A formal Γ-convergence approach for the detection of points in 2-D images[1]

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Abstract. We propose a new variational model to locate points in 2-dimensional biological images. To this purpose we introduce a suitable functional whose minimizers are given by the points we want to detect. In order to provide numerical experiments we replace this energy with a sequence of a more treatable functionals by means of the notion of Γ-convergence.

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1. Introduction

Detecting fine structures, like points or curves in two or three dimensional images respectively, is an important issue in image analysis. In biological images a point may represent a viral particle whose visibility is compromised by the presence of other structures like cell membranes or some noise.

From a variational point of view, the problem of point detection is a difficult task, since it is not clear how these singularities must be classified in terms of some differential operator. Indeed, since these are usually defined as discontinuity without jump, we cannot use the gradient operator as in the classical problem of contour detection. As a consequence the functional framework may be not clear.

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One possible strategy to overcome this obstacle is considering this kind of pathology as a k-codimension object, meaning that they should be regarded as a singularity of a map $U : \mathbb{R}^{k+m} \to \mathbb{R}^k$ (see [7] for a complete survey on this subject) with $k \geq 2$ and $m \geq 0$, where $k + m$ is the dimension of the ambient space and k is the codimension of the singularity to detect. The detecting points case corresponds to the case $k = 2$ and $m = 0$.

In this direction the authors in [5] have suggested a variational approach based on the theory of Ginzburg-Landau systems. In their work the isolated points in 2-D images are regarded as the topological singularities of a map $U : \mathbb{R}^2 \to \mathbb{S}^1$, where $\mathbb{S}^1$ is a unit sphere of $\mathbb{R}^2$. So that it is crucial to construct, starting from the initial image $I : \mathbb{R}^2 \to \mathbb{R}$, an initial vector field $U_0 : \mathbb{R}^2 \to \mathbb{S}^1$ with a topological singularity of degree 1, where the intensity of the initial image I is high. How to do this in a rigorous way, it is still unclear.

Therefore here our purpose is to provide a lighter variational formulation, in which the singular points in the image is directly given in terms of a proper differential operator defined on vector fields. Another important difference is that in [5] points and curves are detected both as singularities, while in the present paper our aim is to isolate and in the same time remove any other singularities.

In order to detect the singularities of the image, we have to find a functional space whose elements generate, in a suitable sense, a measure concentrated on points. Such a space is $\mathcal{DM}^p(\Omega)$ introduced in [4], where $1 < p < 2$ and Ω is an open set which represents the image domain. $\mathcal{DM}^p(\Omega)$ is the space of vector fields $U : \Omega \to \mathbb{R}^2$ whose distributional divergence is a Radon measure (see subsection 2.2 for definitions and examples). The restriction $1 < p < 2$ is due to the fact for $p \geq 2$ the distributional divergence of U cannot charge isolated points (see [6]).

Unfortunately, even if we are capable of constructing an initial vector field U_0 (see below for such a construction) belonging to the space $\mathcal{DM}^p(\Omega)$, its singular set could contains several structures we want to remove from the original image, like, for instance, curve or some noise. Hence, after the initialization we have to remove all the structures we are not interested in by building up, starting from the initial data U_0, a new vector field U whose singularities are given by the points of the image I we want to isolate.

Thus, from one hand, we have to force the concentration set of the distributional divergence of U_0 to contain only the points we want to catch, and, on the other hand, we have to regularize the initial data U_0 outside the points of singularities. To this end, we propose to minimize an energy involving a competition between a divergence term and the counting Hausdorff measure $\mathcal{H}^0$. More precisely the energy is the following

$$F(U,P) = \int_{\Omega \setminus P} |\mathrm{div} U|^2 dxdy + \lambda \int_{\Omega} |U - U_0|^p dxdy + \mathcal{H}^0(P), \tag{1.1}$$

where $U \in L^{p,2}(\mathrm{div}; \Omega \setminus P)$ is the space of L^p-vector fields whose distributional divergence belongs to $L^2(\Omega \setminus P)$, P is the atomic set we want to target and λ is a positive weight. The first integral forces U

to be regular outside P, while the term $\mathcal{H}^0(P)$ penalizes the presence of singular curves in the image and limits the number of points detected, in order to avoid false detection due to noise.

From a practical point of view, this choice allows us to work with a first order differential operator and permits to formulate the minimization problem in a common functional framework.

For initializing the minimization process, we need to construct, from the initial image, a vector field U_0 belonging to $\mathcal{DM}^p(\Omega)$. Such a vector field can be provided by using the gradient of weak solution of the classical Dirichlet problem with measure data.

$$\begin{cases} -\Delta f = I & \text{on } \Omega \\ f = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

This initialization is presented in section 6.2. Then functional (1.1) must be minimized which is a difficult task due to presence of the variable P which is a 0-dimensional object. In order to provide numerical minimization, we must approximate functional (1.1) by means of a sequence of more convenient functionals. The approximation, we suggest in this paper, is based on the so called Γ-convergence, the notion of variational convergence introduced by De Giorgi (see [13, 14]). This theory is designed to approximate a variational problem by a sequence of different variational problems with more regularity. The most important feature of the Γ-convergence relies on the fact that it implies the convergence of minimizers of the approximating functionals to those of the limiting functional. So far variational approximation techniques such as Γ-convergence or continuation method (see [2, 3] and [20] respectively) have been successfully employed in image and signal processing. For instance in ([2, 3]) Ambrosio and Tortorelli have proven that the classical Mumford-Shah's functional for detecting 1-dimensional smooth boundaries, can be approximated by a sequence of elliptic functionals that are numerically more treatable. In this work we suggest a possible Γ-convergence approach for the detection of points. By the way we stress out that the Γ-convergence result is only conjectured in this paper, whose purpose is to test a new variational method from an experimental point of view. For a rigorous variational approximation in a particular case, we refer the reader to [6].

The main difficulty here is related to the presence of a codimension 2 object, which is not a contour: the set P. In order to obtain a variational approximation close to the one provided in ([2, 3]), the crucial step is then to replace the term $\mathcal{H}^0(P)$ of functional (1.1) by a more regular, from a variational point of view, functional involving a smooth boundary and his perimeter given by the 1-dimensional Hausdorff measure $\mathcal{H}^1$. Following some suggestion from [9, 10] such a functional is given by:

$$G_{\beta_\varepsilon}(D) = \frac{1}{4\pi} \int_{\partial D} \big(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2(x,y)\big) d\mathcal{H}^1(x,y),$$

where D is a proper regular set containing the atomic set P, κ is the curvature of its boundary, the constant $\frac{1}{4\pi}$ is a normalization factor, and β_ε infinitesimal as $\varepsilon \to 0$. Roughly speaking the minima of this functional are achieved on the union of balls of small radius, so that when $\beta_\varepsilon \to 0$ the functional shrinks to the atomic measure $\mathcal{H}^0(P)$. On the other hand the introduction of a curvature term requires a non trivial and convenient, for a numerical point of view, approximation of the curvature-dependent functional. Such an approximation is based on a celebrated conjecture due to De Giorgi (see [12]). By means of this argument it is possible to substitute the curvature-depending functional with an integral functional involving the Laplacian operator of smooth functions. Then it remains to approximate the $\mathcal{H}^1$-measure and this can be done by retrieving a classical gradient approach used in [15, 16]. This strategy allows to deal with a functional whose Euler-Lagrange equations can be discretized. A simple and intuitive explanation of the construction of the complete approximating functionals will be given in section 3.

