Around Model Risk in Finance

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Abstract

In these notes we introduce some mathematical material to define, analyze, estimate and control model risk in Finance. We start with explaining why statistical and calibration techniques may not avoid facing model risk. We then present a couple of model risk measures and control strategies. Finally, we propose a mathematical framework to rigorously study the performances of strategies which, based on technical analysis, are not sensitive to model risk, and we compare these strategies to those which are based on mathematical models and therefore are subject to model risk.

1 Introduction

To price derivatives, or to construct portfolio management strategies, practitioners use mathematical models or technical analysis techniques. Mathematical models are introduced in order to derive prices and strategies from the non arbitrage theory or the stochastic control theory combined with the machinery of the stochastic calculus and the analysis of partial differential equations. Technical analysis avoids models and proposes investment rules deduced from historical observations of the behavior of the market. At a first glance, it seems that the use of mathematical models is much more suitable to avoid risk than the use of recipes which are difficult to justify rigorously by a standard mathematical approach. However the situation is far from being so simple at it seems.

Indeed, to design perfectly hedging strategies one needs a lot of stringent conditions on the market which are not guaranteed to be satisfied in practice: in addition to non arbitrage conditions, one assumes that the dynamics of the stock is a semi-martingale; in order to get numerical values for the quantities of stocks or bonds to invest at each time, one usually restricts the model to the class of solutions of stochastic differential equations, which allows one to express the perfectly hedging strategy in terms of solutions of partial differential equations. If the model is extremely simple, such as the Black and Scholes model, one gets explicit expressions for the desired strategies. Otherwise, numerical approximations are necessary, which induces hedging errors because of the time and space discretizations, the finite number of discretization steps, and even the misspecified artificial boundary conditions which are necessary to keep the discretization space into reasonable limits. Therefore, misspecifications of financial strategies are partially due to numerical errors. They are also due to calibration errors. As we will see below, statistical procedures, and calibration algorithms based on the observation of a lot of market prices related to the particular financial object under study, may not avoid substantial model misspecifications. These misspecifications do not only concern parameters in the model, or coefficients in the stochastic differential equations under consideration: they also concern the nature of the driving noises (which may be continuous or have jumps), and the dimension of these noises.

On the other hand, technical analysis does not prescribe to choose a particular model: the strategies derive from recipes based on indices which are computed by observing more or less recent market prices only. As important volumes are exchanged by following these rules, it seems worth studying technical analysis mathematically. Recent papers propose to study the efficiency of the rules when applied to trajectories of stochastic processes, and to determine when they may become more efficient than strategies which would be optimal if the models were perfectly known but are rather sensitive to model misspecifications.

The notes below tackle all these issues in a coarse-grained way. Actually model risk analysis requires various tools from numerical analysis, statistics, stochastic control, stochastic analysis, optimization. We have chosen to introduce a few aspects; some of them are elementary but illustrative, the other ones are rather advanced and original.

2 An Example of Model Risk: Hedging With Misspecified Securities

The contents of this section are elementary. We estimate the effects of misspecified volatilities on hedging strategies for European options, and express the Profit and Loss process in terms of estimation errors on the volatility of the underlying stock.

2.1 A laboratory example

Let $T > 0$ be the maturity of a European option.

Consider the Black and Scholes paradigm, that is, a portfolio consisting in holding two assets: a non risky asset whose price at time t is denoted by S_t^0 and solves the deterministic ordinary differential equation

$$S_t^0 = 1 + \int_0^t rS_\theta^0 d\theta,$$

where $r > 0$ is the instantaneous interest rate; a risky asset, whose price at time t is denoted by S_t and solves the linear stochastic differential equation

$$S_t = S_0 + \int_0^t \mu S_\theta d\theta + \int_0^t \sigma S_\theta dW_\theta,$$

where S_0, μ and σ are strictly positive constants, and (W_t) is a Brownian motion on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the natural augmented filtration of (W_t).

Define a strategy as a pair of continuous processes, (H_t^0) and (H_t), real valued and $(\mathcal{F}_t)$-adapted. The process H_t^0 represents the quantity of non risky assets held by the investor at time t, and H_t represents the quantity of risky assets held at time t; the values of H_t^0 or H_t may be negative: this occurs when the investor borrows the amount $H_t^0 S_t^0$ or, respectively, $H_t S_t$. The value V_t of the investor's portfolio at time t is

$$V_t := H_t^0 S_t^0 + H_t S_t, \ \forall t \leq T.$$

The strategy is said self-financing if it satisfies the following condition:

$$V_t = V_0 + r\int_0^t H_\theta^0 S_\theta^0 d\theta + \mu\int_0^t H_\theta S_\theta d\theta + \sigma\int_0^t H_\theta S_\theta dW_\theta,\ \forall t \le T. \quad (1)$$

In order to ensure that the preceding integrals are well defined, we constrain (H_t^0) and (H_t) to satisfy

$$\int_0^T |H_t^0 S_t^0| dt + \int_0^T |H_t S_t|^2 dt < +\infty,\ \mathbb{P} - a.s. \quad (2)$$

A European option with pay-off function f is a contract which delivers the quantity $f(S_T)$ at the maturity T. By definition, a hedging portfolio for such an option is the value at time t of a portfolio constructed by means of a self-financing strategy and satisfying

$$V_T = f(S_T),\ \mathbb{P}\text{-a.s.}$$

To simplify our presentation, we suppose here that that the positive function f is of class $\mathcal{C}^\infty(\mathbb{R})$, is bounded and has bounded derivatives. The celebrated Black-Scholes formula tells us that a hedging portolio has value

$$V_t = F(t, S_t) \text{ for all } 0 \le t \le T,$$

where F is the unique solution of class $\mathcal{C}_b^{1,2}([0,T]\times\mathbb{R})$ to

$$\begin{cases} \dfrac{\partial F}{\partial t}(t,x) + rx\dfrac{\partial F}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 F}{\partial x^2}(t,x) - rF(t,x) = 0, \\ \hfill 0 \le t < T,\ x \in \mathbb{R}, \\ F(T,x) = f(x),\ x \in \mathbb{R}. \end{cases} \quad (3)$$

Actually, given such a smooth solution, Itô's formula implies that

$$V_t = V_0 + \int_0^t \frac{\partial F}{\partial \theta}(\theta, S_\theta) d\theta + \int_0^t \left(\frac{\partial F}{\partial x}(\theta, S_\theta)\mu S_\theta + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\theta, S_\theta)\sigma^2 S_\theta^2 \right) d\theta$$
$$+ \int_0^t \frac{\partial F}{\partial x}(\theta, S_\theta)\sigma S_\theta dW_\theta.$$

In view of the self-financing condition (1), we deduce that the process (H_t), that is, the delta of the option, necessarily satisfies

$$H_\theta = \frac{\partial F}{\partial x}(\theta, S_\theta) \text{ for all } 0 \le \theta \le T,$$

and that

$$\begin{aligned} H_\theta \mu S_\theta + r H^0_\theta r e^{r\theta} &= \frac{\partial F}{\partial \theta}(\theta, S_\theta) + \frac{\partial F}{\partial x}(\theta, S_\theta)\mu S_\theta + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\theta, S_\theta)\sigma^2 S_\theta^2 \\ &= \frac{\partial F}{\partial x}(\theta, S_\theta)(\mu - r)S_\theta + rF(\theta, S_\theta), \end{aligned}$$

since F solves the PDE (3). Consequently,

$$H^0_\theta = e^{-r\theta}F(\theta, S_\theta) - e^{-r\theta}\frac{\partial F}{\partial x}(\theta, S_\theta)S_\theta \text{ for all } 0 \le \theta \le T.$$

Reciprocally, since

$$V_t = H^0_t e^{rt} + H_t S_t,$$

substituting H^0_t and H_t with the above expressions leads to $V_t = F(t, S_t)$ for all $0 \le t \le T$, and to

$$\begin{aligned} V_t = V_0 &+ \int_0^t \frac{\partial F}{\partial \theta}(\theta, S_\theta)d\theta + \int_0^t \left(\frac{\partial F}{\partial x}(\theta, S_\theta)\mu S_\theta + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\theta, S_\theta)\sigma^2 S_\theta^2\right)d\theta \\ &+ \int_0^t \frac{\partial F}{\partial x}(\theta, S_\theta)\sigma S_\theta dW_\theta \\ = V_0 &+ \int_0^t H_\theta dS_\theta + \int_0^t H^0_\theta dS^0_\theta, \end{aligned}$$

which means that the portfolio under consideration is self-financed. Notice that our hypotheses on f imply smoothness properties on F which in turn implies (2).

Now consider an investor who does not know σ exactly. She/he thus uses a parameter $\bar{\sigma}$ issued from a calibration procedure (e.g., $\bar{\sigma}$ may be an implied volatility or the result of a volatility estimator applied to historical data) and, at all time t she/he buys or sells stocks according to the rule

$$\bar{H}_t := \frac{\partial \bar{F}}{\partial x}(t, S_t),$$

where $\bar{F}$ is the solution of the PDE

$$\begin{cases} \dfrac{\partial \bar{F}}{\partial t}(t,x) + rx\dfrac{\partial \bar{F}}{\partial x}(t,x) + \dfrac{1}{2}\bar{\sigma}^2 x^2 \dfrac{\partial^2 \bar{F}}{\partial x^2}(t,x) - r\bar{F}(t,x) = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad 0 \le t < T,\ x \in \mathbb{R}, \\ \bar{F}(T,x) = f(x),\ x \in \mathbb{R}. \end{cases} \tag{4}$$

Let $\bar{V}_t$ be the value at time t of the corresponding self-financing portfolio:

$$\bar{V}_t = \bar{H}^0_t S^0_t + \bar{H}_t S_t.$$

The self-financing condition implies

$$\bar{V}_t = \bar{V}_0 + \int_0^t \bar{H}^0_\theta r S^0_\theta d\theta + \int_0^t \bar{H}_\theta \mu S_\theta d\theta + \int_0^t \bar{H}_\theta \sigma S_\theta dW_\theta.$$

We now aim to express $\bar{V}_t$ without stochastic integral. Apply Itô's formula to $u(t, S_t) := \exp(-rt)\bar{F}(t, S_t)$. It comes:

$$\begin{aligned} e^{-rt}\bar{F}(t, S_t) = \bar{F}(0, S_0) - r\int_0^t e^{-r\theta}\bar{F}(\theta, S_\theta)d\theta + \int_0^t e^{-r\theta}\frac{\partial \bar{F}}{\partial \theta}(\theta, S_\theta)d\theta \\ + \int_0^t e^{-r\theta}\frac{\partial \bar{F}}{\partial x}(\theta, S_\theta)\mu S_\theta d\theta + \int_0^t e^{-r\theta}\frac{\partial \bar{F}}{\partial x}(\theta, S_\theta)\sigma S_\theta dW_\theta \\ + \frac{1}{2}\int_0^t e^{-r\theta}\frac{\partial^2 \bar{F}}{\partial x^2}(\theta, S_\theta)\sigma^2 S^2_\theta d\theta. \end{aligned}$$

As

$$e^{-rt}\bar{V}_t = \bar{V}_0 + \sigma\int_0^t e^{-r\theta}\bar{H}_\theta S_\theta dW_\theta + (\mu - r)\int_0^t e^{-r\theta}\frac{\partial \bar{F}}{\partial x}(\theta, S_\theta)S_\theta d\theta,$$

we have

$$\begin{aligned} e^{-rt}\bar{V}_t = \bar{V}_0 + e^{-rt}\bar{F}(t, S_t) - \bar{F}(0, S_0) + r\int_0^t e^{-r\theta}\bar{F}(\theta, S_\theta)d\theta \\ - \int_0^t e^{-r\theta}\frac{\partial \bar{F}}{\partial \theta}(\theta, S_\theta)d\theta - \frac{1}{2}\int_0^t e^{-r\theta}\frac{\partial^2 \bar{F}}{\partial x^2}(\theta, S_\theta)\sigma^2 S^2_\theta d\theta \\ - r\int_0^t e^{-r\theta}\frac{\partial \bar{F}}{\partial x}(\theta, S_\theta)S_\theta d\theta. \end{aligned}$$

In addition, $\bar{F}(T, x) = f(x)$ for all x. In view of (4), the value of the P& L at time T of the portfolio is thus

$$\begin{aligned} P\&L_T &:= \bar{V}_T - f(S_T) \\ &= \exp(rT)(\bar{V}_0 - \bar{F}(0, S_0)) \\ &\quad + \frac{1}{2}(\bar{\sigma}^2 - \sigma^2)\int_0^T \exp(r(T - \theta))\frac{\partial^2 \bar{F}}{\partial x^2}(\theta, S_\theta)S^2_\theta \, d\theta. \end{aligned} \tag{5}$$

Remark 2.1. *If the function* $\bar{F}(\theta, \cdot)$ *is convex for all* θ *and if* $\bar{\sigma} > \sigma$*, then, given an initial endowment* $\bar{V}_0 = \bar{F}(0, S_0)$*, one has* $\bar{V}_T > f(S_T)$*. Apparently a model misspecification may lead to an arbitrage. However it should be difficult to find a buyer at the price* $\bar{F}(0, S_0)$ *which reflects an overestimation of the volatility.*

Remark 2.2. *Notice that we have worked under the historical probability* $\mathbb{P}$*. So will we do in all the sequel. We actually are concerned by markets which are intrinsically incomplete since the investor cannot have a perfect knowledge of the market model. In such markets there is no a clear option price. In all the sequel the investor is allowed to freely choose the methodology to compute the initial value* V_0 *of the portfolio and the investment strategies: for example, when her/his objective is to hedge an option, she/he may suppose that the true market is perfectly described by a complete model, or she/he may prefer to follow the surreplication methodology, or to determine* V_0 *by mimimizing a certain risk measure, etc.*

2.2 An extension of our laboratory example

Consider a primary asset with price process S and a deterministic saving account with price process F. To the saving account corresponds a change of numeraire. We denote by S_t^F the price of the primary asset expressed in this numeraire. For example, in the context of the Black and Scholes model, S^F is the discounted stock price and is a martingale under the risk neutral probability.

Consider an option on the primary asset with maturity T^O and payoff function ϕ. Suppose that, in the real market, this option can be perfectly hedged owing to a self-financing strategy; as above, denote by H_t^0 the number of units of the saving account and by H_t units of the primary asset. Expressed in the numeraire F the value of the hedging portfolio is

$$V_t^F = H_t^0 + H_t S_t^F,$$

and the self-financing condition implies

$$V_t^F = V_0^F + \int_0^t H_\theta dS_\theta^F.$$

The investor needs to compute (H_t). Therefore she/he chooses a model which reflects the information available on the market and allows one to get explicit formulae, or to develop numerical approximation methods, for (H_t). In all cases, a common choice consists in supposing that (S_t^F) is a (possibly non homogeneous) Markov process with infinitesimal generator $\bar{\mathcal{L}}_t^F$ such that the following parabolic PDE has a smooth solution $\bar{\pi}(t,x)$:

$$\frac{\partial\bar{\pi}}{\partial t}(t,x) + \bar{\mathcal{L}}_t^F \bar{\pi}(t,x) = 0$$

with boundary condition (remember that $V_T^F = \frac{1}{F_T}\phi(F_T\, S_T^F)$ and that we have supposed that (F_t) is deterministic)

$$\bar{\pi}(T,x) = \frac{\phi(F_T x)}{F_T}.$$

Notice that, in the classical Black and Scholes context, we have $\bar{\pi}(t,x) = e^{-rt}v(t,e^{rt}x)$, where $v(t,x)$ is the solution of the standard Black and Scholes PDE. The investor uses the strategy

$$\bar{H}_t = \frac{\partial\bar{\pi}}{\partial x}(t,S_t^F),$$

and the value of the portfolio satisfies

$$d\overline{V}_t^F = \overline{H}_t dS_t^F.$$

Now define the *model risk P&L function* as

$$P\&L_t^F = \overline{V}_t^F - V_t^F. \tag{6}$$

Suppose that, *in the true world*, the process (S_t^F) is a (not necessarily Markov) Itô process whose dynamics under $\mathbb{P}^F$ is

$$dV_t^F = \beta_t dt + \gamma_t dW_t^F$$

for some adapted processes β and γ. Set

$$\mathcal{L}_t^F \bar{\pi}(t,S_t^f) := \beta_t \frac{\partial\overline{\pi}}{\partial x}(t,S_t^F) + \frac{1}{2}(\gamma_t)^2 \frac{\partial^2\overline{\pi}}{\partial x^2}(t,S_t^F).$$

Apply Itô's theorem to $\overline{\pi}(t,S_t^F)$. It comes:

$$\begin{aligned} d\overline{\pi}(t,S_t^F) &= \frac{\partial\overline{\pi}}{\partial t}(t,S_t^F)dt + \mathcal{L}^F\overline{\pi}(t,S_t^F)dt + \frac{\partial\overline{\pi}}{\partial x}(t,S_t^F)\gamma_t dW_t^F \\ &= (\mathcal{L}_t^F - \overline{\mathcal{L}}_t^F)\overline{\pi}(t,S_t^F)dt + d\overline{V}_t^F, \end{aligned}$$

so that

$$\overline{V}_t^F - \overline{V}_0^F = \overline{\pi}(t, S_t^F) - \overline{\pi}(0, S_0^F) + \int_0^t (\overline{\mathcal{L}}_\theta^F - \mathcal{L}_\theta^F)\overline{\pi}(\theta, S_\theta^F) d\theta.$$

We thus have

$$\begin{aligned} P\&L_t^F &= \overline{V}_t^F - V_t^F \\ &= \overline{V}_0^F - \overline{\pi}(0, S_0^F) + \overline{\pi}(t, S_t^F) - V_t^F + \int_0^t (\overline{\mathcal{L}}_\theta - \mathcal{L}_\theta)\overline{\pi}(\theta, S_\theta^F) d\theta. \end{aligned}$$

At maturity T, this equality simplifies to

$$\begin{aligned} P\&L_T^F &= \overline{V}_T^F - V_T^F \\ &= \overline{V}_0^F - \overline{\pi}(0, S_0^F) + \int_0^T (\overline{\mathcal{L}}_\theta^F - \mathcal{L}_\theta^F)\overline{\pi}(\theta, S_\theta^F) d\theta. \end{aligned}$$

Notice that, if (S_t^F) is a Markov process, that is, if $\beta_t = \beta(t, S_t^F)$ and $\gamma_t = \gamma(t, S_t^F)$ for some functions β and γ, then $\mathcal{L}_t^F$ is the infinitesimal generator of (S_t^F) and $V_t^F = \pi(t, S_t^F)/F_t$ where $\pi(t, x)$ solves a parabolic PDE driven by the operator $\mathcal{L}_t^F$. For a more general discussion on such representations of the P & L process and numerical simulations, see Bossy et al. [18].

