimate from below on $p_T^d(x,\rho(x,\delta))$. We do not know a result which can be applied under our rather weak assumptions (see Remark 2.5). Nevertheless, in the last section of this paper, we give an example of a situation where one can explicit a lower bound which seems accurate enough for numerical purposes.

In the companion paper [18] we show that the condition (M) is satisfied in various financial applications such as the computation of the VaR of a portfolio and the computation of a model risk measurement for the Profit and Loss of a misspecified hedging strategy.

Notation. In all the paper, φ being a smooth function, the notation $\partial_\alpha\varphi(t,x)$ means that the multi-index α concerns the derivation with respect to the coordinates of x, the variable t being fixed.

We will use the same notation K, q, Q, λ, etc, for different functions and positive real numbers which may vary from line to line, having the common property to be independent of the approximation parameter n.

Finally, $< \cdot, \cdot >$ denotes the inner product in $L^2(0,T)$.

2 Approximation of quantiles of diffusion processes

In this section we get the convergence rate of the Euler scheme under two different conditions: either the diffusion is 'uniformly hypoelliptic', or its satisfies Condition (M) below.

2.1 Uniformly hypoelliptic and time homogeneous diffusions

In this subsection we consider time homogeneous coefficients $A_0, A_1, \dots, A_r$. Let $(X_t(x))$ be the solution of

$$X_t(x) = x + \int_0^t A_0(X_s(x))ds + \sum_{i=1}^r \int_0^t A_i(X_s(x))dW_s^i. \tag{6}$$

We start this section by recalling a convergence rate estimate for the Euler scheme. Let

$$X^n_{(p+1)T/n}(x) = X^n_{pT/n}(x) + A_0(X^n_{pT/n}(x))\frac{T}{n} + \sum_{i=1}^r A_i(X^n_{pT/n}(x))(W^i_{(p+1)T/n} - W^i_{pT/n}) \tag{7}$$

for $p = 0, \dots, n-1$, and

$$X^n_t(x) = X^n_{pT/n}(x) + A_0(X^n_{pT/n}(x))(t - \frac{pT}{n}) + \sum_{i=1}^r A_i(X^n_{pT/n}(x))(W^i_t - W^i_{pT/n}) \tag{8}$$

for $\frac{pT}{n} \le t < \frac{(p+1)T}{n}$.

We need to fix some notation. We identify the functions $A_0, A_1, \dots, A_r$ and the vector fields

$$A_0(x) := \sum_{j=1}^d A_0^j(x)\partial_j,$$
$$A_i(x) = \sum_{j=1}^d A_i^j(x)\partial_j, \ i = 1, \dots, r.$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \{0, 1, \dots r\}^k$, we define the vector fields A_i^α $(1 \le i \le r)$ by induction: $A_i^\emptyset = A_i$ and, for $0 \le j \le r$, $A_i^{(\alpha,j)} := [A_j, A_i^\alpha]$, where $[A, A']$ denotes the Lie bracket of the two vector fields A and A'. Let $|\alpha|$ denote the length of the multi-index α, and set

$$V_L(x,\eta) := \sum_{i=1}^{r} \sum_{|\alpha| \leq L-1} < A_i^\alpha(x), \eta >^2,$$

and

$$V_L(x) := 1 \wedge \inf_{\|\eta\|=1} V_L(x,\eta).$$

Theorem 2.1. *Suppose that the uniformly hypoellipticity condition*

(UH) $C_L := \inf_{x\in\mathbb{R}^d} V_L(x) > 0$ *for some integer* L,

holds, as well as

(C) *The coefficients* A_i^j, $i = 0, \dots, r$, $j = 1, \dots, d$ *of (6) are of class* $\mathcal{C}_b^\infty(\mathbb{R}^d)$ *(the* A_i^j*'s may be unbounded).*

Let f *be a measurable and bounded function. The Euler scheme error satisfies*

$$\mathbb{E}\, f(X_T(x)) - \mathbb{E}\, f(X_T^n(x)) = \frac{C_f(T,x)}{n} + \frac{Q_n(f,T,x)}{n^2}. \tag{9}$$

The constants $C_f(T,x)$ *and* $Q_n(f,T,x)$ *have the following property: there exists an integer* m*, a non decreasing function* $K(T)$ *depending on the coefficients* A_i $(0 \leq i \leq r)$ *and their derivatives up to order* m*, and positive real numbers* q *and* Q*, such that*

$$|C_f(T,x)| + \sup_n |Q_n(f,T,x)| \leq \frac{K(T)}{T^q}(1 + \|x\|^Q)\|f\|_\infty. \tag{10}$$

For a proof, see Bally & Talay [3, Theorem 3.1]. Usually the function $\frac{K(T)}{T^q}$ grows to infinity when T goes to infinity. The presence of T^q in the denominator means that, when T is small, the discretization step needs to be chosen small to get good accuracy since the law of $X_T(x)$ is close to a Dirac measure at x.

Observing that the proof of (10) involves no information of f other than its $L^\infty(\mathbb{R}^d)$ norm, it is straigntforward to get the following slight extension.

Corollary 2.2. *Let (f_n) be measurable and bounded functions such that*

$$\sup_n \|f_n\|_\infty < \infty.$$

Under the hypotheses (UH) and (C), the Euler scheme error satisfies

$$\mathbb{E}\, f_n(X_T(x)) - \mathbb{E}\, f_n(X_T^n(x)) = \frac{C_{f_n}(T,x)}{n} + \frac{Q_n(f_n,T,x)}{n^2}, \tag{11}$$

for some constants $C_{f_n}(T,x)$ and $Q_n(f_n,T,x)$ which have the following property: there exists an integer m, a non decreasing function $K(T)$ depending on the coefficients A_i $(0 \leq i \leq r)$ and their derivatives up to the order m, and positive real numbers q and Q, such that

$$|C_{f_n}(T,x)| + \sup_n |Q_n(f_n,T,x)| \leq \frac{K(T)}{T^q}(1 + \|x\|^Q) \sup_n \|f_n\|_\infty. \tag{12}$$

We now consider the approximation of quantiles problem. Under the hypotheses (UH) and (C), we know that the law of $X_T(x)$ has a smooth density $p_T(x,x')$. Thus, the d-th marginal distribution of $X_T(x)$ also has a smooth density $p_T^d(x,y)$. In addition, in view of Proposition 4.1.2 in Nualart [13], the density $p_T^d(x,y)$ of the one dimensional random variable $X_T^d(x)$ is strictly positive at all points y in the interior of its support.

Set

$$\rho(x,\delta) := \inf\{\rho \in \mathbb{R};\ \mathbb{P}\,[X_T^d(x) \leq \rho] = \delta\}$$

for all positive real $0 < \delta < 1$.

We now define our approximation of $\rho(x,\delta)$. The random variable $X_T^n(x)$ may not have a density if the diffusion matrix of $(X_t(x))$ does not satisfy a uniformly elliptic condition. We thus introduce the same slight perturbation of $X_T^n(x)$ as in Bally & Talay [4]. Let Z^n be a $\mathbb{R}^d$-valued random vector independent of $(W_t, 0 \leq t \leq T)$ whose components are i.i.d. and whose law is $\gamma_{1/n}(\xi)d\xi$ where, γ_0 being a smooth and symmetric probability density function with a compact support in $(-1,1)$, we have set

$$\gamma_\epsilon(\xi) := \prod_{i=1}^{d} \frac{\gamma_0(\xi^i/\epsilon)}{\epsilon} \tag{13}$$

for all $\epsilon > 0$ and $\xi \in \mathbb{R}^d$. We set

$$\tilde{X}_T^n(x) = X_T^n(x) + Z^n. \tag{14}$$

We denote by $\tilde{X}_T^{n,d}(x)$ the d-*th* component of $\tilde{X}_T^n(x)$, and by $\tilde{p}_T^{n,d}(x,\cdot)$ the density of its law w.r.t. Lebesgue measure on $\mathbb{R}$. We set

$$\tilde{\rho}^n(x,\delta) := \inf\{\rho \in \mathbb{R};\ \mathbb{P}\,[\tilde{X}_T^{n,d}(x) \leq \rho] = \delta\}.$$

To get an estimate on $|\rho(x,\delta) - \tilde{\rho}^n(x,\delta)|$ we will use the following lemma.