The paper is organized as follows: section 2 contains some mathematical tools, which are used in the following. In section 3 we address the existence result for the functional $F(U, P)$ defined in (1.1). In section 4 we state the two well-known Γ-convergence results we need in the sequel. In section 5 we build in a formal way the approximating sequence. In section 6 we present the discrete model and the whole point detection procedure. Finally the last section is devoted to some computer examples.

2. Preliminaries

2.1. **Convergence for a set of points.** For our purpose it will be crucial dealing with a notion of convergence for finite sets of points introduced in [10].

Definition 2.1. *We say that a sequence of a finite set of points* $\{P_h\} \subset \overline{\Omega}$ *converges to a set* $P \subset \overline{\Omega}$ *if each of the sets* P_h *contains a number* N *of points* $\{x_h^1, ..., x_h^N\}$*, with* N *independent of* h*, such that* $x_h^i \to x^i$ *for any* $i = 1, ..., n$ *and* $\bigcup_{i=1}^N \{x_i\} = P$.

Lemma 2.1. *Let* $\{P_h\}$ *be a sequence of a finite set of points such that* $\mathcal{H}^0(P_h) \le N_0$ *for every* h *with* $N_0 \in \mathbb{N}$*. Then there exists a subsequence* $\{P_{h_k}\} \subset \{P_h\}$ *and a set of points* $P \subset \overline{\Omega}$ *such that* P_{h_k} *converges with respect to the convergence 2.1 to the set* P.

Proof. Since $\mathcal{H}^0(P_h) \le N_0$, we may find $N_1 \le N_0$ such that every set P_h contains at least N_1 points. For every $i = 1, ..., N_1$ and there exists a subsequence $x_{h_k}^i \subset x_h^i$ converging to $x^i \in \overline{\Omega}$.

Then by setting $P_{h_k} = \bigcup_{i=1}^{N_1} x_{h_k}^i$ and $P = \bigcup_{i=1}^{N_1} x^i$, the thesis is achieved.

Lemma 2.2. *Let* $\{P_h\} \subset \overline{\Omega}$ *be a sequence of finite set of points converging to a finite set of points* P*. Then*

$$\mathcal{H}^0(P) \le \liminf_{h \to +\infty} \mathcal{H}^0(P_h) \tag{2.1}$$

Proof. From definition 2.1 it follows that

$$\liminf_{h \to +\infty} \mathcal{H}^0(P_h) \ge \liminf_{h \to +\infty} \mathcal{H}^0(\{x_h^1, ..., x_h^N\}) = N = \mathcal{H}^0(P). \qquad \square$$

2.2. **Distributional divergence.** In this subsection we recall the definition of the space $L^{p,q}(\mathrm{div}; \Omega)$ and $\mathcal{DM}^p(\Omega)$, introduced in [4].

Let $\Omega \subset \mathbb{R}^2$ be an open set and let $U : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field.

Definition 2.2. *We say that* $U \in L^{p,q}(\mathrm{div};\Omega)$ *if* $U \in L^p(\Omega;\mathbb{R}^2)$ *and if its distributional divergence* $\mathrm{div}U \in L^q(\Omega)$*. If* $p = q$ *the space* $L^{p,q}(\mathrm{div};\Omega)$ *will be denoted by* $L^p(\mathrm{div};\Omega)$*.*

Definition 2.3. *For* $U \in L^p(\Omega;\mathbb{R}^2)$, $1 \le p \le +\infty$, *set*

$$|\mathrm{divU}|(\Omega) := \sup\{\int_\Omega U \cdot \nabla\varphi dxdy \ : \ \varphi \in C_0^1(\Omega), \ |\varphi| \le 1\}.$$

We say that U *is an* L^p*-divergence measure field, i.e.* $U \in \mathcal{DM}^p(\Omega)$ *if*

$$\|U\|_{\mathcal{DM}^p(\Omega)} := \|U\|_{L^p(\Omega;\mathbb{R}^2)} + |\mathrm{divU}|(\Omega) < +\infty.$$

Remark 2.1. *If* $U \in \mathcal{DM}^p(\Omega)$ *then via Riesz Theorem it is possible to represent the distributional divergence of* U *by a Radon measure. More precisely there exists a Radon measure* μ *such that for every* $\varphi \in C_0^1(\Omega)$ *the following equality holds:*

$$\int_\Omega U \cdot \nabla\varphi dxdy = -\int_\Omega \varphi d\mu.$$

For instance the field $U(x,y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ *belongs to* $\mathcal{DM}^1_{loc}(\mathbb{R}^2)$ *and its divergence measure is given by* $-2\pi\delta_0$*, where* δ_0 *is the Dirac mass.*

Such a result can be proven by approximation. Let us define the following map:

$$U_\varepsilon(x,y) := \begin{cases} U(x,y) & \text{if } |x| \ge \varepsilon \\ (\frac{x}{\varepsilon^2}, \frac{y}{\varepsilon^2}) & \text{if } |x| < \varepsilon. \end{cases}$$

It is not difficult to check that u_ε *is Lipschitz-map with divergence given by* $\frac{2}{\varepsilon^2}\chi_{B(0,\varepsilon)}$*.*

Then for every test function $\varphi \in C_0^1(\mathbb{R}^2)$ *we have*

$$\int U_\varepsilon \cdot \nabla\varphi dxdy = -\int \frac{2}{\varepsilon^2}\chi_{B(0,\varepsilon)}\varphi dxdy.$$

By applying the change of variables $x = \frac{x_1}{\varepsilon}$, $y = \frac{y_1}{\varepsilon}$ *we obtain*

$$\int U_\varepsilon \cdot \nabla\varphi dxdy = -2\int \chi_{B(0,1)}\varphi(\frac{x_1}{\varepsilon}, \frac{y_1}{\varepsilon})dx_1dy_1,$$

so that, letting $\varepsilon \to 0$*, by the dominated convergence theorem we obtain*

$$\int_\Omega U \cdot \nabla\varphi dxdy = -2\pi\varphi(0,0) = -2\pi\int_\Omega \varphi d\delta_0$$

2.3. **The Dirichlet problem with measure data.** For the initialization of our algorithm we must build a vector field U_0 which should be such that its divergence is singular on points of the image I. Therefore we will use the gradient of the solution of the following Dirichlet problem (applied with $\mu = I$)

$$\begin{cases} -\Delta f = \mu & \text{on } \Omega \\ f = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

where μ is a Radon measure. Classical results (see [19]) guarantee the existence of a unique solution of problem (2.2). Concerning the regularity it is known that $f \in W^{1,p}(\Omega)$ with $p < 2$.

