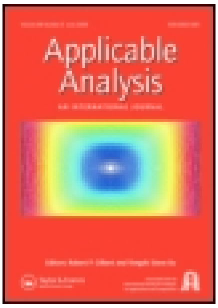

Applicable Analysis: An International Journal
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gapa20

Existence result for a free-discontinuity energy governing the detection of spots in image processing

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Available online: 09 Feb 2012

Applicable Analysis
2012, 1–10, iFirst

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Existence result for a free-discontinuity energy governing the detection of spots in image processing

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Communicated by S. Carl

(*Received 17 May 2011; final version received 26 December 2011*)

We prove an existence result for a new integral functional of image processing made up by a convex volume term and a counting measure term.

Keywords: direct methods; points detection; biological images; divergence-measure fields; p-capacity

AMS Subject Classifications: 49J45; 49Q20

1. Introduction

In this article we study the existence of a minimizer for the integral functional

$$F(u) = \int_{\Omega} f(x, u, \nabla u, \operatorname{div} \nabla u) \mathrm{d}x + \mathcal{H}^0(P_{\nabla u}), \tag{1.1}$$

where $u \in \Delta\mathcal{M}^p(\Omega)$ is the space of $W_0^{1,p}$-functions whose gradient has distributional divergence given by a measure $\operatorname{Div} \nabla u$ with singular part concentrated on points (Section 2.4). Here $\Omega \subset \mathbb{R}^2$ is an open set with Lipschitz boundary, $1 < p < 2$, f is a Carathéodory function, $\operatorname{div} \nabla u$ is the absolutely continuous part of the measure $\operatorname{Div} \nabla u$, $P_{\nabla u}$ denotes the support of the singular (with respect to the p-capacity) part of the measure of $\operatorname{Div} \nabla u$. $\mathcal{H}^0$ is the Hausdorff counting measure (see Section 2 for preliminaries and definitions).

We assume that integrand is convex in the last variable and satisfies the weak coercivity condition:

$$c_1|t|^p + c_2|\xi| - \xi_0 \cdot \xi - a(x) \leq f(x, s, \xi, t) \quad \text{for } \mathcal{L}^2\text{-a.e. } x \in \Omega \text{ and } \forall (s, \xi, t) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \tag{1.2}$$

with c_1, $c_2 > 0$ positive constants, $\xi_0 \in \mathbb{R}^2$, a is a positive $L^1(\Omega)$-function, $1 < p < 2$. Thus if the integrand fulfils assumption (1.2), we are able to prove an existence result for functional (1.1).

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ISSN 0003–6811 print/ISSN 1563–504X online

http://dx.doi.org/10.1080/00036811.2011.653794
http://www.tandfonline.com

The study of this problem was motivated by a recent variational model for the detection of spots in image processing investigated in [1,2]. In those works, in order to detect point-like singularities in the observed image, one seeks for a function space whose elements generate, in a suitable sense, a measure concentrated on points. The natural framework is the space $\mathcal{DM}^p(\Omega)$ of vector fields whose distributional divergence is a Radon measure [3–5]. The restriction on p is due to the fact that when $p \geq 2$ the distributional divergence of an L^p-vector field U cannot be a measure concentrated on points (see Section 3 in [1]). For initializing the minimization process, it is necessary to construct, from the initial image, a vector field U_0 belonging to $\mathcal{DM}^p(\Omega)$. To do this, one can solve the classical Dirichlet problem with measure data.

$$\begin{cases} -\Delta f = I & \text{on } \Omega, \\ f = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

and set $U_0 = -\nabla f$. Hence, after the initialization the task is to keep nothing else but points in the restored image. Thus one has to build up, starting from the initial data U_0, a new vector field U whose singularities are given by the points of the image I to be detected. Nevertheless, the support of the measure Div U_0 is far from containing nothing else but points. Indeed, in general, there could be several geometric structures, such as curves or fractals, in the support the measure Div U_0. Therefore it is crucial to distinguish the different type of singularities, in order to detect isolate points in the detection process.