Once the dynamics of the P & L process is identified, a natural issue consists in estimating numerically related risk measures, or to develop worst case analyses. Before tackling these issues, we explain why calibration errors cannot be neglicted in finance.

3 Statistical Explanations for Model Risk

A huge literature exists on the statistics of random processes: see, e.g., the textbooks by Prakasa Rao ([56] and [57]), Lipster and Shiryayev [49], Kutotants [46], the references therein, and the references below. We limit ourselves to an introductive discussion aimed to explain why, were financial prices perfectly described by diffusion processes, estimation procedures based on historical data only should not be expected to provide good accuracies on the unknown coefficients of the stochastic differential equations supposed to model the prices. For the sake of simplicity, we distinguish the estimation of volatilities and the estimation of trends; the simultaneous estimation of volatilities and trends is complex: see, e.g., the sudy of minimum contrast estimators by Génon-Catalot and Jacod [33], and of simulated moment methods by Duffie and Singleton [28], Clément [21], Duffie and Glynn [27].

The classical framework in parametric Statistics is as follows.

Definition 3.1. *A statistical model is a measurable space* $(S, \mathcal{S})$ *and a collection of probability measures* $\{\mathcal{Q}^{\theta}, \theta \in \Theta\}$ *on that space. The set* Θ *of the possible values of the parameter* θ *is an open set in* $\mathbb{R}^{\ell}$.

An estimator *is a measurable functional defined on* $(S, \mathcal{S})$.

An *estimation procedure* is an algorithm which, given an observation π, allows one to select one the possible values of θ.

Before studying properties of standard parametric estimation procedures when applied to stock prices, we discuss a modeling issue which was tackled in the literature recently only.

3.1 The Brownian dimension of a stochastic model

The contents of this subsection come from Jacod et al. [40].

When calibrating a diffusion model from historical data, the first step consists in choosing the dimension of the diffusion matrix, that is, the dimension of the Brownian motion driving the stochastic differential equation. This is a difficult issue in practice which actually is an ill-posed problem since the observations are made at discrete times and the diffusion coefficient is unknown. We rather need to think of an **explicative Brownian dimension**, that is, a minimal dimension of the noise process which allows one to explain in a satisfying way the estimated quadratic variation of the observed process. For example, the model

$$X_t = X_0 + \int_0^t b_s ds + W_t^1 + \epsilon W_t^2$$

where W^1 and W^2 are two independent Brownian motions and $\epsilon << 1$ has an explicative Brownian dimension equal to 1.

To be more specific, consider the model

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s \, dW_s,$$

where W is a standard q–dimensional Brownian Motion, a is a predictable $\mathbb{R}^d$–valued locally bounded process, σ is a $d \times q$ matrix–valued adapted and càdlàg process which is Hölder continuous with index $\rho > 1/2$:

$$\sup_{0 \le s < t \le T} \frac{\|\sigma_t - \sigma_s\|}{(t-s)^\rho} < \infty \text{ a.s.}$$

A particular case is the pure diffusion case where σ_s is of the type $\sigma(X_s)$ for some function σ. Set $c_s := \sigma_s \sigma_s^\star$. Thus $c_s = \sigma\sigma^\star(X_s)$ in the pure diffusion case. We aim to estimate the maximal rank of c_s on the basis of the observation of $X_{iT/n}$ for $i = 0, 1, \ldots, n$, where T is some fixed time horizon - or, more reasonably, we aim to estimate the number of eigenvalues of c_s which, during a large enough time interval, are significantly larger than the other ones.

We need some linear algebra material. Let $\mathcal{A}_r$ be the family of all subsets of $\{1, \ldots, d\}$ with r elements. For all $K \in \mathcal{A}_r$ and all $d \times d$ symmetric nonnegative matrix Σ, denote by determinant$_K(\Sigma)$ the determinant of the $r \times r$ sub–matrix $(\Sigma^{kl} : k, l \in K)$, and set

$$\text{determinant}(r; \Sigma) := \sum_{K \in \mathcal{A}_r} \text{determinant}_K(\Sigma).$$

Observe that determinant$(d; \Sigma)$ = determinant(Σ), and determinant$(1; \Sigma)$ is the trace of Σ.

Lemma 3.2. *The matrix Σ has eigenvalues*

$$\lambda(1) \ge \ldots \lambda(d) \ge 0,$$

and, for $r = 1, \ldots, d$:

$$\frac{1}{d(d-1)\ldots(d-r+1)} \textit{determinant}(r; \Sigma) \\ \le \lambda(1)\lambda(2)\ldots\lambda(r) \ \le \ \textit{determinant}(r; \Sigma).$$

In addition,

$$1 \le r \le d \quad \Longrightarrow \begin{cases} r \le \textit{rank}(\Sigma) \Longrightarrow \textit{determinant}(r; \Sigma) > 0 \\ r > \textit{rank}(\Sigma) \Longrightarrow \textit{determinant}(r; \Sigma) = 0, \end{cases}$$

and, for all $2 \leq r \leq d$*,*

$$\frac{r!}{d!} \frac{\textit{determinant}(r;\Sigma)}{\textit{determinant}(r-1;\Sigma)} \leq \lambda(r) \leq \frac{d!}{(r-1)!} \frac{\textit{determinant}(r;\Sigma)}{\textit{determinant}(r-1;\Sigma)}.$$

Set

$$L(r)_T := \int_0^T \lambda(r)_s \, ds.$$

Our aim is to determine the largest integer r such that $L(r)_T$ is significantly larger than $L(r+1)_T$. For example, if, say, $L(2)_T$ is fifty times larger than $L(3)_T$, the explicative Brownian motion can reasonably be chosen as 2. We cannot be more precise: the meaning of 'significantly' actually depends on each particular application and on the desired accuracy on the model. Anyhow, the issue is close to the following one: determine the largest integer r such that

$$\bar{L}(r)_T := \int_0^t \lambda(1)_s \dots \lambda(r)_s \, ds$$

is significantly larger than $\bar{L}(r+1)_T$. From a numerical point of view, $\bar{L}(r)_T$ is not easier to approximate than $L(r)_T$ by using observations of a trajectory of (X_t) because eigenvalues are difficult to compute with a good accuracy. However, Lemma 3.2 shows that $\bar{L}(r)_T$ has the same order of magnitude as

$$\mathcal{L}(r)_T := \int_0^t \text{determinant}(r; c_s) \, ds,$$

which is easier to estimate because determinants are easier to compute than eigenvalues. However, in practice, one does not observe the process (c_t): we only observe the process (X_t) at discrete times iT/n. We thus need to construct estimators of $\mathcal{L}(r)_T$ which solely depend on such observations. To this end we set

$$\zeta(r)_i^n := \sum_{j=1}^r (\Delta_{i+j-1}^n X) \, (\Delta_{i+j-1}^n X)^*,$$

with

$$\Delta_i^n X := X_{iT/n} - X_{(i-1)T/n}.$$

Denoting by $[x]$ the integer part of x we also set

$$\mathcal{L}(r)_t^n := \frac{n^{r-1}}{T^{r-1} \, r} \sum_{i=1}^{[nt/T]-r+1} \text{determinant}(r; \zeta(r)_i^n).$$

The following theorem shows that the estimator $\mathcal{L}(r)^n_t$ converges to $\mathcal{L}(r)_t$ and precises the convergence rate.

Theorem 3.3. *The variables $\mathcal{L}(r)^n_t$ converge in probability to $\mathcal{L}(r)_t$ uniformly in $t \in [0, T]$.*

The processes

$$V(r)^n_t := \sqrt{n}\,(\mathcal{L}(r)^n_t - \mathcal{L}(r)_t)$$

converge stably in law to a non-homogeneous Brownian motion $(V(r)_t)_{1 \le r \le d}$ defined on an extension of the original space.

The preceding theorem allows us to develop estimators of explicative Brownian dimensions based on relative thresholds. Set

$$R(\omega)_t := \sup_{s \in [0,t]} \operatorname{rank}(c_s(\omega)).$$

Consider the following scale invariant estimator of R_t:

$$\mathcal{R}_{n,t} := \inf \left(r \in \{0 \dots, d-1\} : \ \mathcal{L}(r+1)^n_t < \rho_n t^{-1/r} (\mathcal{L}(r)^n_t)^{(r+1)/r} \right),$$

where ρ_n is a given sequence of thresholds tending to 0 when n goes to infinity. We have the following convergence theorem:

Theorem 3.4. *For all r, r' in $\{1, \dots, d\}$, provided $\mathbb{P}(R_t = r') > 0$, we have*

$$\mathbb{P}(\mathcal{R}_{n,t} \neq r \mid R_t = r') \longrightarrow \begin{cases} 1 \text{ if } r \neq r', \\ 0 \text{ if } r = r'. \end{cases}$$

From a practical point of view, this result is less useful as it seems to be because, so far, the convergence rate of $\mathbb{P}(\mathcal{R}_{n,t} \neq r \mid R_t = r')$ is an open problem. In [40] one can find other examples of tests, all of them suffering from similar lacks of knowledge on their accuracies, and some numerical examples which illustrate the sensitivity of reasonable choices of explicative Brownian dimensions to the frequency T/n of the observations. Consequently, the difficulty to extract the number of sources of randomness from historical data must be taken into account when evaluating the robustness of strategies to model uncertainties.

3.2 Testing whether the noise has jumps

In all this paper, as in the preceding subsection, we will suppose that the market and the market model obey dynamics driven by Brownian motions. Of course, this hypothesis is questionable. A remarkable work by Aït-Sahalia and Jacod [3] allows one to test whether a price process observed at discrete times is continuous or jumps. The observed process (X_t) is supposed to be of the following type:

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dB_s + \int_0^t \int_{\mathbb{R}} \kappa \circ \delta(s,x)(\mu - \nu)(ds, dx)$$
$$+ \int_0^t \int_{\mathbb{R}} (\delta(s,x) - \kappa \circ \delta(s,x))\mu(ds, dx).$$

Here, B is a Brownian motion, μ is a Poisson random measure with an intensity measure of the form $\nu(ds, dx) = ds \otimes dx$; the function κ is continuous and locally equal to x around the origin; the processes (b_s), (σ_s) are optional, and the random function $\delta(s, \cdot)$ is predictable and uniformly bounded in ω and time by a deterministic function γ such that $\int_{\mathbb{R}} \min(\gamma(x)^2, 1)dx < \infty$. The authors require a few more technical conditions which are not limitative for applications in Finance (for example, the process (σ_t) is supposed to be of the same type as (X_t) itself).

Let Δ_n be a sequence of time steps decreasing to 0. Given the observations of (X_t) at the times $i\Delta n$, Aït-Sahlia and Jacod's test statistics is

$$\hat{C}(p, \Delta_n)_t := \frac{\sum_{i=1}^{t/\Delta_n} |X_{2i\Delta_n} - X_{2(i-1)\Delta_n}|^p}{\sum_{k=1}^{t/\Delta_n} |X_{i\Delta_n} - X_{(i-1)\Delta_n}|^p}.$$

Theorem 3.5. *Under the above assumptions, for all $t > 0$ and $p > 2$ the variables $\hat{C}(p, \Delta_n)_t$ converge in probability when n goes to infinity to*

$$\mathbb{I}_{\{\omega; s \to X_s(\omega) \text{ is continuous on } [0,t]\}} + 2^{\frac{p}{2}-1} \mathbb{I}_{\{\omega; s \to X_s(\omega) \text{ is discontinuous on } [0,t]\}}.$$

We deduce from this theorem that a reasonable decision rule consists in accepting the hypothesis "the process (X_t) is discontinuous" when $\hat{C}(p, \Delta_n)_t < \frac{1+2^{p/2-1}}{2}$, and rejecting it when $\hat{C}(p, \Delta_n)_t \geq \frac{1+2^{p/2-1}}{2}$. For precise critical regions, asymptotic levels, power functions, and illustrative empirical studies, see [3]. When applied to real historical data (Dow Jones Industrial Average stock prices in 2005), observations each 5 seconds lead to the conclusion that most of the prices should be modelled by models with jumps. However, as predicted by the theoretical results, observations each 30 seconds do not allow one to get a significant information from the test. Therefore, model risk studies should include computations of sensitivities of portfolios w.r.t. jump components in the driving noise of the model.

In conclusion, whereas Brownian models are commonly used to compute prices and deltas, it seems that driving noises with jumps should also be considered – especially for prices or physical variables observed at low frequencies since, in such a case, it is impossible to test the (dis)continuity hypothesis.

3.3 Estimation of constant volatilities

In this subsection we will see that, in the case of the log-normal model, estimators of the volatility have a relative quadratic mean error of order $\frac{1}{\sqrt{n}}$; therefore the estimation error may be large with high probabilities when the prices are observed hourly.

Given a time interval $[0, T]$ and a subdivision of the time interval $[0, T]$ with step $\frac{T}{n}$, the *discrete quadratic variation of* (W_t) *between 0 and* T is defined as

$$V_T^n := \sum_{i=0}^{n-1} (W_{(i+1)T/n} - W_{iT/n})^2.$$

Proposition 3.6. *For all* $T > 0$*,* V_T^n *tends to* T *almost surely when* n *tends to infinity.*

Proof. In view of the Borel–Cantelli lemma, it is enough to prove that

$$\sum_{n=1}^{\infty} \mathbb{E}(V_T^n - T)^4 < \infty.$$

Let (G_i) be a sequence of independent Gaussian random variables with zero mean and unit variance. Let $\alpha_i := G_i^2 - 1$. We have:

$$\begin{aligned}
\mathbb{E}(V_T^n - T)^4 &= \frac{T^4}{n^4}\mathbb{E}\left[\sum_{i=1}^n \alpha_i\right]^4 \\
&= \frac{T^4}{n^4}\left(\sum_{i=1}^n \mathbb{E}\alpha_i^4 + \sum_{\substack{i,j=1\\ i\neq j}}^n \mathbb{E}\alpha_i^2\mathbb{E}\alpha_j^2\right) \\
&= \frac{T^4}{n^4}\left(n\mathbb{E}\alpha_1^4 + n(n-1)(\mathbb{E}\alpha_1^2)^2\right) \\
&\leq CT^4\left(\frac{1}{n^2}+\frac{1}{n^3}\right).
\end{aligned}$$

□

Now consider the Black and Scholes model with parameters μ and σ, and lt $X_t^{\mu,\sigma}$ be the logarithm of the stock

$$X^{\mu,\sigma} := \log(S_0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t.$$

The statistical structure is the space $\mathcal{C}([0,T],\mathbb{R})$ of continuous functions equipped with its Borel sigma–field and the family of probability measures

$$\left\{\mathbb{P}_T^{X^\theta} := \mathbb{P}_T^{X^{\mu,\sigma}}, \theta := (\mu,\sigma) \in \Theta := \mathbb{R}\times\mathbb{R}_+\right\}.$$

Denote by $\mathbb{E}_T^{X^{\mu,\sigma}}$ the expectation under the probability law $\mathbb{P}_T^{X^{\mu,\sigma}}$.

The next proposition shows that the estimation of σ^2 can be accurate only when the price is observed at high frequencies.

Proposition 3.7. *For all function π in $\mathcal{C}([0,T],\mathbb{R})$, set*

$$\Sigma_T^n(\pi) := \frac{1}{n-1}\left\{\frac{n}{T}\sum_{i=0}^{n-1}(\pi((i+1)T/n) - \pi(iT/n))^2 - \frac{(\pi(T)-\pi(0))^2}{T}\right\}.$$

The estimator Σ_T^n is unbiased, that is,

$$\forall(\mu,\sigma)\in\Theta,\ \mathbb{E}_T^{X^{\mu,\sigma}}(\Sigma_T^n) = \sigma^2.$$

In addition, the variance $\mathbb{E}_T^{X^{\mu,\sigma}}|\Sigma_T^n - \sigma^2|^2$ *of the estimation error is equal to* $\dfrac{2\sigma^2}{n-1}$*, which is the minimal error variance within the class of the estimators of* σ^2 *requiring* n *observations in the time interval* $[0,T]$*.*

Finally, the estimation is strongly consistent: for $\mathbb{P}_T^{X^{\mu,\sigma}}$*-almost all continuous map* π*,*

$$\lim_{n\to\infty} \Sigma_T^n(\pi) = \sigma^2.$$

The proof of the previous proposition easily results from

$$\sum_{i=0}^{n-1}(X^{\mu,\sigma}_{(i+1)T/n} - X^{\mu,\sigma}_{iT/n})^2 = \left(\mu - \frac{\sigma^2}{2}\right)^2 \frac{T^2}{n} + 2\left(\mu - \frac{\sigma^2}{2}\right)\sigma W_T \frac{T}{n} + \sigma^2 V_T^n.$$

3.4 Estimation of non constant volatilities

In this subsection we present non parametric estimation procedures for stochastic volatilies of Markov type.

Consider a $\mathbb{R}^r$ valued Brownian motion (W_t) and the $\mathbb{R}^d$ valued process diffusion process

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(X_s)dW_s.$$

We aim to approximate the *function* $\sigma(\cdot)^2$. From the observation of a single trajectory in a fixed time interval $[0,T]$,

Let (h_n) be a sequence of real numbers tending to 0. Consider the estimator

$$\Sigma_T^n(\pi, x) := n\frac{\sum_{i=0}^{n-1} \mathbb{I}_{[|\pi(iT/n)-x|<h_n]}\left[\pi((i+1)T/n) - \pi(iT/n)\right]\left[\pi((i+1)T/n) - \pi(iT/n)\right]^*}{T\sum_{i=0}^{n-1} \mathbb{I}_{[|\pi(iT/n)-x|<h_n]}},$$

where * stands for the transposition of a vector in $\mathbb{R}^d$.

