Worst case model risk management

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Abstract: We are interested in model risk control problems. We study a strategy for the trader which, in a sense, guarantees good performances whatever is the unknown model for the assets of his/her portfolio. The trader chooses trading strategies to decrease the risk and therefore acts as a minimizer; the market systematically acts against the interest of the trader, so that we consider it acts as a maximizer. Thus we consider the model risk control problem as a two players (Trader versus Market) zero-sum stochastic differential game problem. Therefore our construction corresponds to a 'worst case' worry and, in this sense, can be viewed as a continuous-time extension of discrete-time strategies based upon prescriptions issued from VaR analyses at the beginning of each period. In addition, the initial value of the optimal portfolio can be seen as the minimal amount of money which is needed to face the worst possible damage.

We give a proper mathematical statement for such a game problem. We prove that the value function of this game problem is the unique viscosity solution to an Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation, and satisfies the Dynamic Programming Principle.

Key words: Model risk, stochastic differential game, Hamilton-Jacobi-Bellman-Isaacs equation.

JEL Classification: G11

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1 Introduction

In this note we describe the financial strategy which a trader can follow in order to manage his/her model risk.

Suppose that the trader precisely *knows* the model followed by the real market, and that this model is given by a system of stochastic differential equations. Then, in a complete market, she/he is able to construct a strategy which perfectly hedges the option. For options written on discount bonds, such an assumption means that the trader knows perfectly the description of the random evolution of the term structure of the interest rates. This is unrealistic for various reasons, for example: lack of sufficient information from the market, choice of the number of factors which drive the term structure of interest rates, choice of the modelling stochastic processes, estimates of the parameters required to use the model, complexity of the estimation procedure, etc.

When one has a rather precise information on the model of the market, then one can take advantage of the robustness of formulae of Black and Scholes type (see, e.g., El Karoui, Jeanblanc–Picqué & Shreve [8] and Romagnoli & Vargiolu [15]). When one has only a vague information on the model of the market, can one find a strategy, which, in a sense, guarantees tolerable performances whatever the unknown model is? To address this question several authors have proposed super–replication techniques: see, e.g., Avellaneda, Levy & Paras [2], Avellaneda & Paras [3], Touzi [17] and references therein. Here we adopt a 'worst case' stochastic game approach which can be summarized as follows:

$$\boxed{\text{Trader} = \text{Minimizer of Risk.}}$$
$$\boxed{\text{Market} = \text{Maximizer of Risk.}}$$
$$\boxed{\text{Trader vs Market.}}$$

We introduce a cost function which describes the risk faced by the trader. Instead of supposing that the trader knows the exact model followed by the real market, we assume that the trader knows that the correct model of the market belongs to a wide class of models. The trader chooses trading strategies from a set of admissible strategies to decrease the risk and therefore acts as a minimizer of the risk; on the other hand, we suppose that the market systematically behaves against the interest of the trader, and therefore we consider it acts as a maximizer of the risk. Thus the model risk control problem can be set up as a two players (Trader versus Market) zero-sum stochastic differential game problem and the corresponding strategies, in a sense, are continuous-time versions of discrete-time procedures based upon prescriptions issued from VaR analyses at the beginning of each period.

In Section 2, we introduce our framework: the misspecified model for the asset prices and our definition of model risk. In Section 3, we state the stochastic differential game problem which is the key tool of our approach; we slightly extend results by Fleming & Souganidis [10] to prove that the value function of this game problem is the unique viscosity solution to an Hamilton-Jacobi-Bellman-Isaacs equation; we also prove that the value function satisfies the Dynamic Programming Principle. In Section 4, we illustrate our methodology by considering the particular case of the hedging of European bond options. We conclude by showing a numerical result.

Our paper is deeply related to the paper [7] by Cvitanić and Karatzas who have introduced and studied dynamic measure of risks such as

$$\inf_{\pi(\cdot)\in\mathcal{A}(x)} \sup_{\nu\in\mathcal{D}} \mathbb{E}_{\nu}(F(X^{x,\pi}(T))),$$

where $\mathcal{A}(x)$ denotes the class of admissible portfolio strategies, and $\mathbb{E}_{\nu}$ denotes the expectation under the probability $\mathbb{P}_{\nu}$ for all ν in a suitable set. All the measures $\mathbb{P}_{\nu}$ have the same risk-neutral equivalent martingale measure, which implies that the trader (or the regulator) is concerned by model risk on stock appreciation rates. Here, we include model risk on volatilities and prove the relationship between the dynamic hedging under the worst case scenario and viscosity solutions of fully nonlinear PDEs, which allows us to develop numerical methods. We thus hopefully answer to a part of the open problems listed in the Conclusion in [7].

We finally emphasize that the solution at time 0 of our stochastic game problem can be viewed as the minimal amount of money that the financial institution needs to "contain the worst possible damage" as said in [7].

2 Our financial model

Suppose that the financial market consists in n risky financial assets, whose prices at time t are denoted by $P_i(t)$, and one instantaneously risk-free asset, the money market, whose price at time t is denoted by $P_0(t)$. We use the following stochastic differential equation to describe this financial market.

$$\begin{cases} dP_i(t) = P_i(t)[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_t^j], \ 1 \le i \le n, \\ dP_0(t) = r(t)P_0(t)dt. \end{cases} \tag{1}$$

Here we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and processes $b_i(\cdot)$, $\sigma_{ij}(\cdot)$, $r(\cdot)$ which are progressively measurable with respect to the $\mathbb{P}$ augmented filtration of the d dimensional Brownian motion (W_t). We also suppose that there exists a unique strong solution to (1).

Consider an economic agent whose decisions do not affect the prices in the market (a small investor). We denote by $X(t)$ the wealth of this agent at time t, by $H_i(t)$ the amount he/she invests in the i^{th} risky financial asset at that time ($1 \leq i \leq n$), and by $H_0(t)$ the amount he/she invests in the risk-free asset. Then the value of his/her portfolio is

$$X(t) = \sum_{i=1}^{n} H_i(t)P_i(t) + H_0(t)P_0(t).$$

Suppose that the portfolio is self-financing. Then the process $X(t)$ satisfies the following stochastic differential equation

$$dX(t) = \sum_{i=1}^{n} H_i(t)P_i(t)\left[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_t\right] + H_0(t)r(t)P_0(t)dt.$$

If $X(t) \neq 0$, set

$$\pi_i(t) := \frac{H_i(t)P_i(t)}{X(t)},\ i = 1, \ldots, n,$$

and

$$\pi_0(t) := \frac{H_0(t)P_0(t)}{X(t)}.$$

If $X(t) = 0$, set $\pi_i(t) = \pi_0(t) = 0$. With the above notation, we have

$$dX(t) = X(t)\sum_{i=1}^{n} \pi_i(t)\left[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_t^j\right] + X(t)r(t)\left(1 - \sum_{i=1}^{n} \pi_i(t)\right)dt. \tag{2}$$

3 Model risk control

To simplify the notation, in this section we suppose that the process $(r(t))$ is constant and equal to $r > 0$. It is straightforward to include it in the list of the controls, or to more specifically include model risks on a fixed type of yield curves models (see Section 4 for such an example).

Suppose that the market chooses the risk premium and the volatility process to increase the risk of the position.

