Existence and intermediate variational approximation results for atoms-like target energies with blur operator

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Abstract The goal of the present work is to analyze, from a theoretical point of view, a new variational formulation for the detection of points in 2-d images in presence of blur operator. We define a new energy whose minimizers give the target set of points. We prove an existence result for this functional and we also provide a variational approximation with functionals defined on smooth sets.

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1. Introduction

1.1. **The variational model.** Suppose to observe in the real word a natural image μ, which contains several geometric structures: isolated points, open curves, edges, textures, homogeneous zones (that is set with positive Lebesgue measure). From a very general point of view μ can be a modeled as a positive Radon measure with atomic part (points), and a non atomic part which contains all the other geometric structures.

The image μ is acquired by a camera system by applying a blur operator T, whose features depends on the system. It means the measure μ is smoothed by convolution with a regularizing kernel. Finally a Gaussian noise b, due to the data transmission, is added. The final observed image is now a function u_0 given by

$$u_0 = T(\mu) + b.$$

One of the important tasks in image processing, for instance in biological images, is to retrieve nothing else but the isolated points, that were contained inside the image domain, lost in the acquisition process.

This is a difficult task. The reason is twofold. First of all, due to the compactness of T, one has to solve an ill posed inverse problem to retrieve μ. Then the atomic part of μ has to be, somehow, extrapolate from whole the support of μ.

Let us note that in the particular case $\mu = v$ with v BV-function, one obtains the classical Rudin-Osher-Fatemi model (see [23]). Without claiming of being exhaustive we refer the reader to [2, 4, 8] for a, still incomplete, survey on classical variational problems in image processing; such as, for instance, restoration by total variation minimization, edge detection, segmentation, inpainting, free discontinuity problems. We point out that the problem we deal with, is deeply different, whereas we are interested in restoring/detecting atoms-like singularities.

To this purpose we define and study a new variational model to detect isolated points in blurred and noisy image. We confine ourselves to a pure theoretical analysis.

We consider as image domain $\Omega \subset \mathbb{R}^2$ an open bounded set with Lipschitz boundary. The blur operator T is an integral operator with kernel ρ_σ, where ρ_σ is a standard Friedrichs mollifier. The natural image μ is a Radon measure on Ω. Then for $x \in \Omega_\sigma = \{x \in \Omega \text{ such that } dist(x, \partial\Omega) > \sigma\}$ the convolution of ρ_σ with a Radon measure is well defined (see subsection 2.2). By following the standard approach to solve ill-posed inverse problems (see [17] and reference therein on this subject), we intend to minimize functionals of type:

$$F(\mu) = \|\rho_\sigma * \mu - u_0\|^2_{L^2(\Omega_\sigma)} + \underbrace{R(\mu)}_{\text{prior term}}$$

where $R(\mu)$ has to be chosen according to the singularities, we want to extrapolate from the support of μ. Moreover in all the paper we assume $u_0 \in L^\infty(\Omega)$. Then in order to force the minimizers of F to be atomic measures, we consider $R(\mu) = \mathcal{H}^0(\text{supp}\mu)$, where $\mathcal{H}^0$ is the counting Haussdorf measure.

1.2. **Related works.** Let us mention some related works based on variational techniques. For detecting point-like target a new approach has been proposed in [6]. In that papers the natural image is still considered as a Radon measure. Then one of the key point is to transform the natural image μ in the divergence measure of a suitable L^p-vector field, in order to define a convenient variational framework. This is done by solving the classical Dirichlet problem with measure data:

$$\begin{cases} -\Delta u = \mu & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

This point of view makes possible the construction of functional of type:

$$\mathcal{F}(U) = \int_\Omega |\mathrm{div} U|^2 dx + \lambda \int_\Omega |U - U_0|^p dx + \mathcal{H}^0(P_U);$$

the initial vector field U_0 is given by $U_0 = -\nabla v$ where v is solution of problem (1.1) with $\mu = I_0$. I_0 is the observed data. U is an L^p vector field (with $p < 2$), whose distributional divergence div$U$ is a Radon measure. The unkown target set is identified with P_U, which is the support of the singular part, with the respect to the p-capacity, of the measure div$U$.

Later on, they follow some suggestions from [10, 11], where the counting measure term is approximated in the sense of the De Giorgi's Γ-convergence (see [15, 16]), by means of curvature depending functionals. They so obtain, under the constraint that U is a gradient of a Sobolev function in $W_0^{1,p}(\Omega)$ with $p < 2$, a variational approximation of functional $\mathcal{F}$ with more convenient energies from a numerical point of view (see [20]).

Another interesting strategy has been developed in [5]. In that paper, in connection with the theory of Ginzburg-Landau systems (see [1] and reference therein on this topic) the isolated points in 2-D images are considered as the topological singularities of a map $U : \mathbb{R}^2 \to \mathbb{S}^1$, where $\mathbb{S}^1$ is the unit sphere of $\mathbb{R}^2$. Then, after a delicate construction of an initial map field $U_0 : \mathbb{R}^2 \to \mathbb{S}^1$, they minimize a family of Ginzburg-Landau's type energy, in order to detect isolated atoms.

By the way none of the previous works takes into account the blur operator. Furthermore both approaches require a preliminary and delicate construction of a proper initial map U_0, related to the initial observed image.

1.3. **Main contributions.** By dealing with the blur operator we define a more realistic model. Moreover the presence of the smoothing kernel ρ_σ allows for a direct and natural variational formulation. We define indeed the energy directly on the space of Radon measures. More precisely we introduce first a functional $F : A\mathcal{M}(\Omega) \to [0, +\infty]$ defined by

$$F(\mu) = \mathcal{H}^0(\mathrm{supp}\mu) + \|\rho_\sigma * \mu - u_0\|^2_{L^2(\Omega_\sigma)}$$

where $A\mathcal{M}(\Omega)$ denotes the set of purely atomic Radon measures (see subsection 2.2 for definitions and standard properties of Radon measures).

Then, by keeping in mind the association of every Radon measure with the solution of Dirichlet problem (1.1), we introduce a functional $G : HS\Delta\mathcal{M}^p(\Omega) \to [0, +\infty]$ defined by

$$G(u) = \mathcal{H}^0(P_{\nabla u}) + \|\rho_\sigma * \delta_{P_{\nabla u}} - u_0\|^2_{L^2(\Omega_\sigma)};$$

where $HS\Delta\mathcal{M}^p(\Omega)$ is the space of $W_0^{1,p}(\Omega)$-functions whose -Laplacian measure has no non atomic part, while $P_{\nabla u}$ denotes the support of its atomic part. So that $\delta_{P_{\nabla u}} = \sum_{x_i \in P_{\nabla u}} a_i \delta_{x_i}$

with $a_i \in [0,1]$ (we refer the reader to subsection 2.3 for precise definitions of all these quantities).

Concerning the two minimum problems associated to functional F and G, it can be seen that they are equivalent. So in most of the paper we focus on functional G.