Theorem 3.8. *Suppose that the function* b *and* σ *are Lipschitz,* b *is bounded, and* $\sigma(x)\sigma(x)^*$ *is a bounded and invertible matrix for all* $x \in \mathbb{R}^d$*. Suppose that*

$$\lim_{n\to\infty} \frac{nh_n^4}{(\log(h_n))^2} = +\infty.$$

Set

$$T_x(\omega) := \inf\{t > 0; X_t(\omega) = x\}.$$

Then

$$\forall \delta > 0, \ \lim_{n\to\infty} \mathbb{P}\left[\omega \in [T_x < T]; \ |\Sigma_T^n(X.(\omega), x) - \sigma(x)\sigma(x)^*| > \delta\right] = 0.$$

The estimator is said *weakly consistent.* Its slow convergence rate is precised in the following statement

Theorem 3.9. *Suppose that the function b and σ are Lipschitz, infinitely differentiable, and bounded. Suppose that, for all $x \in \mathbb{R}^d$, the matrix $\sigma(x)\sigma(x)^*$ is invertible with a bounded inverse matrix. Set*

$$N_n := \sum_{i=0}^{n-1} \mathbb{I}_{[|\pi(iT/n)-x|<h_n]}.$$

Assume that the sequence of windowings h_n satisfies:

$$\begin{cases} \lim_{n\to\infty} nh_n^3 = +\infty \textit{ if } d = 1, \\ \lim_{n\to\infty} nh_n^2 = +\infty \textit{ and } \lim_{n\to\infty}(n\alpha_n h_n^2) = 0 \textit{ if } d \geq 2, \end{cases}$$

where

$$\alpha_n := \begin{cases} 2h_n \textit{ if } d = 1, \\ h_n^2 \ln\left(\frac{1}{h_n^2}\right) \textit{ if } d = 2, \\ h_n^2 \textit{ if } d \geq 3. \end{cases}$$

Then, conditionnaly to $X_0 = x$, $\sqrt{N_n}\left[\{(\sigma\sigma^*)(x)\}^{-1}\Sigma_T^n(\pi, x) - Id_{\mathbb{R}^d}\right]$ converges in distribution to $\sqrt{2}\mathcal{N}\left(0, Id_{\mathbb{R}^d}\right)$.

For proofs and limit theorems, see Florens-Zmirou [31] for the one-dimensional case, and Brugière ([19], [20]) for the multi-dimensional case. Other techniques exist, for example wavelet estimators whose convergence rate in a minimax sense cannot be better as n^{-2} when observations are made at times iT/n: see, e.g., Hoffmann ([36] and [37]).

3.5 Introduction to the estimation of drift parameters

We have just seen that estimation procedures for the volatility have weak convergence rates. In this subsection we will see that the situation is not better when one aims to estimate drift parameters, which is an important issue for optimal portfolio management problems, and also for option hedging problems since the simulation of P & L processes of the type (5) requires to know the dynamics of (S_t) under the historical probability.

We briefly describe a classical statistical procedure, namely, the maximum likelihood estimator (see, e,g. Lipster and Shiryayev [49] or Kutotants [46]).

Consider the case of a diffusion process whose diffusion coefficient is perfectly known but drift coefficient depends on an unknown parameter θ (think of the instantaneous rate of return of an asset under the historical probability measure, or of the drift parameters of a mean reverting model for short term interest rates). Therefore we are given a family of functions $\{b_\theta(x), \theta \in \Theta\}$ and an observation time interval $[0, T]$. The corresponding statistical structure consists in the set $S := \mathcal{C}([0, T], \mathbb{R})$ equipped with the Borel sigma–field $\mathcal{B}_T$ and the collection of probability measures $\{\mathbb{P}_T^{X^\theta}, \theta \in \Theta\}$ where $\mathbb{P}_T^{X^\theta}$ denotes the law $\mathbb{P}^{X^\theta}$ on the time interval $[0, T]$ of the unique solution of

$$X_t = X_0 + \int_0^t b_\theta(X_s)ds + \int_0^t \sigma(X_s)dW_s.$$

One is given an open set $\Theta \subset \mathbb{R}^\ell$, a function b defined on $\Theta \times \mathbb{R}_+ \times \mathbb{R}$, a function σ defined on $\mathbb{R}_+ \times \mathbb{R}$, a real number X_0 and a standard Wiener process (W_t), and

$$X_t^\theta = X_0 + \int_0^t b(\theta, s, X_s^\theta)ds + \int_0^t \sigma(s, X_s^\theta)dW_s, \ t \geq 0.$$

Notice that, in this model, the diffusion coefficient σ does not depend on the parameter θ. This hypothesis is necessary to construct maximum likelihood estimators because they are derived from Girsanov's theorem. The statistical structure is

$$\{\mathcal{C}(0, T; \mathbb{R}), \mathcal{B}_T, \{\mathbb{P}_T^{X^\theta}\}, \theta \in \Theta\}.$$

Construct a reference probability measure as follows: choose a particular value $\theta_0 \in \Theta$ and set $Y_t := X_t^{\theta_0}$; the reference probability measure is the law

P_T^Y of the process (Y_t) which, by definition, solves

$$Y_t = X_0 + \int_0^t b(\theta_0, s, Y_s)ds + \int_0^t \sigma(s, Y_s)dW_s.$$

More generally one may choose (Y_t) as the solution of

$$Y_t = X_0 + \int_0^t \beta(s, Y_s)ds + \int_0^t \sigma(s, Y_s)dW_s,$$

where β is a function so simple as possible which may not be of the form $b(\theta_0, s, x)$: for example, one often chooses $\beta \equiv 0$.

Suppose that X_0 and the functions $b_Y(t,x) := \beta(t,x)$, $b_X(t,x) := b(\theta, t, x)$ and $\sigma_X(t,x) := \sigma(x)$ satisfy the hypotheses of Girsanov's theorem for all θ in Θ. Then $\mathbb{P}_T^{X^\theta}$ is absolutely continuous with respect to $\mathbb{P}_T^Y$. The model is said *dominated* by $\mathbb{P}_T^Y$.

For all fixed continuous function π of $[0, T]$ to $\mathbb{R}$, the (possibly empty) set of θ in Θ where the likelihood ratio

$$\frac{d\mathbb{P}_T^{X^\theta}}{d\mathbb{P}_T^Y}(\pi)$$

reaches it maximum value, is denoted by

$$\operatorname*{Argmax}_{\theta \in \Theta} \left\{ \frac{d\mathbb{P}_T^{X^\theta}}{d\mathbb{P}_T^Y}(\pi) \right\}.$$

By definition, a *likelihood estimator* is a functional $\hat{\theta}_T$ from $\mathcal{C}(0, T; \mathbb{R})$ to Θ such that

$$\forall \pi \in \mathcal{C}(0, T; \mathbb{R}), \hat{\theta}_T(\pi) \in \operatorname{Argmax} \left\{ \frac{d\mathbb{P}_T^{X^\theta}}{d\mathbb{P}_T^Y}(\pi) \right\}.$$

From Girsanov's theorem, for almost every π

$$\frac{d\mathbb{P}_T^{X^\theta}}{d\mathbb{P}_T^Y}(\pi)$$

is strictly positive. Therefore it holds that

$$\hat{\theta}_T(\pi) \in \operatorname*{Argmax}_{\theta \in \Theta} \left\{ \log \frac{d\mathbb{P}_T^{X^\theta}}{d\mathbb{P}_T^Y}(\pi) \right\}.$$

The map

$$\pi \longrightarrow L_T(\pi) := \log \frac{d\mathbb{P}_T^{X^\theta}}{d\mathbb{P}_T^Y}(\pi)$$

is called the *log–likelihood ratio.*

In practice, to get an explicit expression for maximum likelihood estimators, one may have to integrate by parts stochastic integrals. The mechanism is as follows. Consider the S.D.E.

$$dX_t = \alpha(X_t)dt + dW_t.$$

Suppose that the drift α is such that there exists a strong solution X_t which does not explode in finite time. Then the Girsanov theorem holds true: for all $T > 0$, the law $\mathbb{P}^X$ of $(X_t, t \leq T)$ is absolutely continuous w.r.t. the law $\mathbb{P}^W$ of $(W_t, t \leq T)$, and the Radon-Nikodym density $\frac{d\mathbb{P}^X}{\mathbb{P}^W}$ is

$$\begin{aligned} Z_T(W_\bullet(\omega)) &:= \frac{d\mathbb{P}^X}{d\mathbb{P}^W}(W_\bullet(\omega)) \\ &= \exp\{\int_0^T \alpha(W_s)dW_s(\omega) - \frac{1}{2}\int_0^T \alpha^2(W_s(\omega))ds\}. \end{aligned}$$

Set $A(x) := \int_0^x \alpha(z)dz$. Assuming that the function α is smooth, we have:

$$Z_T(W_\bullet) = \exp\{A(W_T) - A(0) - \frac{1}{2}\int_0^T (\alpha^2(W_s) + \alpha'(W_s))ds\},$$

and thus one can define a map Z_T on the whole space of continuous functions by

$$Z_T(\pi) = \exp\{A(\pi_T) - A(0) - \frac{1}{2}\int_0^T (\alpha^2(\pi_s) + \alpha'(\pi_s))ds\}.$$

Example 1: Lognormal model. Consider

$$X_t = \log(S_0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t.$$

Suppose that the volatility σ is known. In view of an observation of the stock price (or, equivalently, of the logarithm of the stock price), one wants to estimate the parameter $\theta := \mu - \frac{1}{2}\sigma^2$. The statistical structure is

$$\{\mathcal{C}(0, T; \mathbb{R}), \mathcal{B}_T, \{\mathbb{P}_T^{X^\theta}, \theta \in \mathbb{R}\}\},$$

where $\mathbb{P}^{X^\theta}$ denotes the law of the process X^θ:

$$X_t^\theta = X_0 + \theta t + \sigma W_t.$$

A natural reference law is the law $\mathbb{P}_T^Y$ of

$$Y_t = X_0 + \sigma W_t.$$

The corresponding *likelihood ratio* is

$$L_T(\pi) = \frac{\theta}{\sigma^2}(\pi(T) - \pi(0)) - \frac{\theta^2 T}{2\sigma^2},$$

since

$$\begin{aligned}
\frac{d\mathbb{P}_T^{X^\theta}}{d\mathbb{P}_T^Y}(Y_\cdot(\omega)) &= \exp\left\{\left(\int_0^T \frac{\theta}{\sigma^2} dY_s\right)(\omega) - \frac{1}{2}\int_0^T \frac{\theta^2}{\sigma^2} ds\right\} \ \mathbb{P} - \text{a.s.} \\
&= \exp\left\{\frac{\theta}{\sigma^2}(Y_T(\omega) - Y_0) - \frac{\theta^2 T}{2\sigma^2}\right\} \ \mathbb{P} - \text{a.s.}
\end{aligned}$$

Thus

$$\hat{\theta}_T(\pi) := \frac{\pi(T) - \pi(0)}{T}$$

is a *maximum likelihood estimator* of $\theta = \mu - \frac{\sigma^2}{2}$. Notice that the estimation error

$$\hat{\theta}_T - \theta = \frac{X_T^\theta - X_0}{T} = \sigma \frac{W_T}{T}, \ \mathbb{P}_T^{X^\theta} - \text{a.s}$$

tends to 0 in view of the Iterated Logarithm Law for the Brownian Motion. Notice also that, under $\mathbb{P}_T^{X^\theta}$, $\sqrt{T}(\hat{\theta}_T - \theta)$ converges in distribution to a Gaussian law: the convergence rate is low.

Example 2: A simplified Vasicek model. Consider

$$X_t^\theta = x - \theta \int_0^t X_s^\theta ds + dW_t.$$

Set $(Y_t) := (W_t)$. The corresponding log-likelihood ratio satisfies

$$\begin{aligned}
L_T(Y_\cdot(\omega)) &= -\theta\left(\int_0^T Y_s dY_s\right)(\omega) - \frac{1}{2}\int_0^T \theta^2 Y_s^2(\omega) ds \\
&= -\theta\left(\int_0^T W_s dW_s\right)(\omega) - \frac{\theta^2}{2}\int_0^T W_s^2(\omega) ds \ \mathbb{P} - \text{a.s.}
\end{aligned}$$

Consequently,

$$\hat{\theta}_T(W.(\omega)) = \frac{-\left(\int_0^T W_s dW_s\right)(\omega)}{\int_0^T W_s^2(\omega)ds}$$
$$= \frac{T - W_T^2(\omega)}{2\int_0^T W_s^2(\omega)ds} \ \mathbb{P} - \text{a.s.},$$

and

$$\hat{\theta}_T(\pi) := \frac{T - \pi(T)^2}{2\int_0^T \pi(s)^2 ds}.$$

The estimation error satisfies

$$\hat{\theta}_T - \theta = \frac{\int_0^T X_s^\theta dW_s}{\int_0^T (X_s^\theta)^2 ds}, \ \mathbb{P}_T^{X^\theta} - \text{a.s.}$$

The a.s. convergence when T goes to infinity and the asymptotic normality of the normalized estimation error readily follow from the theorems 3.10 and 3.11 in the next subsection.

3.6 Convergence of maximum likelihood stimators

In the preceding subsections we have examined the convergence and the fluctuations of maximul likelihood estimators when the observation time length T goes to infinity. This point of view is open to criticism in finance where models cannot remain unchanged during large periods. In addition, the analysis of the estimators require structure hypotheses such as linearity (as in the last example in the preceding subsection) or ergodic properties (whose study is far from the objective of the present notes and, except for interest rate models for which a kind a stationarity is natural, cannot reasonably be assumed): these structure hypotheses are aimed to be in a position to apply the two following limit theorems for normalized Brownian stochastic integrals: see, e.g., Basawa and Prakasa Rao [13].

Theorem 3.10. *Let (W_t) be a standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$. Let (f_t) be a $(\mathcal{F}_t)$-adapted process satisfying*

$$\int_0^T f_t^2 dt < \infty \ \mathbb{P} - a.s \text{ for all } T > 0,$$

and

$$\int_0^\infty f_t^2 dt = \infty \, \mathbb{P} - a.s.$$

Then

$$\lim_{t\to\infty} \frac{\int_0^t f_\theta dW_\theta}{\int_0^t f_\theta^2 du} = 0 \, \mathbb{P} - a.s.$$

Theorem 3.11. *Let (W_t) be a m-dimensional standard Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$. Let (F_t) be a $(\mathcal{F}_t)$-adapted process taking values in the set of the matrices of dimension $n \times m$ and satisfying*

$$\mathbb{E} \int_0^T |F_t^{ik}|^2 dt < \infty \textit{ for all } T > 0, \; i = 1, \ldots, n, \; k = 1, \ldots, m.$$

Suppose that there exist constants c_{ij} such that

$$T^{-1} \int_0^T \sum_{k=1}^m F_t^{ik} F_t^{jk} dt \xrightarrow[T\to\infty]{P} c_{ij} \textit{ for all } i, j = 1, \ldots, n.$$

Then

$$\lim_{T\to\infty} T^{-1/2} \int_0^T F_s dW_s = \mathcal{N}((0, \ldots, 0), C := (c_{ij})_{n,n}),$$

where $\mathcal{N}((0, \ldots, 0), C)$ denotes the Gaussian law on $\mathbb{R}^n$ of zero mean and covariance matrix equal to C.

For the statistical applications of the two preceding theorems, we refer to the references at the beginning of this section.

We now focus on the vanishing volatilities asymptotics. Consider models with *small volatilities*. Let Θ be an interval (α, β), and let $(X_t^\theta(\epsilon))$ be the one dimensional process solution of

$$X_t^\theta(\epsilon) = x + \int_0^t b(\theta, s, X_s^\theta(\epsilon)) ds + \epsilon W_t,$$

where (W_t) is a one dimensional Wiener process and θ belongs to Θ.

Let $(Y_t(\epsilon))$ be the solution to

$$Y_t(\epsilon) = x + \epsilon W_t.$$

Denote by $\hat{\theta}_T(\epsilon, \pi)$ the maximum likelihood estimator defined as a value of θ which maximizes

$$\frac{d\mathbb{P}_T^{X^\theta(\epsilon)}}{d\mathbb{P}_T^{Y(\epsilon)}}(\pi).$$

Denote by $(X_t^\theta(0))$ the solution to the ordinary differential equation

$$X_t^\theta(0) = x + \int_0^t b(\theta, s, X_s^\theta(0))ds,$$

and set

$$J_T^\theta := \int_0^T \left| \frac{\partial}{\partial \theta} b(\theta, s, X_s^\theta(0)) \right|^2 ds.$$

The proof of the following result can be found, e.g., in Kutoyants [46].

Theorem 3.12. *Suppose that the functions $b(\theta, t, x)$, $\theta \to b(\theta, t, x)$, and $\frac{\partial}{\partial \theta} b(\theta, t, x)$ are smooth. In addition, suppose*

(i) *For all $\theta \in \Theta$, $J_T(\theta) > 0$.*

(ii) *For all $\theta_1, \theta_2 \in \Theta$, if $\theta_1 \neq \theta_2$ then*

$$\int_0^T |b(\theta_1, s, Y_s(\epsilon)) - b(\theta_2, s, Y_s(\epsilon))|^2 \, ds > 0.$$

Then all maximum likelihood estimator of θ in Θ is weakly consistent: $\delta > 0$ and for all $\theta \in \Theta$,

$$\lim_{\epsilon \to 0} \mathbb{P}_T^{X^\theta(\epsilon)} \left[\pi \in \mathcal{C}([0,T]; \mathbb{R});\ |\hat{\theta}_T(\epsilon, \pi) - \theta| > \delta \right] = 0.$$

In addition, for all $y \in \mathbb{R}$ and for all $\theta \in \Theta$,

$$\lim_{\epsilon \to 0} \mathbb{P}_T^{X^\theta(\epsilon)} \left[\pi \in \mathcal{C}([0,T]; \mathbb{R}); \frac{\hat{\theta}_T(\epsilon, \pi) - \theta}{\epsilon} \leq y \right] = F_T^\theta(y),$$

where F_T^θ is the distribution function of the Gaussian law with zero mean and variance equal to $1/J_T^\theta$.