Lemma 2.3. *For all n large enough,*

$$|\rho(x,\delta) - \tilde{\rho}^n(x,\delta)| \le 1. \tag{15}$$

Proof. If Inequality (15) were not true one would have

$$\begin{aligned}
&\left|\mathbb{P}\left(X_T^d(x) \le \rho(x,\delta)\right) - \mathbb{P}\left(X_T^d(x) \le \tilde{\rho}^n(x,\delta)\right)\right| \\
&\qquad = \left| \int_{\rho(x,\delta)}^{\tilde{\rho}^n(x,\delta)} p_T^d(x,y)dy \right| \\
&\qquad \ge \min\left(\int_{\rho(x,\delta)-1}^{\rho(x,\delta)} p_T^d(x,y)dy, \int_{\rho(x,\delta)}^{\rho(x,\delta)+1} p_T^d(x,y)dy \right).
\end{aligned} \tag{16}$$

Our aim is to exhibit a contradiction by showing that the left side tends to 0 when n goes to infinity.

Set

$$f^1(y) := \begin{cases} 1 \text{ for } y \le \rho(x,\delta), \\ 0 \text{ for } y > \rho(x,\delta). \end{cases}$$

By Theorem 2.1 we have

$$\mathbb{E}\, f^1(\tilde{X}_T^{n,d}(x)) = \delta + \frac{C_{f^1}(T,x)}{n} - \frac{Q_n(f^1,T,x)}{n^2}.$$

Similarly set

$$f_n^2(y) := \begin{cases} 1 \text{ for } y \le \tilde{\rho}^n(x,\delta), \\ 0 \text{ for } y > \tilde{\rho}^n(x,\delta). \end{cases}$$

By definition we have

$$\mathbb{E}\, f_n^2(\tilde{X}_T^{n,d}(x)) = \delta.$$

We set $f_n^3 = f_n^2 - f^1$. In view of Theorem 2.1, we have

$$|\mathbb{E}\, f_n^3(\tilde{X}_T^{n,d}(x))| = \left| -\frac{C_{f^1}(T,x)}{n} + \frac{Q_n(f^1,T,x)}{n^2} \right| \le \frac{K(T)}{T^q}(1 + \|x\|^Q)\frac{1}{n}. \tag{17}$$

Moreover, in view of Corollary 2.2 we have

$$|\mathbb{E}\, f_n^3(\tilde{X}_T^{n,d}(x)) - \mathbb{E}\, f_n^3(X_T^d(x))| \le \frac{K(T)}{T^q}(1 + \|x\|^Q)\frac{1}{n}. \tag{18}$$

In view of (17) and (18) we deduce that

$$|\mathbb{E}\, f_n^3(X_T^d(x))| \le \frac{K(T)}{T^q}(1 + \|x\|^Q)\frac{1}{n} \tag{19}$$

for some new function K, which contradicts (16). $\square$

Theorem 2.4. *Under Conditions (UH) and (C) there exist positive numbers q and Q and an increasing function K such that*

$$|\rho(x,\delta) - \tilde{\rho}^n(x,\delta)| \leq \frac{K(T)}{T^q} \cdot \frac{1 + \|x\|^Q}{\overline{p}_T^d(\rho(x,\delta))} \cdot \frac{1}{n}, \tag{20}$$

where

$$\overline{p}_T^d(\rho(x,\delta)) = \inf_{y\in(\rho(x,\delta)-1,\rho(x,\delta)+1)} p_T^d(x,y).$$

Proof. In view of (19) we have

$$\begin{aligned}
\frac{K(T)}{T^q}(1 + \|x\|^Q)\frac{1}{n} &\geq |\mathbb{E}\, f_n^3(X_T^d(x))| \\
&= \left| \int_{\rho(\delta)}^{\tilde{\rho}^n(\delta)} p_T^d(x,y)dy \right| \\
&\geq \inf_{y\in(\rho(x,\delta)-1,\rho(x,\delta)+1)} p_T^d(x,y)|\rho(x,\delta) - \tilde{\rho}^n(x,\delta)|.
\end{aligned}$$

That ends the proof. □

Remark 2.5. *Combined with the statistical error of the Monte Carlo method with N simulations, Estimate (20) leads to the following result: the global error on the quantile is of order*

$$\mathcal{O}\left(\frac{1}{\overline{p}_T^d(\rho(x,\delta))n}\right) + \mathcal{O}\left(\frac{1}{\tilde{p}_T^{n,d}(x,\rho(x,\delta))\sqrt{N}}\right),$$

where $\tilde{p}_T^{n,d}(x,\xi)$ denotes the density of $\tilde{X}_T^{n,d}(x)$. It is reasonable to expect that $\tilde{p}_T^{n,d}(x,\xi) - p_T^d(x,\xi)$ is of order $1/n$ (this rate holds under conditions of the type (UH) and (C): see Bally & Talay [4]). In view of our estimates, in order to choose the number of simulations and the discretization step in terms of a desired accuracy, one needs accurate estimates from below of $p_T^d(x,\rho(x,\delta))$. Such estimates are available when the generator of (X_t) is strictly uniform elliptic (see, e.g., Azencott [2]) and, in the hypoelliptic case, under restrictive assumptions on b (the generator more or less needs to be in divergence form: see Kusuoka & Stroock [11]). In Section 3 we face this problem in a particular situation which satisfies none of these conditions.

2.2 Diffusions satisfying Condition (M)

We now return to the general inhomogeneous stochastic differential equation (3).

Let $(X_s^t(x'),\ 0 \leq s \leq T-t)$ be a smooth version of the flow solution to

$$X_s^t(x') = x' + \int_0^s A_0(t+\theta, X_\theta^t(x'))d\theta + \sum_{i=1}^r \int_0^s A_i(t+\theta, X_\theta^t(x'))dW_{t+\theta}^i. \quad (21)$$

We denote by $M(t,s,x')$ the Malliavin covariance matrix of $X_s^t(x')$.

We suppose:

(C') The functions $A_i^j,\ i=0,\dots,r,\ j=1,\dots,d$ are of class $\mathcal{C}_b^\infty([0,T]\times\mathbb{R}^d)$ (the A_i^j's may be unbounded).

(M) For all $p \geq 1$ there exist a non decreasing function K, a positive real number r, and a positive Borel measurable function Ψ such that

$$\left|\left|\frac{1}{M_d^d(t,s,x')}\right|\right|_p \leq \frac{K(T)}{s^r}\Psi(t,x') \quad (22)$$

for all x' in $\mathbb{R}^d$, t in $[0,T)$ and s in $(0,T-t]$. In addition, Ψ satisfies: for all $\lambda \geq 1$, there exists a function Ψ_λ such that

$$\sup_{t\in[0,T]} \mathbb{E}\left[\Psi(t,X_t(x))^\lambda\right] < \Psi_\lambda(x), \quad (23)$$

and

$$\sup_{n>0}\sup_{t\in[0,T]} \mathbb{E}\left[\Psi(t,X_t^n(x))^\lambda\right] < \Psi_\lambda(x) \quad (24)$$

for all x in $\mathbb{R}^d$.