In this direction the notion of p-capacity of a set plays a key role. Indeed, when $p < 2$ the p-capacity of a point in Ω is zero and one can say, in this sense, that it is a discontinuity without jump (Theorem 2.1 and Definition 2.3). On the contrary, the p-capacity of sets with Hausdorff dimension greater than 1, is strictly positive. Furthermore, every Radon measure can be decomposed in two mutually singular measures: the first one is absolutely continuous with respect to the p-capacity, that is concentrated on sets with positive p-capacity, and the second one is singular with respect to the p-capacity, that is a measure concentrated on sets with zero p-capacity [6]. In dimension two, sets with zero p-capacity, and hence discontinuities without jump, can be isolated points, countable set of points or fractals with Hausdorff dimension $0 \leq \alpha < 1$ (see Section 2.3 for the definition of p-capacity and related properties). Finally to achieve the purpose of isolating points, it is necessary to minimize a suitable energy that must remove all the discontinuities which are not discontinuities without jump and remove all the discontinuities without jump which are not isolated points.

Then, these considerations lead to introduce a free discontinuity energy made up by divergence volume term and the counting Hausdorff measure $\mathcal{H}^0$:

$$F(u) = \int_\Omega |\Delta u|^2 \,\mathrm{d}x + \lambda \int_\Omega |\nabla u - U_0|^p \,\mathrm{d}x + \mu \mathcal{H}^0(P_{\nabla u}), \tag{1.4}$$

where $u \in W_0^{1,p}(\Omega)$, ∇u belongs to the space $\mathcal{SDM}^p(\Omega)$ of vector fields whose divergence measure has no absolutely continuous part with respect to the p-capacity, $1 < p < 2$, λ and μ are positive weights. The first integral forces u to be regular outside $P_{\nabla u}$, while the term $\mathcal{H}^0(P_{\nabla u})$ penalizes the presence of singular curves in the image and limits the number of points detected, in order to avoid false detection due to noise. This approach for detecting point-like target problems seems to be more natural than those based on the Ginzburg–Landau theory (see [7] for instance). We refer the reader to [1,2] for further details on the model and experimental results.

This type of energies is in the spirit of other classical free discontinuity energies of the literature: Mumford–Shah's functional, fracture's energies (see, for instance, [8,9]). However we stress that our case is also deeply different, since we deal with point-like singularities and not segment curves or fractures. This requires the use of a different variational framework.

Following these ideas, the goal of this article is to first provide an analysis of new anisotropic free-discontinuity energies such as (1.1). For the applications it can be crucial to allow a dependence with respect to other variables in the integrand f. Indeed, in the damaged image reconstruction one might like to emphasize the singularities contained in a given region of Ω by giving appropriate values to the entries of the integrand f. For instance, in contour restoration anisotropic total variation has important applications in image processing problems such as edge linking [10]. It is then natural to study anisotropic functionals also for detecting point-like target case.

From a technical of point of view the proof of the existence relies on the fact that, thanks to the growth assumption (1.2), one can control, via elliptic regularity, the L^p-norm of the gradient of the minimizing sequences, without having not even a 1-growth from above in the gradient variable, or a p-growth as in [1]. Indeed, with respect to the gradient variable, only a demi-coercivity assumption is made here (Remark 3.1). In fact, one obtains a control on the L^1-norm of the gradient of the minimizing sequence from boundary condition. So that by combining Sobolev embedding together with elliptic regularity and adapting the techniques used in [1], one is able to get compactness property in the space $\Delta\mathcal{M}^p(\Omega)$. Then by using convex analysis tools and the semicontinuity of $\mathcal{H}^0$ for finite set of points, we also get semicontinuity properties of functional F with respect to $W_0^{1,p}(\Omega)$-weak convergence. Thus, roughly speaking, we obtain the existence of a minimum via direct method of calculus of variations.

This article is organized as follows. Section 2 contains some mathematical tools, which are used in the remainder of this article. In Section 3 we address the existence result for the functional F defined in (1.1).

2. Preliminaries

2.1. *Notation*

Throughout this article $\Omega \subset \mathbb{R}^2$ is an open bounded set with Lipschitz boundary. The Euclidean norm will be denoted by $|\cdot|$, while the symbol $\|\cdot\|$ indicates the norm of some function spaces. The brackets $\langle , \rangle$ denote the duality product in some distributional spaces. $\mathcal{L}^d$ or $\mathrm{d}x$ is the d-dimensional Lebesgue measure and $\mathcal{H}^k$ is the k-dimensional Hausdorff measure. Notation for Sobolev spaces and Lebesgue spaces is standard.

2.2. *Distributional divergence and classical spaces*

In this subsection we recall the definition of the distributional space $L^{p,q}(\mathrm{div};\Omega)$ and $\mathcal{DM}^p(\Omega)$, $1 \le p,\ q \le +\infty$ (see [3,4]).