For expansions of the estimation error, see Yoshida [68].

3.7 Cramer–Rao lower bounds

This subsection shows that, for all $T > 0$, there does exists a strictly positive and universal lower bound for the quadratic norms of the estimation errors resulting from estimators of drift parameters which are based on observations during the time interval $[0, T]$.

We are given an open set $\Theta \subset \mathbb{R}$ and a family of functions $\{b(\theta, \cdot), \theta \in \Theta\}$ from $\mathbb{R}$ to $\mathbb{R}$. Suppose that, for each $\theta \in \Theta$, the function $x \in \mathbb{R} \to b(\theta, x)$ is Lipschitz.

Consider the model

$$X_t^\theta = X_0 + \int_0^t b(\theta, X_s^\theta) ds + W_t,$$

where (W_t) is a standard one dimensional Brownian motion.

Notice that here the diffusion coefficient is identically equal to 1. This simplifies the notation and is not a restriction if the diffusion coefficient σ is differentiable and satisfies

$$\exists \alpha > 0,\ \sigma(x) > \alpha > 0 \text{ for all } x \in \mathbb{R}.$$

Indeed, it then suffices to transform the observations by means of the one-to-one function $\int_0^x \frac{1}{\sigma(z)} dz$.

Suppose that, for each $\theta \in \Theta$, the function $x \in \mathbb{R} \to b(\theta, x)$ is Lipschitz. Let $\mathbb{P}^{X^\theta}$ be the law of (X_t^θ) and let $\mathbb{E}_T^{X^\theta}$ denote the expectation corresponding to $\mathbb{P}_T^{X^\theta}$. Suppose that the function

$$I_T(\theta_1, \theta_2) := \mathbb{E}_T^{X^{\theta_1}} \int_0^T \left| \frac{\partial b}{\partial \theta}(\theta_2, \pi(s)) \right|^2 ds$$

is strictly positive and continuous in (θ, θ) for all θ in Θ.

Under some other weak technical conditions which we do not list here, for all estimator $\hat{\theta}_T$ of θ based upon an observation between times 0 and T such that the function

$$Q_T(\theta) := \mathbb{E}_T^{X^\theta} (\hat{\theta}_T - \theta)^2$$

is bounded on compact sets, the *bias*

$$\beta_T(\theta) := \mathbb{E}_T^{X^\theta} (\hat{\theta}_T - \theta)$$

is *differentiable* w.r.t. Θ, and the *quadratic estimation error* is bounded from below:

$$\mathbb{E}_T^{X^\theta}(\hat{\theta}_T - \theta)^2 \geq \frac{(1+\beta_T'(\theta))^2}{I_T(\theta,\theta)} + \beta_T(\theta)^2 \text{ for all } \theta \in \Theta.$$

The right hand side is the *Cramer–Rao lower bound.* For a proof, see, e.g., Kutoyants [46].

Example. Consider

$$X_t^\theta = X_0 + \theta t + \sigma W_t.$$

Set $Z_t^\theta := \frac{X_t^\theta}{\sigma}$, that is,

$$Z_t^\theta = \frac{X_0}{\sigma} + \frac{\theta}{\sigma} t + W_t.$$

The Cramer–Rao lower bound implies that the estimation of $\frac{\theta}{\sigma}$ based upon the observation of one trajectory of (Z_t^θ) in the time interval $[0,T]$, cannot have an accuracy (in quadratic norm) better than $\frac{\sigma}{\sqrt{T}}$. Therefore the values of volatilities σ and maturities T which are used in practice imply that estimation errors for drift parameters are large.

3.8 On calibration methods in Finance

So far, our discussion on the construction of financial models for stock prices followed a purely statistical point of view: we thus considered that the history of the stock price is the only available information on the market. Of course, this is not true since one can also see on the market the prices of derivatives based on the stock under consideration, the prices of stocks of companies which belong to the same economic sector, etc. In this subsection we present a few attempts to use all these informations to calibrate stochastic models. The data set is now a sample χ of a random vector ξ which represents market prices of products related to the asset under consideration (e.g., forward contracts, derivatives,…). Practitioners have developed complex numerical procedures to make models fit such data sets. We here present a couple of methods without comparing their efficiency, which seems to be an ill-posed problem because of the transitory nature of models in finance. In addition, the empirical results obtained par Schoutens et al. [59] show that calibration methods based on vanilla option prices may not allow one to discreminate very different models substantially. Finally, the optimization steps involved in calibration methods based on inverse PDE problems let the author think that, when the practitioners do not agree on the value of the volatility of a stock to price derivatives on this stock (for example, because of the nervosity of the market), then the calibration methods do likely not allow one to derive from the data models which are robust in the sense that hedging portfolios have weak sensitivities w.r.t. the modelers' degrees of freedom.

Let $\mathbb{X}$ be the state space of ξ. The *set of calibration measures* is

$$\mathcal{P}_\chi := \left\{ \mathbb{Q} \text{ probability on } \mathbb{X} \text{ equivalent to } \mathbb{P},\ \mathbb{E}^{\mathbb{Q}}[\xi] = \chi \right\}.$$

How to choose an 'optimal' element of $\mathcal{P}_\chi$? Different approaches have been developed by various authors: inverse problem techniques for the PDEs associated to option prices, optimization techniques combined with the Dupire PDE, optimisation techniques designed to fit the model and historical data, entropy minimization techniques, etc. The short discussion below shows that, in all cases, it is numerically difficult to obtain the solution(s) with good accuracies.

Calibration from Dupire PDE (from Achdou and Pironneau [2]). Let $C(T,K)$ denote the price of a Call European option with maturity T and strike K. Suppose than an asset has a *local volatility* $\sigma(t,x)$.

The Dupire PDE is

$$\frac{\partial C}{\partial T} - \frac{1}{2}\sigma^2(T,K)K^2\frac{\partial^2 C}{\partial K^2} + r\ K\ \frac{\partial C}{\partial K} = 0$$

with boundary condition

$$C(0,K) = (S_0 - K)_+.$$

Given observed option prices P_j, the choice of a local volatility model may result from a minimizing procedure involving a parameter α and weights ω_j:

$$\min_{\alpha} \left\{ \sum_j \omega_j |C(T_j,K_j) - P_j|^2, \right.$$
$$\left. C \text{ solution of the Dupire PDE governed by } \sigma^2_\alpha(t,x) \right\}.$$

For example, the parameter α may be defined by means of the Fourier decomposition

$$\sigma_\alpha^2(t,x) = \sigma_0 + \sum_\ell \mathrm{Re}(\sigma_\ell e^{2i\pi t})\, e^{-\lambda(x-\rho_\ell)},$$

where σ_0, λ and ρ_j are suitably chosen constants. Numerically, the optimisation procedure may be achieved by using gradient methods. We briefly describe these methods.

Consider a functional J from $\mathbb{R}^N$ to $\mathbb{R}$. When J is lower semicontinuous, bounded from below and coercive (i.e, $\lim_{|x|\to\infty} J(x) = +\infty$), J admits at least one minimum. The minimum is unique when J is strictly convex. Suppose that J is differentiable. Then each minimum a^* satisfies $\nabla J(a^*) = 0$. The standard *Gradient method* is the induction

$$a_{n+1} = a_n - \rho_n \nabla J(a_n),$$

where the weights ρ_n are chosen such that

$$J(a_n - \rho_n \nabla J(a_n))$$

is *as small as possible*: for example, one can solve a new minimization problem at each step n. Conjugate gradient methods are defined as follows. Set

$$a_{n+1} := a_n + \rho_n d_n,$$

where

$$d_n := -\nabla J(a_n) + \gamma_n d_{n-1},$$

and ρ_n minimizes $J(a_n + \rho d_n)$. Possible choices of γ_n are:

$$\text{Fletcher-Reeves: } \gamma_n := \frac{|\nabla J(a_n)|^2}{|\nabla J(a_{n-1})|^2},$$

$$\text{Polak-Ribières: } \gamma_n := \frac{\nabla J(a_n)\cdot(\nabla J(a_n) - \nabla J(a_{n-1}))}{|\nabla J(a_{n-1})|^2},$$

$$\text{Hestenes-Stiefel: } \gamma_n := \frac{\nabla J(a_n)\cdot(\nabla J(a_n) - \nabla J(a_{n-1}))}{d_{n-1}(\nabla J(a_n) - \nabla J(a_{n-1}))}.$$

Avellaneda and Samperi approach for volatility calibration (from Achdou and Pironneau [2]). We briefly describe Avellaneda-Friedman-Holmes-Samperi's approach for the calibration of volatilities (for more details on this approach and other approaches, see, e.g., Avellaneda et al. [6] and the volume edited by Avellaneda [5], and references therein).

Consider an asset whose *volatility process* (σ_t) is progressively measurable and satisfies

$$0 < \underline{\sigma} \leq \sigma_t \leq \overline{\sigma}$$

for some deterministic constants $\underline{\sigma}$ and $\overline{\sigma}$. The set of all such processes is denoted by $\mathcal{H}$.

Suppose that the market is complete and that various European options are priced on the market, all the maturities belonging to the time interval $[0, T]$. Avellaneda's approach for calibration volatility consists in choosing a smooth and *strictly convex* function H defined on $\mathbb{R}_+$ with minimal value 0 at at given value σ_0 (resulting, e.g., from statistics based on historical data), and searching the process (σ_t) which solves

$$\sup_{(\sigma_t)\in\mathcal{H}} -\mathbb{E}^{\sigma}\int_0^T \exp(-r\theta)H((\sigma_\theta)^2)d\theta.$$

Denote the observed option prices by P_k, their maturities by T_k, and their payoff functions by Φ_k. Then set

$$f(\sigma.) := -\mathbb{E}^{\sigma}\int_0^T \exp(-r\theta)H((\sigma_\theta)^2)d\theta,$$
$$g_k(\sigma.) := \mathbb{E}^{\sigma}(\exp(-rT_k)\Phi_k(S_{T_k})).$$

One aims to solve

$$\sup_{(\sigma_t)\in\mathcal{H}} \inf_{\mu_k}(f(\sigma.) + \sum_k \mu_k(g_k(\sigma.) - P_k)).$$

Of course, one has

$$\sup_{(\sigma_t)\in\mathcal{H}} \inf_{\mu_k}(f(\sigma.) + \sum_k \mu_k(g_k(\sigma.) - P_k)) \leq \inf_{\mu_k} \sup_{(\sigma_t)\in\mathcal{H}} (f(\sigma.) + \sum_k \mu_k(g_k(\sigma.) - P_k)).$$

The left hand-side is the *primal problem*, the right hand-side is the *dual problem*. The difference is the *duality gap*. So far, the dual program only is well understood and solved by relying it to a family of stochastic control problems (for fixed μ's), and an optimisation algorithm to minimize w.r.t. μ. The global accuracy of this numerical algorithm seems an open problem.

On El Karoui and Hounkpatin's calibration method. Consider the entropy

$$H(\mathbb{Q},\mathbb{P}) := \int \log\left\{\frac{d\mathbb{Q}}{d\mathbb{P}}\right\} d\mathbb{Q} \text{ if } \mathbb{Q} << \mathbb{P}, \ +\infty \text{ otherwise.}$$

Observe that $H(\mathbb{Q},\mathbb{P})$ is positive, and that $H(\mathbb{Q},\mathbb{P}) = 0$ iff $\mathbb{P} = \mathbb{Q}$. We remind that

$$|\mathbb{P}-\mathbb{Q}|_{\text{Var}} := \int \left|\frac{d\mathbb{P}}{dR} - \frac{d\mathbb{Q}}{dR}\right| dR,$$

where

$$R := \frac{\mathbb{P}+\mathbb{Q}}{2}.$$

Observe also that $\mathbb{P} << R$ and $\mathbb{Q} << R$; if $\mathbb{Q} \sim \mathbb{P}$, then

$$|\mathbb{P}-\mathbb{Q}|_{\text{Var}} = \int \left|\frac{d\mathbb{Q}}{d\mathbb{P}} - 1\right| d\mathbb{P}.$$

In addition, if $\mathbb{Q}$ and $\mathbb{P}$ are *probabilities* such that $\mathbb{Q} << \mathbb{P}$, then

$$|\mathbb{P}-\mathbb{Q}|^2_{\text{Var}} \leq 2H(\mathbb{Q},\mathbb{P}).$$

The following theorem is due to Csiszar [25].

Theorem 3.13. *Let A be a convex set of probabilities on $\mathbb{X}$. Suppose that A is closed for the total variation norm topology, and*

$$\exists \mathbb{Q}_0 \in A, \ H(\mathbb{Q}_0,\mathbb{P}) < \infty.$$

Then there exists a unique $\mathbb{Q}^* \in A$ such that

$$H(\mathbb{Q}^*,\mathbb{P}) = \inf_{\mathbb{Q}\in A} H(\mathbb{Q},\mathbb{P}).$$

In addition, if

$$A := \left\{ \mathbb{Q};\ \int f_i d\mathbb{Q} = a_i \textit{ for all } 1 \leq i \leq N \right\}$$

for some prescribed N, a_i, f_i, *and if* $\exists \mathbb{Q} \in A,\ \mathbb{Q} \sim \mathbb{P}$, *then*

$$\frac{d\mathbb{Q}^*}{d\mathbb{P}} = \frac{\exp(\sum_{i=1}^N \lambda_i^* f_i)}{\int \exp(\sum_{i=1}^N \lambda_i^* f_i) d\mathbb{P}},$$

where λ^* *solves*

$$\max_{\lambda \in \mathbb{R}^N} \left\{ \sum_{i=1}^N \lambda_i a_i - \log \int \exp(\sum_{i=1}^N \lambda_i f_i) d\mathbb{P} \right\}.$$

We may apply Csiszar's theorem to calibrate the probability $\mathbb{Q}$. Indeed, suppose that the asset price solves

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t.$$

For ξ of the form $\phi(X_T)$, set

$$h(t, x, \lambda) := \mathbb{E}^{\mathbb{P}} \left\{ \frac{\exp(\sum_{i=1}^N \lambda_i \xi_T^i)}{\mathbb{E}^{\mathbb{P}} \exp(\sum_{i=1}^N \lambda_i \xi_T^i)} \Big| X_t = x \right\}.$$

Under $\mathbb{Q}^*$, the dynamics of (X_t) is

$$dX_t = (b(t, X_t) + \sigma(t, X_t)^2 \partial_x \log h(t, X_t, \lambda^*))dt + \sigma(t, X_t)dB_t^*,\ t \leq T,$$

where (B_t^*) is a Brownian motion under $\mathbb{Q}^*$, and λ^* solves

$$\max_{\lambda \in \mathbb{R}^N} \left\{ \sum_{i=1}^N \lambda_i \chi_i - \log \mathbb{E}^{\mathbb{P}} \exp(\sum_{i=1}^N \lambda_i \xi_T^i) \right\}.$$

Numerically, a crucial issue concerns the approximation of the unknown λ^*, $h(t, x, \lambda^*)$, $\frac{\partial h}{\partial x}(t, x, \lambda^*)$. So far, this interesting question does not seem to have been addressed in the literature.

4 Approximation of Quantiles of Diffusion Processes

We have just seen that it is a hard issue to calibrate financial models with a good accuracy. Therefore Monte Carlo simulations are commonly used to compute VaR indicators for P & L processes related to misspecified strategies or, more generally, to develop risk analyses. In this section we discuss the accuracy of these simulations when one aims to approximate quantiles of the P & L at maturity. The results come from Talay and Zheng ([66], [65]).

Consider the stochastic differential equation

$$X_t(x) = x + \int_0^t A_0(s, X_s(x))ds + \sum_{i=1}^r \int_0^t A_i(s, X_s(x))dW_s^i,$$

driven by by a r-dimensional Brownian motion (W_s), and the Euler scheme

$$X^n_{(p+1)T/n}(x) = X^n_{pT/n}(x) + A_0(pT/n, X^n_{pT/n}(x))\frac{T}{n} + \sum_{i=1}^r A_i(pT/n, X^n_{pT/n}(x))(W^i_{(p+1)T/n} - W^i_{pT/n}).$$

Here, the functions $A_0, A_1, \dots, A_r$ are smooth functions with bounded derivatives. For technical reasons we consider the perturbed Euler scheme

$$\tilde{X}^n_T(x) = X^n_T(x) + Z^n,$$

We aim to get error estimates for the approximation by the perturbed Euler scheme of the quantile of level δ, $\rho(x,\delta)$, of the law of $X^d_T(x)$. Notice that we constraint themselves to consider the quantile of one component of a diffusion process$(X_t(x))$, and we do not suppose that the Malliavin covariance matrix of $(X_t(x))$ is invertible. Actually, we are motivated by financial situations where $(X^1(t,x),\dots,X^{d-1}(t,x))$ are stock prices (and possibly auxiliary processes involved in interest rate dynamics or stochastic volatilities), and $X^d(t,x)$ is the value at time t of a portfolio whose initial value is x^d. However, we start with the simpler case where the Malliavin covariance matrix of the whole vector $X_T(x)$ has good properties. For a while, consider the system with time homogeneous coefficients:

$$X_t(x) = x + \int_0^t A_0(X_s(x))ds + \sum_{i=1}^r \int_0^t A_i(X_s(x))dW_s^i.$$

For multi-indices $\alpha = (\alpha_1, \ldots, \alpha_k) \in \{0, 1, \ldots r\}^k$ set $A_i^{\emptyset} = A_i$ and, for $0 \leq j \leq r$, $A_i^{(\alpha,j)} := [A_j, A_i^{\alpha}]$. Also set

$$V_L(x, \eta) := \sum_{i=1}^{r} \sum_{|\alpha| \leq L-1} < A_i^{\alpha}(x), \eta >^2,$$

and

$$V_L(x) := 1 \wedge \inf_{\|\eta\|=1} V_L(x, \eta).$$

Suppose

(UH) $C_L := \inf_{x \in \mathbb{R}^d} V_L(x) > 0$ for some integer L,

and

(C) The coefficients $A_i^j,\ i = 0, \ldots, r,\ j = 1, \ldots, d$ are of class $\mathcal{C}_b^{\infty}(\mathbb{R}^d)$ (the A_i^j's may be unbounded).