Equipped with Conditions (M) and (C') we have:

Theorem 2.6. *Let f be a bounded function of class $C^\infty(\mathbb{R})$. Under Conditions (M) and (C') there exist positive numbers λ, q and Q and an increasing function K such that*

$$|\mathbb{E}\, f(X_T^d(x)) - \mathbb{E}\, f(X_T^{n,d}(x))| \leq \frac{K(T)}{T^q}(1+\|x\|^Q)\Psi_\lambda(x)\|f\|_\infty\ \frac{1}{n}. \quad (25)$$

We postpone the lengthy proof of Theorem 2.6 to Section 2.3.

Remark 2.7. *In Theorem 2.6 we suppose that f is smooth. Under that assumption Talay & Tubaro [17] obtain an expansion of the error. The constants in the expansion depend on estimates on the derivatives of f. What is new here, and technically demanding, is the control of the error in terms of $\|f\|_\infty$ as in the statement of Theorem 2.1. However the condition (M) is much less restrictive than the condition (UH), and thus an expansion such as (9) might not hold under Condition (M) only. Nevertheless, in spite of the limitation to the inequality (25) instead of an expansion, Theorem 2.6 provides the key result to get the desired convergence rate for the approximation of quantiles.*

As we shall see, the proof of Theorem 2.6 involves no information of f other than its $L^\infty(\mathbb{R})$ norm. We thus readily get the following slight extension.

Corollary 2.8. *Let (f_n) be bounded functions of class $C^\infty(\mathbb{R})$ such that*

$$\sup_n \|f_n\|_\infty < \infty.$$

Suppose that Conditions (M) and (C') hold. Then the Euler scheme error satisfies: there exist an integer m, a non decreasing function $K(T)$ depending on the coordinates of (3) and on their derivatives up to the order m, and positive real numbers q, Q and λ such that

$$|\mathbb{E}\, f_n(X_T^d(x)) - \mathbb{E}\, f_n(X_T^{n,d}(x))| \leq \frac{K(T)}{T^q}(1 + \|x\|^Q)\Psi_\lambda(x) \sup_n \|f_n\|_\infty \, \frac{1}{n}.$$

Under Condition (M), the d-th marginal distribution of $X_T(x)$ has a smooth density $p_T^d(x,y)$. In addition, again by Proposition 4.1.2 in Nualart [13], the density $p_T^d(x,y)$ of the one dimensional random variable $X_T^d(x)$ is strictly positive at all point y in the interior of its support. Set

$$\rho(x,\delta) := \inf\{\rho \in \mathbb{R};\ \mathbb{P}\,[X_T^d(x) \leq \rho] = \delta\}$$

for all positive real $0 < \delta < 1$. We define the slight perturbation $\tilde{X}_T^n(x)$ of $X_T^n(x)$ as in the preceding section. We set

$$\tilde{\rho}^n(x,\delta) := \inf\{\rho \in \mathbb{R};\ \mathbb{P}\,[\tilde{X}_T^{n,d}(x) \leq \rho] = \delta\}.$$

To get an estimate on $|\rho(x,\delta) - \tilde{\rho}^n(x,\delta)|$ we need the following lemma.

Lemma 2.9. *For all n large enough,*

$$|\rho(x,\delta) - \tilde{\rho}^n(x,\delta)| \le 1. \tag{26}$$

Proof. In order to simplify the notation we assume that $\rho(x,\delta) \le \tilde{\rho}^n(x,\delta)$ (if not, one simply has to introduce an obvious modification of the definition of the functions f_n^1 and f_n^2 below). We slightly modify the proof of Lemma 2.3: in order to apply the theorem 2.6 we have to mollify the functions f_n^1 and f_n^2. We thus define f_n^1 and f_n^2 as follows: they are functions of class $C^\infty(\mathbb{R})$ such that

$$f_n^1(y) := \begin{cases} 1 \text{ for } y \le \rho(x,\delta - \frac{1}{n}), \\ 0 \text{ for } y > \rho(x,\delta), \\ 0 \le f_n^1(y) \le 1 \text{ for } y \in \left[\rho(x,\delta - \frac{1}{n}), \rho(x,\delta)\right] \end{cases}$$

and

$$f_n^2(y) := \begin{cases} 1 \text{ for } y \le \tilde{\rho}^n(x,\delta), \\ 0 \text{ for } y > \tilde{\rho}^n(x,\delta + \frac{1}{n}), \\ 0 \le f_n^2(y) \le 1 \text{ for } y \in \left[\tilde{\rho}^n(x,\delta), \tilde{\rho}^n(x,\delta + \frac{1}{n})\right]. \end{cases}$$

Observe that

$$\delta - \frac{1}{n} \le \mathbb{E}\, f_n^1(X_T^d(x)) \le \delta,$$

and

$$\delta \le \mathbb{E}\, f_n^2(\tilde{X}_T^{n,d}(x)) \le \delta + \frac{1}{n}.$$

It then remains to apply Corollary 2.8 and proceed as in the proof of Lemma 2.3. □

We are now in a position to conclude by using the same arguments as in the proof of Theorem 2.4:

Theorem 2.10. *Under Conditions (M) and (C'), we have*

$$|\rho(x,\delta) - \tilde{\rho}^n(x,\delta)| \le \frac{K(T)}{T^q} \cdot \frac{1 + \|x\|^Q}{\overline{p}_T^d(\rho(x,\delta))} \cdot \Psi_\lambda(x) \cdot \frac{1}{n}, \tag{27}$$

where

$$\overline{p}_T^d(\rho(x,\delta)) = \inf_{y \in (\rho(x,\delta)-1, \rho(x,\delta)+1)} p_T^d(x,y).$$

The preceding theorem would not be interesting if Condition (M) would rarely be satisfied in applied contexts. In Talay and Zheng [18] we study financial problems for which the condition (M) is not restrictive: the computation of quantiles of models with stochastic volatility, the computation of the VaR of a portfolio, and the computation of a model risk measurement for the Profit and Loss of a misspecified hedging strategy.

2.3 Proof of Theorem 2.6

To prove Theorem 2.6 it obviously suffices to apply the estimates of Lemmas 2.11 and 2.12 below to the expansion provided in Lemma 2.13. The same statements appear in [3], but here we need to construct a partially different proof of these two lemmas in order to take into account the fact that Condition (M) does not allow us to control the inverse of the Malliavin covariance matrix of $(X_t(x))$. Lemmas 2.11 and 2.12 will be proven in subsections 2.3.1 and 2.3.2. We do not prove Lemma 2.13: the calculation is the same as in [3].

We set

$$u(t,x') := \mathbb{E}\,[f(X^{t,d}_{T-t}(x'))], \tag{28}$$

where $X^{t,d}_{T-t}(x')$ is defined as in (21). As f is a smooth bounded function we have[1]

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x') + \mathcal{L}_t u(t,x') & = 0,\ \ 0 \le t < T, \\ u(T,x') & = f(x'^d), \end{cases} \tag{29}$$

where $\mathcal{L}_t$ denotes the generator of the non homogeneous Markov process $(X_t(x))$.