3. Existence result

In this section we show the existence of a minimizing pair (U, P) for the functional F defined in (1.1.)

Our argument takes two steps (see also [17] for a similar approach to minimize the classical Mumford-Shah's functional). The first one consists in proving the existence a minimizer of the functional (1.1) when the set P is fixed.

To this end we adopt the following notation:

$$F(U) = F(\cdot, P) = \int_{\Omega\setminus P} |\mathrm{div} U|^2 dxdy + \lambda \int_{\Omega} |U - U_0|^p dxdy + \mathcal{H}^0(P). \tag{3.1}$$

Theorem 3.1. *For every set P there exists a unique minimizer $U_P \in L^{p,2}(\mathrm{div}; \Omega \setminus P)$ of the functional (3.1).*

Proof. Let U_n be a minimizing sequence. Then we have the following bound

$$F(U_n) \leq M. \tag{3.2}$$

From the bound (3.2) and the classical inequality:

$$\|U_n\|^p_{L^p(\Omega\setminus P)} \leq 2^{p-1}\|U_n - U_0\|^p_{L^p(\Omega\setminus P)} + \|U_0\|^p_{L^p(\Omega\setminus P)}$$

it follows that

$$\|U_n\|^p_{L^p(\Omega\setminus P)} \leq M + \|U_0\|^p_{L^p(\Omega\setminus P)} := C.$$

Moreoveor we also have:

$$\|\mathrm{div} U_n\|^2_{L^2(\Omega\setminus P)} \leq F(U_n) \leq M;$$

so that, up to subsequences, we obtain

$$\begin{cases} U_n \rightharpoonup U_P & \text{in } L^p(\Omega \setminus P) \\ \mathrm{div} U_n \rightharpoonup \mathrm{div} U_P & \text{in } L^2(\Omega \setminus P). \end{cases} \tag{3.3}$$

Therefore we can conclude that U_n weakly converges in $L^{p,2}(\Omega \setminus P; \mathrm{div})$ to a vector field $U_P \in L^{p,2}(\Omega \setminus P; \mathrm{div})$.

Then, thanks to semicontinuity properties of the L^p-norm with respect to the weak convergence, we have:

$$\inf_U F(U) \leq F(U_P) \leq \liminf_{n\to+\infty} F(U_n) = \inf_U F(U).$$

Finally the strong convexity of functional (3.1) gives the uniqueness of the minimizer U. $\square$

At once we have obtained the existence of the minimizer U_P for every fixed set P, we focus on the following functional:

$$E(P) := F(U_P, P) = \int_{\Omega\setminus P} |\mathrm{div} U_P|^2 dxdy + \lambda \int_{\Omega} |U_P - U_0|^p dxdy + \mathcal{H}^0(P). \tag{3.4}$$

We extend U and $\mathrm{div}U$ by zero on P. However we keep the integration domain of $\mathrm{div}U$ to be $\Omega \setminus P$. We do that in order to make clear that $\mathrm{div}U$ is the distributional divergence of U on $\Omega \setminus P$ and not on Ω.

The following semicontinuity lemma plays a key role.

Lemma 3.1. *Assume that a sequence of finite sets of points $\{P_n\} \subset \overline{\Omega}$ converges to a finite set of points $P \subset \overline{\Omega}$. Then*

$$E(P) \leq \liminf_{h\to\infty} E(P_n)$$

Proof. Let us set $U_n = U_{P_n}$, we can assume that U_n and $\mathrm{div}U_n$ are both defined on all of Ω in the sense explained above. The sequence U_n is bounded in $L^p(\Omega)$. Indeed, by taking into account that U_n is a minimizer of the functional (3.1),

$$\|U_n\|^p_{L^p(\Omega)} \leq 2^{p-1}\|U_n - U_0\|^p_{L^p(\Omega)} + \|U_0\|^p_{L^p(\Omega)} \leq 2^{p-1}F(0) + \|U_0\|^p_{L^p(\Omega)} = \|U_0\|^p_{L^p(\Omega)}(2^{p-1}+1).$$

In the same way one can show that the sequence $\mathrm{div}U_n$ is bounded in $L^2(\Omega)$. So that, up to subsequences, we may assume

$$\begin{cases} U_n \rightharpoonup U & \text{in } L^p(\Omega) \\ \mathrm{div}U_n \rightharpoonup V & \text{in } L^2(\Omega). \end{cases} \tag{3.5}$$

We claim that $\mathrm{div}U = V$ in $\Omega \setminus P$. In fact, take any test function φ with support in $\Omega \setminus P$, then since $P_n \to P$, we have for n large enough

$$supp(\varphi) \subset \Omega \setminus P_n$$

and, consequently,

$$\int_{supp(\varphi)} U_n \nabla\varphi dxdy = -\int_{supp(\varphi)} divU_n \varphi dxdy.$$

Therefore, by taking the weak limit by (3.5) we get

$$\int_{supp(\varphi)} U \nabla\varphi dxdy = -\int_{supp(\varphi)} V \varphi dxdy.$$

Then since the test function φ is arbitrary, we can conclude that $\mathrm{div}U = V$ on $\Omega \setminus P$.

The thesis follows since, from the lower semicontinuity of the L^p-norm and Lemma 2.2, we have

$$E(P) \leq E(U,P) \leq \liminf_n E(U_n, P_n) \qquad \square$$

We are now in position to prove the main result of this section.

Theorem 3.2. *There exists a minimizing pair (U,P) of the functional F, with $U \in L^{p,2}(\mathrm{div};\Omega)$ and $P \subset \Omega$ finite set of points.*

Proof. For every P let U_P the minimizer of functional (3.1), whose existence is guaranteed by Theorem 3.1. Then we focus on the functional $E(P) = F(U_P, P)$ and we take a minimizing sequence $\{P_n\}$. Then by Lemma 2.1 we have (up to a subsequences) that $P_n \to \overline{P} \subset \overline{\Omega}$ and $U_{P_n} \to U_{\overline{P}}$. By Lemma 3.1 we get

$$E(\overline{P}) \leq \liminf_{n\to+\infty} E(P_n).$$

Therefore

$$\inf_{(U,P)} F(U,P) \le F(U_{\overline{P}},\overline{P}) \le \liminf_{n\to+\infty} E(P_n) \le \liminf_{h\to+\infty} F(U_n,P_n) = \inf_P F(U_P,P) \le F(U,P). \tag{3.6}$$

Now set $\tilde{P} := \overline{P} \setminus \partial\Omega$. Since for every P, U_P is a minimizer we get from (3.6)

$$F(U_{\overline{P}},\tilde{P}) \le F(U_{\overline{P}},\overline{P}) \le F(U_P,P) \le F(U,P),$$

for every (U,P). Hence we conclude that

$$F(U_{\overline{P}},\tilde{P}) \le \inf_{(U,P)} F(U,P). \qquad \square$$

4. Γ-convergence

The key point of our strategy is to replace the functional (1.1) by means of more regular functionals by following a formal Γ-convergence approach.