Under (UH) and (C), the law of $X_T(x)$ has a smooth density $p_T(x, x')$, so that the d-th marginal distribution of $X_T(x)$ also has a smooth density $p_T^d(x, y)$. strictly positive at all point y in the interior of its support (see, e.g., Nualart [52]).

For $0 < \delta < 1$ set

$$\rho(x, \delta) := \inf\{\rho \in \mathbb{R};\ \mathbb{P}[X_T^d(x) \leq \rho] = \delta\}$$

and

$$\tilde{\rho}^n(x, \delta) := \inf\{\rho \in \mathbb{R};\ \mathbb{P}[\tilde{X}_T^{n,d}(x) \leq \rho] = \delta\}.$$

The rate at which the distribution functions of $X_T^n(x)$ converges to the distribution function of $X_T(x)$ is a particular case of the following estimate. Let (f_n) be measurable and bounded functions. Then (cf. Bally and Talay [9] under the above hypotheses, and, for extensions, Gobet and Munos [35] and Kohatsu-Higa [43])

$$\mathbb{E} f_n(X_T(x)) - \mathbb{E} f_n(X_T^n(x)) = -\frac{C_{f_n}(T, x)}{n} + \frac{Q_n(f_n, T, x)}{n^2}, \qquad (7)$$

and

$$|C_{f_n}(T, x)| + \sup_n |Q_n(f_n, T, x)| \leq \frac{K(T)}{T^q}(1 + \|x\|^Q) \sup_n \|f_n\|_{\infty}.$$

This result does not suffice to describe the convergence rates of quantiles of $X_T^n(x)$. However, it allows one to get the next theorem.

Theorem 4.1. *Under Conditions (UH) and (C) we have*

$$|\rho(x,\delta) - \tilde{\rho}^n(x,\delta)| \leq \frac{K(T)}{T^q} \cdot \frac{1 + \|x\|^Q}{\overline{p}_T^d(\rho(x,\delta))} \cdot \frac{1}{n},$$

where

$$\overline{p}_T^d(\rho(x,\delta)) = \inf_{y \in (\rho(x,\delta)-1, \rho(x,\delta)+1)} p_T^d(x,y).$$

We now turn to general stochastic differential equations. Let $(X_s^t(x'),\ 0 \leq s \leq T-t)$ be a smooth version of the flow solution to

$$X_s^t(x') = x' + \int_0^s A_0(t+\theta, X_\theta^t(x'))d\theta + \sum_{i=1}^r \int_0^s A_i(t+\theta, X_\theta^t(x'))dW_{t+\theta}^i.$$

We denote by $M(t,s,x')$ the Malliavin covariance matrix of $X_s^t(x')$.

We now suppose:

(C') The functions $A_i^j,\ i = 0, \ldots, r,\ j = 1, \ldots, d$ are of class $\mathcal{C}_b^\infty([0,T] \times \mathbb{R}^d)$ (the A_i^j's may be unbounded).

and

(M) For all $p \geq 1$ there exist a non decreasing function K, a positive real number r, and a positive Borel measurable function Ψ such that

$$\left\| \frac{1}{M_d^d(t,s,x')} \right\|_p \leq \frac{K(T)}{s^r} \Psi(t,x')$$

for all t in $[0,T)$ and s in $(0,T-t]$. In addition, Ψ satisfies: for all $\lambda \geq 1$, there exists a function Ψ_λ such that

$$\sup_{t \in [0,T]} \mathbb{E}[\Psi(t, X_t(x))^\lambda] < \Psi_\lambda(x),$$

and

$$\sup_{n>0} \sup_{t \in [0,T]} \mathbb{E}[\Psi(t, X_t^n(x))^\lambda] < \Psi_\lambda(x).$$

Under Condition (M), the d-th marginal distribution of $X_T(x)$ has a smooth density $p_T^d(x,y)$ is strictly positive at all point y in the interior of its support.

We have the following error estimate which is analogous to (7): let (f_n) be bounded functions of class $C^\infty(\mathbb{R})$ such that

$$\sup_n \|f_n\|_\infty < \infty.$$

Suppose that Conditions (M) and (C') hold. Then

$$|\mathbb{E}f_n(X_T^d(x)) - \mathbb{E}f_n(X_T^{n,d}(x))| \leq \frac{K(T)}{T^q}(1+\|x\|^Q)\Psi_\lambda(x)\sup_n \|f_n\|_\infty \cdot \frac{1}{n}.$$

Notice that we suppose that f_n's are smooth as in Talay and Tubaro [63] who obtain an expansion of the error. What is new here, and technically demanding, is the control of the error in terms of $\|f\|_\infty$. To obtain (7), the functions f_n were supposed bounded and measurable only because Condition (UH) is much more restrictive than Condition (M). This explains why an expansion might not hold true under Condition (M) only. In spite of the limitation to smooth functions f and an inequality instead of an expansion, the preceding estimate provides the key result to get the desired convergence rate for the approximation of quantiles.

Theorem 4.2. *Under Conditions (M) and (C'), we have*

$$|\rho(x,\delta) - \tilde{\rho}^n(x,\delta)| \leq \frac{K(T)}{T^q} \cdot \frac{1+\|x\|^Q}{\overline{p}_T^d(\rho(x,\delta))} \cdot \Psi_\lambda(x) \cdot \frac{1}{n},$$

where

$$\overline{p}_T^d(\rho(x,\delta)) = \inf_{y\in(\rho(x,\delta)-1,\rho(x,\delta)+1)} p_T^d(x,y).$$

The proof of this theorem is based on the following two key lemmas. The first one results from a Taylor expansion.

Lemma 4.3. *It holds that*

$$\mathbb{E}f\left(X_T^{n,d}(x)\right) - \mathbb{E}f\left(X_T^d(x)\right) = \mathbb{E}f\left(X_T^{n,d}(x)\right) - \mathbb{E}f\left(X_{T/n}^{T-T/n,d}(X_{T-T/n}^n(x))\right) + \frac{T^2}{2n^2}\sum_{k=0}^{n-2}\mathbb{E}\Phi\left(\tfrac{kT}{n}, X_{\frac{kT}{n}}^n(x)\right) + \sum_{k=0}^{n-2} R_k^n,$$

where Φ is a sum of terms, each of them being of the form $\varphi^\flat_\beta(t,x)\partial_\beta u(t,x)$, and R^n_k is a sum of terms, each of them being of the form

$$\mathbb{E}\left[\varphi^\natural_\alpha(kT/n, x^n_{kT/n}(x))\right.$$
$$\left.\int_{kT/n}^{(k+1)T/n}\int_{kT/n}^{s_1}\int_{kT/n}^{s_2}\varphi^\sharp_\alpha(s_3, X^n_{s_3}(x))\partial_\alpha u(s_3, X^n_{s_3}(x))ds_3 ds_2 ds_1\right].$$

The second lemma resembles estimates obtained in Bally and Talay [9] by using the Malliavin integration by parts formula and controling the inverse of the Malliavin covariance matrix of $(X_t(x))$ owing to Condition (UH). In our context, Condition (M) does not allow one to obtain such controls; however, Condition (UH) suffices to integrate by parts derivatives of functions of $X^{t,d}_{T-d}(x')$.

Lemma 4.4. *Set*

$$u(t,x') := \mathbb{E}[f(X^{t,d}_{T-t}(x'))].$$

For all multiindex α whose order w.r.t t is no more than 3, and order w.r.t x is no more than 6, and for any smooth function g with polynomial growth,

$$\forall t\in[0,T],\ |\mathbb{E}[g(X_t(x))\partial_\alpha u(t,X_t(x))]| \le \frac{K(T)}{T^q}(1+\|x\|^Q)\Psi_\lambda(x)\|f\|_\infty$$

and

$$\forall t\in\left[0, T-\frac{T}{n}\right],\ |\mathbb{E}[g(X^n_t(x))\partial_\alpha u(t,X^n_t(x))]| \le \frac{K(T)}{T^q}(1+\|x\|^Q)\Psi_\lambda(x)\|f\|_\infty.$$

Proof. We observe that

$$\mathbb{E}\left[g(X^n_t(x))\partial_\alpha u(t,X^n_t(x))\right] = \mathbb{E}\left[g(X^n_t(x))\left\{\partial_\alpha \mathbb{E} f(X^{t,d}_{T-t}(x'))\right\}|_{x'=X^n_t(x)}\right],$$

and

$$\partial_\alpha \mathbb{E}[f(X^{t,d}_{T-t}(x'))] = \sum_{i=1}^{|\alpha|}\mathbb{E}\left[f^{(i)}\left(X^{t,d}_{T-t}(x')\right)\Theta_i(T-t,x')\right],$$

where $f^{(i)}$ is the i-th order derivative of f, and $\Theta_i(T-t,x')$ are sums of products of $\partial_\beta(X^{t,d}_{T-t}(x'))$ where $|\beta|\le|\alpha|-i+1$. Let $X^{t,d}_{T-t}(X^n_t(x))$ denote the d-th component of the image of $X_t^n(x)$ by the flow $X_\cdot^t$ at time $T-t$, and let $M_d^d(t,T-t;n,x)$ denote the Malliavin covariance of $X_{T-t}^{t,d}(X_t^n(x))$:

$$M_d^d(t,T-t;n,x) :=< D(X_{T-t}^{t,d}(X_t^n(x)), D(X_{T-t}^{t,d}(X_t^n(x)) > .$$

□

We now are in a position to sketch the proof of the theorem 4.2. Using standard inequalities (see, e.g., Nualart [53]) one gets

$$\begin{aligned}
&\left| \mathbb{E}\left[g(X_t^n(x)) \sum_{i=1}^{\alpha} \mathbb{E}\left[f^{(i)}(X_{T-t}^{t,d}(x'))\Theta_i(T-t,x') \right] \right] |_{x'=X_t^n(x)} \right| \\
&\qquad = \left| \mathbb{E}\left[g(X_t^n(x)) \sum_{i=1}^{\alpha} f^{(i)}(X_{T-t}^{t,d}(X_t^n(x)))\Theta_i(T-t,X_t^n(x)) \right] \right| \\
&\qquad \le K(T)(1+\|x\|^Q)\|f\|_\infty \left\| \frac{1}{M_d^d(t,T-t;n,x)} \right\|_k^\ell
\end{aligned}$$

for some integers Q, k and ℓ. As $X_{T-t}^{t,d}(X_t^n(x))$ is a good approximation of $X_T^d(x)$, we can adapt the technique used in Bally and Talay [9] and make use of Condition (M) to obtain the conclusion of the theorem.

We now show examples where the condition (M) is satisfied. We start with the partially strictly elliptic case, which concerns, e.g., some models with stochastic volatilities of the type $\sigma(t,Y_t,S_t)$, where (Y_t) is an auxiliary diffusion process and σ a function bounded from below by a strictly positive constant.

Theorem 4.5. *Suppose that* $\sum_{i=1}^r |A_i^d(t,x)|^2 \ge a > 0$ *for some* t *in* $[0,T]$ *and* x *in* $\mathbb{R}^d$. *Then the d-*th *marginal law of* $X_t(x)$ *has a smooth density and satisfies Condition (M).*

Our second example concerns the case of the VaR a portfolio. Consider the following system, where the $d-1$ first components are stock prices dynamics, and the last one is the value of a self-financing portfolio invested in these stocks.

$$\begin{cases}
X_t^j(x) &= x^j + \int_0^t \sigma_0^j(s,X_s(x))X_s^j(x)ds \\
&\quad + \sum_{i=1}^r \int_0^t \sigma_i^j(s,X_s(x))X_s^j(x)dW_s^i,\ j=1,\dots,d-1, \\
X_t^d(x) &= x^d + \sum_{k=1}^{d-1} \int_0^t \sigma_{0,k}^d(s,X_s(x))X_s^k(x)ds \\
&\quad + \sum_{i=1}^r \sum_{k=1}^{d-1} \int_0^t \sigma_{i,k}^d(s,X_s(x))X_s^k(x)dW_s^i.
\end{cases}$$

Suppose that $\|[\sigma^d_{i,k}(t,x')]\xi\|^2 \geq a\|\xi\|^2$ *for all* ξ *in* $\mathbb{R}^r$, *and* $[x^1,\dots,x^{d-1}] \neq 0$. Then Condition (M) holds true if one uses the 'modified Euler scheme'

$$\begin{cases} X^{n,j}_{(p+1)T/n}(x) & = X^{n,j}_{pT/n}(x)\exp\Big((\sigma^j_0 - \sum_{i=1}^r \frac{1}{2}(\sigma^j_i)^2)(pT/n, X^n_{pT/n}(x))\frac{T}{n} \\ & \quad + \sum_{i=1}^r \sigma^j_i(pT/n, X^n_{pT/n}(x))(W^i_{(p+1)T/n} - W^i_{pT/n})\Big), \\ & \quad j = 1,\dots,d-1, \\ X^{n,d}_{(p+1)T/n}(x) & = X^{n,d}_{pT/n}(x) + \sum_{k=1}^{d-1} \sigma^d_{0,k}(pT/n, X^n_{pT/n}(x))X^{n,k}_{pT/n}(x)\frac{T}{n} \\ & \quad + \sum_{i=1}^r \sum_{k=1}^{d-1} \sigma^d_{i,k}(pT/n, X^n_{pT/n}(x))X^{n,k}_{pT/n}(x) \\ & \quad\ (W^i_{(p+1)T/n} - W^i_{pT/n}). \end{cases}$$

Finally, we examine a model risk problem. The trader desires to hedge a European option $\Phi(B(T_0,T))$ with $T_0 < T$. To hedge the trader uses the bond of maturity T and the bond of maturity T_0. Suppose that the bond prices evolve accordingly to an HJM model governed by a deterministic function $\sigma(t,T)$. The exact hedging strategy is

$$H_t = \frac{\partial \pi_\sigma}{\partial x}(t, x_t(x)),$$

where π_σ solves

$$\begin{cases} \dfrac{\partial \pi_\sigma}{\partial t}(t,x') + \dfrac{1}{2}x'^2(\sigma^*(t,T) - \sigma^*(t,T_0))^2 \dfrac{\partial^2 \pi_\sigma}{\partial x'^2}(t,x') = 0, \\ \pi_\sigma(T,x') = \Phi(x'), \end{cases}$$

where $\sigma^*(t,T) := \int_t^T \sigma(t,r)dr$. Suppose that the trader chooses a deterministic model structure $\overline{\sigma}(t,T)$. Easy calculations show that, for suitable $u_1(s)$, $u_2(s)$, $\varphi(s)$ depending on $\sigma(t,T)$ and $\bar{\sigma}(t,T)$, the forward value of the trader's Profit & Loss is $X^2_t(x^1,x^2)$, where

$$\begin{cases} X^1_t(x^1) & = x^1 + \int_0^t X^1_s(x^1)u_1(s)ds + \int_0^t X^1_s(x^1)u_2(s)dW_s, \\ X^2_t(x^1,x^2) & = x^2 + \int_0^t \varphi(s, X^1_s(x^1))X^1_s(x^1)u_1(s)ds \\ & \quad + \int_0^t \varphi(s, X^1_s(x^1))X^1_s(x^1)u_2(s)dW_s \end{cases} \tag{8}$$

(see Bossy et al. [18]). One can check that, if

$$|\varphi(t,x^1)u_2(t)| \geq a > 0\ \forall t,\ \forall x^1 > 0,$$

then Condition (M) is satisfied.

A lower bound for a marginal density. If the quantile is approximated by a Monte Carlo method with N simulations of the Euler scheme, the global error on the quantile is of order

$$\mathcal{O}\left(\frac{1}{\overline{p}_T^d(\rho(x,\delta)n)}\right) + \mathcal{O}\left(\frac{1}{\tilde{p}_T^{n,d}(x,\rho(x,\delta))\sqrt{N}}\right),$$

where $\tilde{p}_T^{n,d}(x,\xi)$ denotes the density of $\tilde{X}_T^{n,d}(x)$. One has (cf. Bally and Talay [10]) that $\tilde{p}_T^{n,d}(x,\xi) - p_T^d(x,\xi)$ is of order $1/n$. For practical applications, one thus needs accurate a priori estimates from below of $p_T^d(x,\rho(x,\delta))$. Such estimates are available when the generator of (X_t) is strictly uniformly elliptic (see, e.g., Azencott [7]); partial extensions to more general generators have been obtained by Kusuoka and Stroock [45], Bally [8], but the algebraic conditions on the coefficients do not seem to be satisfied by financial models. However financial models for portfolios values have an algebraic structure which one can take advantage of. Let us see an example. For example, suppose that one needs to approximate the quantile at a maturity date T of the P & L process in our preceding example, that is, the second coordinate of the solution to (8). Set

$$\Lambda(t) := \int_0^t u_2^2(s)ds \text{ and } \Upsilon(s,z) := \int_0^z \varphi(\Lambda^{-1}(s),\alpha)d\alpha.$$

Suppose that there exist $a > 0$ and $C > 0$ such that

$$0 \leq |\varphi(t,x)| \leq C \text{ and } |u_2(t)| \geq a > 0,\ \forall t \in [0,T],\ \forall x^1 > 0,$$

and

$$\left| \int_0^{\Lambda(t)} \frac{\partial \Upsilon}{\partial s}(s,z)ds \right| \leq C$$

for all t in $[0,T^O]$ and $z \in \mathbb{R}^+$. Then, for some 'explicit' constant K and all $\rho(x,\delta) > K$, the density of the law of $X_T^2(x)$ satisfies

$$p_T^2(\rho(x,\delta)) \geq \mathbb{E}\left[g_{\Lambda(T)}(\mathrm{H}^{-1}(\rho(x,\delta)))\mathcal{J}(\rho(x,\delta))\right].$$

In the preceding formula, g_ϵ denotes the Gaussian density $N(0,\epsilon)$, and we have set

$$W_t^\Lambda := \sqrt{\Lambda^{-1}(t)}W_{\Lambda^{-1}(t)} \text{ and } h(s,z) := \frac{\partial \Upsilon}{\partial s}(s,z) + \frac{1}{2}\frac{\partial \varphi}{\partial z}(\Lambda^{-1}(s),z),$$

and

$$H(x,z,\omega) := x^2 - \Upsilon(0,x^1) + \Upsilon\left(\Lambda(t), x^1 \exp\left(\overline{U}_t + z\right)\right)$$
$$- \int_0^{\Lambda(t)} h\left(s, x^1 \exp\left(\overline{U}_s + \tilde{W}_s^{\Lambda} - \frac{s}{\Lambda(t)}\tilde{W}^{\Lambda}_{\Lambda(t)} + \frac{z\, s}{\Lambda(t)}\right)\right) ds,$$

where $\mathcal{J}$ is the Jacobian matrix of $\mathrm{H}^{-1}(x,\cdot,\omega)$.

