ISOTROPIC 2D QUADRANGLE MESHING WITH SIZE AND ORIENTATION CONTROL

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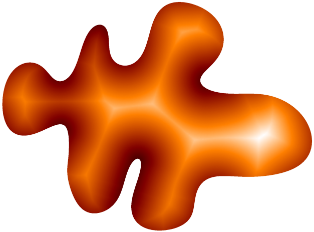
May 4, 2012

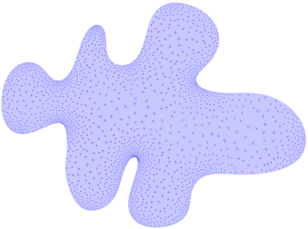
Problem Statement

INPUT

2D domain



Sizing field ($\mathbb{R}^2 \rightarrow \mathbb{R}$)



Cross field ($\mathbb{R}^2 \rightarrow \mathbb{R}^2$)

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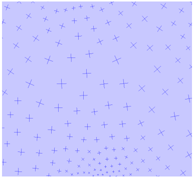

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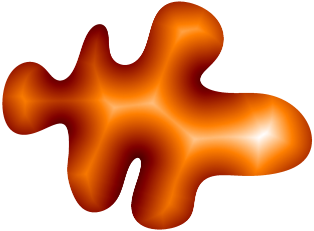

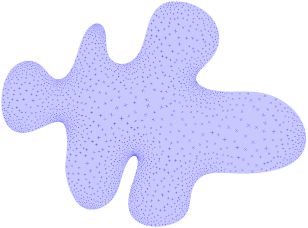

2D domain

Sizing field ($\mathbb{R}^2 \to \mathbb{R}$)

Cross field ($\mathbb{R}^2 \to \mathbb{R}^2$)

Problem Statement

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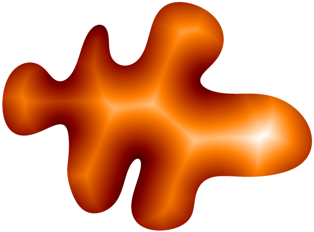

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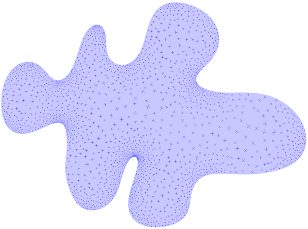

Cross field ($\mathbb{R}^2 \rightarrow \mathbb{R}^2$)

OUTPUT

Quadrangle mesh respecting both size and orientation

Previous Work (2D / Surfaces)

- Quadrangulation: [Bremner2003, Bern2000]
- Square packing: [Shimada1998]
- Advancing front: [Owen1999]
- Conversion: [Bourouchaki1996]
- Medial axis: [Quadros2000]
- Whisker weaving: [Wolfenbarger1998]
- Clustering: [Lévy2010, Boier-Martin2004]
- Local and global operators: [Pietroni2010, Lai2008]
- Parameterization: [Ray2006, Tong2006, Bommes2009]

5 Criteria

1. Size
   element size conforming to input sizing field
2. Shape
   element close to a square
3. Orientation
   edges parallel to input cross field
4. Degree
   quadrangle element: degree 4
5. Regularity
   few irregular vertices (degree $\neq 4$)

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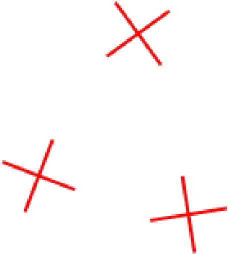
   few irregular vertices (degree $\neq 4$)

Irregular vertices
Varying cross field
Non-uniform sizing field

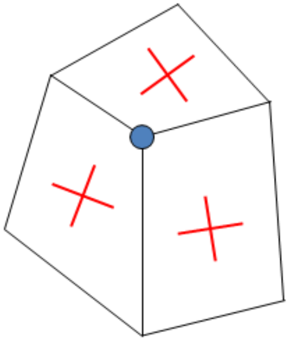
Irregular Vertices

Varying cross field



Irregular Vertices

Varying cross field



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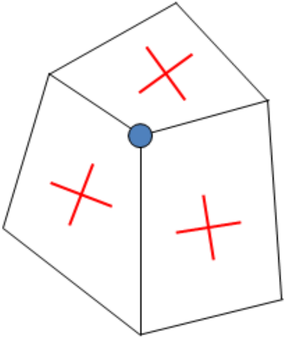
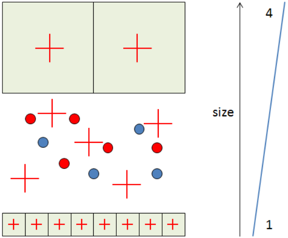


Non-uniform sizing field



Rationale

Delaunay refinement

begin
  while *bad element* **do**
    refine(element)

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for all criteria: size, isotropy

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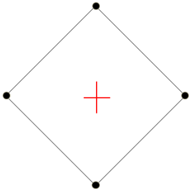
Our algorithm

begin
  Optimize mesh for criterion 1: size
  Optimize mesh for criterion 2: shape
  Optimize mesh for criterion 3: orientation
  Optimize mesh for criterion 4: degree
  Optimize mesh for criterion 5: regularity