Lemma 2.11. *Let the function u be defined by (29). Then, for multiindex α whose order w.r.t t is no more than* 3, *and order w.r.t x is no more than* 6, *and for any smooth function g with polynomial growth, there exist a non decreasing function $K(T)$ and positive constants q, Q and λ uniform with respect to n and T, such that*

$$\forall t \in [0,T],\ \ |\mathbb{E}\,[g(X_t(x))\partial_\alpha u(t,X_t(x))]| \le \frac{K(T)}{T^q}(1+\|x\|^Q)\Psi_\lambda(x)\|f\|_\infty \tag{30}$$

[1] It is at this point of the proof that we need to suppose that f is a smooth function. In [3] the lack of smoothness of f is compensated by Condition (UH).

and

$$\forall t \in \left[0, T - \frac{T}{n}\right],\ |\mathbb{E}\,[g(X_t^n(x))\partial_\alpha u(t, X_t^n(x))]| \leq \frac{K(T)}{T^q}(1+\|x\|^Q)\Psi_\lambda(x)\|f\|_\infty. \tag{31}$$

Lemma 2.12. *For some positive numbers q, Q and λ and some non decreasing function $K(T)$, one has that*

$$|\mathbb{E}\,f(X_T^{n,d}(x)) - \mathbb{E}\,f(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)))| \leq \frac{K(T)}{T^q}\|f\|_\infty(1+\|x\|^Q)\Psi_\lambda(x)\frac{1}{n}. \tag{32}$$

Lemma 2.13. *It holds that*

$$\begin{aligned}\mathbb{E}\,f&\left(X_T^{n,d}(x)\right) - \mathbb{E}\,f\left(X_T^d(x)\right)\\ &= \mathbb{E}\,f\left(X_T^{n,d}(x)\right) - \mathbb{E}\,f\left(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x))\right)\\ &\qquad + \frac{T^2}{2n^2}\sum_{k=0}^{n-2}\mathbb{E}\,\Phi\left(\tfrac{kT}{n}, X_{\frac{kT}{n}}^n(x)\right) + \sum_{k=0}^{n-2} R_k^n,\end{aligned}$$

where Φ is a sum of terms, each of them being of the form $\varphi_\beta^\flat(t,x)\partial_\beta u(t,x)$, and R_k^n is a sum of terms, each of them being of the form

$$\begin{aligned}\mathbb{E}\,&\Big[\varphi_\alpha^\natural(kT/n, x_{kT/n}^n(x))\\ &\int_{kT/n}^{(k+1)T/n}\int_{kT/n}^{s_1}\int_{kT/n}^{s_2}\varphi_\alpha^\sharp(s_3, X_{s_3}^n(x))\partial_\alpha u(s_3, X_{s_3}^n(x))ds_3 ds_2 ds_1\Big],\end{aligned}$$

where $|\beta| \leq 4$, $|\alpha| \leq 6$, and the $\varphi_\alpha^\natural$'s, $\varphi_\alpha^\sharp$'s, $\varphi_\beta^\flat$'s are products of functions which are derivatives up to the order 3 of the A_i^j's.

To prove Lemmas 2.11 and 2.12 we need the following easy technical lemma (see Bally & Talay [3]).

Lemma 2.14. *Under (C'), for any $p > 1$ and $j \geq 0$, there exist an integer Q and a non decreasing function $K(t)$ such that*

$$\sup_{n\geq 1}\|X_t^n(x)\|_{j,p} \leq K(t)(1+\|x\|^Q), \tag{33}$$

and

$$\|X_t(x) - X_t^n(x)\|_{j,p} \leq \frac{K(t)}{\sqrt{n}}(1+\|x\|^Q). \tag{34}$$

2.3.1 Proof of Lemma 2.11

We only prove Estimate (31), Estimate (30) being treated with the same arguments. We need to carefully adapt the technique introduced in Bally & Talay [3]: here, one cannot use the Kusuoka and Stroock [10]'s upper bound estimates on the density and on the inverse of the Malliavin covariance matrix of the hypoelliptic diffusion since the generator of $X_t(x)$ is not hypoelliptic. We thus use the smoothness of the law of $X_t^d(x)$, and the fact that the function f is applied to the sole coordinate $X_t^d(x)$.

In view of (29) it is obvious that we need to consider partial derivatives of $u(t,x')$ only. We observe that

$$\mathbb{E}\left[g(X_t^n(x))\partial_\alpha u(t,X_t^n(x))\right] = \mathbb{E}\left[g(X_t^n(x))\left\{\partial_\alpha \mathbb{E}\, f(X_{T-t}^{t,d}(x'))\right\}\big|_{x'=X_t^n(x)}\right],$$

and

$$\partial_\alpha \mathbb{E}\left[f(X_{T-t}^{t,d}(x'))\right] = \sum_{i=1}^{|\alpha|}\mathbb{E}\left[f^{(i)}\left(X_{T-t}^{t,d}(x')\right)\Theta_i(T-t,x')\right],$$

where $f^{(i)}$ is the i-*th* order derivative of f, and $\Theta_i(T-t,x')$ are sums of products of $\partial_\beta(X_{T-t}^{t,d}(x'))$ where $|\beta| \le |\alpha| - i + 1$. For the sake of simplicity let $X_{T-t}^{t,d}(X_t^n(x))$ denote the d-*th* component of the image of $X_t^n(x)$ by the flow $X_\cdot^t$ at time $T-t$, and let $M_d^d(t,T-t;n,x)$ denote the Malliavin covariance of $X_{T-t}^{t,d}(X_t^n(x))$:

$$M_d^d(t,T-t;n,x) :=< D(X_{T-t}^{t,d}(X_t^n(x)), D(X_{T-t}^{t,d}(X_t^n(x)) > .$$

The proof proceeds as follows: in part I, we prove a useful estimate on the inverse of the Malliavin covariance matrix of $X_{T-t}^{t,d}(X_t^n(x))$; this allows us to develop the calculation of part II, where we use the integration by parts formula to get rid of the derivatives of f, and we use the condition (M) to get the desired estimate (31).

I. We first prove that $(X_{T-t}^{t,d}(X_t^n(x))$ is a smooth functional for all $0 \le t \le T - \frac{T}{n}$ under Condition (M). It is clear that

$$\begin{aligned} M_d^d(t,T-t;n,x) &= \int_0^T (D_\theta(X_{T-t}^{t,d}(X_t^n(x))))^2 d\theta \\ &\ge \int_t^T (D_\theta(X_{T-t}^{t,d}(X_t^n(x))))^2 d\theta \\ &= M_d^d(t,T-t,x')\Big|_{x'=X_t^n(x)}. \end{aligned}$$

In view of Condition (M), for all $p \geq 1$ there exist a $r > 0$ and functions K, Ψ such that

$$\left\| \frac{1}{M_d^d(t, T-t, x')} \right\|_p \leq \frac{K(T)}{(T-t)^r} \Psi(t, x'),$$

and thus

$$\mathbb{P}\left(M_d^d(t, T-t, x') \leq \frac{1}{z} \right) \leq \frac{K(T)}{(T-t)^r} \Psi(t, x') \frac{1}{z^{2p}}$$

for all $z > 0$. We now use the fact that $(X_{T-t}^{t,d}(x'))$ is independent of $(W_\theta, \theta \leq t)$ and, again using Condition (M), we get

$$\begin{aligned} &\mathbb{P}\left(M_d^d(t, T-t; n, x) \leq \frac{1}{z} \right) \\ &\qquad \leq \int \frac{K(T)}{(T-t)^r} \Psi(t, x') \frac{1}{z^{2p}} d\mathbb{P}^{X_t^n(x)}(x') \leq \frac{K(T)}{(T-t)^r} \Psi_1(x) \frac{1}{z^{2p}}, \end{aligned}$$

which induces

$$\left\| \frac{1}{M_d^d(t, T-t; n, x)} \right\|_p \leq \frac{K(T)}{(T-t)^r} \Psi_1(x).$$

The above inequality is not sharp enough for us to obtain the estimate (31), but it allows us to apply Malliavin's integration by parts formula.