We conclude this subsection by emphasizing the importance of methods such as Kohatsu-Higa and Petterson's method [44] to reduce the variance of Monte Carlo simulations to approximate densities of diffusion processes.

5 Artificial Boundary Conditions

In this section we describe a numerical source of model risk. Obviously, the numerical resolution of partial differential equations, or the time discretization of stochastic dynamics combined with the use of Monte Carlo simulations, lead to erroneous approximations of prices and hedgges. We do not discuss here this issue which would deserve a long expository. We focus on a numerical feature which is not so much investigated in the literature so far: the uncertainties on the strategies due to the unavoidable introduction of artificial boundary conditions cannot be avoided.

In practice, the construction of strategies uses numerical methods, particularly discretization methods for PDEs which are, e.g., the parabolic PDEs for European options, the variational inequalities for American options, the Hamilton-Jacobi-Bellman equations for portfolio management, etc. Since the price models usually are processes which evolve in unbounded domains, the integration domains of these PDEs are also unbounded. However, the numerical resolution requires to introduce bounded integration domains and to choose suitable Dirichlet or Neumann artificial boundary conditions. These artificial boundary conditions cannot be exact since the exact solutions in the whole domains are unknown. Therefore the misspecification of these boundary conditions induce inaccurate option prices or sub-optimal strategies. For linear PDEs and European options, see Lamberton and Lapeyre [47], Crépey [24]. More involved estimates can be obtained from Costantini et al.'s results [23]. In this section we are concerned by variational inequalities and American options.

In the sequel, the time origin t is arbitrary in $[0,T]$ and $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ with a smooth boundary.

Consider a d-dimensional Brownian motion $(W_s, s \geq 0)$ on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with the augmented natural filtration $(\mathcal{F}_s, s \geq 0)$ of (W_s), and the following spaces of random variables and processes:

$$\begin{aligned}
\mathcal{L}^2 &:= \{\xi \text{ is } \mathcal{F}_T - \text{measurable and } \mathbb{E}|\xi|^2 < \infty\},\\
\mathcal{S}^2 &:= \{(\psi_s, 0 \leq s \leq T) \text{ is a progressively measurable process s.t.}\\
&\qquad \mathbb{E} \sup_{0 \leq s \leq T} |\psi_s|^2 < \infty\},\\
\mathcal{H}^2 &:= \{(\psi_s, 0 \leq s \leq T) \text{ is a progressively measurable process s.t.}\\
&\qquad \mathbb{E} \int_0^T |\psi_s|^2 ds < \infty\}.
\end{aligned}$$

A laboratory example for nonlinear PDEs. Given the infinitesimal generator A_t of a diffusion process, consider the variational inequality

$$\begin{cases}
\min\Big(\tilde{u}(t,x) - L(t,x);\\
\qquad -\dfrac{\partial \tilde{u}}{\partial t}(t,x) - A_t \tilde{u}(t,x) - f(t,x,\tilde{u}(t,x),(\nabla \tilde{u} \sigma)(t,x))\Big) = 0,\\
\hfill (t,x) \in [0,T) \times \mathbb{R}^d,\\
\tilde{u}(T,x) = \phi(x) \text{ for all } x \in \mathbb{R}^d.
\end{cases} \tag{9}$$

Notice that this equation extends the variational inequalities for American option prices

$$\begin{cases}
\min\left\{v(t,x) - \phi(t,x); -\dfrac{\partial v}{\partial t}(t,x) - A_t v(t,x) - rv(t,x)\right\} = 0,\\
\hfill (t,x) \in [0,T) \times \mathbb{R},\\
v(T,x) = \phi(T,x),\ x \in \mathbb{R},
\end{cases}$$

for which ϕ is the payoff function, A is the infinitesimal generator of the stock price, and r is the instantaneous interest rate.

We localize (9) and choose unhomogeneous Neumann boundary conditions:

$$\begin{cases} \left.\begin{aligned} &\min\{u(t,x)-L(t,x);\\ &\qquad -\frac{\partial u}{\partial t}(t,x)-A_t u(t,x)-f(t,x,u(t,x),(\nabla u\sigma)(t,x))\end{aligned}\right\}=0,\\ \hfill (t,x)\in[0,T)\times\mathcal{O},\\ u(T,x)=\phi(x),\ x\in\overline{\mathcal{O}},\\ (\nabla u(t,x);n(x))+g(t,x)=0,\ (t,x)\in[0,T)\times\partial\mathcal{O}, \end{cases} \tag{10}$$

where, for all x in $\partial\mathcal{O}$, $n(x)$ denotes the inward unit normal vector at point x.

We aim to construct a Reflected Backward Stochastic Differential Equation (RBSDE) coupled with a reflected forward SDE, and to show that the solution of the RBSDE provides the unique viscosity solution $u(t,x)$ of (10); the RBSDE will also allow us to estimate the localization error $|u(t,x)-\tilde{u}(t,x)|$.

We start with recalling the definition of a continuous viscosity solution for nonlinear equations of the type

$$(\mathsf{P}) \qquad F(t,x,v(t,x),D_t v(t,x),Dv(t,x),D^2v(t,x))=0$$

Definition 5.1.

- *The continuous function v is a viscosity sub-solution of (P) if*

  $$F(\bar{t},\bar{x},v(\bar{t},\bar{x}),D_t\varphi(\bar{t},\bar{x}),D\varphi(\bar{t},\bar{x}),D^2\varphi(\bar{t},\bar{x}))\le 0$$

  for all $(\bar{t},\bar{x})$ and all functions φ $C^{1,2}$ such that $(\bar{t},\bar{x})$ is a local maximum of $v-\varphi$.

- *A continuous function v is a viscosity super-solution of (P) if*

  $$F(\bar{t},\bar{x},v(\bar{t},\bar{x}),D_t\varphi(\bar{t},\bar{x}),D\varphi(\bar{t},\bar{x}),D^2\varphi(\bar{t},\bar{x}))\ge 0$$

  for all $(\bar{t},\bar{x})$ and all functions φ $C^{1,2}$ such that $(\bar{t},\bar{x})$ is a local minimum of $v-\varphi$.

- *A continuous function v is a viscosity solution of (P) if it is both a viscosity sub and super-solution.*

For variational inequalities (9) viscosity sub-solutions are defined as follows. A function $u(t,x)$ in $\mathcal{C}([0,T)\times\overline{\mathcal{O}})$ is a viscosity sub-solution if $u(T,x)\le$

$\phi(x)$ for all x in $\bar{\mathcal{O}}$ and, for all function φ in $\mathcal{C}^{1,2}([0,T] \times \overline{\mathcal{O}})$ such that (t,x) is a global maximum of $u - \varphi$ one has, for all $(t,x) \in [0,T] \times \mathcal{O}$,

$$\begin{aligned}
&\min \{u(t,x) - L(t,x); \\
&\qquad \left. -\frac{\partial \varphi}{\partial t}(t,x) - A_t\varphi(t,x) - f(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)) \right\} \leq 0,
\end{aligned}$$

and, for all $(t,x) \in [0,T] \times \partial\mathcal{O}$,

$$\begin{aligned}
&\min \{-(\nabla\varphi(t,x); n(x)) - g(t,x); \\
&\qquad\qquad \min(u(t,x) - L(t,x); \\
&\qquad \left. -\frac{\partial \varphi}{\partial t}(t,x) - A_t\varphi(t,x) - f(t,x,u(t,x),(\nabla\varphi\sigma)(t,x)) \right\} \leq 0.
\end{aligned}$$

Viscosity super-solutions are defined analogously, and viscosity solutions are both viscosity sub-solutions and super-solutions. Viscosity solutions of quasi-linear or semi-linear parabolic and elliptic problems have natural probabilistic interpretations in terms of Backward Stochastic Differential Equations (BSDEs): for a nice introduction to the subject, we advise the reader to study Pardoux's course [54].

Reflected BSDEs with non reflected forward SDEs. Consider the forward stochastic differential equation

$$X_s^{t,x} = x + \int_t^s b(\theta, X_\theta^{t,x})d\theta + \int_t^s \sigma(\theta, X_\theta^{t,x})dW_\theta, \; 0 \leq t \leq s \leq T,$$

where b is a continuous function from $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}^d$ and σ is a continuous function from $[0,T] \times \mathbb{R}^d$ to $\mathbb{R}^{d \times d}$. Both b and σ are supposed Lipschitz w.r.t. the x coordinates.

Consider a continuous function L satisfying

$$\begin{gathered}
L(t,x) \leq K(1 + |x|^p), \; t \in [0,T], \; x \in \mathbb{R}^d, \\
L(T,x) \leq \phi(x), \; x \in \mathbb{R}^d,
\end{gathered}$$

and the BSDE with reflection on the obstacle $(L(s, X_s^{t,x}))$:

$$\begin{cases} Y_s^{t,x} = \phi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + R_T^{t,x} - R_s^{t,x} - \int_s^T Z_r^{t,x} dW_r, \\ Y_s^{t,x} \geq L(s, X_s^{t,x}),\ 0 \leq t \leq s \leq T, \\ (R_s^{t,x}, 0 \leq t \leq s \leq T) \text{ is a continuous increasing process such that} \\ \int_t^T (Y_s^{t,x} - L(s, X_s^{t,x}))dR_s^{t,x} = 0. \end{cases}$$

Suppose that ϕ is a continuous function with an at most polynomial growth at infinity, f is a continuous function satisfying: there exist $K > 0$ and $p \in \mathbb{N}$ such that

$$|f(t, x, 0, 0)| \leq K(1 + |x|^p),$$

and

$$|f(t, x, y, z) - f(t, x, y', z')| \leq K(|y - y'| + |z - z'|)$$

for all $t \in [0, T]$, x, z, z' in $\mathbb{R}^d$, and y, y' in $\mathbb{R}$. El Karoui et al. [29] have shown existence and uniqueness of the triple $(Y_s^{t,x}, Z_s^{t,x}, R_s^{t,x})$ of progressively measurable processes with $Y^{t,x}$ in $\mathcal{S}^2$, $Z^{t,x}$ in $\mathcal{H}^2$, and K_T in $\mathcal{L}^2$. The authors also show that $\tilde{u}(t, x) := Y_t^{t,x}$ is the unique viscosity solution of the variational inequality (9) where A is the infinitesimal generator of the solution of the forward SDE, and have applied this result to represent American option prices.

One can prove that the same result still holds true when Lipschitz conditions on f have been replaced by monotonicity conditions: for all t, x, y_1, y_2 ,z,

$$\begin{cases} |f(t, x, 0, 0)| \leq K(1 + |x|^p),\ t \in [0, T], \\ \exists \gamma \in \mathbb{R},\ (y_1 - y_2)(f(t, x, y_1, z) - f(t, x, y_2, z)) \leq \gamma |y_1 - y_2|^2 \end{cases}$$

(see [14]). Proceeding as in Ma and Cvitanić [50], one can readily get:

$$|\tilde{Y}_{t_1}^{t_1,x_1} - \tilde{Y}_{t_2}^{t_2,x_2}|^2 \leq C(|x_1 - x_2|^2 + t_2 - t_1)$$

for all x_1, x_2 in $\mathbb{R}^d$ and $t \leq t_1 \leq t_2 \leq T$.

Non reflected BSDEs with reflected forward SDEs. Now consider the forward reflected SDE with generator still denoted by A:

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(\theta, X_\theta^{t,x})d\theta + \int_t^s \sigma(\theta, X_\theta^{t,x})dW_\theta + \eta_s^{t,x},\ 0 \leq t \leq s \leq T, \\ \eta_s = \int_t^s n(X_\theta^{t,x})d|\eta|_\theta^{t,x} \text{ with } |\eta|_s^{t,x} = \int_t^s \mathbb{I}_{\{X_\theta^{t,x} \in \partial\mathcal{O}\}} d|\eta|_\theta^{t,x}. \end{cases}$$

Thus, (η_t) is an increasing process which increases only when (X_t) hits the boundary $\mathcal{O}$. When the functions b and σ and the boundary $\mathcal{O}$ are smooth, then there is a unique strong solution to the preceding equation: see, e.g., Menaldi [51], Lions and Sznitman [48].

Let $\underline{\phi}$ be a continuous function from $\mathcal{O}$ (which we continue to suppose bounded) to $\mathbb{R}^p$, and suppose: there exists $\beta > 0$ such that

$$(y - y', g(t, x, y, z) - g(t, x, y', z)) \le \beta |y - y'|^2,$$

for all $t \in [0, T]$, $x \in \bar{\mathcal{O}}$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$.

Pardoux and Zhang [55] have proven that there exists a unique pair $(Y^{t,x}, Y^{t,x})$ of progressively measurable processes taking values in $\mathbb{R} \times \mathbb{R}^d$ and satisfying

$$\mathbb{E}\left(\sup_{t \le s \le T} |Y_s^{t,x}|^2 + \int_t^T |Z_s^{t,x}|^2 ds\right) < \infty,$$

to the BSDE

$$\begin{aligned} Y_s^{t,x} = \underline{\phi}(X_T^{t,x}) &+ \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r^{t,x} dW_r \\ &+ \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}) d\eta_r^{t,x},\ 0 \le t \le s \le T. \end{aligned}$$

In addition, the authors have shown that

$$u(t, x) := Y_t^{t,x} \text{ for all } (t, x) \in [0, T] \times \mathcal{O},$$

is a viscosity solution of the quasi-linear PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) + Au(t, x) + f(t, x, u(t, x), \nabla u \sigma(t, x)) = 0,\ (t, x) \in [0, T) \times \mathcal{O}, \\ u(T, x) = \underline{\phi}(x),\ x \in \bar{\mathcal{O}}, \\ \dfrac{\partial u}{\partial n}(t, x) + g(t, x, u(t, x)) = 0,\ (t, x) \in [0, T) \times \partial\mathcal{O}. \end{cases}$$

For uniqueness of the viscosity solution, see Barles [12].

Reflected BSDEs with reflected forward SDEs. Now consider a forward reflected SDE and the unhomogeneous reflected BSDE with reflected forward

SDE

$$\begin{cases} Y_s^{t,x} = \phi(X_T^{t,x}) + \int_s^T f(\theta, X_\theta^{t,x}, Y_\theta^{t,x}, Z_\theta^{t,x})d\theta - \int_s^T (Z_\theta^{t,x}; dW_\theta) \\ \qquad\qquad + R_T^{t,x} - R_s^{t,x} + \int_s^T g(\theta, X_\theta^{t,x}) d|\eta|_\theta^{t,x},\ t \le s \le T, \\ Y_s^{t,x} \ge L(s, X_s^{t,x}),\ t \le s \le T, \\ (R_s^{t,x}, t \le s \le T) \text{ is an increasing continuous process s.t.} \\ \int_t^T (Y_\theta^{t,x} - L(\theta, X_\theta^{t,x}))dR_\theta^{t,x} = 0. \end{cases}$$

We make the same assumptions as above on ϕ, g, f and $L(s, x)$. Berthelot et al. [14] have shown that for all $0 \le t \le T$ there exists a unique triple $(Y_s^{t,x}, Z_s^{t,x}, R_s^{t,x}, t \le s \le T)$ of progressively measurable processes which solves the preceding BSDE, and that the function $u(t, x) := Y_t^{t,x}$ is a viscosity solution of the localized PDE with Neumann boundary conditions.

Let $\tilde{u}$ and u be as above. From the smoothness property and Rademacher's theorem we know that $\tilde{u}$ is differentiable almost everywhere. Suppose that one can find a smooth boundary $\partial\mathcal{O}$ such that $\nabla\tilde{u}(t, \cdot)$ is a continuous function.

Then $v(t, x) := \tilde{u}_{|\mathcal{O}}(t, x)$ is the unique viscosity solution of

$$\begin{cases} \min\Big\{ v(t,x) - L(t,x); -\dfrac{\partial v}{\partial t}(t,x) \\ \qquad -A_t v(t,x) - f(t,x,v(t,x),(\nabla v\sigma)(t,x))\} = 0,\ (t,x) \in [0,T) \times \mathcal{O}, \\ v(T,x) = \phi(x),\ x \in \overline{\mathcal{O}}, \\ (\nabla v(t,x); n(x)) = (\nabla\tilde{u}(t,x); n(x)),\ (t,x) \in [0,T) \times \partial\mathcal{O}. \end{cases}$$

Under the above conditions and under the above assumption on $\partial\mathcal{O}$, there exists $C > 0$ such that, for all $0 \le t \le T$ and $x \in \mathcal{O}$,

$$\begin{aligned} &|u(t,x) - \tilde{u}(t,x)| \\ &\qquad \le C \left\{ \mathbb{E} \max_{t \le s \le T} \left| g(s, X_s^{t,x}) - (\nabla\tilde{u}(s, X_s^{t,x}); n(X_s^{t,x})) \right|^4 \mathbb{I}_{[X_s^{t,x} \in \partial\mathcal{O}]} \right\}^{1/4}. \end{aligned}$$

In a recent paper, Bossy et al. [17] have extended the above analysis in the one-dimensional case ($d = 1$), and got estimates on $|\frac{\partial u}{\partial x}(t,x) - \frac{\partial \tilde{u}}{\partial x}(t,x)|$, and therefore on hedging portfolios for American options subject to model risk due to misspecified boundary conditions.

