A posteriori estimates and mesh adaptation for the thermistor problem

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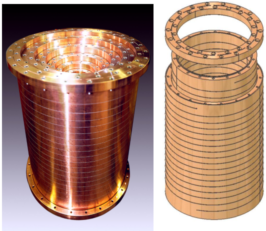
Outline

1. The Grenoble High Magnetic Field Lab
2. The thermistor problem: study and numerical resolution
3. A posteriori estimate
4. Application to adaptation

1. The Grenoble High Magnetic Field Lab

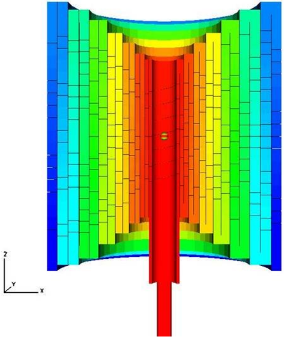
- Resistive magnets ($34 Tesla, 30000A, 20MW$)
- Water cooling, high flow ($20l/s$)

Polyhelice

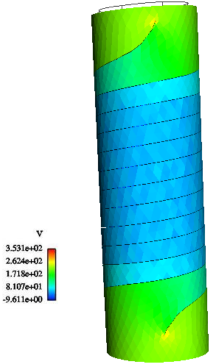


- Geometrical optimization [1].
- Heating effect and mechanical stress: Joule, Lorentz.

1. The Grenoble High Magnetic Field Lab



Potential ϕ in a cut of a 3D Magnet...
under some simplifications.



... on an helix

- Magnetic field **b** and current density **j**:

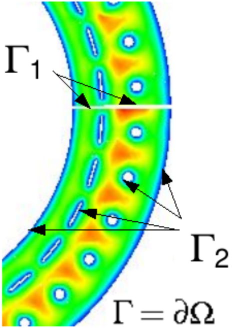
$$\forall x \in \Omega, \quad \mathbf{j}(x) = \sigma(u)\, \nabla\, \phi(x),$$
$$\forall x \in \omega, \quad \mathbf{b}(x) = \mu \int_{\Omega} \mathbf{j}(y) \wedge \nabla G(x,y)\, dy,$$

2. The thermistor problem

Model

Find $(\phi, u) : \Omega \to \mathbb{R}$ *s.t.:*

$$(C) \left\{ \begin{array}{rcll} -div\ (\sigma(u)\ \nabla\phi) & = & f & \text{in } \Omega, \\ -div\ (\kappa(u)\ \nabla u) & = & \sigma(u)|\nabla\phi|^2 & \text{in } \Omega, \\ \phi & = & \phi_0 & \text{on } \Gamma_1, \\ -\sigma(u)\nabla\phi.\mathbf{n} & = & 0 & \text{on } \Gamma_2, \\ u & = & u_w & \text{on } \Gamma_2, \\ -\kappa(u)\nabla u.\mathbf{n} & = & 0 & \text{on } \Gamma_1. \end{array} \right.$$



- $\Omega, \omega \subset \mathbb{R}^2$, $\omega \cap \Omega = \varnothing$.
- σ et κ bounded, Lipschitz-continuous on $\mathbb{R}^{+*}$.

Difficulties

- Geometry: highly non convex, fissures, holes.
- Numeric: mesh, method?
- A posteriori estimate on **b**?

2. The thermistor problem

Model

Let Ω be a polygonal domain, and ω_i the interior angle between two consecutive edges of Ω. Let ω_i^* s.t.:
$\omega_i^* = \omega_i$ if the two edges have the same BC,
$\omega_i^* = 2\omega_i$ else.
Let $\omega^* = max_i(\omega_i^*)$

If $\kappa,\ \sigma:\ \Omega \mapsto \mathbb{R}, \in C^m(\overline{\Omega})$ and $f \in L^s(\Omega)$, $s > 1$.

$$\Longrightarrow u, \phi \in H^{1+2/q}(\Omega),\ q > max(q^*, 2),\ q^* = \frac{2}{\pi}\omega_i^*$$

(C_h) : *Find $u_h \in V_h$ and $\phi_h \in W_h$ s.t.*

$$\begin{aligned} \int_\Omega \kappa(u_h)\, \nabla u_h . \nabla v_h \, dx &= \int_\Omega \sigma(u_h)\, |\nabla \phi_h|^2\, v_h \, dx,\ \forall v_h \in V_h \\ \int_\Omega \sigma(u_h) \nabla \phi_h . \nabla \psi_h \, dx &= \int_\Omega f\, \psi_h \, dx, \qquad \forall \psi_h \in W_h \end{aligned}$$

2. The thermistor problem

Model

If more general conditions on Ω, well-posed problem, associated to the limit-problem [2,3]:

Let $\tau(\theta) = \sigma\ o\ \kappa^{-1}(u)$,

(C') : *Find* $\theta \in L^2(\Omega)$ *and* $\phi \in H_0^1(\Omega)$ *s.t.*

$$\begin{aligned} \int_\Omega \theta . \xi \, dx &= \int_\Omega \tau(\theta) \, |\nabla\phi|^2 \, (\Delta^{-1}\xi) \, dx, \ \forall \xi \in L^2(\Omega) \\ \int_\Omega \tau(\theta) \nabla\phi . \nabla\psi \, dx &= \int_\Omega f \psi \, dx, \ \forall \psi \in H^1(\Omega) \end{aligned}$$

$\Longrightarrow \exists (\theta, \phi) \in H^s(\Omega) \times H_0^1(\Omega)$, $s < 1/2$ solutions of (C').