Therefore this section is devoted to a very simple presentation of the two results we need: Modica-Mortola's theorem (see [15, 16]) concerning the approximation of the perimeter and De Giorgi's conjecture (see [12]) about the approximation of curvature depending functionals. For the definition of the Γ-convergence and its main properties we refer the reader to [8, 11] and references therein.

4.1. **Modica Mortola's approach.** Modica-Mortola theorem states that it is possible to approximate, in the Γ-convergence sense, a perimeter by means of the following sequence of functionals

$$F_\varepsilon^1(u) := \begin{cases} \int_\Omega \left(\varepsilon|\nabla u|^2 + \frac{V(u)}{\varepsilon}\right)dxdy & \text{if } u \in W^{1,2}(\Omega), \\ +\infty & \text{otherwise,} \end{cases}$$

where $V(u) = u^2(1-u)^2$ is a double well potential. Besides, since the minimizers of the functional F_ε^1 may be trivial, some constraint on the functions u_ε must be added. Usually a volume constraint of the type $\int_\Omega u dxdy = m$, is assumed.

Let us give an intuitive explanation of such a result. Since V has two absolute minimizers at $u = 0, 1$, when ε is small, a local minimizer u_ε is closed to 1 on a part of Ω and close to 0 on the other part, making a rapid transition of order ε between 0 and 1. When $\varepsilon \to 0$ the transition set shrinks to a set of dimension 1, so that u_ε goes to a function taking values u into $\{0,1\}$ and the family of functionals Γ-converges to the measure of the perimeter of the discontinuity set of u. Modica-Mortola's Theorem is the following.

Theorem 4.1. *The functionals $F_\varepsilon^1 : L^1(\Omega) \to [0,+\infty]$ $\Gamma-$converge, with respect to the L^1-convergence, to the following functional*

$$F^1(u) = \begin{cases} C_V \mathcal{H}^1(S_u) & \textit{if } u \in \{0,1\} \\ +\infty & \textit{otherwise} \end{cases}$$

where, as usual, S_u denotes the set of discontinuities of u and C_V is a suitable constant depending on the potential V.

4.2. **De Giorgi's conjecture.** The aim of De Giorgi was finding a variational approximation of a curvature depending functional of the type:

$$F^2(D) = \int_{\partial D} (1+\kappa^2) d\mathcal{H}^1;$$

where D is a regular set and κ is a curvature of its boundary ∂D.

Since ∂D can be represented as the discontinuity set of the function $u_0 = 1 - \chi_D$, by Modica-Mortola's Theorem it follows that there is a sequence of non constant local minimizers such that $u_\varepsilon \to u_0$, with respect to the L^1-convergence, and

$$\lim_{\varepsilon \to 0} F^1_\varepsilon(u_\varepsilon) := C_V \mathcal{H}^1(\partial D).$$

Furthermore looking at the Euler-Lagrange equation associated to a contour length term, it yields a contour curvature term κ, while the Euler-Lagrange equations for the functional $F^1_\varepsilon(u)$ contains the term $2\varepsilon\Delta u - \frac{V'(u)}{\varepsilon}$.

Then De Giorgi suggested to approximate the functional F^2 by adding to Modica-Mortola's approximating functionals the term

$$F^2_\varepsilon(u) = \int_\Omega (2\varepsilon\Delta u - \frac{V'(u)}{\varepsilon})^2 (\varepsilon|\nabla u|^2 + \frac{V(u)}{\varepsilon}) dxdy.$$

In [18] the authors have proven a simplified version of the De Giorgi's conjecture, where the integral above is replaced by the functional

$$F^2_\varepsilon(u) = \int_\Omega (2\varepsilon\Delta u - \frac{V'(u)}{\varepsilon})^2 dxdy.$$

5. The approximating functionals

In this section we present the energy we deal with and the construction of the approximating sequence.

The energy we are interested in is given by

$$\int_{\Omega \setminus P} |\mathrm{div} U|^2 dxdy + \lambda \int_\Omega |U - U_0|^p dxdy + \mathcal{H}^0(P).$$

where $U \in L^{p,2}(\mathrm{div}; \Omega\setminus P)$, $U_0 \in \mathcal{DM}^p_{loc}(\mathbb{R}^2)$ and finally P is an atomic set consisting of a finite number N of points, i.e. $P = \{x_1, ..., x_N\}$.

As pointed out in the introduction, the first step is to substitute the counting measure $\mathcal{H}^0(P)$ with a more treatable term given by:

$$G_{\beta_\varepsilon}(D) = \frac{1}{4\pi} \int_{\partial D} \big(\frac{1}{\beta_\varepsilon} + \beta_\varepsilon \kappa^2(x,y)\big) d\mathcal{H}^1(x,y);$$

where D is an union of regular simply connected sets $\{D_i\}$ with $i = 1, .., N$, such that $x_i \in D_i$, $D_i \bigcap D_j = \emptyset$ for $i \neq j$, κ is the curvature of the boundary of the set D, the constant $\frac{1}{4\pi}$ is a normalization factor and β_ε is infinitesimal as $\varepsilon \to 0$.

To understand why we can approximate $\mathcal{H}^0(P)$ with $G_{\beta_\varepsilon}(D)$, one should note that the solution of the following minimum problem

$$\min_{D\supset P} G_{\beta_\varepsilon}(D) \tag{5.1}$$

is given by $D = \bigcup_i^N B(x_i, \beta_\varepsilon)$, where x_i are the points of P. We give an idea of a possible proof in the case of a single point.

By the Young's inequality we have

$$G_{\beta_\varepsilon}(D) \geq \frac{1}{4\pi}\int_{\partial D} 2\kappa d\mathcal{H}^1$$

and by applying the Gauss-Bonnet Theorem

$$G_{\beta_\varepsilon}(D) \geq \frac{1}{4\pi}(2)(2\pi) = 1 = \mathcal{H}^0(P).$$

Finally a simple calculation shows that, if we evaluate the functional G_{β_ε} on $B(x_1, \beta_\varepsilon)$, we obtain the value 1, i.e. the number of points in P, i.e. $\mathcal{H}^0(P)$. The N point case can be recovered with minor changes by the same argument.