We adopt the definition of admissible controls and strategies of Fleming & Souganidis [10] and, as in [10], we introduce the canonical sample spaces for the underlying Brownian motion in (1) and (2). For each $\theta \in [0, T]$ we set

$$\Omega_\theta := (\omega \in C([\theta, T]; \mathbb{R}^d) : \omega_\theta = 0).$$

We denote by $(\mathcal{F}_{\theta,s})$ the filtration generated by the canonical process from time θ to time s. Equipped with the Wiener measure $\mathbb{P}_\theta$ on $\mathcal{F}_{\theta,T}$, the filtered probability space $(\Omega_\theta, \mathcal{F}_{\theta,T}, \mathbb{P}_\theta, (\mathcal{F}_{\theta,s}, \theta \leq s \leq T))$ is the canonical sample space.

We define the admissible controls and the admissible strategies by:

Definition 3.1. *An admissible control process $u(\cdot) := (b(\cdot), \sigma(\cdot))$ for the market on $[\theta, T]$ is an $(\mathcal{F}_{\theta,s})$-progressively measurable process taking value in a compact subset K_u of $\mathbb{R}^n \times \mathbb{R}^{nd}$. An admissible control process $\pi(\cdot)$ for the investor on $[\theta, T]$ is an $(\mathcal{F}_{\theta,s})$-progressively measurable process taking value in a compact subset K_π of $\mathbb{R}^n$. The set of all admissible controls for the market on $[\theta, T]$ is denoted by $Ad_u(\theta)$ and the set of all admissible controls for the investor on $[\theta, T]$ is denoted by $Ad_\pi(\theta)$. We say that two admissible control processes u and $\overline{u}$ in $Ad_u(\theta)$ are the same on $[\theta, s]$ if $P_\theta(u(\cdot) = \overline{u}(\cdot)$ a.e. in $[\theta, s]) = 1$. A similar convention is assumed for elements of $Ad_\pi(\theta)$. We write $u(\cdot) \approx \overline{u}(\cdot)$ on $[\theta, s]$ (respectively, $\pi(\cdot) \approx \overline{\pi}(\cdot)$ on $[\theta, s]$) when u and $\overline{u}$ (respectively, π and $\overline{\pi}$) are the same on $[\theta, s]$.*

Definition 3.2. *An admissible strategy Π for the investor on $[\theta, T]$ is a mapping $\Pi : Ad_u(\theta) \to Ad_\pi(\theta)$ such that if $u(\cdot) \approx \overline{u}(\cdot)$ on $[\theta, s]$, then $\Pi(u(\cdot))(\cdot) \approx \Pi(\overline{u}(\cdot))(\cdot)$ on $[\theta, s]$ for every $s \in [\theta, T]$. The set of all admissible strategies for the investor on $[\theta, T]$ is denoted by $Ad_\Pi(\theta)$.*

For given $\Pi \in Ad_\Pi(\theta)$ and $u(\cdot) = (b(\cdot), \sigma(\cdot)) \in Ad_u(\theta)$, we set $\pi(\cdot) := \Pi(u(\cdot))(\cdot)$.

The controlled system of prices and value of the portfolio is defined as follows:

$$\begin{cases} dP_i(t) &= P_i(t)[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW_t^j] \text{ for } 0 \leq i \leq n, \\ dX(t) &= X(t) \sum_{i=1}^n \pi_i(t) \left[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW_t^j \right] + rX(t) \left(1 - \sum_{i=1}^n \pi_i(t)\right) dt. \end{cases} \tag{3}$$

Given a suitable function F the cost function is now defined as follows:

$$J(\theta, p, x, \Pi, u(\cdot)) := \mathbb{E}_{\theta,p,x} F(P(T), X(T)), \tag{4}$$

where the symbol $\mathbb{E}_{\theta,p,x}$ stands for the conditional expectation under $\mathbb{P}_\theta$ knowing that $(P(\theta) = p, X(\theta) = x)$[1]. In this section we do not discuss the choice of F in practice, which highly depends on the financial objective of the trader or of the regulator. We thus make a weak assumption on F (see (8) below), and postpone some discussion to Section 4.

We then introduce the value function of our game problem.

[1] For the sake of simplicity our notation does not emphasize that the process $(P(t), X(t))$ is parametered by $(b(t), \sigma(t), \pi_t)$.

Definition 3.3. *The lower value of the model risk control problem with initial data* (θ, p, x) *is*

$$V(\theta, p, x) := \inf_{\Pi \in Ad_\Pi(\theta)} \sup_{u(\cdot) \in Ad_u(\theta)} J(\theta, p, x, \Pi, u(\cdot)). \tag{5}$$

3.1 Our main result

Define the $(n+1) \times (n+1)$ symmetric matrix $a(p, x, \sigma, \pi)$ as

$$\begin{cases} a_{ij}(p, x, \sigma, \pi) & := \sum_{k=1}^{d} (p_i \sigma_{ik} p_j \sigma_{jk}) \ \text{ for } 1 \leq i, j \leq n, \\ a_{n+1,j}(p, x, \sigma, \pi) & := \sum_{k=1}^{d} \sum_{l=1}^{n} (x \pi_l \sigma_{kl} p_j \sigma_{jk}) \ \text{ for } 1 \leq j \leq n, \\ a_{n+1,n+1}(p, x, \sigma, \pi) & := \sum_{k=1}^{d} \sum_{l=1}^{n} (x^2 \pi_l^2 \sigma_{kl}^2). \end{cases} \tag{6}$$

Let $q(p, x, b, \pi)$ denote the $n+1$ dimensional vector

$$q(p, x, b, \pi) := \left(p_1 b_1, \ldots, p_n b_n, x \left(r + \sum_{i=1}^{d} \pi_i (b_i - r) \right) \right). \tag{7}$$

For all $u = (b, \sigma) \in K_u$ and $\pi \in K_\pi$ we set

$$\mathcal{H}_{u,\pi}(A, z, p, x) := \left[\frac{1}{2} \mathrm{Tr} \left(a(p, x, \sigma, \pi) A \right) + z \cdot q(p, x, b, \pi) \right],$$

and

$$\mathcal{H}^-(A, z, p, x) := \max_{u \in K_u} \min_{\pi \in K_\pi} \left[\mathcal{H}_{u,\pi}(A, z, p, x) \right].$$

Our main result is

Theorem 3.4. *Suppose that* F *is a continuous function such that*

$$|F(p, x) - F(\overline{p}, \overline{x})| \leq Q(|p|, |x|, |\overline{p}|, |\overline{x}|)(|p - \overline{p}| + |x - \overline{x}|), \tag{8}$$

where $Q(|p|, |x|, |\overline{p}|, |\overline{x}|)$ *is a polynomial function.*

Then the value function $V(\theta, p, x)$ *defined in (5) is the unique viscosity solution in the space*

$$S := \{\varphi(t, p, x) \text{ is continuous on } [0, T] \times \mathbb{R}^n \times \mathbb{R}; \exists \overline{A} > 0, \lim_{|p|^2 + x^2 \to \infty} \varphi(t, p, x) \exp(-\overline{A} |\log(|p|^2 + x^2)|^2) = 0 \text{ for all } t \in [0, T]\}$$

to the Hamilton-Jacobi-Bellman-Isaacs Equation

$$\begin{cases} \dfrac{\partial v}{\partial t}(t, p, x) + \mathcal{H}^-(D^2 v(t, p, x), Dv(t, p, x), p, x) = 0 \ \text{ in } [0, T) \times \mathbb{R}^{n+1}, \\ v(T, p, x) = F(p, x). \end{cases} \tag{9}$$

In addition, $V(\theta,p,x)$ satisfies the Dynamic Programming Principle, that is,

$$V(\theta,p,x)=\inf_{\Pi\in Ad_\Pi(\theta)}\ \sup_{u(\cdot)\in Ad_u(\theta)}\ \mathbb{E}_{\theta,p,x}[V(t,P(t),X(t))] \text{ for all } \theta<t<T. \tag{10}$$

3.2 Proof of Theorem 3.4

We start the proof of Theorem 3.4 by an easy lemma which implies that the value function V belongs to the space S.