II. We again use the fact that $(X_{T-t}^{t,d}(x'))$ is independent of $(W_\theta, \theta \leq t)$, and apply Malliavin's integration by parts formula. In view of Condition (C'), standard calculations show: for all $p > 1$ and $j \geq 1$ there exist an integer Q'' and a non decreasing function K such that

$$\|\partial_\beta (X_{T-t}^{t,d}(x'))\|_{j,p} < K(T-t)(1 + \|x'\|^{Q''}),$$

so that

$$\|\Theta_i(T-t, x')\|_{j,p} < K(T-t)(1 + \|x'\|^Q)$$

for all $1 \leq i \leq |\alpha|$. Standard inequalities (see, e.g., Nualart [13, Prop.3.2.2]) then imply

$$
\begin{aligned}
&\left|\mathbb{E}\left[g(X_t^n(x))\sum_{i=1}^{\alpha}\mathbb{E}\left[f^{(i)}(X_{T-t}^{t,d}(x'))\Theta_i(T-t,x')\right]\right]\Big|_{x'=X_t^n(x)}\right| \\
&\qquad = \left|\mathbb{E}\left[g(X_t^n(x))\sum_{i=1}^{\alpha}f^{(i)}(X_{T-t}^{t,d}(X_t^n(x)))\Theta_i(T-t,X_t^n(x))\right]\right| \\
&\qquad \le K(T)(1+\|x\|^Q)\|f\|_\infty\left\|\frac{1}{M_d^d(t,T-t;n,x)}\right\|_k^\ell
\end{aligned}
\tag{35}
$$

for some integers Q, k and ℓ. As $X_{T-t}^{t,d}(X_t^n(x))$ is a good approximation of $X_T^d(x)$, we can adapt the technique used in Bally & Talay [3]. To this end, we set

$$
r_T^{n,t} := \frac{M_d^d(t,T-t;n,x)-M_d^d(0,T,x)}{M_d^d(0,T,x)},
$$

and we choose a function $\phi\in\mathcal{C}_b^\infty(\mathbb{R})$ such that $\phi(x)=1$ for $|x|\le\frac{1}{4}$, $\phi(x)=0$ for $|x|\ge\frac{1}{2}$ and $0<\phi(x)<1$ for $|x|\in(\frac{1}{4},\frac{1}{2})$. One then has

$$
\begin{aligned}
&\mathbb{E}\left[g(X_t^n(x))\partial_\alpha u(t,X_t^n(x))\right] \\
&\qquad \le K(T)(1+\|x\|^Q)\left\|\frac{1}{M_d^d(t,T-t;n,x)}(1-\phi(r_T^{n,t})+\phi(r_T^{n,t}))\right\|_k^\ell\|f\|_\infty \\
&\qquad \le K(T)(1+\|x\|^Q)\left\|\frac{1}{M_d^d(t,T-t;n,x)}(1-\phi(r_T^{n,t}))\right\|_{2k}^\ell\|f\|_\infty \\
&\qquad\quad + K(T)(1+\|x\|^Q)\left\|\frac{1}{M_d^d(t,T-t;n,x)}\phi(r_T^{n,t})\right\|_{2k}^\ell\|f\|_\infty \\
&\qquad =: A+B.
\end{aligned}
$$

In view of Condition (M) one has

$$
\begin{aligned}
|B| &\le K(T)(1+\|x\|^Q)\left\|\frac{1}{M_d^d(t,T-t;n,x)}\mathbb{I}_{M_d^d(t,T-t;n,x)\ge M_d^d(0,T,x)/2}\right\|_{2k}^\ell\|f\|_\infty \\
&\le K(T)(1+\|x\|^Q)\left\|\frac{1}{M_d^d(0,T,x)}\right\|_{2k}^\ell\|f\|_\infty \\
&\le \frac{K(T)}{T^r}(1+\|x\|^Q)\Psi_\lambda(x)\|f\|_\infty
\end{aligned}
$$

for some $\lambda \geq 1$. On the other hand,

$$
\begin{aligned}
|A| &\leq K(T)(1+\|x\|^Q) \left\| \frac{1}{M_d^d(t, T-t; n, x)} \right\|_{4k}^{\ell} \|(1-\phi(r_T^{n,t}))\|_{4k}^{\ell} \|f\|_\infty \\
&\leq \frac{K(T)}{(T-t)^q}(1+\|x\|^Q)\Psi_\lambda(x)\|(1-\phi(r_T^{n,t}))\|_{4k}^{\ell}\|f\|_\infty.
\end{aligned}
$$

Using Inequality (34) and the fact that $T-t \geq \frac{T}{n}$ by hypothesis, we then proceed as in Bally & Talay [3, Subsection 5.1.2] to deduce

$$
|A| \leq K(T)(1+\|x\|^Q)\Psi_\lambda(x)\|f\|_\infty,
$$

Inequality (31) follows.

□

2.3.2 Proof of Lemma 2.12

The proof of Lemma 2.12 is based on the following two inequalities:

- In view of the proof of Lemma 2.11, for any $p > 1$ there exist positive numbers q, Q, λ and a non decreasing function $K(t)$ such that

$$
\begin{aligned}
\left\| \frac{1}{M_d^d(T-T/n, T/n; n, x)} \right\|_p &\leq \left\| \frac{1}{M_d^d(T-T/n, T/n; n, x)}\phi(r_T^{n,t}) \right\|_p \\
&\quad + \left\| \frac{1}{M_d^d(T-T/n, T/n; n, x)}(1-\phi(r_T^{n,t})) \right\|_p \\
&\leq \frac{K(T)}{T^q}(1+\|x\|^Q)\Psi_\lambda(x).
\end{aligned}
\tag{36}
$$

- Under (C'), for any $p > 1$ and $j \geq 0$ there exist an integer Q and a non decreasing function $K(t)$ such that

$$
\left\| X_T^{n,d}(x) - X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)) \right\|_{j,p} \leq K(T)(1+\|x\|^Q)\frac{1}{n}. \tag{37}
$$

Set $F(x') := \int_0^{x'} f(y)dy$ and $M_d^{n,d}(0,T,x) =< DX_T^{n,d}(x), DX_T^{n,d}(x) >$. Set

$$
r_T^n := \frac{M_d^{n,d}(0,T,x) - M_d^d(T-T/n, T/n; n, x)}{M_d^d(T-T/n, T/n; n, x)}.
$$

We proceed as in the proof of Lemma 2.11.

$$\begin{aligned}
&\left|\mathbb{E}\left[f(X_T^{n,d}(x)) - f(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)))\right]\right| \\
&\qquad\le \left|\mathbb{E}\left[\left(f(X_T^{n,d}(x)) - f(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)))\right)(1-\phi(r_T^n))\right]\right| \\
&\qquad\quad + \left|\mathbb{E}\left[\left(F'(X_T^{n,d}(x)) - F'(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)))\right)\phi(r_T^n)\right]\right| \\
&\qquad =: A + B.
\end{aligned}$$

We first proceed as in Bally & Talay [3, Subsection 5.1.2] to deduce

$$A \le 2\|f\|_\infty \mathbb{E}\left[1-\phi(r_T^n)\right] \le K(T)(1+\|x\|^Q)\|f\|_\infty \frac{1}{n^2}.$$

Let us now consider B. We aim to take advantage of the smoothing effect induced by the condition (M). The purely technical difficulty comes from the fact that the Euler scheme does not necessarily inherit this smoothing effect. However, applying Malliavin's integration by parts formula, we have

$$\begin{aligned}
B = \Big|\mathbb{E}\Big[&F(X_T^{n,d}(x))H_1(X_T^{n,d}(x),\phi(r_T^n)) \\
&-F(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)))H_1(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)),\phi(r_T^n))\Big]\Big|,
\end{aligned}$$

where, denoting by L the Ornstein-Uhlenbeck operator, we have set

$$\begin{aligned}
H_1(X_T^{n,d}(x),\phi(r_T^n)) := -\Bigg\{&\phi(r_T^n) < D\left(M_d^{n,d}(0,T,x)\right)^{-1}, DX_T^{n,d}(x) > \\
&+\left(M_d^{n,d}(0,T,x)\right)^{-1} < D\phi(r_T^n), DX_T^{n,d}(x) > \\
&+\left(M_d^{n,d}(0,T,x)\right)^{-1}\phi(r_T^n)LX_T^{n,d}(x)\Bigg\},
\end{aligned}$$

and

$$\begin{aligned}
&H_1(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)),\phi(r_T^n)) \\
&:= -\Big\{\phi(r_T^n) < D\left(M_d^d(T-T/n,T/n;n,x)\right)^{-1}, DX_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)) > \\
&\quad+\left(M_d^d(T-T/n,T/n;n,x)\right)^{-1} < D\phi(r_T^n), DX_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)) > \\
&\quad+\left(M_d^d(T-T/n,T/n;n,x)\right)^{-1}\phi(r_T^n)LX_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x))\Big\}.
\end{aligned}$$