Definition 2.1 We say that $U \in L^{p,q}(\mathrm{div};\, \Omega)$ if $U \in L^p(\Omega;\, \mathbb{R}^2)$ and if its distributional divergence $\mathrm{Div}\, U = \mathrm{div}\, U \in L^q(\Omega)$. If $p = q$, the space $L^{p,q}(\mathrm{div};\, \Omega)$ will be denoted by $L^p(\mathrm{div};\, \Omega)$.

We say that a function $u \in W^{1,p}(\Omega)$ belongs to $W^{1,p,q}(\mathrm{div};\, \Omega)$ if $\nabla u \in L^{p,q}(\mathrm{div};\, \Omega)$. We say that a function $u \in W_0^{1,p}(\Omega)$ belongs to $W_0^{1,p,q}(\mathrm{div};\, \Omega)$ if $\nabla u \in L^{p,q}(\mathrm{div};\, \Omega)$.

Definition 2.2 For $U \in L^p(\Omega;\, \mathbb{R}^2)$, $1 \le p \le +\infty$, set

$$|\mathrm{Div}\, U|(\Omega) := \sup\{\langle U, \nabla\varphi\rangle : \varphi \in C_0^\infty(\Omega),\ |\varphi| \le 1\}.$$

We say that U is an L^p-divergence measure field, i.e. $U \in \mathcal{DM}^p(\Omega)$, if

$$\|U\|_{\mathcal{DM}^p(\Omega)} := \|U\|_{L^p(\Omega;\mathbb{R}^2)} + |\mathrm{Div}\, U|(\Omega) < +\infty.$$

We recall that $U \in L^p(\Omega;\, \mathbb{R}^N)$ belongs to $\mathcal{DM}^p(\Omega)$ if and only if there exists a Radon measure denoted by $\mathrm{Div}\, U$ such that

$$\langle U, \nabla\varphi\rangle = -\int_\Omega \mathrm{Div}\, U\varphi \quad \forall \varphi \in C_0^\infty(\Omega),$$

and the total variation of the measure $\mathrm{Div}\, U$ is given by $|\mathrm{Div}\, U|(\Omega)$.

2.3. *p-Capacity*

The notion of p-capacity of sets will be crucial to find a convenient functional framework to deal with. If $K \subset \mathbb{R}^2$ is a compact set and χ_K denotes its characteristic function, we define

$$\mathrm{Cap}_p(K, \Omega) = \inf\left\{\int_\Omega |\nabla f|^p\, \mathrm{d}x,\ f \in C_0^\infty(\Omega), f \ge \chi_k\right\}.$$

If $U \subset \Omega$ is an open set and $K \subset U$ is a compact set, its p-capacity is given by

$$\mathrm{Cap}_p(U, \Omega) = \sup_{K \subset U} \mathrm{Cap}_p(K, \Omega).$$

Finally, if $A \subset U \subset \Omega$ with A Borel set and U open then

$$\mathrm{Cap}_p(A, \Omega) = \inf_{A \subset U \subset \Omega} \mathrm{Cap}_p(U, \Omega).$$

We recall the following result (see, for instance, [11, Theorem 2.27]) that explains the relationship between p-capacity and Hausdorff measure. Such a result is crucial to have geometric information on null p-capacity sets.

THEOREM 2.1 *Assume* $1 < p < 2$. *If* $\mathcal{H}^{2-p}(A) < \infty$ *then* $\mathrm{Cap}_p(A, \Omega) = 0$.

For a general survey, we refer the reader to [11–13].

2.4. *p-capacity decomposition*

If $U \in \mathcal{DM}^p(\Omega)$, According to the Radon–Nikodym decomposition of the measure $\mathrm{Div}\, U$, we have

$$\mathrm{Div}\, U = \mathrm{div} U\, \mathrm{d}x + \mathrm{div}^s U,$$

where $\operatorname{div} U \in L^1(\Omega)$ and $\operatorname{div}^s U$ is a singular measure with respect to $\mathcal{L}^2$.