Lemma 3.5. *$V(\theta,p,x)$ is time continuous and locally Lipschitz continuous w.r.t. (p,x) in $\mathbb{R}^{n+1}$. There exists a positive polynomial function Q of (p,x) such that $|V(\theta,p,x)|\le Q(p,x)$ for all θ in $[0,T]$.*

Proof. The continuity in θ of V results from the classical estimates on the solution of stochastic differential equations: see, e.g., Fleming & Soner [9, Chap.IV, Sec.6].

We now check that, for each pair of admissible control and strategy, the cost function is locally Lipschitz continuous w.r.t. (p,x) in $\mathbb{R}^{n+1}$. Let $(P(t),X(t))$ (respectively $(\overline{P}(t),\overline{X}(t))$) be the solution to (3) with initial condition (p,x) (respectively $(\overline{p},\overline{x})$) at time θ. In view of (8), we have

$$\begin{aligned}&|F(P(T),X(T))-F(\overline{P}(T),\overline{X}(T))|\\&\quad\le Q(|P(T)|,|X(T)|,|\overline{P}(T)|,|\overline{X}(T)|)(|P(T)-\overline{P}(T)|+|X(T)-\overline{X}(T)|).\end{aligned}$$

An elementary calculation and classical estimates lead to

$$\begin{aligned}&\left|\mathbb{E}_{\theta,p,x}[F(P(T),X(T))]-\mathbb{E}_{\theta,\overline{p},\overline{x}}[F(\overline{P}(T),\overline{X}(T))]\right|\\&\qquad\qquad\le\overline{Q}(|p|,|x|,|\overline{p}|,|\overline{x}|)\sqrt{(|p-\overline{p}|^2+|x-\overline{x}|^2)},\end{aligned}$$

for some polynomial function $\overline{Q}$. The Lipschitz continuity of V w.r.t. (p,x) is a then straightforward consequence of the following inequalities: for all bounded functions f_1,f_2 and set Z,

$$\begin{cases}|\sup_{z\in Z}f_1(z)-\sup_{z\in Z}f_2(z)|\le\sup_{z\in Z}|f_1(z)-f_2(z)|,\\|\inf_{z\in Z}f_1(z)-\inf_{z\in Z}f_2(z)|\le\sup_{z\in Z}|f_1(z)-f_2(z)|.\end{cases}\tag{11}$$

A similar computation proves the polynomial growth in (p,x) of $V(\theta,p,x)$. □

If the controlled system had bounded coefficients and F were a bounded Lipschitz function, Theorem 2.4 would result from Fleming & Souganidis [10, Thm.1.6,Thm.2.6]. As the coefficients of (3) are unbounded and F is locally Lipschitz only, we have to use localization techniques.

3.2.1 Existence of a viscosity solution

We set $B_k := \{(p,x) \in \mathbb{R}^{n+1},\ |p|^2 + x^2 < k^2\}$. Choose a function ϕ_k in $\mathcal{C}_b^\infty(\mathbb{R}^{n+1})$ such that $\phi_k(p,x) = 1$ on B_k, and $\phi_k(p,x) = 0$ outside B_{k+1}, and the Lipschitz constant of ϕ_k is less than 2. Consider the following S.D.E

$$\begin{cases} dP_i^k(t) &= \phi_k(P^k(t), X^k(t))P_i^k(t)[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW_t^j] \text{ for } 0 \le i \le n, \\ dX^k(t) &= \phi_k(P^k(t), X^k(t)) \Big[X^k(t)\sum_{i=1}^n \pi_i(t) \Big[b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t)dW_t^j\Big] \\ & \quad +rX^k(t)(1-\sum_{i=1}^n \pi_i(t))dt\Big] . \end{cases} \tag{12}$$

Observe that the coefficients of (12) are uniformly Lipschitz and bounded.

Our objective is to use the stability lemma for viscosity solutions (Lions [12, Prop.I.3]) in order to construct a viscosity solution of Equation (9). Define a cost function J^k as in (4) and a lower value function $V^k(\theta,p,x)$ as in (5) by substituting $(P^k(t), X^k(t))$ to $(P(t), X(t))$ and $F^k(p,x) := \phi_{k+2}(p,x)F(p,x)$ to $F(p,x)$.

Consider the HJBI equation associated with (12), that is,

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,p,x) + \mathcal{H}_k^-(D^2v(t,p,x), Dv(t,p,x), p, x) = 0 \text{ in } [0,T) \times \mathbb{R}^{n+1}, \\ v(T,p,x) = F^k(p,x), \end{cases} \tag{13}$$

where, the functions a, q being defined in (6) and (7),

$$\begin{aligned} &\mathcal{H}_k^-(A,z,p,x) \\ &:= \max_{u\in K_u} \min_{\pi \in K_\pi} \left[\frac{1}{2}\mathrm{Tr}\left(\phi_k(p,x)^2 a(p,x,\sigma,\pi)A\right) + \phi_k(p,x)z\cdot q(p,x,b,\pi)\right]. \end{aligned} \tag{14}$$

All the assumptions of Fleming & Souganidis [10, Thm.2.6] are satisfied, so that V^k is the unique viscosity solution to (13). In addition, as the right hand side of (12) is null as soon as $(P^k(t), X^k(t))$ is outside B_{k+1}, and as $F^k(p,x) \equiv F(p,x)$ inside B_{k+2}, in view of the definition of V^k as a value function one has

$$V^k(\theta,p,x) = F(p,x) \tag{15}$$

for all $\theta \in [0,T]$ and $(p,x) \in B_{k+2} - B_{k+1}$.

Set

$$\bar{V}^k(t,p,x) := \begin{cases} V^k(t,p,x) \text{ in } [0,T] \times B_{k+1}, \\ F(p,x) \text{ in } [0,T] \times \mathbb{R}^{n+1} - B_{k+1}. \end{cases} \tag{16}$$

In view of (15) and the continuity of V^k, $\bar{V}^k$ is a continuous function on $[0,T] \times \mathbb{R}^{n+1}$. Moreover, as $F^k \equiv F$ in $\overline{B_{k+1}}$ and $\mathcal{H}_k^-(A,z,p,x) = 0$ for all

(p,x) in $\mathbb{R}^{n+1} - B_{k+1}$, we have that $\bar{V}^k$ is a viscosity solution to

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,p,x) + \mathcal{H}_k^-(D^2 v(t,p,x), Dv(t,p,x), p, x) = 0 \text{ in } [0,T) \times \mathbb{R}^{n+1}, \\ v(T,p,x) = F(p,x). \end{cases} \tag{17}$$

Moreover, in view of (12), for all (p,x) outside B_{k+1} one has

$$\mathbb{E}_{\theta,p,x} F(P^k(T), X^k(T)) = F(p,x).$$

Thus, for all $(p,x) \in \mathbb{R}^{n+1}$ one has

$$\bar{V}^k(t,p,x) = \inf_{\Pi \in Ad_\Pi(\theta)} \sup_{u(\cdot) \in Ad_u(\theta)} \mathbb{E}_{\theta,p,x}[F(P^k(T), X^k(T))].$$

Observe that $\mathcal{H}_k^-(A,z,p,x)$ converges to $\mathcal{H}^-(A,z,p,x)$ locally on compact subsets of $S^2 \times \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$. Therefore, owing to Lions [12, Prop.I.3], if the viscosity solutions $\bar{V}^k$ to (17) converge on compact subsets of $\mathbb{R}^+ \times \mathbb{R}^{n+1}$ to the function V, then V is a viscosity solution to (9). We now prove the convergence.