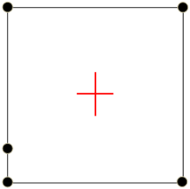
Bad Elements

Cases where a single criterion is not met

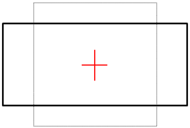
Orientation



Degree



Shape



Algorithm and Rationale

Input: $2D$ domain, sizing field, cross field

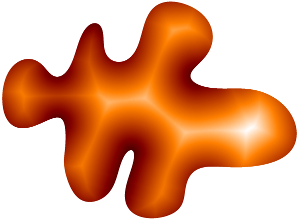
STEPS / CRITERIA	*Size*	*Shape*	*Orientation*	*Degree*	*Regularity*
❶ Initialization	●	•			
❷ Relaxation	○	●	●		
❸ Conforming Relaxation	○	○	○	•	•
❹ Local parameterizations	○	○	○	•	●
❺ Barycentric subdivision	○	×	○	●	○
❻ Smoothing	○	●	○	○	○

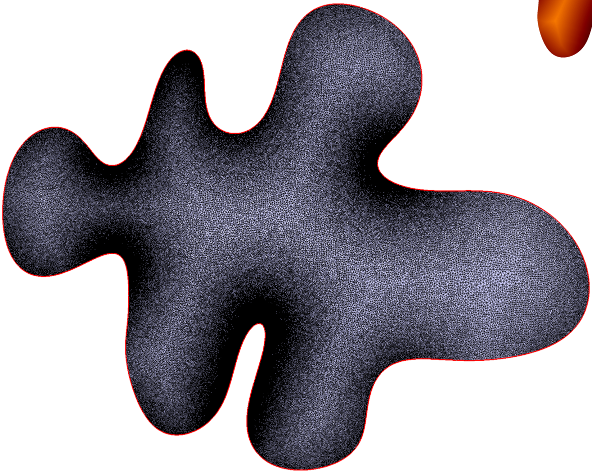
•: partially met, ●: met, ○: preserved

Output: Quadrangle mesh

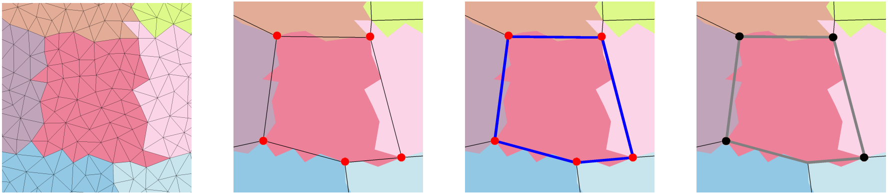
Steps ❶-❹ Act on Background Mesh

Obtained through Delaunay refinement



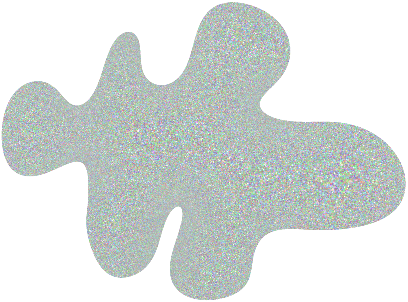


Notations



Tile / Tiling   Meta-vertices   Meta-edges   Sides

Initialization



one tile per triangle

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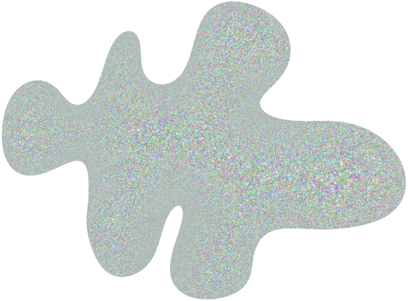

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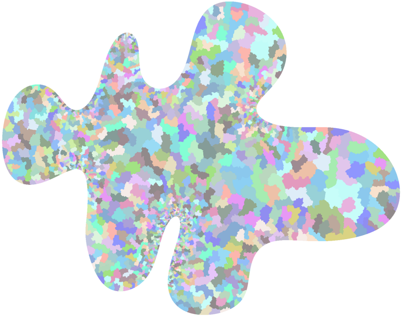
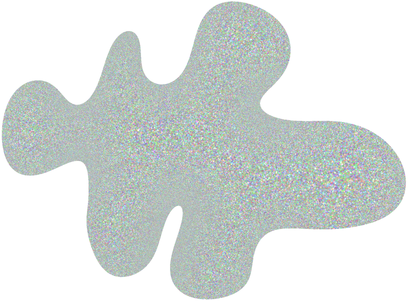


conformance to sizing + isotropy

Initialization



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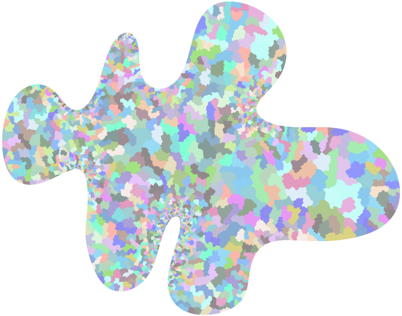

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- Fill priority queue (pqueue) with pairs of mergeable tiles
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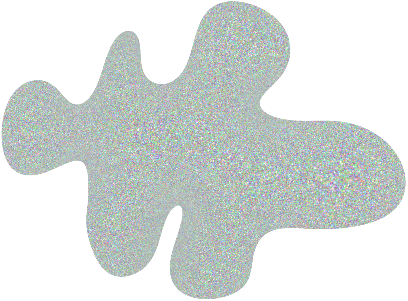