For what follows it is convenient to split the functional G_{β_ε} in two terms:

$$G_{\beta_\varepsilon}(D) = G^1_{\beta\varepsilon}(D) + G^2_{\beta\varepsilon}(D)$$

where

$$G^1_{\beta_\varepsilon}(D) = \frac{1}{4\pi}\int_{\partial D}\frac{1}{\beta\varepsilon} d\mathcal{H}^1(x,y);$$

and

$$G^2_{\beta_\varepsilon}(D) := \frac{1}{4\pi}\int_{\partial D}\beta_\varepsilon \kappa^2(x,y) d\mathcal{H}^1(x,y).$$

We can write an intermediate approximation of energy (1.1):

$$E_\varepsilon(U,D) = G^1_{\beta_\varepsilon}(D) + G^2_{\beta_\varepsilon}(D) + \int_\Omega (1-\chi_D)|\mathrm{div}(U)|^2 dxdy + \lambda\int_\Omega |U-U_0|^p dxdy. \tag{5.2}$$

The advantage of such a formulation is that we know how to provide a variational approximation of the perimeter measure $\mathcal{H}^1\lfloor\partial D$. Following Modica-Mortola's approach such an approximation can be obtained by using the following measure:

$$\mu_\varepsilon(w,\nabla w)dxdy = \big(\varepsilon|\nabla w|^2 + \frac{V(w)}{\varepsilon}\big)dxdy,$$

where $V(w) = w^2(1-w)^2$ is a double well functional.

Next step is expressing the curvature term by means of the function w. Thanks to the simplified version of the De Giorgi's conjecture we can replace the term κ by the term $2\varepsilon\Delta w - \frac{V'(w)}{\varepsilon}$.

So that we can formally write the complete approximating functional:

$$\begin{aligned}\Phi_\varepsilon(U,w) : \ &= \int_\Omega w^2|\mathrm{div}(U)|^2dxdy + \frac{1}{4\pi}\int_\Omega \beta_\varepsilon(2\varepsilon\Delta w - \frac{V'(w)}{\varepsilon})^2dxdy + \frac{1}{\beta_\varepsilon}\int_\Omega \mu_\varepsilon(w,\nabla w)dxdy \\ &+ \lambda\int_\Omega |U-U_0|^p dxdy + \frac{1}{\mu_\varepsilon}\int_\Omega (1-w)^2dxdy,\end{aligned} \tag{5.3}$$

where $U \in L^{p,2}(\mathrm{div};\Omega)$ is equal to 0 on the $\partial\Omega$ and w is smooth function equal to 1 on the boundary, i.e. $1-w \in C_0^\infty(\Omega)$, $\mu_\varepsilon \to 0$ when ε goes to 0. The last integral is a penalization term which prevents w_ε from converging to the function constantly equal to 0 as $\varepsilon \to 0$.

Then if $(U_\varepsilon, w_\varepsilon)$ is a minimizing sequence of Φ_ε, then w_ε must be very close to the values 1 when ε goes to 0, since the double well potential is positive except for $w_\varepsilon = 0, 1$ and w must be equal to 1 on $\partial\Omega$. On the other hand, near the points where the divergence is very high w_ε must be close to 0.

Therefore, while the functions U_ε approximate a minimizer U of the original functional, the level set $\{w_\varepsilon = 0\}$ approximate the original singular set P.

Remark 5.1. *We point out that the Γ-convergence result is not proved in this paper, but only conjectured. A complete proof of the Γ-convergence result and the equicoerciveness of the sequence Φ_ε, in the particular case where the vector field U is a gradient, has been provided by the first and third author in* [6].

The first variation of this functional leads to the following gradient flow system

$$\begin{aligned}\frac{\partial U}{\partial t} &= 2\nabla(w^2\mathrm{div}U) + \lambda p|U-U_0|^{p-2}(U-U_0) \\ \frac{\partial w}{\partial t} &= -4\frac{\Delta h}{\beta_\varepsilon} + \beta_\varepsilon h + \frac{2}{\varepsilon^2}\frac{1}{\beta_\varepsilon}V''(w)h - 2w|\mathrm{div}U|^2 + \frac{2}{\mu_\varepsilon}(1-w),\end{aligned} \tag{5.4}$$

where h is given by the equation

$$h = 2\varepsilon\Delta w - \frac{1}{\varepsilon}V'(w).$$

6. Complete procedure for point detection

In our model the image contains an atomic Radon measure. Thus, in order to find an initial vector field which copies the singularities of the initial image, we consider $U_0 = -\nabla f$ gradient of the solution of the following Dirichlet problem:

$$\begin{cases} -\Delta f = I & \text{on } \Omega \\ f = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

In this way we obtain a vector field whose divergence is singular on a proper set which contains the points we want to detect. In general this set could contain other structures. For instance if the initial image is a Radon measure concentrated both on points and curves, the divergence of ∇f will be singular on points and curves. Besides if there is some noise in the image, it could be not clear how to differentiate the singular points due to the noise, from those we want to catch. As a consequence, by solving problem (6.1), we obtain a predetection, which must be refined. We do this by searching for a minimizer of the energy $\Phi_\varepsilon(U, w)$ via solving equations (5.4), with initial data U_0 given by ∇f. So that we obtain a vector field U whose divergence is relevant only on the set P and a function w whose zeros are given by the set P.

6.1. **Discretization.** The image is an array of size N^2. We endowed the space $R^{N\times N}$ with the standard scalar product and standard norm. The gradient $\nabla I \in (R^{N\times N}) \times (R^{N\times N})$ is given by:

$$(\nabla I)_{i,j} = ((\nabla I)^1_{i,j}, (\nabla I)^2_{i,j})$$

where

$$(\nabla I)^1_{i,j} = \begin{cases} I_{i+1,j} - I_{i,j} & \text{if } i < N \\ 0 & \text{if } i = N, \end{cases}$$

$$(\nabla I)^2_{i,j} = \begin{cases} I_{i,j+1} - I_{i,j} & \text{if } j < N \\ 0 & \text{if } j = 0. \end{cases}$$

We also introduce the discrete version of the divergence operator simply defined as the adjoint operator of the gradient: $\text{div} = -\nabla^*$. More in details if $v \in (R^{N\times N}) \times (R^{N\times N})$, we have

$$(\text{div}v)_{i,j} = \begin{cases} v^1_{i,j} + v^2_{i,j} & \text{if } i,j = 1 \\ v^1_{i,j} + v^2_{i,j} - v^2_{i-1,j} & \text{if } i = 1, 1 < j < N \\ v^1_{i,j} - v^1_{i-1,j} + v^2_{i,j} - v^2_{i-1,j} & \text{if } 1 < i < N, 1 < j < N \\ -v^1_{i-1,j} + v^2_{i,j} - v^2_{i-1,j} & \text{if } i = N, 1 < j < N \\ v^1_{i,j} - v^1_{i-1,j} + v^2_{i,j} & \text{if } 1 < i < N, j = 1 \\ v^1_{i,j} - v^1_{i-1,j} - v^2_{i-1,j} & \text{if } 1 < i < N, j = N \\ -(v^1_{i-1,j} + v^2_{i-1,j}) & \text{if } i,j = N. \end{cases}$$

Then we can define the discrete version of the Laplacian operator as $\Delta I = \text{div}(\nabla I)$.