2. The thermistor problem

Numerical simulation

Relaxed Fixed-Point Algorithm:

$$\begin{aligned}
\int_\Omega u^{n+1}\, v\, \mathrm{d}x + \Delta t \int_\Omega \kappa(u^n)\, \nabla u^{n+1} \nabla v\, \mathrm{d}x &= \Delta t \int_\Omega \sigma(u^n) |\nabla \phi^{n+1}|^2\, v \mathrm{d}x \\
&\quad + \int_\Omega u^n\, v \mathrm{d}x \quad \forall v \in V_h, \\
\int_\Omega \sigma(u^n)\, \nabla \phi^{n+1} \nabla w \mathrm{d}x &= \int_\Omega f\, w\, \mathrm{d}x \quad \forall w \in W_h,
\end{aligned}$$

$u^{n+1} = \sum_i u_i^{n+1} v_i$, v_i finite element of order k on quads.
(gmsh, deal.ii)
Stop when $\|u^{n+1} - u^n\|_{L^2} < \varepsilon$.

3. A posteriori estimates

Looking forward an estimate η s.t. $\|u-u_h\|_{H^1}+\|\phi-\phi_h\|_{H^1}\lesssim\eta\lesssim h^p$.

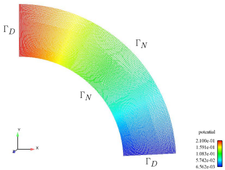
- Theoretical estimate [5] with residual and edge terms.
- *Kelly estimate of u defined on an element K [6]:*

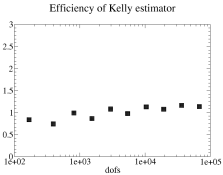
$$\eta_K^2(u)=\frac{h}{24}\int_{\partial K}\left[\frac{\kappa(u_h)\,\partial u_h}{\partial \mathbf{n}}\right]^2 ds \qquad \Longrightarrow \textit{convergence order } \gamma$$

- *Comparison of $\gamma+1$ with λ solution of [3]:*

$$(1) \qquad \frac{\|u_{h_{n-1}}-u_{h_{n-2}}\|_{L^2(\Omega)}}{\|u_{h_n}-u_{h_{n-1}}\|_{L^2(\Omega)}}=\frac{(h_{n-2}/h_{n-1})^{\lambda}-1}{1-(h_n/h_{n-1})^{\lambda}}$$

$\Longrightarrow$ Validation of Kelly estimate.
$\Longrightarrow$ Numerical indicators for the error: γ et λ.

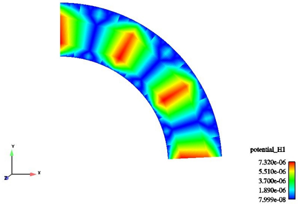


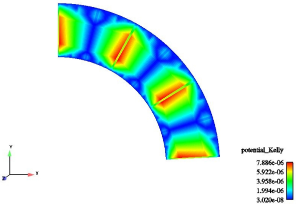


3. A posteriori estimates

Comparison with an analytical solution

Error $\frac{|\phi - \phi_h|_{H^1(\Omega)}}{|\phi_h|_{H^1(\Omega)}}$. Estimate $\frac{\eta^{\phi}}{|\phi_h|_{H^1(\Omega)}}$.





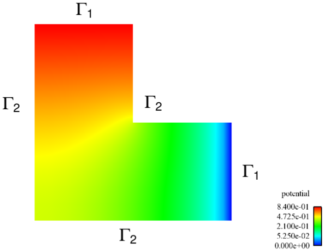
	Q_1			Q_2			Q_3		
	rel. err.	λ	γ	rel. err.	λ	γ	rel. err.	λ	γ
ϕ	1.99	1.97	0.84	2.97	2.95	2.73	3.99	3.51	2.85
u	2.00	2.00	0.98	2.98	2.54	2.49	3.96	2.41	2.96

Convergence orders for ϕ and u and several Q_k.

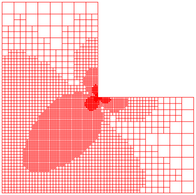
$$\text{rel. err.} = \frac{\| u - u_h \|_{L^2}}{\| u \|_{L^2}}.$$

4. Application to adaptation

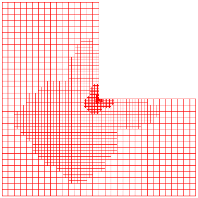
Non convex geometry



η^ϕ

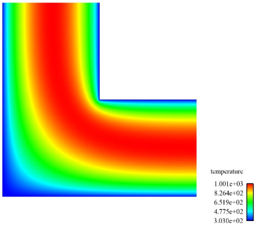


$\eta^\phi + \eta^u$

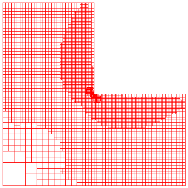


Refinment criteria:

u



η^u



4. Application to adaptation

Non convex geometry

	Q_1		Q_2		Q_3	
	γ	λ	γ	λ	γ	λ
ϕ	1.42	0.70	1.43	0.67	1.45	0.67
u	1.52	0.55	1.82	0.53	1.76	0.54

Convergence orders for ϕ and u and several Q_k.

$\Longrightarrow$ Ok with the theory, u and $\phi \in H^s$, with $s > 1.66$.

Conclusion and perspectives

- Numerical study of an efficient A posteriori estimate for the thermistor problem [3].
- Efficiency of the adaptive scheme.
- Limitation by hanging nodes on quads?
- Future developpements: A posteriori error estimate for the magnetic field, by means of a optimal control approach [7].

References

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