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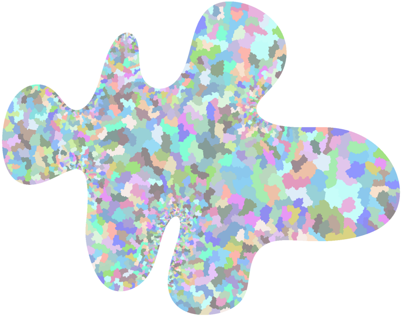

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Isoperimetric inequality:

$$4\pi\mathcal{A} \leq \mathcal{L}^2$$

$\mathcal{L}$: length
$\mathcal{A}$: area

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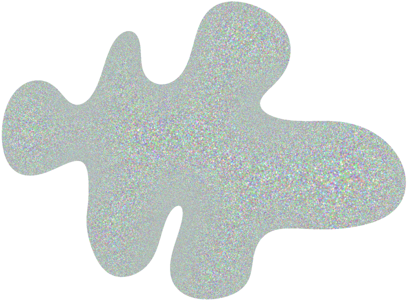

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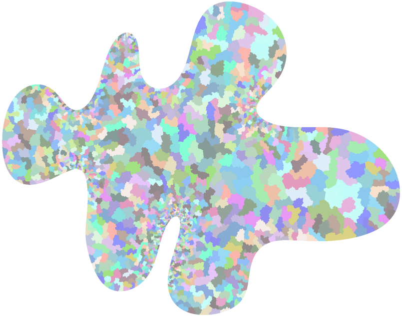

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$\mathcal{L}_\infty^\mathcal{R}$ Relaxation

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$\mathcal{L}_\infty^\mathcal{R}$ Relaxation

Discrete Voronoi diagram
distance : $d_\infty^\mathcal{R}(p,q) = \max\left(|(q-p)\circ \mathbf{u}|, |(q-p)\circ \mathbf{v}|\right)$
$\mathcal{R} = (\mathbf{u},\mathbf{v})$: local Cartesian coordinate frame

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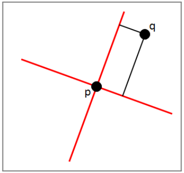

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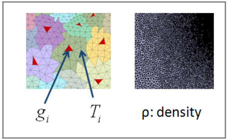
Minimizers

continuous: $\mathcal{G}\left(\{z_i\}_{i=1}^N, \{V_i\}_{i=1}^N\right) = \sum_{i=1}^N \int_{V_i} \rho(x)\ d_\infty^\mathcal{R}(x, z_i)\ dx$

discrete: $\mathcal{H}\left(\{g_i\}_{i=1}^N, \{T_i\}_{i=1}^N\right) = \sum_{i=1}^N \left(\sum_{t_j \in T_i} \rho(c(t_j))\mathrm{area}(t_j) d_\infty^\mathcal{R}(c(t_j), c(\mu(T_i))) \right)$



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$\mathcal{L}_\infty^\mathcal{R}$ Relaxation

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❷ Relaxation	○	●	●		

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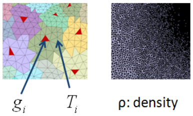

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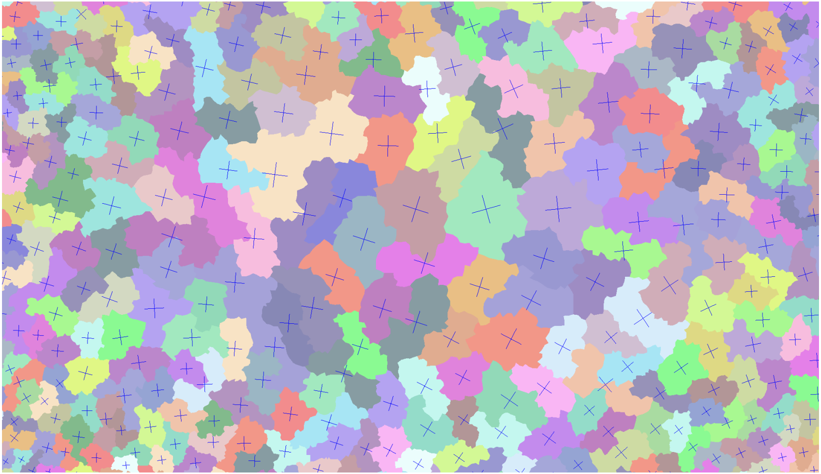
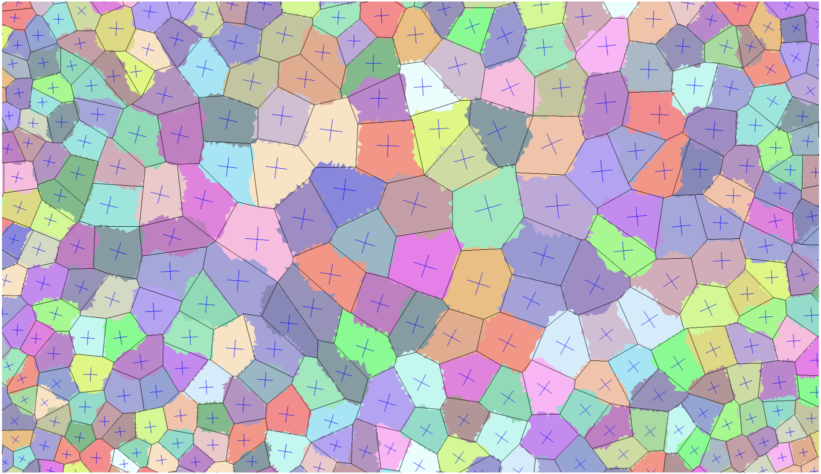
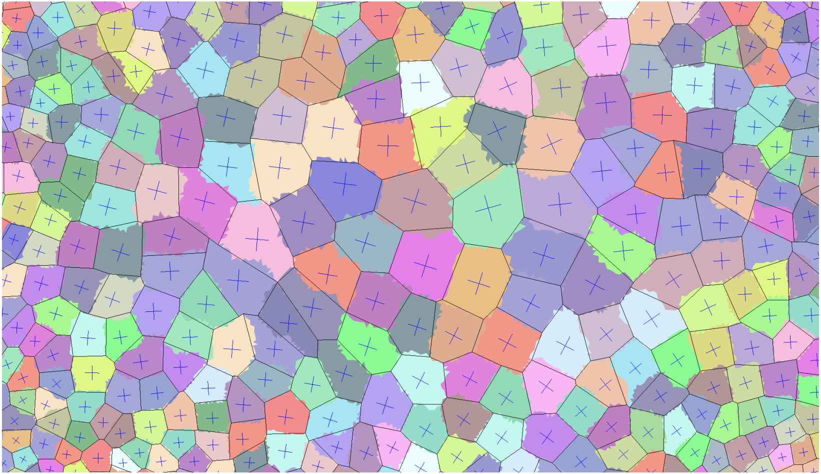
$\mathcal{L}_\infty^\mathcal{R}$ Relaxation: Initial