Arbitrarily fix a compact subset of $\mathbb{R}^{n+1}$. This set is included in the ball B_k for all k large enough. Let (p,x) be in the compact set under consideration. Fix an arbitrary admissible control $u.$ and an arbitrary admissible strategy Π. Let τ_k be the minimum of T and the first exit time of $(P(t), X(t))$ from B_k. Before the stopping time τ_k, the solutions to (3) and (12) coincide. Thus

$$\begin{aligned} &|\mathbb{E}_{\theta,p,x}[F(P(T),X(T)) - F(P^k(T),X^k(T))]| \\ &= |\mathbb{E}_{\theta,p,x}[F(P(T),X(T)) - F(P^k(T),X^k(T))]\mathbb{I}_{\tau_k<T}]| \\ &\leq \mathbb{E}_{\theta,p,x}\left[Q(|P(T)|,|X(T)|,|P^k(T)|,|X^k(T)|)\right. \\ &\qquad \left.(|P(T)|+|X(T)|+|P^k(T)|+|X^k(T)|)\mathbb{I}_{\tau_k<T}\right]. \end{aligned}$$

From

$$\mathbb{E}_{\theta,p,x}[|P^k(T)|^{2m} + (X^k(T))^{2m} + |P(T)|^{2m} + (X(T))^{2m}] \leq C(T)(|p|^{2m} + x^{2m}), \tag{18}$$

and

$$\begin{aligned} \mathbb{P}_{\theta,p,x}(\tau_k < T) &= \mathbb{P}_{\theta,p,x}\left[\sup_{\theta \leq t \leq T}(|P(t)|^2 + X(t)^2) \geq k^2\right] \\ &\leq \frac{(|p|^2 + x^2)C(T)}{k^2}, \end{aligned} \tag{19}$$

for some constant $C(T)$ independent from k, using Cauchy–Schwarz's inequality we deduce

$$|\mathbb{E}_{\theta,p,x}[F(P(T),X(T)) - F(P^k(T),X^k(T))]| \leq \frac{\Phi(|p|,|x|)}{k},$$

where Φ is a polynomial function determined by the set $K_u \times K_\pi$, the function F, and the time T only. Using Inequality (11), for all θ in $[0,T]$ and (p,x) in the compact set under consideration, we deduce that

$$|\bar{V}^k(\theta,p,x) - V(\theta,p,x)| \leq \frac{\Phi(|p|,|x|)}{k}. \tag{20}$$

Thus $\bar{V}^k$ converges to V uniformly on compact subsets of $\mathbb{R}^+ \times \mathbb{R}^{n+1}$, from which we deduce that V is a viscosity solution to (9).

3.2.2 Uniqueness of a viscosity solution

The next statement and its proof are appropriate modifications to the uniqueness result of Barles, Buckdahn & Pardoux [4, Thm.3.5].

Theorem 3.6. *Suppose that there exist a viscosity subsolution $v(t,p,x)$ and a viscosity supersolution $w(t,p,x)$ to (9) such that*

$$\lim_{|p|^2+x^2\to\infty} v(t,p,x)\exp(-\overline{A}|\log(|p|^2+x^2)|^2) = 0 \text{ for all } t \in [0,T],$$

and

$$\lim_{|p|^2+x^2\to\infty} w(t,p,x)\exp(-\overline{A}|\log(|p|^2+x^2)|^2) = 0 \text{ for all } t \in [0,T]$$

for some $\overline{A} > 0$.

Moreover suppose that $v(T,p,x) \leq w(T,p,x)$ for all $(p,x) \in \mathbb{R}^{n+1}$. Then $v(t,p,x) \leq w(t,p,x)$ for all $(t,p,x) \in [0,T] \times \mathbb{R}^{n+1}$.

We start with two technical lemmas.

Lemma 3.7. *Let $v(t,p,x)$ be a viscosity subsolution and $w(t,p,x)$ a viscosity supersolution to (9). Then the function $\overline{v} := v - w$ is a viscosity subsolution to*

$$\frac{\partial v}{\partial t}(t,p,x) + \mathcal{H}^+(D^2v(t,p,x), Dv(t,p,x),p,x) = 0 \text{ in } [0,T) \times \mathbb{R}^{n+1}, \tag{21}$$

where

$$\mathcal{H}^+(A,z,p,x) := \max_{u=(b,\sigma)\in K_u} \max_{\pi\in K_\pi} \left(\frac{1}{2}\mathrm{Tr}\,(a(p,x,\sigma,\pi)A) + z\cdot q(p,x,b,\pi)\right),$$

for all $(n+1)\times(n+1)$ symmetric matrix A and all vector z in $\mathbb{R}^{n+1}$.

Proof. Let $\varphi \in C^2([0,T]\times\mathbb{R}^{n+1})$ and let (t,p,x) be a strict global maximum point of $\overline{v}-\varphi$.

We introduce the following function

$$\begin{aligned}&\psi_{\epsilon,\alpha}(t,p,x,s,\overline{p},\overline{x})\\ &\qquad := v(t,p,x)-w(s,\overline{p},\overline{x})-\frac{|p-\overline{p}|^2+|x-\overline{x}|^2}{\epsilon^2}-\frac{|t-s|^2}{\alpha^2}-\varphi(t,p,x),\end{aligned}$$

where ϵ,α are positive real numbers which are devoted to tend to zero.

By a standard argument in viscosity solution theory, (see Crandall, Ishii & Lions [6]), there exists a sequence $(t_{\epsilon,\alpha},p_{\epsilon,\alpha},x_{\epsilon,\alpha},s_{\epsilon,\alpha},\overline{p}_{\epsilon,\alpha},\overline{x}_{\epsilon,\alpha})$ such that

- $(t_{\epsilon,\alpha},p_{\epsilon,\alpha},x_{\epsilon,\alpha},s_{\epsilon,\alpha},\overline{p}_{\epsilon,\alpha},\overline{x}_{\epsilon,\alpha})$ is a global maximum point of $\psi_{\epsilon,\alpha}$ in $([0,T]\times\overline{B}_R)^2$ where B_R is a large ball in $\mathbb{R}^{n+1}$.
- $(t_{\epsilon,\alpha},p_{\epsilon,\alpha},x_{\epsilon,\alpha},s_{\epsilon,\alpha},\overline{p}_{\epsilon,\alpha},\overline{x}_{\epsilon,\alpha})$ tends to (t,p,x,t,p,x) when $(\epsilon,\alpha)\to 0$.
- $\dfrac{|p_{\epsilon,\alpha}-\overline{p}_{\epsilon,\alpha}|^2+|x_{\epsilon,\alpha}-\overline{x}_{\epsilon,\alpha}|^2}{\epsilon^2}+\dfrac{|t_{\epsilon,\alpha}-s_{\epsilon,\alpha}|^2}{\alpha^2}$ is bounded and tends to zero when $(\epsilon,\alpha)\to 0$.