6.2. **Discretization in time.** We simply replace $\frac{\partial U}{\partial t}$ and $\frac{\partial w}{\partial t}$ by $\frac{U^{n+1}_{i,j} - U^n_{i,j}}{\delta t}$ and $\frac{w^{n+1}_{i,j} - w^n_{i,j}}{\delta t}$ respectively. Then we write system (5.4) in the form (for simplicity we omit the dependence on ε)

$$\begin{cases} U^{n+1}_1 = & -\delta t \Phi_{U_1}(U_n, w_n) \\ U^{n+1}_2 = & -\delta t \Phi_{U_2}(U_n, w_n) \\ w^{n+1} = & -\delta t \Phi_w(U_n, w_n). \end{cases}$$

6.3. **initialization.** In order to compute $U(0) = U_0 = -\nabla f$, where f is the solution of problem (6.1), we need to solve a Dirichlet problem with data measure I, therefore we regularize the image by convolution with a Gaussian kernel G_σ with very small σ and then we solve, by a classical finite differences method, the problem:

$$\begin{cases} -\Delta f = I_\sigma & \text{on } \Omega \\ f = 0 & \partial\Omega, \end{cases} \tag{6.2}$$

where $I_\sigma = I * G_\sigma$.

To initialize our algorithm, we also need of an initial guess on w. We choose $w(0) = 1$.

7. Computer examples

7.1. **Parameter settings.** Before running our algorithm all the parameters have to be fixed. The most important are ε, β_ε and μ_ε, which govern the set D approximating points we want to detect. Those parameters are related, as in [10], by the conditions $\lim_{\varepsilon\to 0}\dfrac{\varepsilon|\log(\varepsilon)|}{\beta_\varepsilon}=0$, $\lim_{\varepsilon\to 0}\dfrac{\beta_\varepsilon}{\mu_\varepsilon}=0$. Furthermore, since the mesh grid size is 1 and β_ε gives the radius of a ball centered in the singular point we want to detect, from a discrete point of view the smallest value we can take is $\frac{\sqrt{2}}{2}$. Then we use the values 0.1 for ε, 0.7 for β_ε, and 0.8 for μ_ε. As exponent p of the discrepancy term we always take $p = 1.5$

Concerning the parameter λ we mainly used the value $\lambda = 0.1$, in order to force the algorithm to regularize as much as possible the initial data U_0.

Since we deal with small values of ε, in order to have some stability, we must take a small discretization time step. Practically we mainly used the value $\delta t = 1 \times 10^{-6}$.

Concerning the stopping criterion we iterate the algorithm until $\max\left\{\frac{\|U_1^{n+1}-U_1^n\|_1}{\|U_1^n\|_1}, \frac{\|U_2^{n+1}-U_2^n\|_1}{\|U_2^n\|_1}, \frac{\|w^{n+1}-w^n\|_1}{\|w^n\|_1}\right\}$ $\leq 1\times 10^{-2}$.

In all the computer examples the points are detected by means of the function w_ε, by displaying the level-set $\{w_\varepsilon \simeq 0\}$.

7.2. **Commentaries.** The figure 1 shows how resistant to noise our model is. When the noise is large the parameter ε must be as close as possible to the ideal value 0. More in details in the first row we display the initial image obtained by adding a Gaussian noise to a binary image of five points. The second row shows the behavior of w_ε for small values of ε and β_ε.

Looking at the histograms of the gray level of I and w_ε, one can see that it is easier fixing a threshold value starting from the function w_ε than from the initial image I. In the last row we display the set $\{w_\varepsilon \simeq 0\}$ obtained by plotting the set $\{w_\varepsilon \leq \alpha\}$ with threshold value $\alpha = 0.5$.

In figure 2 we test our algorithm on curves and points at the same time. In the first row we have a sequence of points and a curve with boundary inside Ω. In the second row we display the function w_ε and the level set $\{w_\varepsilon \simeq 0\}$ once again obtained by fixing a threshold value $\alpha = 0.5$. The result is that, as desired, our algorithm is capable of eliminating the curve from the initial image. According to the continuous setting when ε takes values close to 0 the approximating energy (5.3) behaves similarly to the limit energy (1.1), so that the presence of the curve is penalized in the minimization process. Then the set $\{w_\varepsilon \cong 0\}$ contains nothing else but points.

Finally in figure (a),(b),(c) and (d) we deal with a biological image. Our task is catching the finest structure present in the image. In figure (d) the isolated points are quite well detected, while the branches of the cell are not. Nevertheless due to the small time discretization step the computation time is quite large. To test the image in figure (a) of size 500×500, our algorithm takes 140 iterations and about 17mm on running on an Intel(R) Xeon(R) CPU 5120 @ 1.86GHz.

Certainly the algorithm can be accelerated by using more sophisticated techniques such as multigrid methods. Such a faster algorithm is the subject of our current investigation.

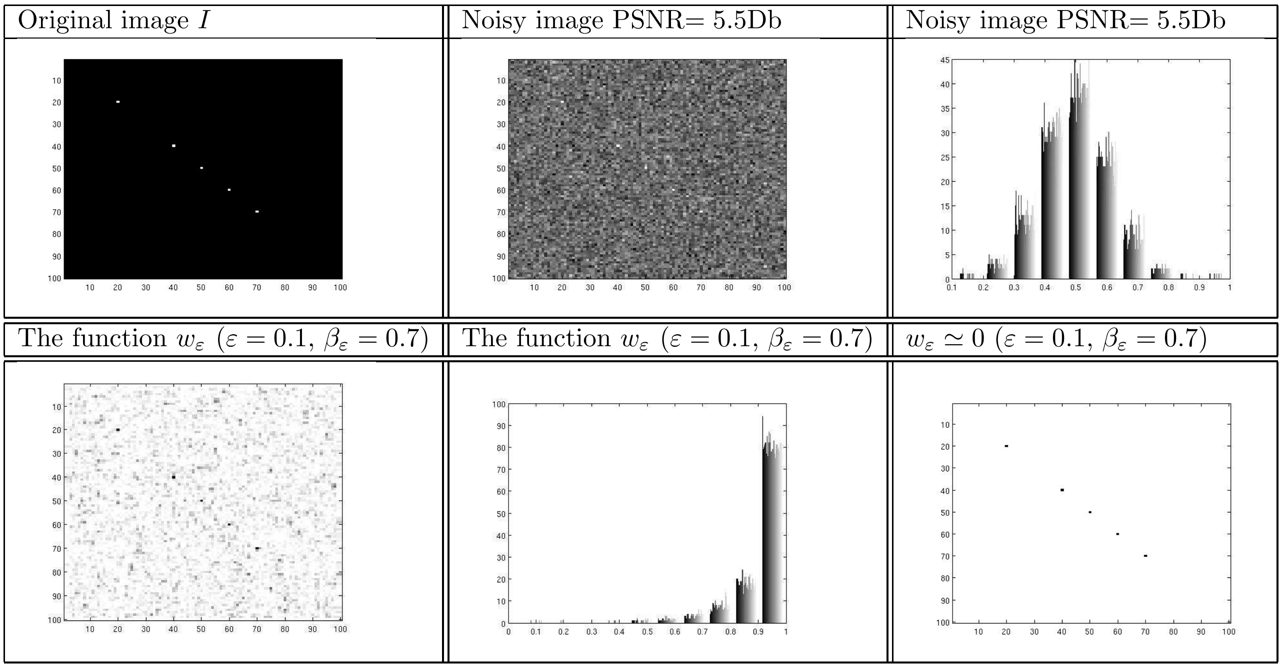

FIGURE 1. *Synthetic image: we test our algorithm on noisy images. When the parameters ε and β_ε are small as much as possible the detection is finer. The detection is refined by fixing a threshold value $\alpha = 0.5$ for the function w_ε. Top left: Original image. Top center: Noisy image. Top right: the histogram of the noisy image. Bottom left: the function w_ε. Bottom center: the histogram of the function w_ε. Bottom right: the level set $w_\varepsilon \simeq 0$ obtained by fixing a threshold value $\alpha = 0.5$*

.