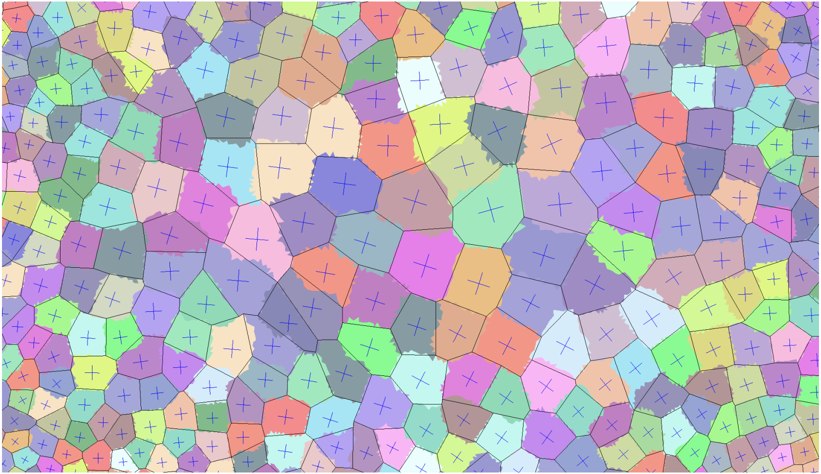
$\mathcal{L}_\infty^\mathcal{R}$ Relaxation: Iteration 1



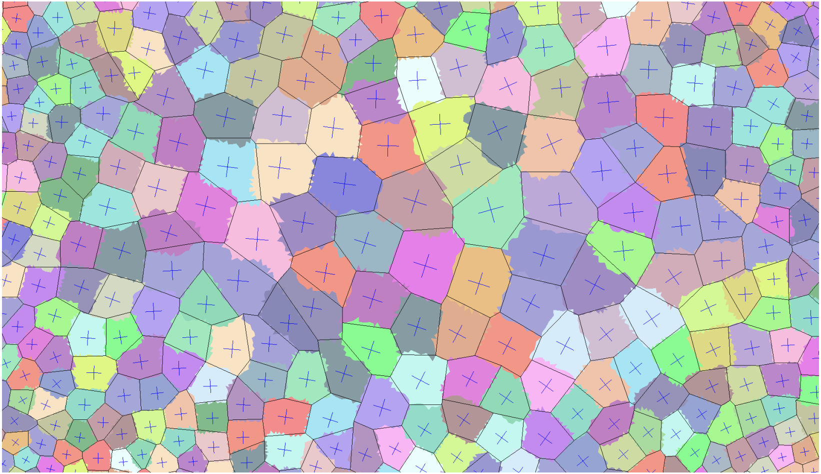
$\mathcal{L}_\infty^\mathcal{R}$ Relaxation: Iteration 2



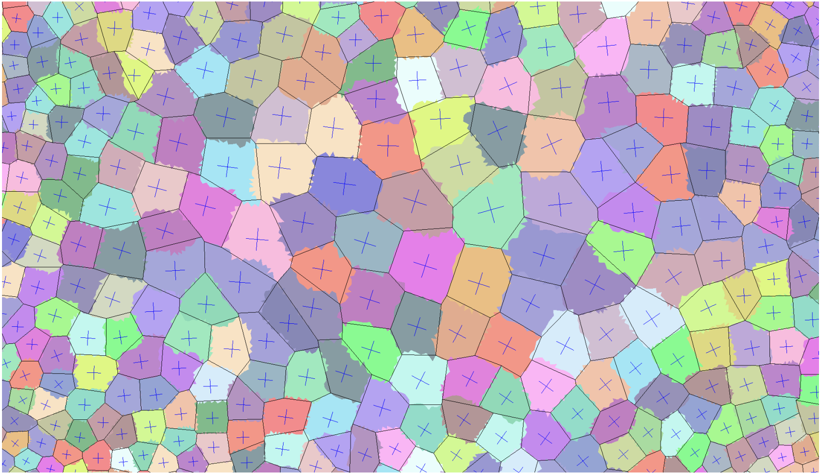
$\mathcal{L}^{\mathcal{R}}_{\infty}$ Relaxation: Iteration 3