So that in the first part the main result is to provide, via direct methods, an existence result for functional G (see Theorem 4.1). Roughly speaking, lower semicontinuity a compactness of a minimizing sequence $\{u_n\}$ of G follows by carefully combining two tools. The first one is the well known (see [24]) a priori estimation for the $W_0^{1,p}(\Omega)$-norm of the weak solution of problem (1.1). The second one is a uniform bound on the counting measure term, which gives the convergence for sets of points in the sense of Definition 2.1 (see section 2).

In the second part, as in [6], inspired by the techniques used in a different context by the authors of [11] (see also [10]), we investigate the variational approximation for functional G via depending curvature functionals defined on smooth sets.

The first step is to replace, as in [6, 11], the counting measure by curvature depending energies defined on regular sets, whose minimizers are given by small disk $B_\epsilon(x_i)$ with $x_i \in P_{\nabla u}$. That is formally

$$\mathcal{H}^0(P_{\nabla u}) \cong \frac{1}{4\pi}\int_{\partial D}\big(\frac{1}{\epsilon}+\epsilon\kappa^2\big)d\mathcal{H}^1, \tag{1.2}$$

where k is the curvature of D. Then we turn our attention on the last term of functional G.

We look at the associate density measures $\theta_\epsilon d\mathcal{H}^1\lfloor D = \big(\frac{1}{\epsilon}+\epsilon\kappa^2\big)d\mathcal{H}^1\lfloor D$. We observe that (1.2), if we assumed all the coefficients $a_i \in [0,1]$ of the atomic measure $\delta_{P_{\nabla u}}$ equal to 1, would read in term of total variation of Radon measures as

$$\big|\delta_{P_{\nabla u}}\big|(\Omega) \cong \frac{1}{4\pi}\big|\theta_\epsilon d\mathcal{H}^1\lfloor D\big|(\Omega). \tag{1.3}$$

The key point of our approach is then to replace, formally, the whole functional G with:

$$G_\epsilon(D) = \frac{1}{4\pi}\int_{\partial D}\big(\frac{1}{\epsilon}+\epsilon\kappa^2\big)d\mathcal{H}^1 + \frac{1}{\epsilon}\mathcal{L}^2(D) + \int_{\Omega_\sigma}|\rho_\sigma * \theta_\epsilon d\mathcal{H}^1\lfloor D - u_0|^2 dx, \tag{1.4}$$

where D is a regular set (see subsection 2.1 for notation and definition of regular sets) with small Lebesgue measure (see subsection 5.1 for the definition of the family of energies G_ϵ). The goal of the second part is to prove that such a replacement can be actually performed in the sense of Γ-convergence. In line with [6, 11] we define the Γ-convergence with respect to an ad hoc convergence for smooth sets. Such a notion involves the Haussdorf convergence of boundaries of regular sets to finite sets of points (we refer to section 5 for definition and rigorous statements). It is known that a uniform bound, with respect to ϵ, on the first term of functionals G_ϵ guarantees compactness properties with respect to this notion of convergence (in section 5.3 we state a very simplified version of the compactness result proven in [11]). Nevertheless such a nice property does not allow to take the limit directly in the last term of functionals (1.4). It makes the proof the Γ-convergence more than a simple adaption of the argument used in [11]. Indeed, to prove the so-called $\Gamma-\liminf$ inequality, we have to deduce, up to subsequences, the weak star convergence of the density measure to $\delta_{P_{\nabla u}}$. Such a results will follow from the association of every Radon measure with the solution of Dirichlet problem (1.1), combined with properties of Haussdorf convergence.

Concerning the so-called $\Gamma-\limsup$ inequality, the optimal sequence $\{D_\epsilon\}$, as in [6, 11], will be given by disk of small radius. For such a sequence, it can be seen that the whole sequence of density measures $\theta_\epsilon d\mathcal{H}^1$ converges, with respect to the weak star convergence, to the measure $\delta_{P_{\nabla u}}$.

1.4. **Final remarks.** Let us point out that with respec tot [6, 11] we do not go further in the variational approximation, by replacing the sequence G_ϵ with more convenient functionals defined on suitable smooth functions w. In particular in [11] (see also [7]) the measure $d\mathcal{H}^1$ is replaced by Modica-Mortola's density energies (see [21, 22]). Then by combining Sard's Theorem and coarea formula one can formally replace the integral on ∂D by an integral computed over the level sets of w, whose curvature κ becomes $\mathrm{div}\frac{\nabla w}{|\nabla w|}$.

Very roughly speaking, these considerations lead to variational approximation for the counting measure term of this type:

$$\mathcal{H}^0(P_{\nabla u}) \cong \frac{1}{8\pi C}\int_{\Omega\setminus\{|\nabla w|=0\}} \Big(\frac{1}{\beta_\epsilon} + \beta_\epsilon\big(\mathrm{div}(\frac{\nabla w}{|\nabla w|})\big)^2\Big)(\epsilon|\nabla w|^2 + \frac{1}{\epsilon}W(w))dx,$$

W is a double well potential, $C = \int_0^1 \sqrt{W(t)}dt$. The parameters ϵ, β_ϵ are such that
$\lim_{\epsilon\to 0^+} \frac{\epsilon|\log(\epsilon)|}{\beta_\epsilon} = 0.$

So that in our case the density measures playing the role of $\theta_\epsilon d\mathcal{H}^1\lfloor D$ should be given by

$$\mu_\epsilon(x)dx = \frac{1}{8\pi C}\Big(\frac{1}{\beta_\epsilon} + \beta_\epsilon\underbrace{\big(\mathrm{div}(\frac{\nabla w}{|\nabla w|})\big)^2}_{\cong k^2}\Big)\underbrace{(\epsilon|\nabla w|^2 + \frac{1}{\epsilon}W(w))dx}_{\cong d\mathcal{H}^1}.$$

Therefore we would like to replace functional G with much more convenient energies defined on smooth functions, of type:

$$F_\epsilon(w) = \frac{1}{8\pi C}\int_{\Omega\setminus\{|\nabla w|=0\}} \Big(\frac{1}{\beta_\epsilon}+\beta_\epsilon\big(\mathrm{div}(\frac{\nabla w}{|\nabla w|})\big)^2\Big)(\epsilon|\nabla w|^2+\frac{1}{\epsilon}W(w))dx+\|\rho_\sigma*\mu_\epsilon dx-u_0\|^2_{L^2(\Omega_\sigma)}.$$

However this type of variational approximation is still subject of our current investigation.

1.5. **Organization of the paper.** The paper is organized as follows. Section 2 is devoted to notation, preliminary definitions and results. In section 3 we introduce the functionals and we show some related properties. In section 4 we address the existence result for functional G. In section 5 we provide the variational approximation of functional G.