It is known (see [6]) that given a Radon measure μ, the following decomposition holds:

$$\mu = \mu_a + \mu_0, \tag{2.1}$$

where the measure μ_a is absolutely continuous with respect to the p-capacity and μ_0 is singular with respect to the p-capacity, that is concentrated on sets with zero p-capacity. Besides it is also known (see [6]) that every measure which is absolutely continuous with respect to the p-capacity can be characterized as an element of $L^1 + W^{-1,p'}$, leading to the finer decomposition:

$$\mu = f - \operatorname{Div} G + \mu_0, \tag{2.2}$$

where $G \in L^{p'}(\Omega; \mathbb{R}^2)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in L^1(\Omega)$.

By applying this decomposition to the measure $\operatorname{div}^s U$, we obtain the following decomposition of the measure $\operatorname{Div} U$:

$$\operatorname{Div} U = \operatorname{div} U \, dx + f - \operatorname{Div} G + (\operatorname{div}^s U)_0, \tag{2.3}$$

with $G \in L^{p'}(\Omega; \mathbb{R}^2)$, $f \in L^1(\Omega)$, $\operatorname{div} U \in L^1(\Omega)$ and $(\operatorname{div}^s U)_0$ is a measure concentrated on a set with zero p-capacity.

According to this decomposition, it is possible to introduce the notion of discontinuity without/with jump.

Definition 2.3 We say that a point $x \in \Omega \subset \mathbb{R}^2$ is a point of discontinuity without jump of U if $x \in \operatorname{supp}(\operatorname{div}^s U)_0$.

Definition 2.4 We say that a point $x \in \Omega \subset \mathbb{R}^2$ is a point of discontinuity with jump of U if $x \in \operatorname{supp}(f - \operatorname{Div} G)$.

Since we are interested in detecting points, we define the space

$$\mathcal{SDM}^p(\Omega) := \{U \in \mathcal{DM}^p(\Omega), \quad f - \operatorname{Div} G = 0\}. \tag{2.4}$$

We state the following result which can be proved as in Proposition 3.1 in [1].

Proposition 2.1 *Let $P \subset \Omega$ be a finite set of points. Let $u \in W_0^{1,p,q}(\mathrm{div}; \Omega \setminus P)$, with $1 < p \le q < 2$. Then $\nabla u \in \mathcal{SDM}^p(\Omega)$, with $(\operatorname{div}^s \nabla u)_0 = P$.*

Finally, as in [1], we restrict our attention to vector fields U which are the gradient of a function $u \in W_0^{1,p}(\Omega)$. This leads to define the space

$$\Delta\mathcal{M}^p(\Omega) := \{u \in W_0^{1,p}(\Omega), \ \nabla u \in \mathcal{SDM}^p(\Omega)\}. \tag{2.5}$$

2.5. *Convergence for a set of points*

We recall the notion of convergence for finite set of points [2,14].

Definition 2.5 We say that a sequence of a finite set of points $\{P_h\} \subset \overline{\Omega}$ converges to a set $P \subset \overline{\Omega}$ if each of the sets P_h contains a number N of points $\{x_h^1, \ldots, x_h^N\}$, with N independent of h, such that $x_h^i \to x^i$ for any $i = 1, \ldots, n$ and $\bigcup_{i=1}^N \{x_i\} = P$.

The following results [2] will be useful.

LEMMA 2.1 *Let $\{P_h\}$ be a sequence of a finite set of points such that $\mathcal{H}^0(P_h) \le N_0$ for every h with $N_0 \in \mathbb{N}$. Then there exists a subsequence $\{P_{h_k}\} \subset \{P_h\}$ and a set of points $P \subset \overline{\Omega}$ such that P_{h_k} converges with respect to the convergence of Definition 2.5 to the set P.*

LEMMA 2.2 *Let $\{P_h\} \subset \overline{\Omega}$ be a sequence of a finite set of points converging to a finite set of points P. Then*

$$\mathcal{H}^0(P) \le \liminf_{h \to +\infty} \mathcal{H}^0(P_h). \tag{2.6}$$

3. The existence result

We shall consider a nonnegative integrand $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ such that

(i) $x \mapsto f(x, s, \xi, t)$ is measurable for any $(s, \xi, t) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$,
(ii) $(s, \xi, t) \mapsto f(x, s, \xi, t)$ is continuous for $\mathcal{L}^2$-a.e. $x \in \Omega$,
(iii) $t \mapsto f(x, s, \xi, t)$ is convex for $\mathcal{L}^2$-a.e. $x \in \Omega$ and for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^2$.