6 A Stochastic Game to Face Model Risk

After having described various sources of model risk, we now examine two extreme ways to manage this risk: in our last section we will study the tools from technical analysis which avoids to choose a model; here we develop a worst case analysis.

Cvitanić and Karatzas [26] have studied the following dynamic measure of risks due to misspecifications on stock appreciation rates:

$$\inf_{\pi(\cdot)\in\mathcal{A}(x)} \sup_{\nu\in\mathcal{D}} \mathbb{E}_{\nu}(F(X^{x,\pi}(T))),$$

where $\mathcal{A}(x)$ denotes the class of admissible portfolio strategies issued from the initial wealth x, and $\mathbb{E}_{\nu}$ denotes the expectation under the probability $\mathbb{P}_{\nu}$ for all ν in a suitable set. All the measures $\mathbb{P}_{\nu}$ are equivalent to the same risk-neutral martingale measure. Gao et al. [32] have developed numerical algorithms to compute these measure of risks.

Another rsik measure appears in Cont [22], namely, a coherent risk measure compatible with market prices of derivatives. Here we present a different approach which allows one to compute, by means of the numerical resolution of a stochastic game PDE, the minimal amount of money that the financial institution needs to contain the worst possible damage due to model uncertainty.

Consider the market

$$\begin{cases} dS_t^i & = S_t^i[b_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j] \text{ for } 0 \le i \le n, \\ \\ dP_t & = P_t \sum_{i=1}^n \pi_t^i \left[b_t^i dt + \sum_{j=1}^d \sigma_t^{ij} dW_t^j \right] + r P_t \left(1 - \sum_{i=1}^n \pi_t^i\right) dt. \end{cases}$$

Here, the S^i's denote stock prices, the π^i's are prescribed strategies, and (P_t) is the value of a portfolio invested in the stocks under consideration and in a bank account with dterministic rate r. In order to take into account that the calibration methods for b and σ are erroneous, and in order to develop a worst case analysis, we consider $u(\cdot) := (b(\cdot), \sigma(\cdot))$ as the market's control process.

The trader acts as a minimizer of the risk; on the other hand, the market is assumed to systematically behave against the interest of the trader, and acts as a maximizer of the risk. Thus the model risk control problem can be set up as a two players (Trader versus Market) zero-sum stochastic differential game problem. Notice that we include model risk on volatilities, stock appreciation rates, yield curves, etc.

Given a suitable function F the cost function for this game is

$$J(t,x,p,\Pi,u(\cdot)) := \mathbb{E}_{t,x,p} F(S_T, P_T),$$

and the value function is

$$V(t,x,p) := \inf_{\Pi \in Ad_\Pi(t)} \sup_{u(\cdot) \in Ad_u(t)} J(t,x,p,\Pi,u(\cdot)).$$

The next theorem comes from Talay and Zheng [64].

Theorem 6.1. *Under an appropriate locally Lipschitz condition on F, the value function $V(t,x,p)$ is the unique viscosity solution in the space*

$$\begin{aligned} S := \{ &\varphi(t,x,p) \textit{ is continuous on } [0,T] \times \mathbb{R}^n \times \mathbb{R}\textit{; } \exists \overline{A} > 0, \\ &\lim_{|p|^2 + x^2 \to \infty} \varphi(t,x,p) \exp(-\overline{A} |\log(|p|^2 + x^2)|^2) = 0 \textit{ for all } t \in [0,T] \} \end{aligned}$$

to the Hamilton-Jacobi-Bellman-Isaacs equation

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x,p) + \mathcal{H}^-(D^2 v(t,x,p), Dv(t,x,p), x, p) = 0 \ \ \textit{in } [0,T) \times \mathbb{R}^{n+1}, \\ \\ v(T,x,p) = F(x,p), \end{cases}$$

where

$$\mathcal{H}^-(A,z,x,p) := \max_{u \in K_u} \min_{\pi \in K_\pi} \left[\frac{1}{2} \mathit{Tr}\,(a(x,p,\sigma,\pi)A) + z \cdot q(x,p,b,\pi) \right].$$

Sketch of the proof. We start with proving the existence of a viscosity solution. If the controlled system had bounded coefficients and F were a bounded Lipschitz function, the theorem would result from Fleming and Souganidis [30]. Here we need to use localization techniques.

Set $B_k := \{(p,x) \in \mathbb{R}^{n+1},\ |p|^2 + x^2 < k^2\}$. Choose a function ϕ_k in $\mathcal{C}_b^\infty(\mathbb{R}^{n+1})$ such that $\phi_k(p,x) = 1$ on B_k, and $\phi_k(p,x) = 0$ outside B_{k+1}, and the Lipschitz constant of ϕ_k is less than 2. Multiply F and all the coefficients of the SDE by ϕ_k. You get a new SDE with solution $({}^k S_t, {}^k W_t)$, a new cost function, a new value function V^k, a new HJBI equation. Apply Fleming and Souganidis's results, and then use the linear structure of the SDE to obtain

$$\bar{V}^k(t,p,x) = \inf_{\Pi \in Ad_\Pi(\theta)} \sup_{u(\cdot)\in Ad_u(\theta)} \mathbb{E}_{\theta,p,x} F({}^kS_T, {}^kW_T).$$

Lions'stability lemma for viscosity solutions allows one to conclude.

To prove uniqueness, we adapt a result and a proof designed by Barles, Buckdahn and Pardoux [11] for other families of PDEs. The objective is to show the following comparison result: suppose that there exist a viscosity subsolution $v(t,p,x)$ and a viscosity supersolution $w(t,p,x)$ to such that

$$\lim_{|p|^2+x^2\to\infty} v(t,p,x)\exp(-\overline{A}|\log(|p|^2+x^2)|^2) = 0 \text{ for all } t \in [0,T],$$

and

$$\lim_{|p|^2+x^2\to\infty} w(t,p,x)\exp(-\overline{A}|\log(|p|^2+x^2)|^2) = 0 \text{ for all } t \in [0,T]$$

for some $\overline{A} > 0$; suppose that $v(T,p,x) \le w(T,p,x)$ for all $(p,x) \in \mathbb{R}^{n+1}$; then $v(t,p,x) \le w(t,p,x)$ for all $(t,p,x) \in [0,T] \times \mathbb{R}^{n+1}$.

The key step consists in proving that, if $v(t,p,x)$ be a viscosity subsolution and $w(t,p,x)$ a viscosity supersolution, then $\overline{v} := v - w$ is a viscosity subsolution to

$$\frac{\partial v}{\partial t}(t,p,x) + \mathcal{H}^+(D^2v(t,p,x), Dv(t,p,x), p, x) = 0 \text{ in } [0,T) \times \mathbb{R}^{n+1},$$

where

$$\mathcal{H}^+(A,z,p,x) := \max_{u=(b,\sigma)\in K_u} \max_{\pi\in K_\pi} \left(\frac{1}{2}\mathrm{Tr}\left(a(p,x,\sigma,\pi)A\right) + z \cdot q(p,x,b,\pi) \right),$$

for all $(n+1)\times(n+1)$ symmetric matrix A and all vector z in $\mathbb{R}^{n+1}$.

7 When One Does Not Control Model Risk

In the financial industry one follows the fundamental approach, where strategies are based on fundamental economic principles, the technical analysis approach, where strategies are based on past prices behavior, the mathematical approach, where strategies are based on mathematical models.

The main advantage of technical analysis is that it avoids model specification. On the other hand, technical analysis techniques have limited theoretical justifications. See Achelis [1] for a presentation of technical analysis indices and rules, and [58], [61] for mathematical studies. It might be useful to compare the performances obtained by using erroneously calibrated mathematical models and the performances obtained by techniques issued from technical analysis.

Our purpose is to present the mathematical complexity of this question, and some preliminary results issued from Blanchet et al. [15].

Here we consider the case of an asset whose instantaneous expected rate of return changes at an unknown random time. We compare the performances of traders who have a logarithmic utility function and respectively use: a strategy which is optimal when the model is perfectly specified and calibrated; strategies derived from statistics of random processes; an investment strategy based on technical analysis.

The *real* market consists in two assets whose prices obey the following dynamics:

$$\begin{cases} dS_t^0 &= S_t^0 r dt, \\ dS_t &= S_t \left(\mu_2 + (\mu_1 - \mu_2) \mathbb{I}_{(t \le \tau)} \right) dt + \sigma S_t dB_t. \end{cases}$$

Notice that

$$S_t = S^0 \exp\left(\sigma B_t + (\mu_1 - \frac{\sigma^2}{2})t + (\mu_2 - \mu_1) \int_0^t \mathbb{I}_{(\tau \le s)} ds \right) =: S^0 \exp(R_t),$$

where the process $(R_t)_{t\ge 0}$ is defined as

$$R_t = \sigma B_t + \left(\mu_1 - \frac{\sigma^2}{2} \right) t + (\mu_2 - \mu_1) \int_0^t \mathbb{I}_{(\tau \le s)} ds.$$

This model was considered by Shiryaev ([60], [61], [62]) who studied the problem of detecting the change time τ as early and reliably as possible when one only observes the process $(S_t)_{t\ge 0}$.

Let $\mathcal{F}_t^S := \sigma\left(S_u, 0 \le u \le t\right)$ be the σ algebra generated by the observations, $(B_t)_{t\ge 0}$ be a Brownian motion independent of the time of change τ whose law is the exponential law with parameter λ, and W_t be the value of the portfolio at time t. Suppose:

$$\mu_1 - \frac{\sigma^2}{2} < r < \mu_2 - \frac{\sigma^2}{2}.$$

The technical analysis strategy. Consider the time partition

$$0 = t_0 < t_1 < \ldots < t_N = T, \quad t_n = n\Delta t.$$

Denote by $\pi_t \in \{0, 1\}$ the proportion of the agent's wealth invested in the risky asset at time t, and by M_t^δ the moving average indicator of the prices. Therefore,

$$M_t^\delta = \frac{1}{\delta} \int_{t-\delta}^{t} S_u \, du.$$

At time 0, the agent knows the history before time 0 and has enough data to compute M_0^δ. As proposed by technical analysis, at each t_n, $n \in [1 \cdots N]$, the agent invests all his/her wealth into the risky asset if $S_{t_n} > M_{t_n}^\delta$. Otherwise, he/she invests all the wealth into the riskless asset. Consequently,

$$\pi_{t_n} = \mathbb{I}_{\left(S_{t_n} \geq M_{t_n}^\delta\right)}.$$

The technical analylist's wealth at time t_{n+1} is

$$W_{t_{n+1}} = W_{t_n} \left(\frac{S_{t_{n+1}}}{S_{t_n}} \pi_{t_n} + \frac{S_{t_{n+1}}^0}{S_{t_n}^0} (1 - \pi_{t_n}) \right),$$

from which ($W_0 = x$)

$$W_T = x \prod_{n=0}^{N-1} \left[\pi_{t_n} \big(\exp(R_{t_{n+1}} - R_{t_n}) - \exp(r\Delta t) \big) + \exp(r\Delta t) \right],$$

and

$$\begin{aligned}
\mathbb{E} \log(W_T) &= \log(x) + rT + \left(\mu_2 - \frac{\sigma^2}{2} - r \right) T p_\delta^{(1)} \\
&\quad + \Delta t \left(\mu_2 - \frac{\sigma^2}{2} - r \right) \frac{1 - e^{-\lambda T}}{1 - e^{-\lambda \Delta t}} \left((p_\delta^{(2)} - p_\delta^{(1)}) e^{\lambda\delta} + p_\delta^{(3)} \right) \\
&\quad - \Delta t (\mu_2 - \mu_1)(e^{-\lambda\Delta t} - \lambda \Delta t) \frac{1 - e^{-\lambda T}}{1 - e^{-\lambda\Delta t}} p_\delta^{(3)},
\end{aligned}$$

where

$$
\begin{aligned}
p_\delta^{(1)} &= \int_0^\infty \int_y^\infty \frac{z^{\mu_2-3/2}}{2y} e^{-\frac{(\mu_2/\sigma-\sigma/2)^2\delta}{2}-\frac{(1+z^2)}{2\sigma^2 y}} i_{\sigma^2\delta/2}\left(\frac{z}{\sigma^2 y}\right) dzdy, \\
p_\delta^{(2)} &= \int_0^\delta \int_{\mathrm{R}^4} \mathbb{I}_{\left\{\delta y_2 \geq \frac{z_1}{y_1}+z_2\right\}} \frac{z_2^{\mu_2-3/2}}{2y_2} e^{-\frac{(\mu_2/\sigma-\sigma/2)^2(\delta-v)}{2}-\frac{(1+z_2^2)}{2\sigma^2 y_2}} \\
&\quad i_{\sigma^2(\delta-v)/2}\left(\frac{z_2}{\sigma^2 y_2}\right) \frac{z_1^{\mu_1-3/2}}{2y_1} e^{-\frac{(\mu_1/\sigma-\sigma/2)^2 v}{2}-\frac{(1+z_1^2)}{2\sigma^2 y_1}} i_{\sigma^2 v/2}\left(\frac{z_1}{\sigma^2 y_1}\right) \\
&\quad e^{-\lambda v} dy_1 dz_1 dy_2 dz_2 dv, \\
p_\delta^{(3)} &= \int_0^\infty \int_y^\infty \frac{z^{\mu_1-3/2}}{2y} e^{-\frac{(\mu_1/\sigma-\sigma/2)^2\delta}{2}-\frac{(1+z^2)}{2\sigma^2 y}} i_{\sigma^2\delta/2}\left(\frac{z}{\sigma^2 y}\right) dzdy, \\
i_y(z) &= \frac{z e^{\pi^2/4y}}{\pi\sqrt{\pi y}} \int_0^\infty e^{-z\cosh(u)-u^2/4y} \sinh(u)\sin(\pi u/2y)du.
\end{aligned}
$$

The tedious calculation involves an explicit formula, due to Yor [67] for the density of $(\int_0^t \exp(2B_s)ds, B_t)$:

Theorem 7.1. *Let B be a real Brownian Motion. Let $\sigma > 0$ and ν be in $\mathbb{R}$. Let V be a geometric Brownian Motion:*

$$
V_s = e^{\sigma^2\nu s+\sigma B_s}.
$$

Then

$$
\mathbb{P}\left(\int_0^t V_s ds \in dy\, ;\, V_t \in dz\right) = \frac{z^{\nu-1}}{2y} e^{-\frac{\nu^2\sigma^2 t}{2}-\frac{(1+z^2)}{2\sigma^2 y}} i_{\frac{\sigma^2 t}{2}}\left(\frac{z}{\sigma^2 y}\right) dydz, \tag{11}
$$

where

$$
i_y(z) := \frac{z e^{\pi^2/4y}}{\pi\sqrt{\pi y}} \int_0^\infty e^{-z\cosh u-u^2/4y} \sinh u \sin(\pi u/2y)du.
$$

Thus the law of $\Phi_t := \frac{F_t}{1-F_t}$ is explicitly known since

$$
\begin{aligned}
\Phi_t = \lambda \exp\left(\frac{\mu_2-\mu_1}{\sigma}\tilde{B}_t + \left(\lambda - \frac{(\mu_2-\mu_1)^2}{\sigma^2}\right)t\right) \\
\times \int_0^t \exp\left(-\frac{\mu_2-\mu_1}{\sigma}\tilde{B}_u - \left(\lambda - \frac{(\mu_2-\mu_1)^2}{\sigma^2}\right)u\right)du.
\end{aligned}
$$

The optimal portfolio allocation strategy. Our aim is to explicit the optimal wealth and strategy of a trader who perfectly knows the parameters μ_1, μ_2, λ and σ, and thus can get optimal financial performances. We impose constraints: a technical analyst is only allowed to invest all his/her wealth in the stock or the bond. Therefore the proportions of the trader's wealth invested in the stock are constrained to lie within the interval $[0, 1]$.

We use the martingale approach developed by Karatzas, Shreve, Cvitanić, etc. Notice that the drift coefficient of the dynamics of the risky asset is not constant over time (since it changes at the random time τ). Notice also that we must face some subtle measurability issues since the trader's strategy needs to be adapted with respect to the filtration generated by (S_t) which, because of τ, is different from the filtration generated by (B_t).