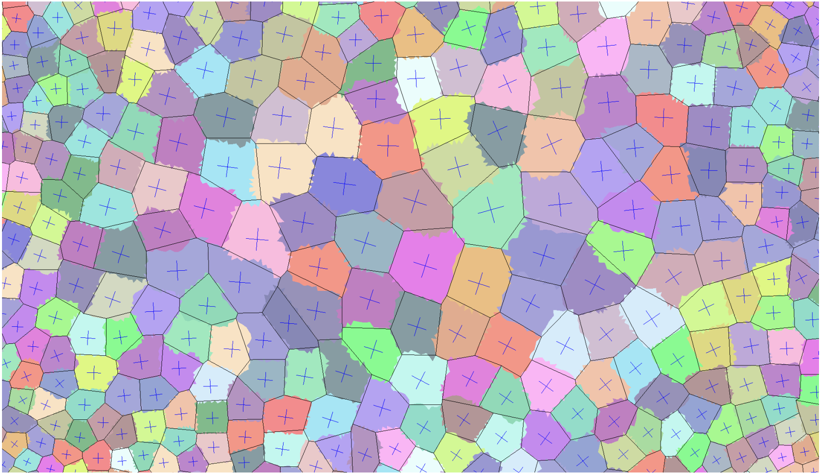
$\mathcal{L}^{\mathcal{R}}_{\infty}$ Relaxation: Iteration 4



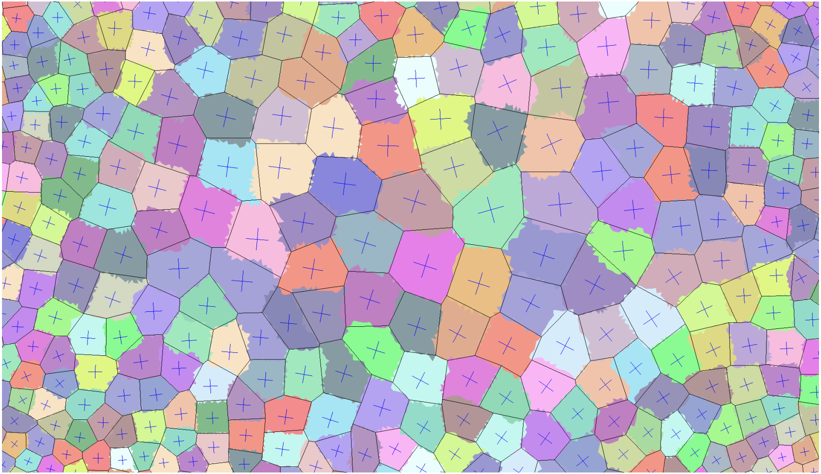
$\mathcal{L}_\infty^{\mathcal{R}}$ Relaxation: Iteration 5



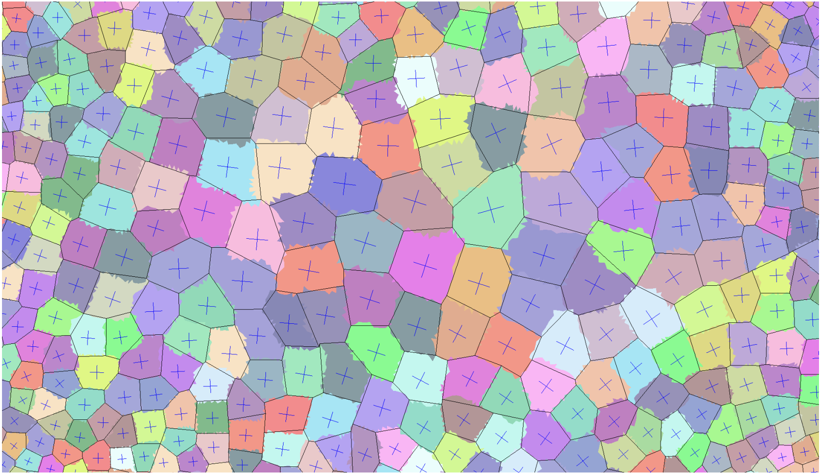
$\mathcal{L}_\infty^\mathcal{R}$ Relaxation: Iteration 6



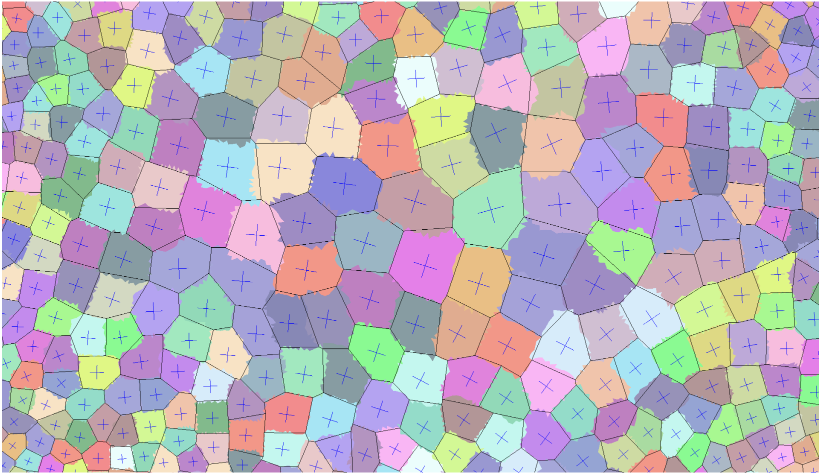
$\mathcal{L}_\infty^\mathcal{R}$ Relaxation: Iteration 7