2. Definition and main properties

2.1. **Notation.** In all the paper $\Omega \subset \mathbb{R}^2$ is an open bounded set with Lipschitz boundary. The Euclidean norm will be denoted by $|\cdot|$, while the symbol $\|\cdot\|$ indicates the norm of some function spaces. The brackets $\langle , \rangle$ denotes the duality product in some distributional spaces. $\mathcal{L}^2$ or dx is the 2-dimensional Lebesgue measure and $\mathcal{H}^k$ is the k-dimensional Hausdorff measure. $B_\rho(x_0)$ is the ball centered at x_0 with radius ρ. We say that a set $D \subset \Omega$ is a regular set if it can be written as $\{F < 0\}$ with $F \in C_0^\infty(\Omega)$. In the following we will denote by $R(\Omega)$ the family of all regular sets in Ω. $\mathcal{B}(\Omega)$ is the family of all Borel set in Ω. We denote by $\mathcal{M}(\Omega)$ the standard space of Radon measures. If $\mu \in \mathcal{M}(\Omega)$ and $B \subseteq \Omega$ is a generic Borel set $|\mu|(B)$ denotes its total variation.

If $D \in R(\Omega)$ is a regular set, the symbol $d\mathcal{H}^1 \lfloor D$ denotes the Radon measure defined for every Borel set $B \in \mathcal{B}(\Omega)$ by $\mathcal{H}^1 \lfloor \partial D(B) = \mathcal{H}^1(\partial D \cap B)$. In particular if is $\phi \in C_0(\Omega)$ we have $\langle d\mathcal{H}^1 \lfloor D, \phi \rangle = \int_{\partial D \cap \Omega} \phi d\mathcal{H}^1$.

Finally Hausdorff distance between two closed sets C and K is defined as $d_H(C, K) = \inf_{r>0}\{C \subset (K)_r \quad K \subset (C)_r\}$ with $(A)_r = \{x \in \mathbb{R}^2 \quad dist(A, r) < r^2\}$ for a generic set $A \subset \mathbb{R}^2$. Notation for Sobolev spaces, Lebesgue spaces and the space of distributions is standard.

2.2. **Radon measures.** In this subsection we collect some well known fact about Radon measures. For more details we refer the reader to [2, 18].

Let σ be a positive parameter and ρ_σ a standard mollifying sequence, with $\Omega_\sigma = \{x \in \Omega \text{ such that } dist(x, \partial\Omega) > \sigma\}$. For $x \in \Omega_\sigma$ we have that the support of $\rho_\sigma(x - \cdot)$ is contained in Ω, that is $\rho_\sigma(x - \cdot) \in C_0(\Omega)$. Then for every $x \in \Omega_\sigma$ is well defined the convolution between a Radon measure $\mu \in \mathcal{M}(\Omega)$ and ρ_σ given by

$$\mu \in \mathcal{M}(\Omega) \mapsto \big(\rho_\sigma * \mu\big)(x) = \int_\Omega \rho_\sigma(x - y) d\mu \in C^\infty(\Omega_\sigma). \tag{2.1}$$

We define linear operator $T : \mathcal{M}(\Omega) \mapsto L^2(\Omega)$ given by:

$$\mu \in \mathcal{M}(\Omega) \to T(\mu) = \int_\Omega \rho_\sigma(x - y) d\mu(y) \in C^\infty(\Omega_\sigma) \subset L^2(\Omega),$$

where $\rho_\sigma * \mu$ is identified with an L^2-function by setting $(\rho_\sigma * \mu)(x) = 0$ outside Ω_σ. It can be seen that T is a compact operator. In particular T no admits a bounded inverse operator.

If $P \subset \Omega$ is a set of points, that is $P = \{x_i\}_{i=1}^{+\infty}$, δ_P denotes the atomic measures $\sum_{i=1}^\infty a_i \delta_{x_i}$, where δ_{x_i} is the Dirac measure concentrated at x_i and $a_i \in [0, 1]$.

We recall that every positive Radon measure $\mu \in \mathcal{M}(\Omega)$ can be decomposed in the following way:

$$\mu = \tilde{\mu} + \sum_{i=1}^{\infty} a_i \delta_{x_i} \tag{2.2}$$

where $\tilde{\mu}$ is non atomic, $a_i \in [0, 1]$, $x_i \in \Omega$, $x_i \neq x_j$ for $i \neq j$.

For latter use we introduce the following auxiliary space of purely atomic positive Radon measures:

$$A\mathcal{M}(\Omega) := \{\mu \in \mathcal{M}(\Omega); \quad \tilde{\mu} = 0\} \tag{2.3}$$

where A stands for atomic.

2.3. **Convergence for sets of points.** We recall the notion of convergence for finite sets of points (see [11, 19, 20]).

Definition 2.1. *We say that a sequence of a finite set of points* $\{P_h\}_h \subset \overline{\Omega}$ *converges as a sequence of sets of points to a set* $P \subset \overline{\Omega}$*, if each of the sets* P_h *contains a number* N *of points* $\{x_h^1, \dots, x_h^N\}$*, with* N *independent of* h*, such that* $x_h^i \to x^i$ *for any* $i = 1, \dots, n$ *and* $\bigcup_{i=1}^N \{x_i\} = P$.

Sometimes we will simply write $P_h \to P$ or say P_h converges to P, if no confusion is possible.

The following results (see, for instance, [20]) will be useful.

Lemma 2.1. *Let* $\{P_h\}_h$ *be a sequence of a finite set of points such that* $\mathcal{H}^0(P_h) \leq N_0$ *for every* h *with* $N_0 \in \mathbb{N}$*. Then there exists a subsequence* $\{P_{h_k}\}_k \subset \{P_h\}_h$ *and a set of points* $P \subset \overline{\Omega}$ *such that* $P_{h_k} \to P$.

Lemma 2.2. *Let* $\{P_h\} \subset \overline{\Omega}$ *be a sequence of a finite set of points converging to a finite set of points* P*. Then*

$$\mathcal{H}^0(P) \leq \liminf_{h \to +\infty} \mathcal{H}^0(P_h). \tag{2.4}$$

2.4. **Distributional divergence and distributional spaces.** In this subsection we recall the definition of the distributional space $\mathcal{DM}^p(\Omega)$, $1 \leq p, q \leq +\infty$, (see [3, 12, 13]). We also introduce some other auxilary spaces.

Definition 2.2. *For* $U \in L^p(\Omega; \mathbb{R}^N)$, $1 \leq p \leq +\infty$*, set*

$$|\mathrm{div} U|(\Omega) := \sup\{\langle U, \nabla\varphi\rangle \ : \ \varphi \in C_0^\infty(\Omega), \ |\varphi| \leq 1\}.$$

We say that U *is an* L^p*-divergence measure field, i.e.* $U \in \mathcal{DM}^p(\Omega)$*, if*

$$\|U\|_{\mathcal{DM}^p(\Omega)} := \|U\|_{L^p(\Omega;\mathbb{R}^N)} + |\mathrm{div} U|(\Omega) < +\infty.$$

We recall that $U \in L^p(\Omega;\mathbb{R}^N)$ belongs to $\mathcal{DM}^p(\Omega)$ if and only if there exists a Radon measure denoted by $\mathrm{div}U$ such that

$$\langle U, \nabla\varphi\rangle = -\int_\Omega \mathrm{divU}\varphi \quad \forall\varphi \in C_0^\infty(\Omega),$$

and the total variation of the measure $\mathrm{div}U$ is given by $|\mathrm{div}U|(\Omega)$.