Thus

$$
\begin{aligned}
B \le & \Big| \mathbb{E} \Big[F(X_T^{n,d}(x)) H_1(X_T^{n,d}(x), \phi(r_T^n)) \\
& - F(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x))) H_1(X_T^{n,d}(x), \phi(r_T^n)) \Big] \Big| \\
& + \Big| \mathbb{E} \Big[F(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x))) H_1(X_T^{n,d}(x), \phi(r_T^n)) \\
& - F(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x))) H_1(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)), \phi(r_T^n)) \Big] \Big| \\
=: & \, B_1 + B_2.
\end{aligned}
$$

In view of the proposition 3.2.2 in Nualart [13] (derived from Meyer's inequalities of the section 2.4 in the same reference) and the inequalities (36), (37), we have

$$
\begin{aligned}
B_1 \le & \sqrt{\mathbb{E} \left[\Big(H_1(X_T^{n,d}(x), \phi(r_T^n)) \Big)^2 \right]} \\
& \sqrt{\mathbb{E} \left[\Big(F(X_T^{n,d}(x)) - F(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x))) \Big)^2 \right]} \\
\le & \frac{K(T)}{T^q} (1 + \|x\|^Q) \left\| \frac{1}{M_d^{n,d}(0,T,x)} \mathbb{I}_{M_d^{n,d}(0,T,x) \ge 1/2 M_d^d(T-T/n, T/n; n, x)} \right\|_p^{\ell} \|f\|_\infty \frac{1}{n} \\
\le & \frac{K(T)}{T^q} (1 + \|x\|^Q) \Psi_\lambda(x) \|f\|_\infty \frac{1}{n}.
\end{aligned}
$$

We now consider B_2. By Schwartz's inequality one has

$$
\begin{aligned}
B_2 = & \Big| \mathbb{E} \Big[F(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x))) \\
& \Big(H_1(X_T^{n,d}(x), \phi(r_T^n)) - H_1(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x))), \phi(r_T^n) \Big) \Big] \Big| \\
\le & \sqrt{\left| \mathbb{E} \left[\Big(F(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x))) \Big)^2 \right] \right|} \\
& \sqrt{\left| \mathbb{E} \left[\Big(H_1(X_T^{n,d}(x), \phi(r_T^n)) - H_1(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x))), \phi(r_T^n) \Big)^2 \right] \right|}.
\end{aligned}
$$

In view of (33) (with $j = 0$) and the definition of F, one has

$$
\left| \mathbb{E} \left[|F(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)))|^2 \right] \right| \le K(T)(1 + \|x\|^Q) \|f\|_\infty^2. \tag{38}
$$

Notice that

$$\begin{aligned}
&H_1(X_T^{n,d}(x), \phi(r_T^n)) - H_1(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)), \phi(r_T^n)) \\
&\quad = \phi(r_T^n) \Big\{ < D\left(M_d^d(T-T/n, T/n; n, x)\right)^{-1}, DX_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)) > \\
&\qquad - < D\left(M_d^{n,d}(0,T,x)\right)^{-1}, DX_T^{n,d}(x) > \Big\} \\
&\qquad + \phi(r_T^n) \Big\{ \left(M_d^d(T-T/n, T/n; n, x)\right)^{-1} LX_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)) \\
&\qquad - \left(M_d^{n,d}(0,T,x)\right)^{-1} LX_T^{n,d}(x) \Big\} \\
&\qquad + \Big\{ \left(M_d^d(T-T/n, T/n; n, x)\right)^{-1} < D\phi(r_T^n), DX_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)) > \\
&\qquad - \left(M_d^{n,d}(0,T,x)\right)^{-1} < D\phi(r_T^n), DX_T^{n,d}(x) > \Big\} \\
&\quad =: B_{21} + B_{22} + B_{23}.
\end{aligned}$$

We first observe that

$$\begin{aligned}
B_{21} &= \phi(r_T^n) \Big\{ < D\left(M_d^d(T-T/n, T/n; n, x)\right)^{-1}, DX_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)) > \\
&\quad - < D\left(M_d^d(T-T/n, T/n; n, x)\right)^{-1}, DX_T^{n,d}(x) > \\
&\quad + < D\left(M_d^d(T-T/n, T/n; n, x)\right)^{-1}, DX_T^{n,d}(x) > \\
&\quad - < D\left(M_d^{n,d}(0,T,x)\right)^{-1}, DX_T^{n,d}(x) > \Big\} \\
&= \phi(r_T^n) \Big\{ < D\left(M_d^d(T-T/n, T/n; n, x)\right)^{-1}, DX_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)) - DX_T^{n,d}(x) > \\
&\quad + < D\left(M_d^d(T-T/n, T/n; n, x)\right)^{-1} - D\left(M_d^{n,d}(0,T,x)\right)^{-1}, DX_T^{n,d}(x) > \Big\}.
\end{aligned} \tag{39}$$

As

$$\begin{aligned}
&D\left(M_d^d(T-T/n, T/n; n, x)\right)^{-1} \\
&\qquad = -\left(M_d^d(T-T/n, T/n; n, x)\right)^{-2} DM_d^d(T-T/n, T/n; n, x),
\end{aligned}$$

and

$$D\left(M_d^{n,d}(0,T,x)\right)^{-1} = -\left(M_d^{n,d}(0,T,x)\right)^{-2} DM_d^{n,d}(0,T,x),$$

one has

$$
\begin{aligned}
&D\left(M_d^d(T-T/n,T/n;n,x)\right)^{-1} - D\left(M_d^{n,d}(0,T,x)\right)^{-1}\\
&=\left(M_d^{n,d}(0,T,x)\right)^{-2} DM_d^{n,d}(0,T,x)\\
&\quad-\left(M_d^{n,d}(0,T,x)\right)^{-2} DM_d^{d}(T-T/n,T/n;n,x)\\
&\quad+\left(M_d^{n,d}(0,T,x)\right)^{-2} DM_d^{d}(T-T/n,T/n;n,x)\\
&\quad-\left(M_d^{d}(T-T/n,T/n;n,x)\right)^{-2} DM_d^{d}(T-T/n,T/n;n,x)\\
&=\left(M_d^{n,d}(0,T,x)\right)^{-2}\left(DM_d^{n,d}(0,T,x)-DM_d^{d}(T-T/n,T/n;n,x)\right)\\
&\quad+\left(M_d^{n,d}(0,T,x)\right)^{-2}\left(M_d^{d}(T-T/n,T/n;n,x)\right)^{-2} DM_d^{d}(T-T/n,T/n;n,x)\\
&\qquad\left((M_d^{d}(T-T/n,T/n;n,x))^2-(M_d^{n,d}(0,T,x))^2\right).
\end{aligned}
\tag{40}
$$