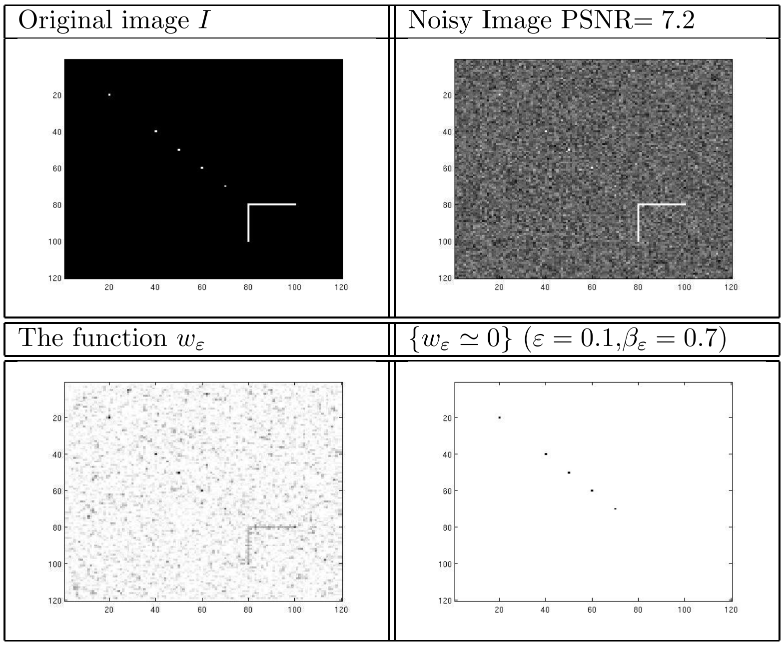

FIGURE 2. *Synthetic image: curve points and noise are present in the initial image. As expected our method is capable of removing the curve from the image. Top left: Original image with five isolate points and a curve. Top right: Noisy image. Bottom left: The function w_ε. Bottom right: the level set $w_\varepsilon \simeq 0$ obtained by fixing a threshold value $\alpha = 0.5$*

8. Conclusion

In this work, a new variational method for point detection in biological images has been proposed and tested. We emphasize that, according to our knowledge, this is the first method which makes possible isolating the spots from a filament in the observed image. Furthermore it permits, in a noisy image, to fix a threshold value in a simple and direct way. Moreover we believe that a suitable generalization of this method for the detection of spots and even filaments in 3-D biological images can be provided. This is a subject of our current investigation. Certainly there are many rooms for improvement both from a theoretical and numerical point of view ,such as a deep variational analysis of the Γ-convergence approximation, as a well as a significant acceleration of the algorithm.