We define the following space

$$\Delta\mathcal{M}^p(\Omega) := \{u \in W_0^{1,p}(\Omega),\ \nabla u \in \mathcal{DM}^p(\Omega)\}. \tag{2.5}$$

Then for $u \in \Delta\mathcal{M}^p(\Omega)$ we will write $-\Delta u$ to denote the measure $-\mathrm{div}\nabla u$.

For latter use, we also define the auxiliary subspace of $W_0^{1,p}(\Omega)$ whose -Laplacian measure $-\Delta u$ does not have non atomic part.

$$HS\Delta\mathcal{M}^p(\Omega) := \{u \in \Delta\mathcal{M}^p(\Omega), \quad -\Delta u \in A\mathcal{M}(\Omega)\} \tag{2.6}$$

where H stands for harmonic and S for special. For $u \in HS\Delta\mathcal{M}^p(\Omega)$, the support $\mathrm{supp} - \Delta u$ of the Radon measure $-\Delta u$ is denoted by $P_{\nabla u}$.

Finally we state the following result, which, up to minor changes, can be proven as Proposition 3.1 of [6].

Proposition 2.1. *Let $P \subset \Omega$ be a set of finite number of points. Let $u \in W_0^{1,p}(\Omega)$, with $-\Delta u = 0$ in $\mathcal{D}'(\Omega \setminus P)$. Then $u \in \Delta\mathcal{M}^p(\Omega)$. Moreover, if the measure $-\Delta u$ is positive, we have $u \in HS\Delta\mathcal{M}^p(\Omega)$, with $P_{\nabla u} = P$.*

3. Functionals and related properties

In this section we study two possible functionals well adapted for the detection of spots. We show that the associate minimum problems are equivalent and that the infimum value is trictly positive.

We consider first the functional $F : A\mathcal{M}(\Omega) \to [0, +\infty]$ given by

$$F(\mu) = \mathcal{H}^0(\mathrm{supp}\mu) + \|\rho_\sigma * \mu - \mathrm{u}_0\|^2_{\mathrm{L}^2(\Omega_\sigma)}. \tag{3.1}$$

Let us verify in the next proposition that the infimum value of F is strictly positive.

Proposition 3.1. *If $u_0 \neq 0$ a.e., then $m_1 = \inf_{A\mathcal{M}(\Omega)} F > 0$.*

Proof. Suppose without loss of generality that $m_1 < +\infty$. Then if $m_1 = 0$, it should be possible to exhibit a sequence $\{\mu_n\} \subset A\mathcal{M}(\Omega)$ such that $F(\mu_n) \to 0$. In particular $\mathcal{H}^0(\mathrm{supp}\mu_n) \to 0$. By Lemmas 2.2 and 2.1 we infer that there exists a subsequence of set of points $\{P_{n_k}\} \subseteq \{\text{supp}\mu_n\}$ converging, as a sequence of sets of points, to a finite set of points $P \subset \overline{\Omega}$ with

$$\mathcal{H}^0(P) \leq \lim_{k\to+\infty} \mathcal{H}^0(P_{n_k}) = \lim_{n\to+\infty} \mathcal{H}^0(\text{supp}\mu_n) = 0.$$

Then $P = \emptyset$. We consider the associated sequence of Radon measure $\nu_k = \delta_{P_{n_k}}$ with $\text{supp}\nu_\text{k} = P_{n_k}$ and we observe the following facts:

(i) $\mathcal{H}^0(\text{supp}\nu_k) = \mathcal{H}^0(P_{n_k}) \to 0$

(ii) since $\text{supp}\nu_k \to \emptyset$ (as a sequence of set of points), for k large enough $\text{supp}\nu_k = \emptyset$.

Then we deduce

$$0 = \lim_{k\to+\infty} F(\nu_k) = \lim_{k\to+\infty} \mathcal{H}^0(P_{n_k}) + \| \underbrace{\rho_\sigma * \nu_k}_{=0} - u_0\|^2_{L^2(\Omega_\sigma)} = \|u_0\|_2^2,$$

but then $u_0 = 0$ a.e., which is a contradiction. So the proof is achieved $\square$

To deal with space of functions, we associate to every $\mu \in \mathcal{M}(\Omega)$ a function $u \in \Delta\mathcal{M}^p(\Omega)$ in the following way. We consider the Dirichlet problem

$$\begin{cases} -\Delta u = \mu & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Classical results (see [24]) ensures the existence of a weak solution $u \in W_0^{1,p}(\Omega)$ with $p < 2$. Then it easy to see that the distributional divergence of $-\nabla u$ is given by μ and we have $u \in \Delta\mathcal{M}^p(\Omega)$. It leads to consider, instead of F, the functional $G : HS\Delta\mathcal{M}^p(\Omega) \to [0, \infty]$

$$G(u) = \mathcal{H}^0(P_{\nabla u}) + \|\rho_\sigma * \delta_{P_{\nabla u}} - u_0\|^2_{L^2(\Omega_\sigma)}.$$

We consider the minimum problems

$$\inf\{F(\mu); \quad \mu \in A\mathcal{M}(\Omega)\} \tag{3.3}$$

and

$$\inf\{G(u) \quad u \in HS\Delta\mathcal{M}^p(\Omega)\}. \tag{3.4}$$

We have the equivalence in the sense of the following proposition.

Proposition 3.2. *For the minimum problems (3.3) and (3.4) the following equality holds:*

$$m_1 = \inf_{A\mathcal{M}(\Omega)} F = \inf_{HS\Delta\mathcal{M}^p(\Omega)} G = m_2. \tag{3.5}$$

Proof. We first prove $m_1 \geq m_2$.

We may assume without loss of generality that $m_1 < +\infty$. Then for every $\mu \in A\mathcal{M}(\Omega)$ we have that $\text{supp}\mu$ is given by a finite set of points. We write $\mu = \delta_{\text{supp}\mu}$.

Moreover there exists $u \in W_0^{1,p}(\Omega)$ with $p < 2$ such that

$$\begin{cases} -\Delta u = \delta_{\text{supp}\mu} & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence $u \in HS\Delta\mathcal{M}^p(\Omega)$, with $P_{\nabla u} = \text{supp}\mu$. Therefore we have:

$$m_1 \geq F(\mu) = G(u) \geq m_2.$$

Now we show that $m_2 \geq m_1$.