In view of (36), (37), (39) and (40), we obtain

$$
\|B_{21}\|_2 \le \frac{K(T)}{T^q}(1+\|x\|^Q)\Psi_\lambda(x)\frac{1}{n}.
$$

Second, from

$$
\begin{aligned}
B_{22}=\phi(r_T^n)\Big\{&\left(M_d^d(T-T/n,T/n;n,x)\right)^{-1} LX_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x))\\
&-\left(M_d^d(T-T/n,T/n;n,x)\right)^{-1} LX_T^{n,d}(x)\\
&+\left(M_d^d(T-T/n,T/n;n,x)\right)^{-1} LX_T^{n,d}(x)\\
&-\left(M_d^{n,d}(0,T,x)\right)^{-1} LX_T^{n,d}(x)\Big\},
\end{aligned}
$$

by noticing that

$$
\begin{aligned}
&\left(M_d^{n,d}(0,T,x)\right)^{-1} - \left(M_d^d(T-T/n,T/n;n,x)\right)^{-1}\\
&\qquad=\left(M_d^{n,d}(0,T,x)\right)^{-1}\left(M_d^d(T-T/n,T/n;n,x)\right)^{-1}\\
&\qquad\qquad\left(M_d^d(T-T/n,T/n;n,x)-M_d^{n,d}(0,T,x)\right),
\end{aligned}
$$

and

$$\left\| L(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)) - X_T^{n,d}(x)) \right\|_p$$
$$\leq C \left\| (X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)) - X_T^{n,d}(x) \right\|_{2,p},$$

in view of (36) and (37) we obtain

$$\|B_{22}\|_2 \leq \frac{K(T)}{T^q}(1+\|x\|^Q)\Psi_\lambda(x)\frac{1}{n}.$$

Finally, from

$$\begin{aligned} B_{23} = & \left(M_d^d(T-T/n, T/n; n, x)\right)^{-1} < D\phi(r_T^n), DX_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x)) > \\ & - \left(M_d^d(T-T/n, T/n; n, x)\right)^{-1} < D\phi(r_T^n), DX_T^{n,d}(x) > \\ & + \left(M_d^d(T-T/n, T/n; n, x)\right)^{-1} < D\phi(r_T^n), DX_T^{n,d}(x) > \\ & - \left(M_d^{n,d}(0, T, x)\right)^{-1} < D\phi(r_T^n), DX_T^{n,d}(x) > \end{aligned}$$

and (36) and (37), we deduce that

$$\|B_{23}\|_2 \leq \frac{K(T)}{T^q}(1+\|x\|^Q)\Psi_\lambda(x)\frac{1}{n}.$$

The result follows. □

3 A lower bound for a marginal density

Recall Remark 2.5. For real applications, an accurate lower bound for $\frac{1}{\overline{p}_T^d(\rho(x,\delta))}$ is desirable. It usually is a difficult task. In this section we give an example where one succeeds to get a rather good lower bound for the marginal density of a process $(X_t^1(x^1), X_t^2(x^1, x^2))$ whose generator does not satisfy the conditions supposed in the references mentioned in Remark 2.5. We have not succeeded to adapt Kusuoka and Stroock's technique [10]: first, Kusuoka and Stroock seek a lower bound for the density of the joint law of all the coordinates; second, they consider generators under divergence form or 'almost' under divergence form, and this property seems crucial in their construction of a lower bound. Here we take advantage of the particular structure of the generator of $(X_t^1(x^1), X_t^2(x^1, x^2))$. It appears that rather simple tools (time change, Brownian bridge) are sufficient. Nevertheless our technique might apply in other situations.

In Talay and Zheng [18] we consider the problem of computing a model of risk measurement for the Profit and Loss of a misspecified European option hedging strategy. One has to approximate the quantile at a maturity date T of the second coordinate of the solution to

$$\begin{cases} X_t^1(x^1) & = x^1 + \int_0^t X_s^1(x^1)u_1(s)ds + \int_0^t X_s^1(x^1)u_2(s)dW_s, \\ X_t^2(x^1,x^2) & = x^2 + \int_0^t \varphi(s, X_s^1(x^1))X_s^1(x^1)u_1(s)ds \\ & \quad + \int_0^t \varphi(s, X_s^1(x^1))X_s^1(x^1)u_2(s)dW_s, \end{cases} \tag{41}$$

where $\varphi(s,z)$ is a prescribed function related to the payoff of the option under consideration.

Supposing that the coefficients of the stochastic differential equation (41) satisfy Condition (C') and that

$$|\varphi(t,x^1)u_2(t)| \geq a > 0 \text{ for all } t \text{ in } [0,T^O] \text{ and } x^1 \in \mathbb{R}^+, \tag{42}$$

one can show that the law of $X_T^2(x)$ has a smooth density p_T^2 which is strictly positive in its support (see [18]). Denote by $\rho(x,\delta)$ the quantile of $X_T^2(x)$ at level δ. We aim to give a lower bound estimate for $p_T^2(\rho(x,\delta))$ and add two assumptions: In addition, we suppose that there exists a constant C such that

$$|\varphi(t,z)| \leq C, \tag{43}$$

and

$$\left| \int_0^{\Lambda(t)} \frac{\partial \Upsilon}{\partial s}(s,z)ds \right| \leq C, \tag{44}$$

for all t in $[0,T^O]$ and $z \in \mathbb{R}^+$.

Define

$$\begin{cases} \Lambda(t) := \int_0^t u_2^2(s)ds, \\ \overline{X}_t^1(x) := X_{\Lambda^{-1}(t)}^1, \\ \overline{X}_t^2(x) := X_{\Lambda^{-1}(t)}^2(x), \\ W_t^\Lambda := \sqrt{\Lambda^{-1}(t)}W_{\Lambda^{-1}(t)}, \\ \mathcal{F}_t^\Lambda := \sigma\{W_s^\Lambda, 0 \leq s \leq t\}. \end{cases}$$

For all s in $[0,\Lambda^{-1}(T^O)]$ set

$$\Upsilon(s,z) := \int_0^z \varphi(\Lambda^{-1}(s),\alpha)d\alpha,$$

and

$$h(s,z) := \frac{\partial \Upsilon}{\partial s}(s,z) + \frac{1}{2}\frac{\partial \varphi}{\partial z}(\Lambda^{-1}(s),z). \tag{45}$$

By the time change formula, (W_t^Λ) is an $(\mathcal{F}_t^\Lambda)$-Brownian motion, and $(\overline{X}_t^1(x), \overline{X}_t^2(x))$ satisfies

$$\begin{cases} \overline{X}_t^1(x) & = x^1 + \int_0^t \dfrac{u_1(\Lambda^{-1}(s))}{u_2(\Lambda^{-1}(s))} \overline{X}_s^1(x) ds + \int_0^t \overline{X}_s^1(x) dW_s^\Lambda, \\ \overline{X}_t^2(x) & = x^2 + \int_0^t \varphi(\Lambda^{-1}(s), \overline{X}_s^1(x)) \dfrac{u_1(\Lambda^{-1}(s))}{u_2(\Lambda^{-1}(s))} \overline{X}_s^1(x) ds \\ & \quad + \int_0^t \varphi(\Lambda^{-1}(s), \overline{X}_s^1(x)) \overline{X}_s^1(x) dW_s^\Lambda. \end{cases}$$

Set $\overline{u}_s := \frac{u_1(\Lambda^{-1}(s))}{u_2(\Lambda^{-1}(s))}$ for all s in $[0, \Lambda^{-1}(T^O)]$. One has

$$\overline{X}_t^1(x) = x^1 \exp\left(\int_0^t (\overline{u}_s - \frac{1}{2}) ds + W_t^\Lambda \right) =: x^1 \exp\left(\overline{U}_t + W_t^\Lambda \right).$$

Observe that

$$X_t^2(x) = \overline{X}_{\Lambda(t)}^2(x^1, x^2) = x^2 - \Upsilon(0, x^1) + \Upsilon(\Lambda(t), \overline{X}_{\Lambda(t)}^1(x^1)) - \int_0^{\Lambda(t)} h(s, \overline{X}_s^1(x^1)) ds,$$

where h is defined as in (45).

Denote by (B_s^z) the Brownian bridge from $(0,0)$ to $(\Lambda(t), z)$. It is identical in law to $\tilde{W}_s^\Lambda - \frac{s}{\Lambda(t)} \tilde{W}_{\Lambda(t)}^\Lambda + \frac{zs}{\Lambda(t)}$, where $(\tilde{W}_s^\Lambda)$ is a $(\mathcal{F}_s^\Lambda)$-Brownian motion.