We now use the jet sets $\overline{D}^{2,+}v(t,p,x)$ and $\overline{D}^{2,-}w(t,p,x)$. It follows from Crandall, Ishii & Lions [6, Thm.8.3] that there exist $X,Y\in S^2$ such that

$$\left(a_{\epsilon,\alpha}+\frac{\partial\varphi}{\partial t}(t_{\epsilon,\alpha},p_{\epsilon,\alpha},x_{\epsilon,\alpha}),z_{\epsilon,\alpha}+D\varphi(t_{\epsilon,\alpha},p_{\epsilon,\alpha},x_{\epsilon,\alpha}),X\right)\in\overline{D}^{2,+}v(t_{\epsilon,\alpha},p_{\epsilon,\alpha},x_{\epsilon,\alpha}),$$

$$(a_{\epsilon,\alpha},z_{\epsilon,\alpha},Y)\in\overline{D}^{2,-}w(s_{\epsilon,\alpha},\overline{p}_{\epsilon,\alpha},\overline{x}_{\epsilon,\alpha}),$$

and

$$\begin{bmatrix} X & 0\\ 0 & -Y\end{bmatrix}\le\frac{4}{\epsilon^2}\begin{bmatrix} I & -I\\ -I & I\end{bmatrix}+\begin{bmatrix} D^2\varphi(t_{\epsilon,\alpha},p_{\epsilon,\alpha},x_{\epsilon,\alpha}) & 0\\ 0 & 0\end{bmatrix},$$

where

$$a_{\epsilon,\alpha}:=\frac{2(t_{\epsilon,\alpha}-s_{\epsilon,\alpha})}{\alpha^2},$$

and

$$z_{\epsilon,\alpha}:=\frac{2((p_{\epsilon,\alpha},x_{\epsilon,\alpha})-(\overline{p}_{\epsilon,\alpha},\overline{x}_{\epsilon,\alpha}))}{\epsilon^2}.$$

Observing that $(t_{\epsilon,\alpha},p_{\epsilon,\alpha},x_{\epsilon,\alpha},s_{\epsilon,\alpha},\overline{p}_{\epsilon,\alpha},\overline{x}_{\epsilon,\alpha})$ is a local maximum point of $\psi_{\epsilon,\alpha}$ and $v(t,p,x)$ is a viscosity subsolution of (9), we obtain

$$a_{\epsilon,\alpha}+\frac{\partial\varphi}{\partial t}(t_{\epsilon,\alpha},p_{\epsilon,\alpha},x_{\epsilon,\alpha})+\mathcal{H}^-(X,z_{\epsilon,\alpha}+D\varphi(t_{\epsilon,\alpha},p_{\epsilon,\alpha},x_{\epsilon,\alpha}),p_{\epsilon,\alpha},x_{\epsilon,\alpha})\ge 0. \tag{22}$$

Similarly, since $w(t,p,x)$ is a viscosity supersolution to (9), we have

$$a_{\epsilon,\alpha} + \mathcal{H}^-(Y, z_{\epsilon,\alpha}, \overline{p}_{\epsilon,\alpha}, \overline{x}_{\epsilon,\alpha}) \leq 0. \tag{23}$$

Thus

$$\frac{\partial \varphi}{\partial t}(t_{\epsilon,\alpha}, p_{\epsilon,\alpha}, x_{\epsilon,\alpha}) + \mathcal{H}^-(X, z_{\epsilon,\alpha} + D\varphi(t_{\epsilon,\alpha}, p_{\epsilon,\alpha}, x_{\epsilon,\alpha}), p_{\epsilon,\alpha}, x_{\epsilon,\alpha}) \\ - \mathcal{H}^-(Y, z_{\epsilon,\alpha}, \overline{p}_{\epsilon,\alpha}, \overline{x}_{\epsilon,\alpha}) \geq 0. \tag{24}$$

Suppose that we have shown

$$\begin{aligned} \mathcal{H}^-(X, z_{\epsilon,\alpha} + D\varphi(t_{\epsilon,\alpha}, p_{\epsilon,\alpha}, x_{\epsilon,\alpha}), p_{\epsilon,\alpha}, x_{\epsilon,\alpha}) - \mathcal{H}^-(Y, z_{\epsilon,\alpha}, \overline{p}_{\epsilon,\alpha}, \overline{x}_{\epsilon,\alpha}) \\ \leq \max_{u \in K_u} \max_{\pi \in K_\pi} \big[\mathrm{Tr}(a(p_{\epsilon,\alpha}, x_{\epsilon,\alpha}, \sigma, \pi) D^2\varphi(t_{\epsilon,\alpha}, p_{\epsilon,\alpha}, x_{\epsilon,\alpha})) \\ + D\varphi(t_{\epsilon,\alpha}, p_{\epsilon,\alpha}, x_{\epsilon,\alpha}) \cdot q(p_{\epsilon,\alpha}, x_{\epsilon,\alpha}, b, \pi)\big] \\ + C\frac{|p_{\epsilon,\alpha} - \overline{p}_{\epsilon,\alpha}|^2 + |x_{\epsilon,\alpha} - \overline{x}_{\epsilon,\alpha}|^2}{\epsilon^2} \\ + C| < z_{\epsilon,\alpha}, ((p_{\epsilon,\alpha}, x_{\epsilon,\alpha}) - (\overline{p}_{\epsilon,\alpha}, \overline{x}_{\epsilon,\alpha})) > | \end{aligned} \tag{25}$$

for some positive real number C. We use the right side of (25) in (24). In view of the definition of $(t_{\epsilon,\alpha}, p_{\epsilon,\alpha}, x_{\epsilon,\alpha}, \overline{p}_{\epsilon,\alpha}, \overline{x}_{\epsilon,\alpha})$, the right side of (25) tends to

$$\max_{u \in K_u} \max_{\pi \in K_\pi} \big[\mathrm{Tr}(a(p,x,\sigma,\pi) D^2\varphi(t,p,x)) + D\varphi(t,p,x) \cdot q(p,x,b,\pi)\big]$$

when (ϵ, α) tends to zero. We then get our desired result:

$$\frac{\partial \varphi}{\partial t}(t,p,x) + \mathcal{H}^+(D^2\varphi(t,p,x), D\varphi(t,p,x), p, x) \geq 0 \text{ in } [0,T) \times \mathbb{R}^{n+1}. \tag{26}$$

We now prove (25). We have

$$\begin{aligned} &\mathcal{H}^-(X, z, p, x) - \mathcal{H}^-(Y, \overline{z}, \overline{p}, \overline{x}) \\ &\quad = \max_{u \in K_u} \min_{\pi \in K_\pi} [\mathcal{H}_{u,\pi}(X, z, p, x)] - \max_{u \in K_u} \min_{\pi \in K_\pi} [\mathcal{H}_{u,\pi}(Y, \overline{z}, \overline{p}, \overline{x})] \\ &\quad \leq \max_{u \in K_u} \max_{\pi \in K_\pi} (\mathcal{H}_{u,\pi}(X, z, p, x) - \mathcal{H}_{u,\pi}(Y, \overline{z}, \overline{p}, \overline{x})) \\ &\quad \leq \max_{u \in K_u} \max_{\pi \in K_\pi} (\mathcal{H}_{u,\pi}(X, \overline{z}, p, x) - \mathcal{H}_{u,\pi}(Y, \overline{z}, \overline{p}, \overline{x}) \\ &\qquad + (\mathcal{H}_{u,\pi}(X, z, p, x) - \mathcal{H}_{u,\pi}(X, \overline{z}, p, x))), \end{aligned}$$

and

$$\mathcal{H}_{u,\pi}(X, z, p, x) - \mathcal{H}_{u,\pi}(X, \overline{z}, p, x) = < q(p,x,b,\pi), z - \overline{z} > .$$

In addition, from the definition of X, Y it results that

$$\begin{bmatrix} X - D^2\varphi(t_{\epsilon,\alpha}, p_{\epsilon,\alpha}, x_{\epsilon,\alpha}) & 0 \\ 0 & -Y \end{bmatrix} \le \frac{4}{\epsilon^2} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}.$$