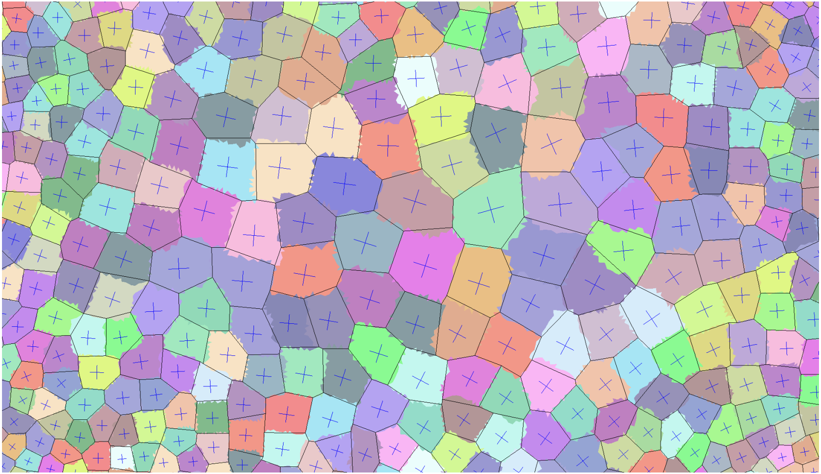
$\mathcal{L}_{\infty}^{\mathcal{R}}$ Relaxation: Iteration 8



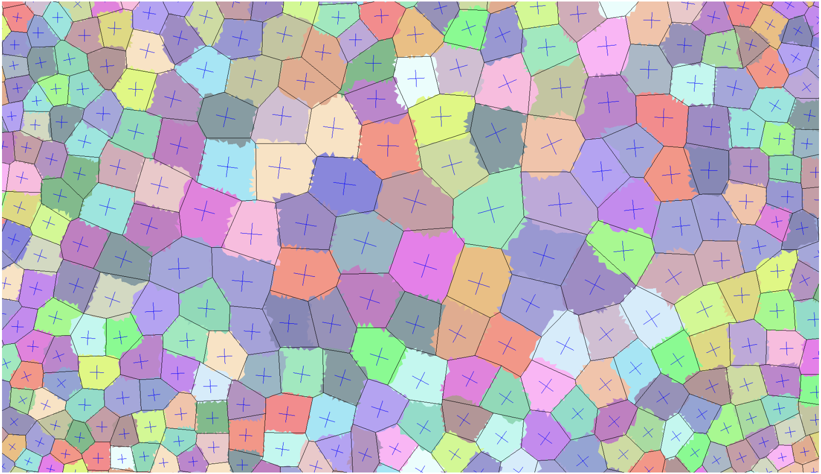
$\mathcal{L}^{\mathcal{R}}_{\infty}$ Relaxation: Iteration 9



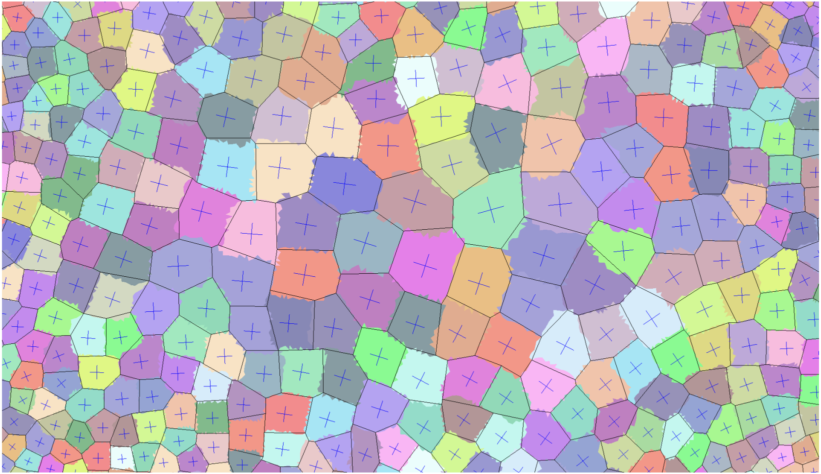
$\mathcal{L}_\infty^\mathcal{R}$ Relaxation: Iteration 10



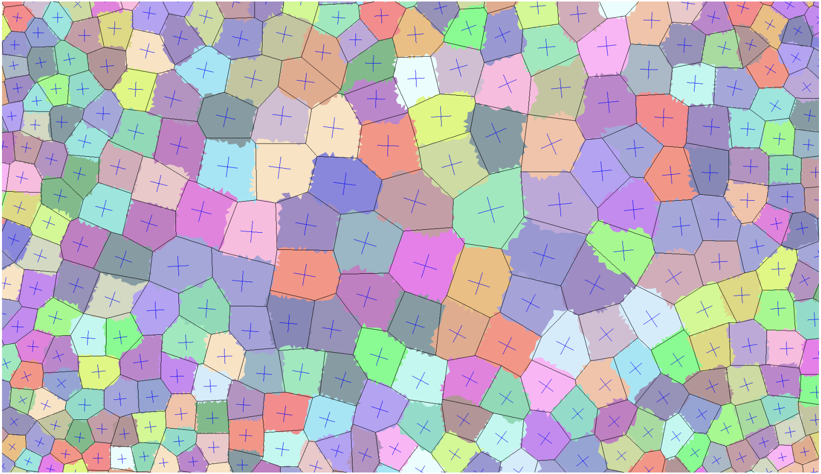
$\mathcal{L}_\infty^{\mathcal{R}}$ Relaxation: Iteration 11