In the proof of the existence result, we will apply the following well-known lower semicontinuity lemma. A proof can be found, for instance, in [15, see Theorem 4.5].

THEOREM 3.1 *Let $g(x, v, z)$ be a nonnegative function from $\Omega \times \mathbb{R}^m \times \mathbb{R}^k$, such that $x \mapsto g(x, v, z)$ is measurable for any $(v, z) \in \mathbb{R}^m \times \mathbb{R}^k$, $(v, z) \mapsto g(x, v, z)$* is continuous for *$\mathcal{L}^d$-a.e. $x \in \Omega$ and convex in z for $\mathcal{L}^d$-a.e. $x \in \Omega$ and for every $v \in \mathbb{R}^m$. Let v_h, $v \in L^1(\Omega; \mathbb{R}^m)$, z_h, $z \in L^1(\Omega; \mathbb{R}^k)$. If v_h converges to v strongly in $L^1(\Omega; \mathbb{R}^m)$ and z_h converges weakly to z in $L^1(\Omega; \mathbb{R}^k)$, then*

$$\int_\Omega g(x, v, z)\mathrm{d}x \le \liminf_{h \to +\infty} \int_\Omega g(x, v_h, z_h)\mathrm{d}x.$$

Finally, we also assume the following weak coerciveness assumption:

$$c_1|t|^p + c_2|\xi| - \xi_0 \cdot \xi - a(x) \le f(x, s, \xi, t) \quad \text{for } \mathcal{L}^2\text{-a.e. } x \in \Omega \text{ and } \forall (s, \xi, t) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \tag{3.1}$$

with c_1, c_2 positive constants, $\xi_0 \in \mathbb{R}^2$, a is a nonnegative $L^1(\Omega)$-function and $1 < p < 2$.

Remark 3.1 Assumption (3.1) is weaker than a typical coercivity assumption of type:

$$c_1|t|^p + c_2|\xi| - a(x) \le f(x, s, \xi, t) \quad \text{for } \mathcal{L}^2\text{-a.e. } x \in \Omega \text{ and for every } (s, \xi, t) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}.$$

For instance, it is not difficult to check that the function $f(\xi, t) = |t|^p + \max\{\xi_1, 0\} + \max\{\xi_2, 0\}$ satisfies (3.1) but does not verify the above standard coercivity assumption. When the integrand does not depend on t, condition (3.1) is known as demicoerciveness and it was introduced in [16].

For $u \in \Delta\mathcal{M}^p(\Omega)$ with $1 < p < 2$, we consider F: $\Delta\mathcal{M}^p(\Omega) \to [0, +\infty]$ given by

$$F(u) = \int_\Omega f(x, u, \nabla u, \operatorname{div} \nabla u) + \mathcal{H}^0(P_{\nabla u}), \tag{3.2}$$

where to simplify the notation we have set $P_{\nabla u} = \operatorname{supp}(\operatorname{div}^s \nabla u)_0$. We consider the following minimum problems:

$$\min_{u \in \Delta\mathcal{M}^p(\Omega)} F(u). \tag{3.3}$$

THEOREM 3.2 *Let f be a nonnegative integrand which satisfies assumptions* (i), (ii), (iii) *and* (3.1). *Then minimum problem* (3.3) *admits at least a solution.*

Proof Without loss of a generality, we may assume that $\inf F < +\infty$.

Let $\{u_n\}_n \subset \Delta\mathcal{M}^p(\Omega)$ be a minimizing sequence such that

$$\liminf_{n\to+\infty} F(u_n) = \lim_{n\to+\infty} F(u_n) = \inf F, \tag{3.4}$$

so that we also have

$$\lim_{n\to+\infty} F(u_n) \leq M, \tag{3.5}$$

where M is a positive constant. From (3.5) we have that for every $n \in \mathbb{N}$, $P_{\nabla u_n}$ is given by a finite set of points and we set $P_{\nabla u_n} = P_n$ with $\mathcal{H}^0(P_n) \leq N_0$ where N_0 is a positive number independent of n.

From (3.1) and (3.5), it follows that

$$c_1 \|\Delta u_n\|^p_{L^p(\Omega\setminus P_n)} + c_2 \|\nabla u_n\|_{L^1(\Omega\setminus P_n;\mathbb{R}^2)} - \xi_0 \cdot \int_{\Omega\setminus P_n} \nabla u \, \mathrm{d}x \leq$$
$$\leq M + \int_\Omega a(x)\mathrm{d}x, \tag{3.6}$$

where we have used the identification $\operatorname{div} \nabla u_n = \Delta u_n$ in $\mathcal{D}'(\Omega \setminus P_n)$.