In view of Fleming & Soner [9, Chap.V, Lemma 6.2], there exists $C > 0$ such that

$$\begin{aligned} &\mathcal{H}_{u,\pi}(X, \overline{z}, p, x) - \mathcal{H}_{u,\pi}(Y, \overline{z}, \overline{p}, \overline{x}) \\ &\quad \le C \frac{|p - \overline{p}|^2 + |x - \overline{x}|^2}{\epsilon^2} + \mathrm{Tr}(a(p, x, \sigma, \pi) D^2\varphi(t_{\epsilon,\alpha}, p_{\epsilon,\alpha}, x_{\epsilon,\alpha})) \\ &\quad\quad + < q(p, x, b, \pi) - q(\overline{p}, \overline{x}, b, \pi), \overline{z} > \end{aligned}$$

for all X, Y, $\overline{z}$, $(p, \overline{p})$ and $(x, \overline{x})$. That ends the proof. □

Lemma 3.8. *Set*

$$\phi(p, x) := \left[\frac{1}{2} \log(|p|^2 + x^2 + 1) + 1 \right]^2 .$$

For any $\overline{A} > 0$, there exists a $C_1 > \frac{\overline{A}}{T}$ such that the function

$$\rho(t, p, x) := \exp[(C_1(T - t) + \overline{A})\phi(p, x)],$$

satisfies

$$\frac{\partial \rho}{\partial t}(t, p, x) + \mathcal{H}^+(D^2\rho(t, p, x), D\rho(t, p, x), p, x) < 0 \text{ in } [t_1, T) \times \mathbb{R}^{n+1},$$

where $t_1 := T - \dfrac{\overline{A}}{C_1}$.

Proof. In view of the definition of t_1, there exists a constant C independent of C_1 such that, for all t in $[t_1, T]$,

$$\begin{aligned} |D\rho(t, p, x)| &\le C\rho(t, p, x) \frac{\phi^{\frac{1}{2}}(p, x)}{(|p|^2 + x^2 + 1)^{\frac{1}{2}}}, \\ |D^2\rho(t, p, x)| &\le C\rho(t, p, x) \frac{\phi(p, x)}{(|p|^2 + x^2 + 1)}. \end{aligned}$$

As, in addition,

$$\frac{\partial \rho(t, p, x)}{\partial t} = -C_1 \rho(t, p, x)\phi(p, x),$$

we have

$$\begin{aligned}
&|\mathcal{H}^+(D^2\rho(t,p,x), D\rho(t,p,x), p, x)| \\
&\qquad \le C((|p|^2 + x^2)|D^2\rho(t,p,x)| + (|p| + |x|)|D\rho(t,p,x)|) \\
&\qquad \le C\rho(t,p,x)\phi(p,x),
\end{aligned}$$

and the result follows by choosing a large enough C_1. □

In order to conclude the proof of Theorem 3.6, we now proceed in three steps: we will first prove that

$$\overline{v}(t,p,x) \le \alpha\rho(t,p,x) \text{ in } [t_1, T) \times \mathbb{R}^{n+1}$$

for any $\alpha > 0$, where ρ and t_1 are defined in Lemma 3.8. Then we will let α tend to zero, and finally we will show that

$$\overline{v}(t,p,x) \le \alpha\rho(t,p,x) \text{ in } [0, T) \times \mathbb{R}^{n+1}.$$

Since by assumption $\frac{\overline{v}}{\alpha\rho}$ tends to zero when $|p|^2 + x^2$ tends to infinity, we have that $\overline{v} - \alpha\rho$ is bounded from above in $[t_1, T] \times \mathbb{R}^{n+1}$, and the global maximum

$$M := \max_{[t_1,T]\times\mathbb{R}^{n+1}} (\overline{v} - \alpha\rho)(t,p,x)$$

is achieved at some point (t_0, p_0, x_0).

We assume that $\overline{v}(t_0, p_0, x_0) > 0$, otherwise we are done.

Because (t_0, p_0, x_0) is the global maximum, we have

$$(\overline{v} - \alpha\rho)(t,p,x) \le (\overline{v} - \alpha\rho)(t_0, p_0, x_0).$$

Thus $\overline{v} - \psi$ has a maximum at (t_0, p_0, x_0) where

$$\psi(t,p,x) := \alpha\rho(t,p,x) + (\overline{v} - \alpha\rho)(t_0, p_0, x_0).$$

We now show that $t_0 = T$. If not, since $\overline{v}$ is a viscosity subsolution to (21), we have

$$\frac{\partial\psi}{\partial t}(t_0, p_0, x_0) + \mathcal{H}^+(D^2\psi(t_0, p_0, x_0), D\psi(t_0, p_0, x_0), p_0, x_0) \ge 0.$$

The left hand side of the preceding inequality is nothing but

$$\alpha\left(\frac{\partial\rho}{\partial t}(t_0, p_0, x_0) + \mathcal{H}^+(D^2\rho(t_0, p_0, x_0), D\rho(t_0, p_0, x_0), p_0, x_0)\right),$$

which is strictly negative by Lemma 3.8. We thus have a contradiction. Therefore, $t_0 = T$. By assumption we have $\overline{v}(T,p,x) \le 0$, we thus have

$$\overline{v}(t,p,x) \le \alpha\rho(t,p,x) \text{ in } [t_1, T) \times \mathbb{R}^{n+1}.$$

Letting α tend to zero, we get

$$\overline{v}(t,p,x) \le 0 \text{ in } [t_1, T) \times \mathbb{R}^{n+1}.$$

We apply successively the same argument on the interval $[t_2, t_1]$, where $t_2 := (t_1 - \overline{A}/C_1)^+$ and if $t_2 > 0$ we continue... We finally obtain

$$\overline{v}(t,p,x) \le 0 \text{ in } [0, T) \times \mathbb{R}^{n+1}.$$

That ends the proof.

3.2.3 The Dynamic Programming Principle

The Dynamic Programming Principle holds for V^k defined as in Subsection 3.2.1 (see Fleming & Souganidis [10, Thm.1.6]). Thus

$$V^k(\theta,p,x) = \inf_{\Pi \in Ad_\Pi(\theta)} \sup_{u(\cdot)\in Ad_u(\theta)} \mathbb{E}_{\theta,p,x}[V^k(t, P^k(t), X^k(t))]. \tag{27}$$

In order to get (10), we now let k tend to infinity. We first observe that, in view of (20), $\bar{V}^k(\theta,p,x)$ tends to $V(\theta,p,x)$. Second, we have $\bar{V}^k(\theta,p,x) = V^k(\theta,p,x)$ for all $(p,x) \in \overline{B_{k+1}}$. Third, if (p,x) is in $\overline{B_{k+1}}$ then $(P^k(t), X^k(t))$ is in $\overline{B_{k+1}}$ for all $t \in [\theta, T]$ almost surely. Thus, for all (p,x), one has