Let π_t be the proportion of the trader's wealth invested in the stock at time t; $W_\cdot^{x,\pi}$ denotes the corresponding wealth process. Let $\mathcal{A}(x)$ denote the set of admissible strategies, that is,

$$\begin{aligned}\mathcal{A}(x) := \{\pi_\cdot - \mathcal{F}_t^S - &\text{progressively measurable process s.t.}\\ &W_0^{x,\pi} = x,\ W_t^{x,\pi} > 0 \text{ for all } t > 0,\ \pi_\cdot \in [0,1]\}.\end{aligned}$$

The value function thus is

$$V(x) := \sup_{\pi_\cdot \in \mathcal{A}(x)} \mathbb{E}\, U(W_T^\pi).$$

As in Karatzas and Shreve [42], we introduce an auxiliary unconstrained market defined as follows. We first decompose the process R in its own filtration as

$$dR_t = \left((\mu_1 - \frac{\sigma^2}{2}) + (\mu_2 - \mu_1)F_t\right) dt + \sigma d\overline{B}_t,$$

where $\overline{B}_\cdot$ is the innovation process, i.e., the $\mathcal{F}_t^S$- Brownian motion defined as

$$\overline{B}_t = \frac{1}{\sigma}\left(R_t - (\mu_1 - \frac{\sigma^2}{2})t - (\mu_2 - \mu_1)\int_0^t F_s ds\right),\quad t \geq 0,$$

where F is the conditional a posteriori probability (given the observation of S) that τ has occurred within $[0, t]$:

$$F_t := \mathbb{P}\left(\tau \leq t / \mathcal{F}_t^S\right).$$

Let $\mathcal{D}$ the subset of the $\{\mathcal{F}_t^S\}-$ progressively measurable processes $\nu : [0,T] \times \Omega \to$ R such that

$$\mathbb{E} \int_0^T \nu^-(t)dt < \infty \text{ , where } \nu^-(t) := -\inf(0, \nu(t)).$$

The bond price process $S^0(\nu)$ and the stock price $S(\nu)$ satisfy

$$S_t^0(\nu) = 1 + \int_0^t S_u^0(\nu)(r + \nu^-(u))du,$$
$$S_t(\nu) = S_0 + \int_0^t S_u(\nu) \left((\mu_1 + (\mu_2 - \mu_1)F_u + \nu(u)^- + \nu(u))du + \sigma d\overline{B}_u\right).$$

For each auxiliary unconstrained market driven by a process ν, the value function is

$$V(\nu, x) := \sup_{\pi. \in \mathcal{A}(\nu,x)} \mathbb{E}_x U(W_T^\pi(\nu)),$$

where

$$dW_t^\pi(\nu) = W_t^\pi(\nu) \left((r + \nu^-(t))dt + \pi_t \left(\nu(t)dt + (\mu_2 - \mu_1)F_t dt \right.\right.$$
$$\left.\left. +(\mu_1 - r)dt + \sigma d\overline{B}_t\right)\right).$$

Karatzas and Shreve have proven: If there exists $\widetilde{\nu}$ such that

$$V(\widetilde{\nu}, x) = \inf_{\nu \in \mathcal{D}} V(\nu, x)$$

then there exists an optimal portfolio π^* for which the optimal wealth (for the *constrained admissible strategies*) is

$$W_t^* = W_t^{\widetilde{\pi}^*}(\widetilde{\nu}).$$

An optimal portfolio allocation strategy is

$$\pi_t^* := \sigma^{-1} \left(\frac{\mu_1 - r + (\mu_2 - \mu_1)F_t + \widetilde{\nu}(t)}{\sigma} + \frac{\phi_t}{H_t^{\widetilde{\nu}} W_t^* \, e^{-rt - \int_0^t \widetilde{\nu}^-(s)ds}} \right),$$

where F_t satisfies

$$F_t = \frac{\lambda e^{\lambda t} L_t \int_0^t e^{-\lambda s} L_s^{-1} ds}{1 + \lambda e^{\lambda t} L_t \int_0^t e^{-\lambda s} L_s^{-1} ds},$$

and $H_t^{\widetilde{\nu}}$ is the exponential process defined by

$$H_t^{\widetilde{\nu}} = \exp\left(- \int_0^t \left(\frac{\mu_1 - r + \widetilde{\nu}(s)}{\sigma} + \frac{(\mu_2 - \mu_1) F_s}{\sigma} \right) d\overline{B}_s \right.$$
$$\left. -\frac{1}{2} \int_0^t \left(\frac{\mu_1 - r + \widetilde{\nu}(s)}{\sigma} + \frac{(\mu_2 - \mu_1) F_s}{\sigma} \right)^2 ds \right),$$

and ϕ is a $\mathcal{F}_t^S$ adapted process which satisfies

$$\mathbb{E}\left(H_T^{\widetilde{\nu}} e^{-rT - \int_0^T \widetilde{\nu}^-(t)dt} (U')^{-1} (\upsilon H_T^{\widetilde{\nu}} e^{-rT - \int_0^T \widetilde{\nu}^-(t)dt}) \;/\; \mathcal{F}_t^S \right) = x + \int_0^t \phi_s d\overline{B}_s.$$

Here, υ is the Lagrange multiplier which makes the expectation of the left hand side equal to x for all x.

If $U(\cdot) = \log(\cdot)$ then

$$W_t^{*,x} = \frac{x e^{r(T-t) + \int_t^T \widetilde{\nu}^-(t)dt}}{H_t^{\widetilde{\nu}}},$$
$$\pi_t^* = \left(\frac{\mu_1 - r + (\mu_2 - \mu_1) F_t + \widetilde{\nu}(t)}{\sigma^2} \right),$$

where

$$\widetilde{\nu}(t) := \begin{cases} -\left(\mu_1 - r + (\mu_2 - \mu_1) F_t\right) & \text{if } \dfrac{\mu_1 - r + (\mu_2 - \mu_1) F_t}{\sigma^2} < 0, \\ 0 & \text{if } \dfrac{\mu_1 - r + (\mu_2 - \mu_1) F_t}{\sigma^2} \in [0,1], \\ \sigma^2 - \left(\mu_1 - r + (\mu_2 - \mu_1) F_t\right) & \text{otherwise,} \end{cases}$$

and, as above,

$$\widetilde{\nu^-}(t) := -\inf\left(0, \widetilde{\nu}(t)\right).$$

Thus the optimal strategies for the constrained problem are the projections on $[0,1]$ of the optimal strategies for the unconstrained problem.

Using again Yor's formula (11), one can obtain an (horrible) explicit formula for the value function corresponding to the optimal strategy. As

$$\begin{aligned}V(x) &= V(\widetilde{\nu}, x)\\ &= \log(x) + rT\\ &\quad + \mathbb{E}_{\mathbb{P}}\left[\int_0^T \left(\mu_1 - r + (\mu_2 - \mu_1)F_t - \frac{\sigma^2}{2}\right)\mathbb{I}_{\left\{F_t > \frac{\sigma^2 - (\mu_1 - r)}{\mu_2 - \mu_1}\right\}} dt\right]\\ &\quad + \frac{1}{2}\mathbb{E}_{\mathbb{P}}\left[\int_0^T \left(\frac{\mu_1 - r + (\mu_2 - \mu_1)F_t}{\sigma}\right)^2 \mathbb{I}_{\left\{\frac{\mu_1 - r}{\mu_2 - \mu_1} < F_t < \frac{\sigma^2 - (\mu_1 - r)}{\mu_2 - \mu_1}\right\}} dt\right],\end{aligned}$$

we deduce, denoting by $g(a,t)$ the density of $\frac{\Phi_t}{\lambda}$, that the value at time T of the optimal portfolio, W_T^*, satisfies

$$\begin{aligned}\mathbb{E}\log(W_T) &= \log(x) + rT\\ &\quad + \int_0^T \int_0^\infty \left[\left(\mu_1 - r + (\mu_2 - \mu_1)\frac{a}{1+a} - \frac{\sigma^2}{2}\right)\mathbb{I}_{\left\{a > \frac{\sigma^2 - \mu_1 + r}{\mu_2 - \sigma^2 + r}\right\}}\right.\\ &\quad \left.\frac{1}{\sigma^2}\left(\mu_1 - r + (\mu_2 - \mu_1)\frac{a}{1+a}\right)^2 \mathbb{I}_{\left\{-\frac{\mu_1 - r}{\mu_2 - r} < a < \frac{\sigma^2 - \mu_1 + r}{\mu_2 - \sigma^2 + r}\right\}}\right]\\ &\quad e^{-\lambda t}(1+a)g(\lambda a, t)\lambda da\, dt.\end{aligned}$$

A model and detect strategy. We now consider the case of a trader who chooses a mathematical model and wants to reinvest the portfolio only once, namely at the time where the change time τ is optimally detected owing to the price history. We suppose that the reinvestment rule is the same as the technical analyst's one: at the detected change time from μ_1 to μ_2, all the portfolio is reinvested in the risky asset.

We consider the optimal stopping rule Θ^K which minimizes the *expected miss*

$$\mathcal{R}(\Theta) := \mathbb{E}|\Theta - \tau|$$

over all stopping rules Θ, where τ is a positive random variable (see Shiryayev [61] and Karatzas [41]). One has: The stopping rule Θ^K which minimizes the expected miss $\mathbb{E}|\Theta - \tau|$ over all the stopping rules $\mathit{\Theta}$ with $\mathbb{E}(\mathit{\Theta}) < \infty$ is

$$\mathit{\Theta}^K := \inf\left\{t \geq 0 \;\middle|\lambda e^{\lambda t} L_t \int_0^t e^{-\lambda s} L_s^{-1} ds \geq \frac{p^*}{1-p^*}\right\},$$

where $(L_t)_{t\geq 0}$ is the exponential likelihood-ratio process

$$L_t := \exp\left\{\frac{\mu_2-\mu_1}{\sigma^2}R_t - \frac{1}{2\sigma^2}\left((\mu_2-\mu_1)^2 + 2(\mu_2-\mu_1)(\mu_1-\frac{\sigma^2}{2})\right)t\right\},$$

and p^* is the unique solution in $(\frac{1}{2},1)$ of the equation

$$\int_0^{1/2}\frac{(1-2s)e^{-\beta/s}}{(1-s)^{2+\beta}}s^{2-\beta}ds = \int_{1/2}^{p^*}\frac{(2s-1)e^{-\beta/s}}{(1-s)^{2+\beta}}s^{2-\beta}ds$$

with $\beta := 2\lambda\sigma^2/(\mu_2-\mu_1)^2$.

The value of the portfolio at maturity T is

$$W_T = \frac{xS^0_{\theta^K}}{S_{\theta^K}}S_T\mathbb{I}_{(\theta^K\leq T)} + xS^0_T\mathbb{I}_{(\theta^K>T)}.$$

For a logarithmic utility function, one can again exhibit an exact formula for $\mathbb{E}(\log(W_T))$ which we do not write here.

It remains an open problem to mathematically compare the exact values of the logarithmic utilities of the portfolios based on technical analysis, optimal allocations, or model and detect strategies. However we can numerically compare them. Fig. 1 illustrates that, when the model is perfectly calibrated, the strategies based on mathematical models have significantly better performances than the technical analyst method.

The performances of the strategies based on misspecified models. In practice, it is extremely difficult to know parameters exactly. If one may hope to calibrate μ_1 and σ relatively well owing to historical data, the value of μ_2 cannot be determined a priori, and data concerning τ miss.

Consider a trader who believes that the stock price is

$$dS_t = S_t\left(\overline{\mu}_2 + (\overline{\mu}_1-\overline{\mu}_2)\mathbb{I}_{t\leq\overline{\tau}}\right)dt + \overline{\sigma}S_t dB_t,$$

where the law of $\overline{\tau}$ is exponential with parameter $\overline{\lambda}$.

Set:

$$\overline{L}_t := \exp\left\{\frac{1}{\overline{\sigma}^2}(\overline{\mu}_2-\overline{\mu}_1)R_t - \frac{1}{2\overline{\sigma}^2}\left((\overline{\mu}_2-\overline{\mu}_1)^2 + 2(\overline{\mu_2}-\overline{\mu}_1)(\overline{\mu}_1-\frac{\overline{\sigma}^2}{2})\right)t\right\},$$

$$\overline{F}_t := \frac{\overline{\lambda}e^{\overline{\lambda}t}\overline{L}_t\int_0^t e^{-\overline{\lambda}s}\overline{L}_s^{-1}ds}{1+\overline{\lambda}e^{\overline{\lambda}t}\overline{L}_t\int_0^t e^{-\overline{\lambda}s}\overline{L}_s^{-1}ds}.$$

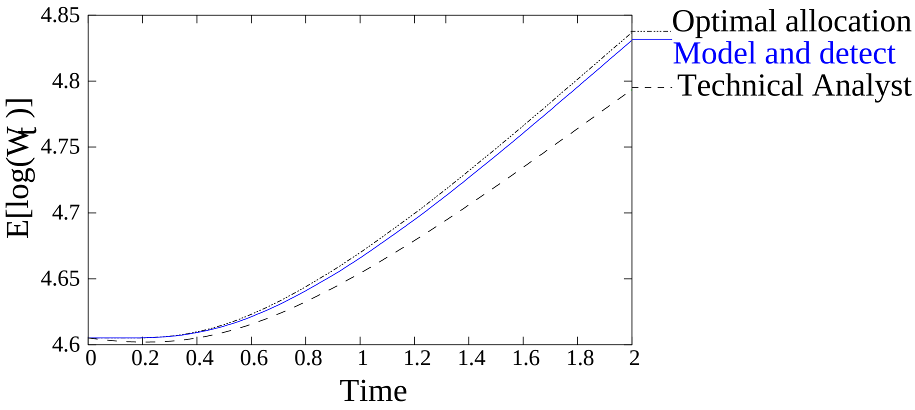

Figure 1: No misspecification

On the misspecified optimal allocation strategy. The trader computes a pseudo optimal allocation by using the erroneous parameters $\overline{\mu}_1$, $\overline{\mu}_2$, $\overline{\sigma}$ and $\overline{\tau}$. Thus the value of his/her misspecified optimal allocation strategy is

$$\overline{\pi}_t^* = \text{proj}_{[0,1]} \frac{(\overline{\mu}_1 - r + (\overline{\mu}_2 - \overline{\mu}_1)\overline{F}_t)}{\overline{\sigma}^2},$$

and the corresponding wealth is

$$\overline{W}_t^* = e^{rt} \exp\left(\int_0^t \overline{\pi}_u^* d(e^{-ru} S_u) \right).$$

On misspecified model and detect strategies. The erroneous stopping rule is

$$\overline{\Theta}^K = \inf \left\{ t \geq 0, \ \ \overline{\lambda} e^{\overline{\lambda} t} \overline{L}_t \int_0^t e^{-\overline{\lambda} s} \overline{L}_s^{-1} ds \geq \frac{\overline{p}^*}{1 - \overline{p}^*} \right\},$$

where $\overline{p}^*$ is the unique solution in $(\frac{1}{2}, 1)$ of

$$\int_0^{1/2} \frac{(1-2s)e^{-\overline{\beta}/s}}{(1-s)^{2+\overline{\beta}}} s^{2-\overline{\beta}} ds = \int_{1/2}^{\overline{p}^*} \frac{(2s-1)e^{-\overline{\beta}/s}}{(1-s)^{2+\overline{\beta}}} s^{2-\overline{\beta}} ds,$$

with $\overline{\beta} = 2\overline{\lambda}\overline{\sigma}^2 / (\overline{\mu}_2 - \overline{\mu}_1)^2$.

The value of the corresponding portfolio is

$$\overline{W}_T = xS^0_{\overline{\Theta}^K} \frac{S_T}{S_{\overline{\Theta}^K}} \mathbb{I}_{(\overline{\Theta}^K \leq T)} + xS^0_T \mathbb{I}_{(\overline{\Theta}^K > T)}.$$

A comparison between misspecified strategies and the technical analysis technique. Our main question is: Is it better to invest according to a mathematical strategy based on a misspecified model, or according to a strategy which does not depend on any mathematical model?

Unfortunately, the analytical representations for the portofolios respectively corresponding to the optimal strategy, the model and detect strategy, or the chartist strategy, are too complex to allow one to easily deduce precise comparisons. Getting such comparisons, for example in the asymptotics of large volatilities, are an open question so far. However Monte Carlo simulations lead to interesting results. Consider the following study case:

Parameters of the model	μ_1	μ_2	λ	σ	r
True values	-0.2	0.2	2	0.15	0.0
Parameters used by the trader	$\overline{\mu}_1$	$\overline{\mu}_2$	$\overline{\lambda}$	$\overline{\sigma}$	r
Misspecified values (case I)	-0.3	0.1	1.0	0.25	0.0
Misspecified values (case II)	-0.3	0.1	3.0	0.25	0.0

Other cases can be exhibited, where the technical analyst overperforms the misspecified optimal allocation strategies. For example, consider the case where the true values of the parameters are in Table 1. Table 2 must be read as follows. For the misspecified values $\overline{\mu}_2 = 0.1$, $\overline{\sigma} = 0.25$, $\overline{\lambda} = 1$, if the trader chooses $\overline{\mu}_1$ in the interval $(-0.5, -0.05)$ then the misspecified optimal strategy is worse than the technical analyst's one.

Other numerical studies show that a single misspecified parameter is not sufficient to allow the technical analyst to overperform the Model and Detect traders. Astonishingly, other simulations show that the technical analyst may overperform the misspecified optimal allocation strategy but not the misspecified model and detect strategy. One can also observe that, when μ_2/μ_1 decreases, the performances of well specified and misspecified model and detect strategies decrease. Theoretical estimates and explanations for these effects are open issues so far.

Table 1: True values of the parameters

Parameter	True Value
μ_1	-0.2
μ_2	0.2
σ	0.15
λ	2

Table 2: Misspecified values and range of the parameters

$\overline{\mu}_1$	(-0.5,-0.05)		$\overline{\mu}_1$	-0.3		$\overline{\mu}_1$	-0.3		$\overline{\mu}_1$	-0.3
$\overline{\mu}_2$	0.1		$\overline{\mu}_2$	(0,0.13)		$\overline{\mu}_2$	0.1		$\overline{\mu}_2$	0.1
$\overline{\sigma}$	0.25		$\overline{\sigma}$	0.25		$\overline{\sigma}$	(0.2,$\rightarrow$)		$\overline{\sigma}$	0.25
$\overline{\lambda}$	1		$\overline{\lambda}$	1		$\overline{\lambda}$	1		$\overline{\lambda}$	(0,1.5)

We have just considered the case where the change of trend occurs at one random time only. In a more realistic situation, several changes of trend may occur at random times τ_n: the trend process is

$$\mu(t) := \begin{cases} \mu_1 \text{ if } \tau_{2n} \leq t < \tau_{2n+1} \\ \mu_2 \text{ if } \tau_{2n+1} \leq t < \tau_{2n+2}. \end{cases} \tag{12}$$

The trader should rebalance his/her portfolio at each change of trend. However the times τ_n cannot be detected exactly. In addition, the filtration generated by the observed prices of the stock is strictly smaller than the filtration generated by the filtration generated by the Brownian motion and the τ_n's. Taking also into account transaction costs, Blanchet et al. [16] recently extended the above comparison between strategies subject to model risk and strategies derived from technical analysis.

A lot of mathematical analysis remains to be done to better understand the practical success of technical analysis, and the effects on the market resulting from the common belief that a lot of agents take their investment decisions by applying the technical analysis rules.

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