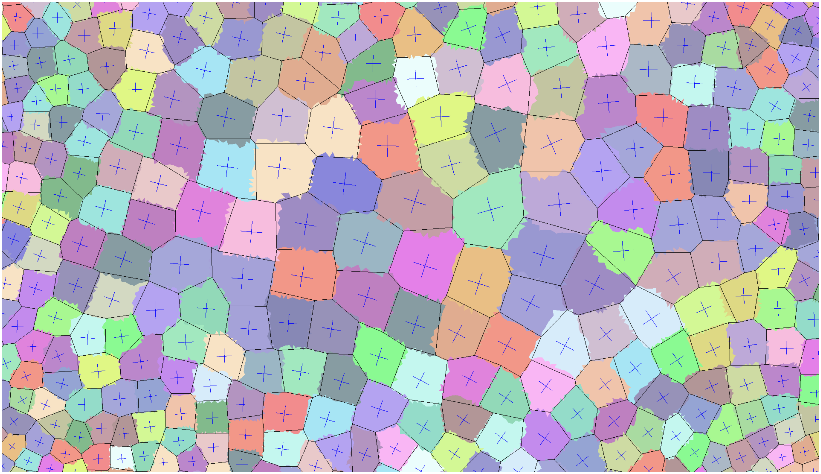
$\mathcal{L}_\infty^\mathcal{R}$ Relaxation: Iteration 12



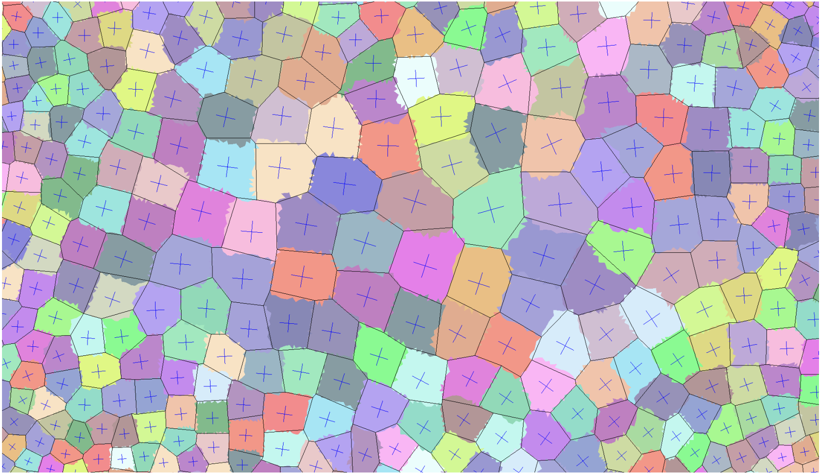
$\mathcal{L}_\infty^\mathcal{R}$ Relaxation: Iteration 13



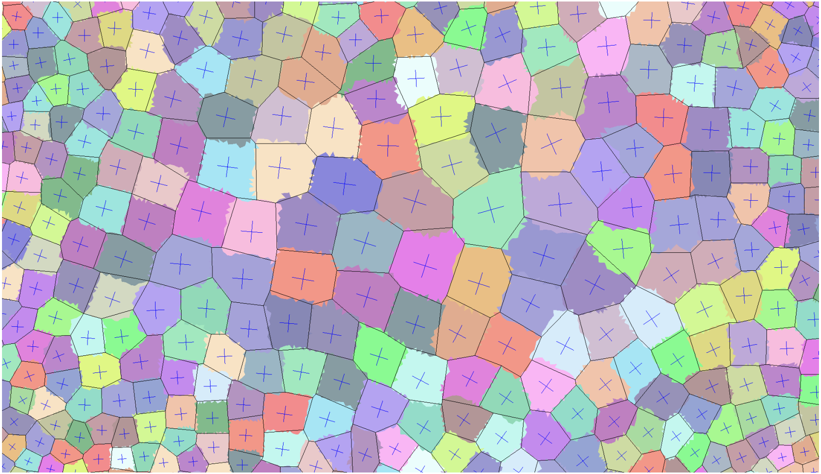
$\mathcal{L}_\infty^\mathcal{R}$ Relaxation: Iteration 14



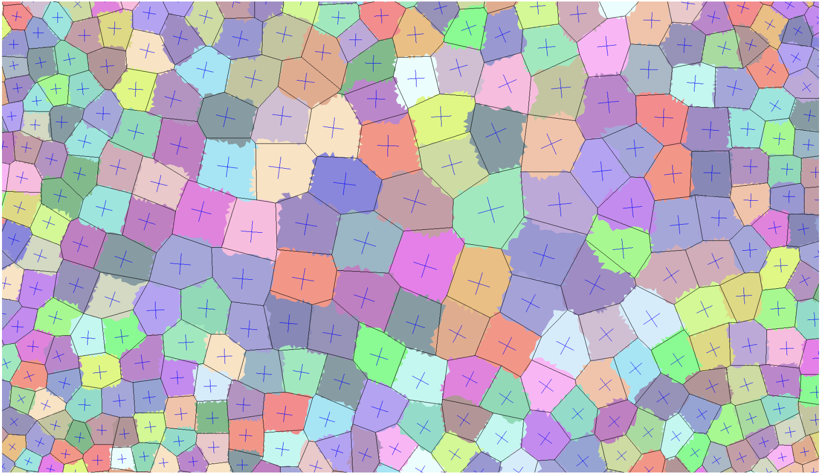
$\mathcal{L}^{\mathcal{R}}_{\infty}$ Relaxation: Iteration 15