$$\begin{aligned}
&\Big| \inf_{\Pi \in Ad_\Pi(\theta)} \sup_{u(\cdot)\in Ad_u(\theta)} \mathbb{E}_{\theta,p,x}[V^k(t, P^k(t), X^k(t))] \qquad (28)\\
&\qquad - \inf_{\Pi \in Ad_\Pi(\theta)} \sup_{u(\cdot)\in Ad_u(\theta)} \mathbb{E}_{\theta,p,x}[V(t, P(t), X(t))]\Big| \\
&\le \sup_{\Pi \in Ad_\Pi(\theta)} \sup_{u(\cdot)\in Ad_u(\theta)} \mathbb{E}_{\theta,p,x}|V^k(t, P^k(t), X^k(t)) - V(t, P(t), X(t))| \\
&\le \sup_{\Pi \in Ad_\Pi(\theta)} \sup_{u(\cdot)\in Ad_u(\theta)} \mathbb{E}_{\theta,p,x}|V^k(t, P^k(t), X^k(t)) - V(t, P^k(t), X^k(t))| \\
&\quad + \sup_{\Pi \in Ad_\Pi(\theta)} \sup_{u(\cdot)\in Ad_u(\theta)} \mathbb{E}_{\theta,p,x}|V(t, P^k(t), X^k(t)) - V(t, P(t), X(t))| \\
&\le \sup_{\Pi \in Ad_\Pi(\theta)} \sup_{u(\cdot)\in Ad_u(\theta)} \mathbb{E}_{\theta,p,x}|(\bar{V}^k(t, P^k(t), X^k(t)) - V(t, P^k(t), X^k(t)))\mathbb{I}_{\tau_k \ge T}| \\
&\quad + \sup_{\Pi \in Ad_\Pi(\theta)} \sup_{u(\cdot)\in Ad_u(\theta)} \mathbb{E}_{\theta,p,x}|(V^k(t, P^k(t), X^k(t)) - V(t, P^k(t), X^k(t)))\mathbb{I}_{\tau_k < T}| \\
&\quad + \sup_{\Pi \in Ad_\Pi(\theta)} \sup_{u(\cdot)\in Ad_u(\theta)} \mathbb{E}_{\theta,p,x}|V(t, P^k(t), X^k(t)) - V(t, P(t), X(t))|.
\end{aligned} \tag{29}$$

In view of (20) and (18), we have

$$
\begin{aligned}
\mathbb{E}_{\theta,p,x} |(\bar{V}^k(t, P^k(t), X^k(t)) - V(t, P^k(t), X^k(t)))\mathbb{I}_{\tau_k \geq T}| \\
&\leq \mathbb{E}_{\theta,p,x} \left[\frac{K(|P^k(t)|, |X^k(t)|, T)\sqrt{|P^k(t)|^2 + (X^k(t))^2}}{k} \right] \\
&\leq \frac{C(|p|, |x|, T)}{k}.
\end{aligned}
$$

In addition, in view of (19),

$$
\begin{aligned}
\mathbb{E}_{\theta,p,x} |V(t, P^k(t), X^k(t)) - V(t, P(t), X(t))| \\
&= \mathbb{E}_{\theta,p,x} [|V(t, P^k(t), X^k(t)) - V(t, P(t), X(t))|\mathbb{I}_{\tau_k < T}] \\
&\leq \frac{K(|p|, |x|, T)}{k}
\end{aligned}
$$

and

$$
\mathbb{E}_{\theta,p,x} |(V^k(t, P^k(t), X^k(t)) - V(t, P^k(t), X^k(t)))\mathbb{I}_{\tau_k < T}| \leq \frac{K(|p|, |x|, T)}{k}.
$$

That ends the proof.

4 Example: worst case model risk management for bond options

In this section we illustrate our result by considering the model risk control when the trader aims to hedge a European option. The case of options on stocks readily appears as a particular case of the general setting introduced in the preceding sections. We thus describe a less elementary application, namely the case of options on bonds.

4.1 The Heath-Jarrow-Morton model

Suppose that the yield curve of the financial market follows the Heath-Jarrow-Morton (HJM) [11] model, and therefore we have:

Definition 4.1. *For all time T^*, the instantaneous forward rate $f(t, T^*)$ satisfies the stochastic differential equation*

$$
f(t, T^*) = f(0, T^*) + \int_0^t \alpha(s, T^*) ds + \int_0^t \sigma(s, T^*) dW_s, \tag{30}
$$

for a given Borel measurable function $f(0,\cdot):\mathbb{R}^+\to\mathbb{R}$, and given random maps $\alpha: C\times\Omega\to\mathbb{R}$, $\sigma: C\times\Omega\to\mathbb{R}$, where $C:=((s,t)\mid 0\le s\le t)$. We suppose that $\alpha(\cdot,T^*)$ and $\sigma(\cdot,T^*)$ are adapted processes such that

$$\int_0^{T^*}|\alpha(s,T^*)|ds+\int_0^{T^*}|\sigma(s,T^*)|^2ds<\infty\ \ \mathbb{P}-a.s.$$

When the market has no arbitrage, there exists an adapted process $(\lambda(t))$ such that

$$\alpha^*(t,T^*)=\frac{1}{2}|\sigma(t,T^*)|^2-\sigma^*(t,T^*)\lambda(t),$$

where

$$\alpha^*(t,T^*):=\int_t^{T^*}\alpha(t,s)ds$$

(see, e.g., Musiela & Rutkowski [13, Chap.13]).

The HJM model is an infinite dimensional factor model. Special cases are the one factor Merton model, the Vasicek model, the Cox-Ingersoll-Ross model, the Long & Schwartz model, the Hull & White model, the Ho & Lee model, etc.

The price of a discount bond maturing at the date T^* is

$$B(t,T^*)=\exp\left(-\int_t^{T^*}f(t,s)ds\right). \tag{31}$$

As shown in Musiela & Rutkowski [13, Lemma 13.1.1], the discount bond price $B(t,T^*)$ satisfies

$$\begin{aligned}B(t,T^*)=1&-\int_t^{T^*}(r(s)+\sigma^*(s,T^*)\lambda(s))B(s,T^*)ds\\&+\int_t^{T^*}\sigma^*(s,T^*)B(s,T^*)dW_s\end{aligned}$$

for all $0\le t\le T^*$, where $r(t):=f(t,t)$ is the short term rate.

In order to simplify notation we use forward prices. Given a price P_t, define the forward price P_t^F by

$$P_t^F:=\frac{P_t}{B(t,T^O)},$$

where $B(t,T^O)$ is the price of the bond of maturity T^O in the model driven by $\sigma(t,T^O)$. In particular, the forward price of the discount bond maturing at T is

$$B^F(t,T):=\frac{B(t,T)}{B(t,T^O)}.$$

It is easy to check that $B^F(t,T)$ satisfies

$$\begin{aligned} dB^F(t,T) &= B^F(t,T)(\sigma^*(t,T^O) - \lambda(t))(\sigma^*(t,T^O) - \sigma^*(t,T))dt \\ &\quad + B^F(t,T)(\sigma^*(t,T^O) - \sigma^*(t,T))dW_t. \end{aligned} \tag{32}$$

4.2 Exact hedging strategies

We consider the following situation. A trader wants to hedge a European option written on $B(T^O,T)$. The payoff at maturity T^O is denoted by $f(B(T^O,T))$. The trader uses two bonds to hedge the option: the bond of maturity T^O and the bond of maturity T.

At each date $0 \le t \le T^O$, a self-financing strategy consists in buying or selling a quantity H_t^O of discount bonds of maturity T^O and a quantity H_t of discount bonds of maturity T such that

(i) The portfolio is self-financing, which means that, if

$$V_t = H_t B(t,T) + H_t^O B(t,T^O) \tag{33}$$

is the value of portfolio at time t, then

$$V_t = V_0 + \int_0^t H_s dB(s,T) + \int_0^t H_s^O dB(s,T^O). \tag{34}$$

(ii) The processes (H_t) and (H_t^O) are adapted and satisfy technical assumptions which ensure that all the stochastic integrals in the sequel, where (H_t) and (H_t^O) are integrated, are well defined and are martingales.