If $u \in HS\Delta\mathcal{M}^p(\Omega)$ we have

$$m_2 \geq G(u) = F(\delta_{P_{\nabla u}}) \geq m_1.$$

Thus the equality (3.5) holds. $\square$

4. Existence

We focus our attention on the minimum problem (3.4). We show the existence, via direct methods, of a minimizer in the class $HS\Delta\mathcal{M}^p(\Omega)$ for the functional G. We start by proving the compactness property.

Theorem 4.1. *Let $\{u_n\}_n \subset HS\Delta\mathcal{M}^p(\Omega)$ be a sequence such that*

$$\mathcal{H}^0(P_{\nabla u_n}) \leq M < +\infty. \tag{4.1}$$

Then there exist $\{u_{n_k}\}_k \subset HS\Delta\mathcal{M}^p(\Omega)$ and $u \in HS\Delta\mathcal{M}^p(\Omega)$ such that

$$\begin{cases} u_{n_k} \to u \quad \textit{strongly in } L^2(\Omega) \\ \liminf\limits_{k\to+\infty} \mathcal{H}^0(P_{\nabla u_{n_k}}) \geq \mathcal{H}^0(P_{\nabla u}). \end{cases} \tag{4.2}$$

Proof.

Let $\{u_n\}_n \subset HS\Delta\mathcal{M}^p(\Omega)$ be a sequence such that bound (4.1) holds.

For every $n \in \mathbb{N}$ u_n is the solution of the dirichlet problem with measure data

$$\begin{cases} -\Delta u = \delta_{P_{\nabla u_n}} & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover we can always assume $P_{\nabla u_n} = \{x_n^1, ..., x_n^{J(n)}\}$ and $\delta_{P_{\nabla u_n}} = \sum_i^{J(n)} a_i \delta_{x_i}$ with $a_i \in [0,1]$. Then by Theorem 9.1 of [24] and (4.1) we have the estimate

$$\|u_n\|_{W_0^{1,p}} \leq C|\delta_{P_{\nabla u_n}}|(\Omega) = C\sum_{i=1}^{J(n)} |a_i| \leq \; C\mathcal{H}^0(P_{\nabla u_n}) \leq CM := C_1 \tag{4.3}$$

where the constant C_1 does not depend on n and $1 < p < 2$.

Sobolev's embedding together star weak compactness of measures, give immediately the existence of a subsequence $\{u_{n_l}\}_l$ and a positive measure μ such:

$$\begin{cases} u_{n_l} \to u & \text{in } L^2(\Omega) \text{ and a.e.} \\ \nabla u_{n_l} \rightharpoonup \nabla u & \text{in } L^p(\Omega;\mathbb{R}^2) \\ -\Delta u_{n_l} dx \rightharpoonup 0 & \text{in } L^2(\Omega) \\ -\Delta u_{n_l} \overset{*}{\rightharpoonup} \mu & \text{in } \mathcal{M}(\Omega). \end{cases}$$

Moreover from (4.1) and Lemma 2.1 we infer the existence of a subsequence $\{P_{n_l}\}_l \subset \{P_{\nabla u_n}\}_n$ and a finite set of points $P \subset \overline{\Omega}$ such that $P_{n_l} \to P$, in the sense of Definition 2.1.

By diagonal argument we have the existence of a subsequence $\{(u_{n_{l(k)}}, P_{n_{l(k)}})\}_k$ such that :

$$\begin{cases} u_{n_{l(k)}} \to u & \text{in } L^2(\Omega) \text{ and a.e.} \\ \nabla u_{n_{l(k)}} \rightharpoonup \nabla u & \text{in } L^p(\Omega;\mathbb{R}^2) \\ -\Delta u_{n_{l(k)}} dx \rightharpoonup 0 & \text{in } L^2(\Omega) \\ -\Delta u_{n_{l(k)}} \overset{*}{\rightharpoonup} \mu & \text{in } \mathcal{M}(\Omega) \\ P_{n_{l(k)}} \to P. \end{cases}$$

We claim that $-\Delta u = 0$ in $\mathcal{D}'(\Omega \setminus P)$. Indeed let ϕ be a test function with support in $\Omega \setminus P$. Since $P_{n_{l(k)}} \to P$ we have that for k large enough suppϕ is contained in $\Omega \setminus P_{n_{l(k)}}$.

Thus we have

$$\int_{\text{supp}\phi} \nabla u_{n_{l(k)}} \nabla \phi dx = -\int_{\text{supp}\phi} \underbrace{\Delta u_{n_{l(k)}}}_{\to 0} \phi dx.$$

By taking as $k \to +\infty$, we get

$$\int_{\text{supp}\phi} \nabla u \nabla \phi dx = 0$$

and being the test function arbitrary, the claim follows.

Set now $\tilde{P} = P \setminus \partial\Omega$. Then we have $u \in W_0^{1,p}(\Omega)$ with $-\Delta u = 0$ in $\mathcal{D}'(\Omega \setminus \tilde{P})$, since $\mathcal{D}'(\Omega \setminus \tilde{P}) \subset \mathcal{D}'(\Omega \setminus P)$.

So, by the Proposition 2.1, we conclude that $u \in \Delta\mathcal{M}^p(\Omega)$. It remains to prove that the measure $-\Delta u$ is positive.

We show that it coincides with the weak limit μ of the sequence of positive measures $\{-\Delta u_{n_{l(k)}}\}$. Indeed, if ϕ is a test function with support in Ω, we have

$$\int_\Omega \nabla u_{n_{l(k)}} \nabla \phi dx = \int_\Omega \phi d\Delta u_{n_{l(k)}} = \int_\Omega \phi d\delta_{P_{\nabla u_{n_{l(k)}}}},$$

and taking the limit we have the equality $-\Delta u = \mu$ in $\mathcal{D}'(\Omega)$ and in $\mathcal{M}(\Omega)$.

Hence, by the Proposition 2.1, we conclude that $u \in HS\Delta\mathcal{M}^p(\Omega)$ with $P_{\nabla u} = \tilde{P}$.

Finally by taking into account the lim inf is the same for all the subsequences extracted from the sequence $\{P_{\nabla u_n}\}$, we also have from Lemma 2.2:

$$\mathcal{H}^0(P_{\nabla u}) \le \mathcal{H}^0(P) \le \liminf_{k\to+\infty} \mathcal{H}^0(P_{n_{l(k)}}) \le \liminf_{k\to+\infty} \mathcal{H}^0(P_{\nabla u_{n_{l(k)}}}). \tag{4.4}$$

So that (4.2) holds and the proof is complete. □

We pass now to prove of the lower semicontinuity property. The proof, up to an integration by parts, will follow in practice from the compactness-continuity of the operator T defined in section 2.2.

Theorem 4.2. *Let $\{u_n\}_n, u \subset HS\Delta\mathcal{M}^p(\Omega)$ such that*

$$\begin{cases} u_n \to u \quad \textit{in } L^2(\Omega) \\ \liminf_{n\to+\infty} \mathcal{H}^0(P_{\nabla u_n}) \ge \mathcal{H}^0(P_{\nabla u}). \end{cases} \tag{4.5}$$

Then

$$\liminf_{n\to+\infty} G(u_n) \ge G(u). \tag{4.6}$$

Proof. Without loss of generality we assume

$$\lim_{n\to+\infty} G(u_n) \le M < +\infty, \tag{4.7}$$

where M is a positive constant.