Since $u_n \in W^{1,p}_0(\Omega)$ and P_n is a set with 0 p-capacity, we have from the divergence theorem:

$$\xi_0 \cdot \int_{\Omega\setminus P_n} \nabla u_n \, \mathrm{d}x = \xi_0 \cdot \int_\Omega \nabla u_n \, \mathrm{d}x = \xi_0 \cdot \int_{\partial\Omega} u_n \nu \mathrm{d}\mathcal{H}^1 = 0.$$

Then (3.6) gives

$$c_1 \|\Delta u_n\|^p_{L^p(\Omega\setminus P_n)} + c_2 \|\nabla u_n\|_{L^1(\Omega\setminus P_n;\mathbb{R}^2)} \leq$$
$$\leq M + \int_\Omega a(x)\mathrm{d}x.$$

Then, by the Sobolev inequality and taking into account that the p-capacity of P_n is 0, we obtain

$$\|u_n\|_{L^2(\Omega\setminus P_n)} = \|u_n\|_{L^2(\Omega)} \leq$$
$$\leq S(\Omega)\|\nabla u_n\|_{L^1(\Omega;\mathbb{R}^2)} = S(\Omega)\|\nabla u_n\|_{L^1(\Omega\setminus P_n;\mathbb{R}^2)} \leq M_1,$$

where we have used that in space dimension 2 the Sobolev exponent $1^* = \frac{N}{N-1}$ is equal to 2 and M_1 is a suitable positive constant independent on n.

Thus, via standard elliptic regularity, we deduce that $\{u_n\} \subset W^{2,p}_{\text{loc}}(\Omega)$ with the estimate

$$\|u_n\|_{W^{2,p}_{\text{loc}}(\Omega \setminus P_n)} \leq C\Big(\|u_n\|_{L^p(\Omega \setminus P_n)} + \|\Delta u_n\|_{L^p(\Omega \setminus P_n)}\Big) \leq M_2, \tag{3.7}$$

where M_2 is a suitable constant which does not depend on n.

From (3.7) it follows that for every open set $\Omega' \subset\subset \Omega \setminus P_n$

$$\|\nabla u_n\|_{L^p(\Omega')} \leq M_2$$

and

$$\|u_n\|_{L^p(\Omega')} \leq M_2,$$

where M_2 is independent of Ω' and n. Therefore via monotone convergence for every n, we have

$$\|u_n\|_{W^{1,p}(\Omega \setminus P_n)} = \sup_{\Omega' \subset\subset \Omega \setminus P_n} \|u_n\|_{W^{1,p}(\Omega')} \leq M_2.$$

Hence since P_n is a set of 0, p-capacity for every n,

$$\|u_n\|_{W^{1,p}_0(\Omega)} \leq \|u_n\|_{W^{1,p}(\Omega)} = \|u_n\|_{W^{1,p}(\Omega \setminus P_n)} \leq M_2. \tag{3.8}$$

Thus, by the Sobolev embedding we may extract a subsequence $\{u_{n_l}\}_l \subset \Delta\mathcal{M}^p(\Omega)$ such that

$$\begin{cases} u_{n_l} \to u & \text{in } L^p(\Omega) \text{ and a.e.} \\ \nabla u_{n_l} \rightharpoonup \nabla u & \text{in } L^p(\Omega; \mathbb{R}^2). \end{cases}$$

From Lemma 2.1 we infer the existence of a subsequence P_{n_l} and a finite set of points $P \subset \overline{\Omega}$ such that $P_{n_l} \to P$. Let $\{\Omega_i\}$ be a sequence of open sets invading $\Omega \setminus P$. Since P_{n_l} converges to P, for any i there exists l_i such that for all $l \geq l_i$

$$\Omega_i \subset \Omega \setminus P_{n_l} \quad \text{for all } i.$$

Thus (3.1) implies that for every $n \geq n_i$

$$\int_{\Omega_i} |\Delta u_{n_l}|^p \, \mathrm{d}x \leq \int_{\Omega \setminus P_{n_l}} |\Delta u_{n_l}|^p \, \mathrm{d}x \leq M_2, \tag{3.9}$$

therefore we have $\{u_{n_l}\}_l \subset W^{1,p,p}(\text{div}; \Omega_i)$ and there exists a subsequence still denoted by $\{u_{n_l}\}$ such that

$$\begin{cases} u_{n_l} \to u & \text{in } L^p(\Omega_i) \text{ and a.e.} \\ \nabla u_{n_l} \rightharpoonup \nabla u & \text{in } L^p(\Omega_i; \mathbb{R}^2) \\ \Delta u_{n_l} \rightharpoonup \Delta u & \text{in } L^p(\Omega_i). \end{cases}$$

Since $u_{n_l} \to u$ almost everywhere, by applying diagonal procedure we get that subsequence is the same for every i.