Let V_t^F denote the forward price of the trader's portfolio:

$$V_t^F := \frac{V_t}{B(t,T^O)}.$$

For any self-financing strategy we have

$$dV_t^F = H_t dB^F(t,T) \tag{35}$$

(see Bossy et al. [5] for the proof). An exact hedging (replicating) strategy is a self-financing strategy (H, H^O) such that

$$V_{T^O} = H_{T^O} B(T^O,T) + H_{T^O}^O = f(B(T^O,T)).$$

For a HJM model with a deterministic parameter σ, the exact hedging strategy satisfies

$$H_t = \frac{\partial \pi_\sigma}{\partial x}(t, B^F(t,T)),$$

where π_σ is the solution to

$$\begin{cases} \dfrac{\partial \pi_\sigma}{\partial t}(t,x) + \dfrac{1}{2}x^2(\sigma^*(t,T^O) - \sigma^*(t,T))^2 \dfrac{\partial^2 \pi_\sigma}{\partial x^2}(t,x) = 0, \\ \pi_\sigma(T,x) = f(x). \end{cases}$$

The quantity H_t^O is chosen according to the self-financing condition: in view of (33) and (34) one has

$$H_t^O = V_0^F + \int_0^t H_s dB^F(s,T) - H_t B^F(t,T).$$

For details, see Bossy et al. [5].

4.3 Model risk

Bossy et al. [5] propose examples of model risk analysis based on Monte Carlo simulations of the Profit & Loss of self-financing misspecified strategies. In this subsection we give an outline of their results.

Suppose that the trader does not know the random map $\sigma(s,T)$. Instead, he or she chooses a deterministic model structure $\overline{\sigma}(s,T)$ and tries to hedge the contingent claim according to this model. The quantity $\overline{H}_t$ of bonds of maturity T is determined according to the rule

$$\overline{H}_t = \frac{\partial \pi_{\overline{\sigma}}}{\partial x}(t, B^F(t,T)),$$

where $\pi_{\overline{\sigma}}$ is the solution to

$$\begin{cases} \dfrac{\partial \pi_{\overline{\sigma}}}{\partial t}(t,x) + \dfrac{1}{2}x^2(\overline{\sigma}^*(t,T^O) - \overline{\sigma}^*(t,T))^2 \dfrac{\partial^2 \pi_{\overline{\sigma}}}{\partial x^2}(t,x) = 0, \\ \pi_{\overline{\sigma}}(T,x) = f(x). \end{cases}$$

As above, the quantity $\overline{H}_t^O$ is then determined by the self-financing condition.

Let $\overline{V}_t$ be the value of the trader's portfolio at time t. The option seller's Profit & Loss at time $t < T^O$ is

$$P\&L_t := \overline{V}_t - V_t,$$

and the forward price of the Profit & Loss is

$$P\&L_t^F := \frac{P\&L_t}{B(t,T^O)}.$$

One can study the probability law of the Profit & Loss at maturity T^O, and compute some of its statistics (moments, quantiles, etc.) by fixing $\sigma(\cdot,\cdot)$ and $\overline{\sigma}(\cdot,\cdot)$. See Bossy et al. [5]. This methodology is however restricted to the comparison of one (potentially incorrect) model against one or several (possible true) models among a class of univariate Markov term structure models (with deterministic parameters). This class does not contain all possible term structure models.

Consider the quantity

$$F\left(B^F(T^O,T),V^F_{T^O}\right) := \mathbb{E}\left[G(f(B^F(T^O,T)) - V^F_{T^O})\right],$$

where G is a given continuous function. It might be useful to choose the function G in order to be able to justify that our game problem corresponds to a kind of a measure of risk (to be defined). Of course such a measure of risk would be weaker than the coherent measures of risks introduced by Artzner, Delbaen, Eber and Heath [1]: for a discussion on a related topics, see Section 2 in [7] (where it must be understood that $C/S_0(T)$ plays the role of $f(B^F(T^O,T))$).

We assume that the function $F(p,x) := G(f(p)-x)$ satisfies (8). The control processes of the market are

$$\begin{aligned} u_1(t) &:= \sigma^*(t,T^O) - \lambda(t),\\ u_2(t) &:= \sigma^*(t,T) - \sigma^*(t,T^O). \end{aligned}$$

The control process of the trader is $\pi(t) := \frac{H_t B^F(t,T)}{V^F(t)}$ if $V^F(t) \neq 0$ and $\pi(t) := 0$ if $V^F(t) = 0$. In view of (32) and (35), $(P(t),X(t)) := (B^F(t,T),V^F(t))$ satisfies a S.D.E of type (3) with $r=0$, namely,

$$\begin{cases} dP(t) &= P(t)[u_1(t)u_2(t)dt + u_2(t)dW_t],\\ dX(t) &= X(t)\pi(t)\left[u_1(t)u_2(t)dt + u_2(t)dW_t\right]. \end{cases}$$

The Hamilton-Jacobi-Bellman-Isaacs equation (9) then writes

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,p,x) + \mathcal{H}^-(D^2v(t,p,x),Dv(t,p,x),p,x) = 0 \ \text{ in } [0,T^O)\times\mathbb{R}^2,\\ v(T^O,p,x) = G(f(p)-x), \end{cases} \tag{36}$$

where

$$\mathcal{H}^-(A,z,p,x) := \max_{u\in K_u}\min_{\pi\in K_\pi} \left[\frac{1}{2}u_2^2p^2A_{11} + u_2^2px\pi A_{12} + \frac{1}{2}u_2^2x^2\pi^2A_{22} + z_1u_1u_2p + z_2u_1u_2\pi x\right],$$

for all 2×2 symmetric matrix A and all vector z in $\mathbb{R}^2$.

4.4 A numerical illustration

We finally illustrate that the value functions of our stochastic game problems, and the corresponding optimal strategies, can be numerically approximated. In our numerical example below, except for $\rho = 0$, the function F does not satisfy the requirements for a coherent measure of risk (again see Remark 2.3 in [7]); this is to emphasize the flexibility of our approach. We consider a call option and set

$$F(p,x) := ((p-K)_+ - x)_+ + \rho(((p-K)_+ - x)_+)^2.$$

The maturity T^O of the option is 6 months and the option is written on a discount bond of maturity T equal to 5 years. The trader uses two bonds to hedge this option: the bond of maturity 6 months and the bond of maturity 5 years. The strike of the option is $K = 0.509156$. We set $\rho = 20$, $\pi \in [-1,1]$, $u_2 \in [0,0.6]$ and $u_1 \in [-0.08, 0.06]$.

The finite difference method is used for the numerical resolution of the viscosity solution to (36) numerically. Because (36) is strongly degenerate, the monotone condition is not satisfied in the standard explicit finite difference approximation. We apply Kushner's technique to overcome this difficulty, that is, there exists a (p,x,t,u_1,u_2,π)- dependent scale transformation wich yields a locally consistent approximation at (p,x,t,u_1,u_2,π). For details, see Kushner [14].

Our approximate value function is drawn in Fig.1. For other numerical results (e.g., strategies obtained through the numerical resolution of the HJBI equation) and comments, see Talay & Zheng [16].

5 Conclusion

In this paper we have studied a stochastic game problem related to a worst case model risk management. We have proven that the value function of this game problem is the unique viscosity solution to an Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation, and satisfies the Dynamic Programming Principle.

Empirical studies should be developed to check whether the strategies obtained by solving the HJBI equation numerically are realistic in practice. The question of the accuracy of the numerical method should also be addressed.

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Figure 1: The Value Function: Graph of $V(0, p, x)$