Let us first consider the term

$$\int_{\Omega_\sigma} |\rho_\sigma * \delta_{P_{\nabla u_n}} - u_0|^2 dx = \int_{\Omega_\sigma} |\int_\Omega \rho_\sigma(x-y) d\delta_{P_{\nabla u_n}} - u_0|^2 dx.$$

Let us set

$$g_n(x) = \int_\Omega \rho_\sigma(x-y) d\delta_{P_{\nabla u_n}}; \quad g(x) = \int_\Omega \rho_\sigma(x-y) d\delta_{P_{\nabla u}}.$$

For $x \in \Omega_\sigma$ the support of the function $\rho_\sigma(x-\cdot)$ is contained in Ω. Then, by considering $\rho_\sigma(x-\cdot)$ as a test function, we have by performing an integration by parts:

$$g_n(x) = \int_\Omega \nabla_y \rho_\sigma(x-y) \nabla u_n(y) dy = \int_\Omega \Delta(-\rho_\sigma(x-y)) u_n(y) dy.$$

Thus, since $u_n \to u$ in $L^2(\Omega)$ with $u \in HS\Delta\mathcal{M}^p(\Omega)$, we have, by performing another integration by parts,

$$g_n(x) \to \int_\Omega \Delta(-\rho_h(x-y)) u(y) dy = g(x) \quad \text{almost everywhere in } \Omega_\sigma.$$

Then we also have

$$|g_n(x) - u_0|^2 \to |g(x) - u_0|^2 \quad \text{almost everywhere in } \Omega_\sigma. \tag{4.8}$$

By using standard properties of convolution together with bound (4.7) we get:

$$
\begin{aligned}
|g_n(x)-u_0|^2 &\le |g_n(x)|^2+|u_0|^2 \le \|\rho_\sigma\|_\infty^2\big(|\delta_{P_{\nabla u_n}}|(\Omega)\big)^2+|u_0|^2 \\
&\le \big(\mathcal{H}^0(P_{\nabla u_n})\big)^2+|u_0|^2 \le M^2+\|u_0\|_\infty^2 := K
\end{aligned} \tag{4.9}
$$

with K independent of n. Then by (4.8) and (4.9), we can apply the dominated convergence theorem. So that

$$
|g_n(x)-u_0|^2 \to |g(x)-u_0|^2 \quad \text{strongly in } L^1(\Omega_\sigma)
$$

$$\Downarrow$$

$$
\lim_{n\to+\infty}\int_{\Omega_\sigma}|\rho_\sigma * \delta_{P_{\nabla u_n}}-u_0|^2dx=\int_{\Omega_\sigma}|\rho_\sigma * \delta_{P_{\nabla u}}-u_0|^2dx. \tag{4.10}
$$

By taking into account the superlinearity property of the lim inf operator, from assumption (4.5) it follows (4.6). So the proof is complete. □.

As a consequence of Theorems 4.1 and 4.2, we obtain the following existence result.

Theorem 4.3. *There exists a solution $u \in HS\Delta\mathcal{M}^p(\Omega)$ of problem (3.4).*

Remark 4.1. *As a consequence of the previous theorem and Proposition 3.2, we obtain that $\mu=\delta_{P_{\nabla u}}$ with u solution of problem (3.4), is a solution of problem (3.3).*

5. Variational approximation with smooth sets

Inspired by [6, 11] we investigate the variational approximation for functional G via depending curvature functionals defined on smooth sets.

5.1. Definition of the approximating sequence.

As described in the introduction the first step is the following formal substitution:

$$
\mathcal{H}^0(P_{\nabla u}) \cong \frac{1}{4\pi}\int_{\partial D}\big(\frac{1}{\epsilon}+\epsilon\kappa^2\big)d\mathcal{H}^1, \tag{5.1}
$$

where $D \in R(\Omega)$ and k denotes its curvature. Next we consider the associate density measures $\theta_\epsilon d\mathcal{H}^1\lfloor D=\big(\frac{1}{\epsilon}+\epsilon\kappa^2\big)d\mathcal{H}^1\lfloor D$. Consequently (5.1) gives in term of total variation of Radon measures

$$
\big|\delta_{P_{\nabla u}}\big|(\Omega) \cong \frac{1}{4\pi}\big|\theta_\epsilon d\mathcal{H}^1\lfloor D\big|(\Omega) \tag{5.2}
$$

Therefore we want to approximate the whole functional G with:

$$
G_\epsilon(D)=\frac{1}{4\pi}\int_{\partial D}\big(\frac{1}{\epsilon}+\epsilon\kappa^2\big)d\mathcal{H}^1+\frac{1}{\epsilon}\mathcal{L}^2(D)+\int_{\Omega_\sigma}|\rho_\sigma * \theta_\epsilon d\mathcal{H}^1\lfloor D-u_0|^2dx \quad \text{on } R(\Omega), \tag{5.3}
$$

where the second term forces D to have small Lebesgue measure.

5.2. **Ad hoc convergence and Γ-convergence.** As in [6, 11] we adopt an analogous specific notion of convergence to deal with regular sets and and functions belonging to a distributional space. Let us set $X^p(\Omega) = \{u \in HS\Delta\mathcal{M}^p(\Omega); \quad \mathcal{H}^0(P_{\nabla u}) < +\infty)$. For the convenience of the reader we recall that functional $G : X^p(\Omega) \to [0, +\infty)$ is defined by

$$G(u) = \mathcal{H}^0(P_{\nabla u}) + \|\rho_\sigma * \delta_{P_{\nabla u}} - u_0\|^2_{L^2(\Omega_\sigma)}. \tag{5.4}$$

Definition 5.1. *We say that a sequence $\{D_h\}_h \subset R(\Omega)$ H-converges to $u \in X^p(\Omega)$ if the following three conditions hold*

(i) $\mathcal{L}^2(D_h) \to 0$;

(ii) $\{\partial D_h\}_h \to P \subset \Omega$ *in the Hausdorff metric, where P is a finite set of points;*

(iii) $P_{\nabla u} = P \setminus \partial\Omega$.

Then Γ-convergence is then defined according to the ad hoc convergence.

Definition 5.2. *We say that G_ϵ Γ-converges to G if for every sequence of positive numbers $\{\epsilon_h\}_h \to 0$ and for every $u \in X^p(\Omega)$ we have:*

(i) *for every sequence $\{D_h\}_h \subset R(\Omega)$ H-converging to u*

$$\liminf_{h\to+\infty} G_{\epsilon_h}(D_h) \ge G(u);$$

(ii) *there exists a sequence $\{D_h\}_h \subset R(\Omega)$ H-converging to u such that*

$$\limsup_{h\to+\infty} G_{\epsilon_h}(D_h) \le G(u).$$

5.3. **Compactness and Γ-convergence.** The compactness and Γ-convergence theorems, proven in [11], will play an important role for our variational approximation result.