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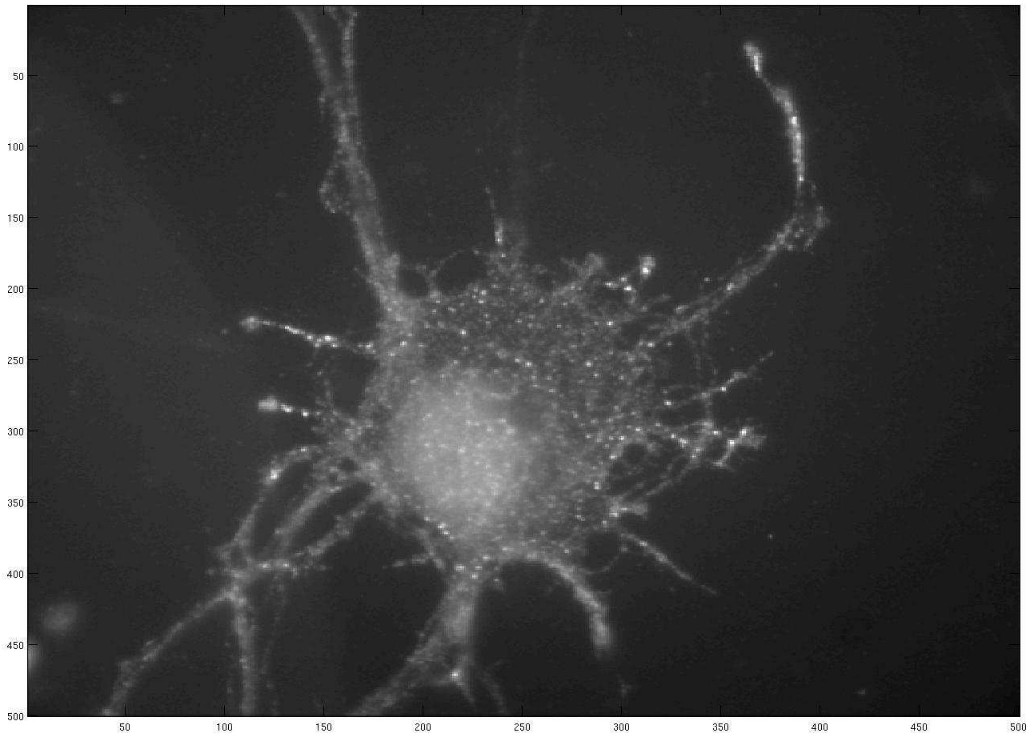
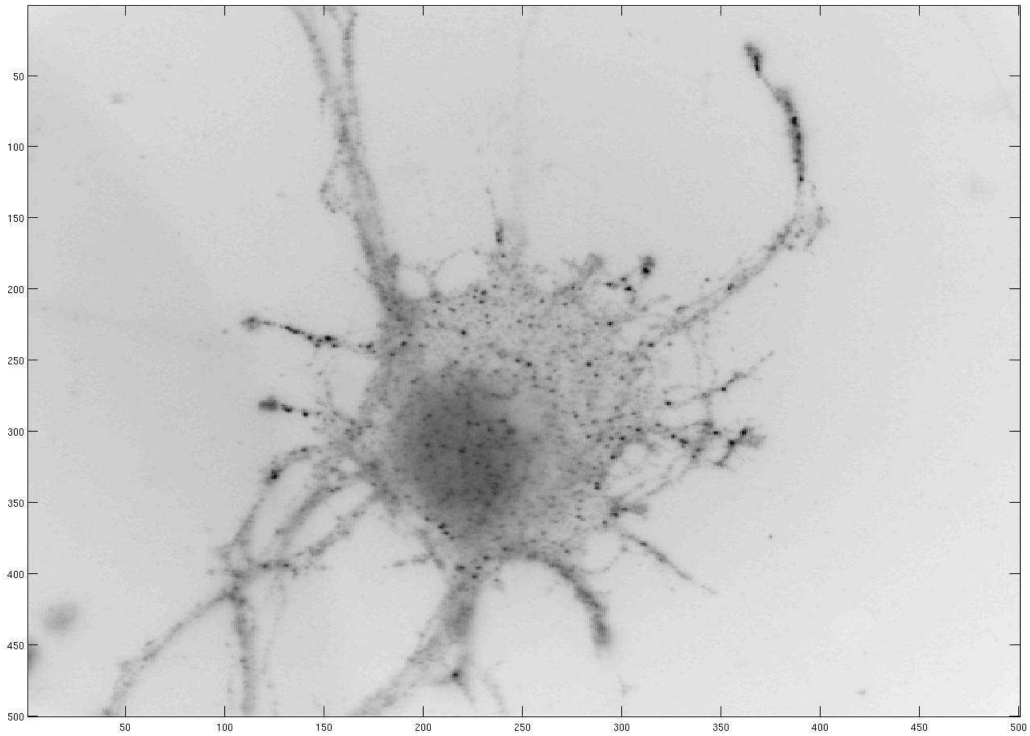
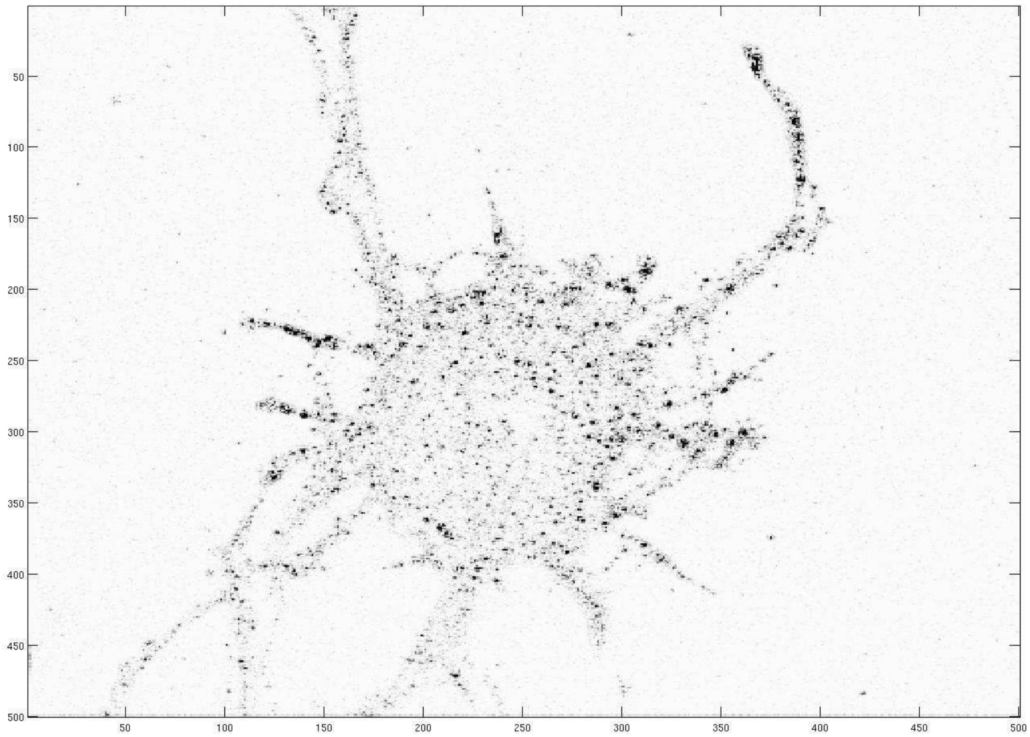


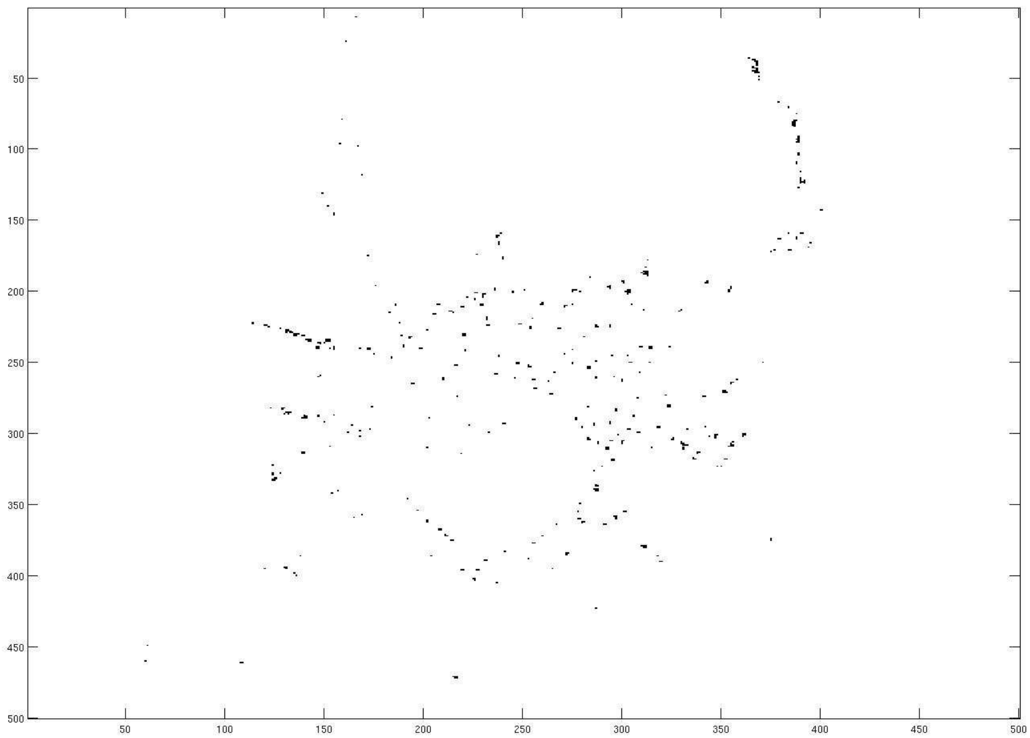
(a) Original Image ©Pasteur. A biological cell of filaments with points (spots) to detect.



(b) Superposition of the original image with the set $\{w_\varepsilon \simeq 0\}$ ($\varepsilon = 0.1$,$\beta_\varepsilon = 0.7$), $\alpha = 0.5$



(c) The function w_ε ($\varepsilon = 0.1$, $\beta_\varepsilon = 0.7$)



(d) the set $\{w_\varepsilon \simeq 0\}$ ($\varepsilon = 0.1$, $\beta_\varepsilon = 0.7$), $\alpha = 0.5$

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