Thus by using the semicontinuity of the L^p-norm from (3.9) it follows, for every i, that

$$\int_{\Omega_i} |\Delta u|^p \, dx \le \liminf_{l+\infty} \int_{\Omega_i} |\Delta u_{n_l}|^p \, dx \le M_2,$$

and by monotone convergence

$$\int_{\Omega \setminus P} |\Delta u|^p \, dx = \sup_i \int_{\Omega_i} |\Delta u|^p \, dx \le \sup_i \liminf_{l+\infty} \int_{\Omega_i} |\Delta u_{n_l}|^p \, dx \le M_2.$$

Now set $\tilde{P} = P \setminus \partial\Omega$. Then we have $u \in W_0^{1,p,p}(\text{div}; \Omega \setminus \tilde{P})$. So, thanks to Proposition 2.1, we conclude that $u \in \Delta\mathcal{M}^p(\Omega)$ with $P_{\nabla u} \subseteq P$.

Moreover, from Theorem 3.1, applied with

$$v_{n_l} = u_{n_l},$$

$$z_{n_l} = (\nabla u_{n_l}, \Delta u_{n_l}) = (\nabla u_{n_l}, \text{div}\, \nabla u_{n_l}) \text{ in } \mathcal{D}'(\Omega_i),$$

we get

$$\begin{aligned}\int_{\Omega_i} f(x, u, \nabla u, \text{div}\, \nabla u) dx &\le \liminf_{l \to +\infty} \int_{\Omega_i} f(x, u_{n_l}, \nabla u_{n_l}, \text{div}\, \nabla u_{n_l}) dx \\ &\le \liminf_{l \to +\infty} \int_{\Omega \setminus P_{n_l}} f(x, u_{n_l}, \nabla u_{n_l}, \text{div}\, \nabla u_{n_l}) dx \qquad (3.10)\end{aligned}$$

and in fact via monotone convergence

$$\begin{aligned}\int_{\Omega} f(x, u, \nabla u, \text{div}\, \nabla u) dx &= \int_{\Omega \setminus P} f(x, u, \nabla u, \text{div}\, \nabla u) dx = \sup_i \int_{\Omega_i} f(x, u, \nabla u, \text{div}\, \nabla u) dx \\ &\le \liminf_{l \to +\infty} \int_{\Omega} f(x, u_{n_l}, \nabla u_{n_l}, \text{div}\, \nabla u_{n_l}) dx. \qquad (3.11)\end{aligned}$$

We also have from Lemma 2.2:

$$\mathcal{H}^0(P_{\nabla u}) \le \mathcal{H}^0(P) \le \liminf_{l \to +\infty} \mathcal{H}^0(P_{n_l}). \qquad (3.12)$$

In the end by (3.11), (3.12) and taking into account the superlinearity property of the lim inf operator, we have obtained

$$F(u) \le \liminf_{l \to +\infty} F(u_{n_l}) \qquad (3.13)$$

with $u_{n_l}, u \in \Delta\mathcal{M}^p(\Omega)$.

Finally, by taking into account (3.4) we have

$$\inf_{\Delta\mathcal{M}^p(\Omega)} F \le F(u) \le \liminf_{l \to +\infty} F(u_{n_l}) = \inf_{\Delta\mathcal{M}^p(\Omega)} F.$$

So u is minimizer for F and the proof is achieved. ■

Acknowledgements

I am grateful to Prof. Gilles Aubert for his help. His advice has permitted me to improve this article significantly. I warmly thank anonymous referees for their careful reading of this article and numerous remarks. Their advice has permitted to improve my work significantly. The research of Daniele Graziani was supported by CNRS under the research project 'Gyrovision' (Laboratory I3S, Université de Nice Sophia antipolis).

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