In order to make the paper self-contained, we state, in a very simplified form, the result proven in [11]. (see Theorem 4.1 of [11] for a complete statement and proof). We just recall the part concerning the counting measure, which we will use in the sequel.

Theorem 5.1. *Let $\{\epsilon_h\}_h \to 0^+$. Then the following properties holds.*

(i) *Let $\{D_h\}_h \subset R(\Omega)$ be such that*

$$\frac{1}{4\pi}\int_{\partial D_h}\Big(\frac{1}{\epsilon_h} + \epsilon_h\kappa^2\Big)d\mathcal{H}^1 + \frac{1}{\epsilon_h}\mathcal{L}^2(D_h) \le M < +\infty.$$

Then there exists $\{D_{h_k}\}_k \subset R(\Omega)$ and $P \subset \overline{\Omega}$ such that $\partial D_{h_k} \to P$, with respect to Haussdorf distance.

(ii) *for every sequence $\{D_h\}_h \subset R(\Omega)$ and a set of points $P \subset \overline{\Omega}$ such that $\partial D_h \to P$, with respect to the Haussdorf distance, we have:*

$$\liminf_{h\to+\infty}\frac{1}{4\pi}\int_{\partial D_h}(\frac{1}{\epsilon_h}+\epsilon_h\kappa^2)d\mathcal{H}^1\geq\mathcal{H}^0(P)$$

(iii) *for every set $P\subset\overline{\Omega}$ there exists a sequence $\{D_h\}_h\subset R(\Omega)$ converging to P, with respect to the Haussdorf distance, such that*

$$\limsup_{h\to+\infty}\frac{1}{4\pi}\int_{\partial D_h}(\frac{1}{\epsilon_h}+\epsilon_h\kappa^2)d\mathcal{H}^1=\mathcal{H}^0(P).$$

In the next compactness theorem we adapt property (i) of the previous theorem to our framework. Moreover we prove the weak star convergence of the measures $d\mathcal{H}^1\lfloor D_{h_k}$ to the measure $\delta_{P_{\nabla u}}$.

Theorem 5.2. *Let $\{\epsilon_h\}_h\to 0^+$ be such that*

$$G_{\epsilon_h}(D_h)\leq M, \tag{5.5}$$

then

(i) *there exists a subsequence $\{D_{h_k}\}_k\subset R(\Omega)$ and a function $u\in X^p(\Omega)$ such that $\{D_{h_k}\}_k$ H-converges to u.*

(ii) *the sequence of Radon measures $\{\theta_{\epsilon_{h_k}}d\mathcal{H}^1\lfloor D_{h_k}\}_k$, possibly passing to a subsequence, converges, with respect to the weak star convergence, to the measure $\delta_{P_{\nabla u}}$.*

Proof.

(i). From bound (5.5) and Theorem 5.1 we have that there exists a subsequence $\{D_{h_k}\}_k$, which converges, with respect tho the Haussdorf distance, to a finite set of points $P\subset\overline{\Omega}$ with $\mathcal{L}^2(D_{h_k})\to 0$. We thus consider the Dirichlet problem

$$\begin{cases}-\Delta u=\delta_{P\setminus\partial\Omega} & \text{on } \Omega\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

There exists a weak solution $u\in W_0^{1,p}$ with $p<2$. Moreover $-\Delta u=0$ in $\mathcal{D}'(\Omega\setminus(P\setminus\partial\Omega))$. Then by Proposition 2.1 we have $u\in HS\Delta\mathcal{M}^p(\Omega)$, with $P_{\nabla u}=P\setminus\partial\Omega$. Then $u\in X^p(\Omega)$, since $\mathcal{H}^0(P_{\nabla u})\leq\mathcal{H}^0(P)<+\infty$. So that $\{D_{h_k}\}_k$ H-converges to u and the proof of (i) is complete.

We now prove (ii).

First of all we note that from bound (5.5) we have:

$$|\theta_{\epsilon_{h_k}}d\mathcal{H}^1\lfloor D_{h_k}|(\Omega)\leq M. \tag{5.6}$$

Then up to subsequences it follows that:

$$\theta_{\epsilon_h}d\mathcal{H}^1\lfloor D_h\overset{*}{\rightharpoonup}\mu \tag{5.7}$$

for some positive measure $\mu \in \mathcal{M}(\Omega)$. The idea is then to show that $\mu = \delta_{P_{\nabla u}}$ in $\mathcal{M}(\Omega)$. The argument is similar to the one used to prove compactness in Theorem 4.1. We consider the Dirichlet problem with measure data

$$\begin{cases} -\Delta w_k = \theta_{\epsilon_h} d\mathcal{H}^1 \lfloor D_h & \text{on } \Omega \\ w_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.8}$$

By Theorem 9.1 of [24] and (5.6) we have the estimate

$$\|w_k\|_{W_0^{1,p}} \leq C$$

where the constant C does not depend on k.

Sobolev's embedding implies that, possibly passing to a subsequence, $\{w_k\}_k$ is such that:

$$\begin{cases} w_k \to w & \text{in } L^2(\Omega) \text{ and a.e.} \\ \nabla w_k \rightharpoonup \nabla w & \text{in } L^p(\Omega; \mathbb{R}^2). \end{cases} \tag{5.9}$$

We claim that for k large enough we have $-\Delta w = 0$ in $\mathcal{D}'(\Omega \setminus P)$. Let $\phi \in C_0^\infty(\Omega \setminus P)$ be a test function. Then since w_k is a weak solution of problem (5.8) and ϕ is also a test function in $C_0^\infty(\Omega)$, we have

$$\int_{\text{supp}\phi} \nabla w_k \nabla \phi = \int_{\text{supp}\phi} \phi \theta_{\epsilon_{h_k}} d\mathcal{H}^1 \lfloor D_{h_k}. \tag{5.10}$$

Now we know that ∂D_{h_k} converges in the Haussdorf metric to P. Therefore for k large enough we have $\text{supp}\phi \cap \partial D_k = \emptyset$, whereas the support of ϕ is contained in $\Omega \setminus P$.

By taking the limit as $k \to +\infty$ in (5.10), we obtain $-\Delta w = 0$ in $\mathcal{D}'(\Omega \setminus P)$. In particular $-\Delta w = 0$ in $\mathcal{D}'(\Omega \setminus P_{\nabla u})$, being $P_{\nabla u} = P \setminus \partial\Omega$. So, by Proposition 2.1, we have $w \in \Delta\mathcal{M}^p(\Omega)$.