Denote by g_ϵ the Gaussian density $N(0, \epsilon)$. One has

$$\begin{aligned}
&\mathbb{E}\left[g_\epsilon(X_t^2(x^1, x^2) - \rho(x, \delta)) \right] \\
&= \mathbb{E}\left[g_\epsilon \left(x^2 - \Upsilon(0, x^1) + \Upsilon(\Lambda(t), \overline{X}_{\Lambda(t)}^1(x^1)) - \int_0^{\Lambda(t)} h(s, \overline{X}_s^1(x)) ds - \rho(x, \delta) \right) \right] \\
&= \mathbb{E}\left[\int \left[g_\epsilon \left(x^2 - \Upsilon(0, x^1 + \Upsilon\left(\Lambda(t), x^1 \exp\left(\overline{U}_t + z \right) \right) \right. \right. \right. \\
&\quad \left. \left. \left. - \int_0^{\Lambda(t)} h(s, x^1 \exp\left(\overline{U}_s + B_s^z \right)) ds - \rho(x, \delta) \right) \right] g_{\Lambda(t)}(z) dz \right] \\
&= \mathbb{E}\left[\int \left[g_\epsilon \left(x^2 - \Upsilon(0, x^1) + \Upsilon\left(\Lambda(t), x^1 \exp\left(\overline{U}_t + z \right) \right) \right. \right. \right. \\
&\quad \left. \left. \left. - \int_0^{\Lambda(t)} h\left(s, x^1 \exp\left(\overline{U}_s + \tilde{W}_s^\Lambda - \frac{s}{\Lambda(t)} \tilde{W}_{\Lambda(t)}^\Lambda + \frac{z\ s}{\Lambda(t)} \right) \right) ds - \rho(x, \delta) \right) \right] g_{\Lambda(t)}(z) dz \right] \\
&=: \mathbb{E}\left[\int g_\epsilon \left(\mathrm{H}(x, z, \omega) - \rho(x, \delta) \right) g_{\Lambda(t)}(z) dz \right],
\end{aligned}$$

where we have set

$$H(x,z,\omega) := x^2 - \Upsilon(0,x^1) + \Upsilon\left(\Lambda(t), x^1 \exp\left(\overline{U}_t + z\right)\right) - \int_0^{\Lambda(t)} h\left(s, x^1 \exp\left(\overline{U}_s + \tilde{W}^\Lambda_s - \frac{s}{\Lambda(t)}\tilde{W}^\Lambda_{\Lambda(t)} + \frac{z\ s}{\Lambda(t)}\right)\right) ds. \tag{46}$$

For all x and $\omega \in \Omega$, one has

$$\frac{\partial \mathrm{H}}{\partial z}(x,z,\omega) = \varphi\left(\Lambda(t), x^1 \exp\left(\overline{U}_t + z\right)\right) x^1 \exp\left(\overline{U}_t + z\right) - \int_0^{\Lambda(t)} \frac{\partial h}{\partial z}\left(s, x^1 \exp\left(\overline{U}_s + \tilde{W}^\Lambda_s - \frac{s}{\Lambda(t)}\tilde{W}^\Lambda_{\Lambda(t)} + \frac{z\ s}{\Lambda(t)}\right)\right) ds.$$

By the assumption on ϕ one has

$$\left|\int_0^{\Lambda(t)} \frac{\partial h}{\partial z}\left(s, x^1 \exp\left(\overline{U}_s + \tilde{W}^\Lambda_s - \frac{s}{\Lambda(t)}\tilde{W}^\Lambda_{\Lambda(t)} + \frac{z\ s}{\Lambda(t)}\right)\right)\ ds\right| \le C\Lambda(t).$$

Thus, for all x and $\omega \in \Omega$,

$$\frac{\partial \mathrm{H}}{\partial z}(x,z,\omega) \ge x^1 \exp\left(\overline{U}_t + z\right) \inf_{t,\alpha} \varphi(t,\alpha) - C\Lambda(t),$$

from which

$$\frac{\partial \mathrm{H}}{\partial z}(x,z,\omega) > 0 \text{ for all } z > \log\left(\frac{C\Lambda(t)}{ax^1}\right) - \overline{U}_t.$$

We deduce

$$\begin{aligned}
&\mathbb{E}\left[g_\epsilon(X^2_t(x^1,x^2) - \rho(x,\delta))\right] \\
&\qquad \ge \mathbb{E}\left[\int_{\log\left(\frac{C\Lambda(t)}{ax^1}\right) - \overline{U}_t}^{\infty} g_\epsilon\left(\mathrm{H}(x,z,\omega) - \rho(x,\delta)\right) g_{\Lambda(t)}(z) dz\right] \\
&\qquad = \mathbb{E}\left[\int_{\mathrm{H}\left(x, \log\left(\frac{C\Lambda(t)}{ax^1}\right) - \overline{U}_t, \omega\right)}^{\infty} g_\epsilon(\xi - \rho(x,\delta)) g_{\Lambda(t)}(\mathrm{H}^{-1}(\xi)) \mathcal{J}(\xi) d\xi\right],
\end{aligned}$$

where $\mathcal{J}$ is the Jacobian matrix of $\mathrm{H}^{-1}(x,\cdot,\omega)$. Let us make ϵ tend to zero. Then

$$p^2_T(\rho(x,\delta)) \ge \mathbb{E}\left[g_{\Lambda(t)}(\mathrm{H}^{-1}(\rho(x,\delta)))\mathcal{J}(\rho(x,\delta)) \mathbb{I}_{\mathrm{H}\left(x, \log\left(\frac{C\Lambda(t)}{ax^1}\right) - \overline{U}_t, \omega\right) < \rho(x,\delta)}\right].$$

Set

$$K := \sup_{\omega} \mathrm{H}\left(x, \log(\tfrac{C\Lambda(t)}{ax^1}) - \overline{U}_t, \omega\right). \tag{47}$$

In view of (44), K is finite. Thus for all $\rho(x,\delta) > K$ we get

$$p_T^2(\rho(x,\delta)) \geq \mathbb{E}\left[g_{\Lambda(T)}(\mathrm{H}^{-1}(\rho(x,\delta)))\mathcal{J}(\rho(x,\delta))\right]. \tag{48}$$

We thus have proved:

Theorem 3.1. *Suppose that Condition (C') holds. Suppose that (42), (43) and (44) hold.*

Let K be defined as in (47). Then, for all $\rho(x,\delta) > K$, the density of the law of $X_T^2(x)$ is bounded from below by the right handside of (48).

4 Extensions

During the refereeing process of this paper, Gobet and Munos [7] have developed sensitivity analysis techniques for parametered diffusion processes. One of their techniques is based upon Malliavin calculus, and is especially designed for processes which are partially hypoelliptic in the sense that the inverse of the Malliavin covariance matrix γ_T of some coordinates of the vector space at time T belongs to $L^p(\Omega)$ for all integer $p \geq 1$. In their subsection 3.2, they obtain a result similar to our above theorem (2.6), except that the right side of (25) becomes

$$K(T)\|f\|_\infty \, \|1/\det(\gamma_T)\|_p^q \, \frac{1}{n}$$

for some function $K(T)$ and some real numbers p, q. The method of proof used by Gobet and Munos is derived from an idea originally introduced by Kohatsu–Higa and Pettersson [9] who, instead of using Markovian tools as in our subsection 2.3, apply the method of variations of constants to an equation satisfied by the process $(X_t - X_t^n)$: that trick allows one to develop an error analysis which requires Malliavin integration by parts formulae at time T only. Thus our convergence rates on the approximation of quantiles seem to also hold true under the hypothesis: the inverse of $M(0,T,x)$ (respectively, $M_d^d(0,T,x)$) belongs to $L^p(\Omega)$ for all integer $p \geq 1$, instead of (UH) (respectively, (M)).

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