On the other hand, by (5.7) and (5.9), we can pass to limit in problem (5.8) to get as $k \to +\infty$

$$\int_\Omega \nabla w_k \nabla \phi = \langle \theta_{\epsilon_k} d\mathcal{H}^1 \lfloor D_h, \phi \rangle$$
$$\downarrow$$
$$\int_\Omega \nabla w \nabla \phi = \langle \mu, \phi \rangle.$$

So $-\Delta w = \mu$ in $\mathcal{D}'(\Omega)$ and therefore in $\mathcal{M}(\Omega)$. Being the measure $-\Delta w$ positive, we obtain by Proposition 2.1, $w \in HS\Delta\mathcal{M}^p(\Omega)$. with $P_{\nabla w} = P_{\nabla u}$ and therefore $-\Delta w = \delta_{P_{\nabla u}}$. So that $\mu = \delta_{P_{\nabla u}}$ in $\mathcal{M}(\Omega)$ and (ii) is proven. The proof is now complete. □

Theorem 5.3. *Let $\{\epsilon_h\}_h \to 0^+$. Let G_ϵ, G be defined by (5.3) and (5.4). Then the sequence G_ϵ Γ-converge to G.*

Proof.

Lower bound. We prove property (i) of Definition 5.2. Without loss of generality we can assume

$$\liminf_{h\to+\infty} G_{\epsilon_h}(D_h) = \lim_{h\to+\infty} G_{\epsilon_h}(D_h) \le M < +\infty. \tag{5.11}$$

Then, by (i) of Theorem 5.2, up to subsequences we can assume D_h H-converges to $u \in X^p(\Omega)$. So that $\mathcal{H}^0(P_{\nabla u}) \le \mathcal{H}^0(P)$.

By (ii) of Theorem 5.1 we have

$$\liminf_{h\to+\infty} \frac{1}{4\pi}\int_{\partial D_h} (\frac{1}{\epsilon_h} + \epsilon_h \kappa^2) d\mathcal{H}^1 \ge \mathcal{H}^0(P) \ge \mathcal{H}^0(P_{\nabla u}) \tag{5.12}$$

We now focus on the third term.

By using to the compactness properties (ii) of Theorem 5.1 we have possibly passing to a subsequence

$$\theta_{\epsilon_h} d\mathcal{H}^1 \lfloor D_h \stackrel{*}{\rightharpoonup} \delta_{P_{\nabla u}}.$$

Therefore by the compactness of the operator $T(\mu) = \rho_\sigma * \mu$ defined in section 2.2, we obtain

$$\liminf_{h\to+\infty} \int_{\Omega_\sigma} \left|\rho_\sigma * \theta_{\epsilon_h} d\mathcal{H}^1 \lfloor D_h - u_0\right|^2 dx \ge \int_{\Omega_\sigma} |\rho_\sigma * \delta_{P_{\nabla u}} - u_0|^2 dx. \tag{5.13}$$

Hence from (5.12),(5.13) and by superlinearity property of the lim inf operator it follows property (i) of Definition 5.2.

Upper bound. Let $u \in X^p(\Omega)$.

Up to a slight modification, we take as optimal sequence $\{D_h\}_h$, the same one considered to prove (iii) of Theorem 5.1 in [11].

Let n be the number of points x_i in $P_{\nabla u}$. Therefore we have $\delta_{P_{\nabla u}} = \sum_{i=1}^n a_i \delta_{x_i}$ with $a_i \in [0,1]$. Then we take $D_h = \bigcup_{i=1}^n B_{a_i\epsilon_h}(x_i)$. So that $\mathcal{L}^2(D_h) \to 0$, $\frac{1}{\epsilon_h}\mathcal{L}^2(D_h) \to 0$ and ∂D_h converges with respect to the Hausdorff distance to $P_{\nabla u}$. Then the sequence $\{D_h\}$ H-converges to u.

Moreover for h large enough we may assume $B_{a_i\epsilon_h}(x_i) \cap B_{a_j\epsilon_h}(x_j) = \emptyset$ for $i \ne j$.

Thus we obtain

$$\lim_h \frac{1}{4\pi}\int_{\partial D_h} (\frac{1}{\epsilon_h} + \epsilon_h k^2) d\mathcal{H}^1 = \lim_h \sum_{i=1}^n \frac{1}{4\pi}\int_{\partial B_{a_i\epsilon_h}(x_i)} \frac{2}{\epsilon_h} d\mathcal{H}^1 = \sum_{i=1}^n a_i \le n = \mathcal{H}^0(P_{\nabla u}). \tag{5.14}$$

We show now that $\theta_{\epsilon_h} d\mathcal{H}^1 \lfloor D_h \stackrel{*}{\rightharpoonup} \delta_{P_{\nabla u}}$. Let $\phi \in C_0(\Omega)$ a test function. Let (x_i^1, x_i^2) be the coordinates of the point x_i for $i = 1, .., n$. Then by writing the curvilinear integral

$$\langle \theta_{\epsilon_h} d\mathcal{H}^1 \lfloor D_h, \phi\rangle = \sum_{i=1}^n \frac{1}{4\pi}\int_0^{2\pi} \phi(x_i^1 + a_i\epsilon_h \cos\theta, x_i^2 + a_i\epsilon_h \sin\theta)\frac{2}{\epsilon_h} a_i \epsilon_h d\theta \to \sum_{i=1}^n a_i \phi(x_i),$$

that is $\theta_{\epsilon_h} d\mathcal{H}^1 \lfloor D_h \stackrel{*}{\rightharpoonup} \delta_{P_{\nabla u}}$.

Hence the compactness of the operator T defined in section 2.2 implies

$$\lim_{h\to+\infty}\int_{\Omega_\sigma}|\rho_\sigma*\theta_{\epsilon_h}d\mathcal{H}^1\lfloor D_h-u_0|^2=|\int_{\Omega_\sigma}\rho_\sigma*\delta_{P_{\nabla u}}-u_0|^2. \tag{5.15}$$

Finally we know that:

$$\frac{1}{\epsilon_h}\mathcal{L}^2(D_h)\to 0. \tag{5.16}$$

By collecting (5.14), (5.15), (5.16) and recalling that the lim sup is a sublinear operation we achieve (ii) of Definition 5.2. The proof is complete. □

We conclude by properly stating the relaxed version of the Fundamental Theorem of Γ-convergence which is a direct consequence of Theorem 5.2, and Theorem 5.3. The proof can be achieved by a classical argument (see [9], Section 1.5). See also Theorem 4.4 of [6] for a selfcontained proof in a similar context.

Theorem 5.4. *Let G_ϵ, G be given respectively by (5.3) and (5.4). If $\{\epsilon_h\}_h$ is a sequence of positive numbers converging to zero and $\{D_h\}_h\subset R(\Omega)$ such that*

$$\lim_{h\to+\infty}(G_{\epsilon_h}(D_h)-\inf_{R(\Omega)}G_{\epsilon_h}(D))=0,$$

then there exist a subsequence $\{D_{h_k}\}_k\subset R(\Omega)$ and a minimizer $\overline{u}$ of $G(u)$ with $\overline{u}\in X^p(\Omega)$, such that D_{h_k} H-converges to $\overline{u}$.

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