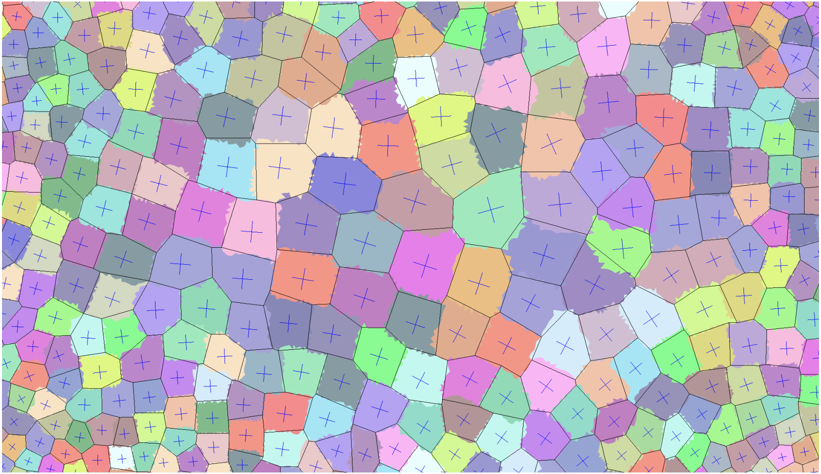
$\mathcal{L}_{\infty}^{\mathcal{R}}$ Relaxation: Iteration 16



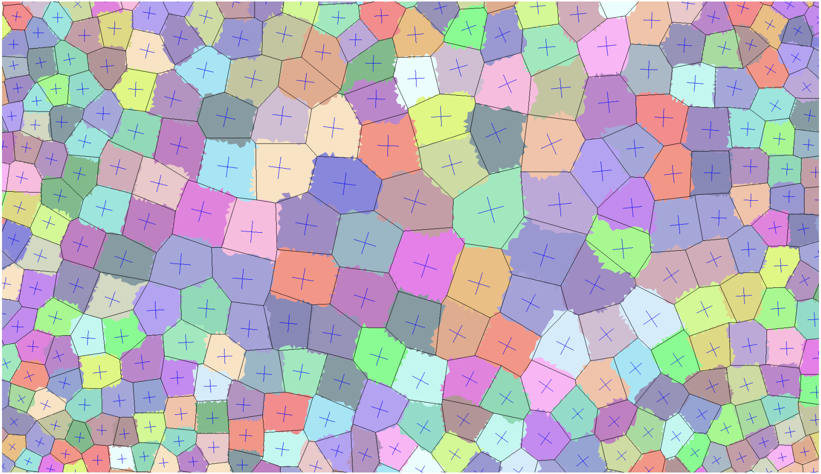
$\mathcal{L}_\infty^\mathcal{R}$ Relaxation: Iteration 17



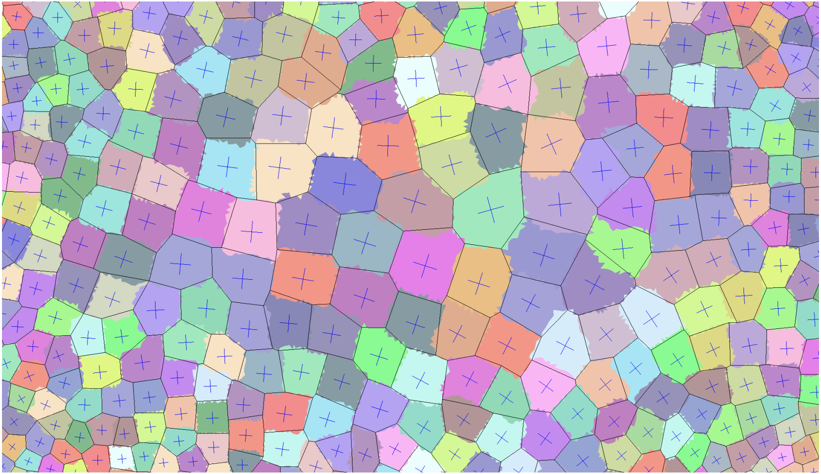
$\mathcal{L}_\infty^{\mathcal{R}}$ Relaxation: Iteration 18



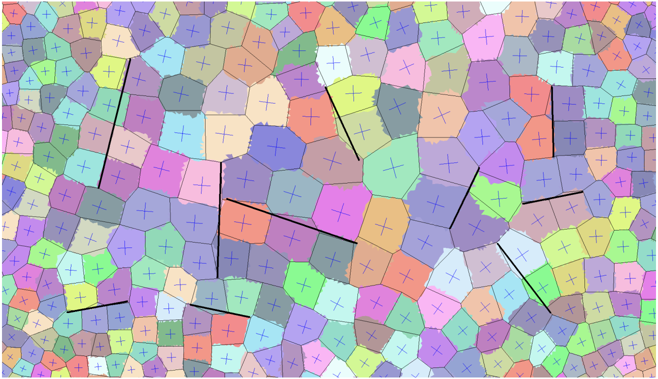
$\mathcal{L}_{\infty}^{\mathcal{R}}$ Relaxation: Iteration 19



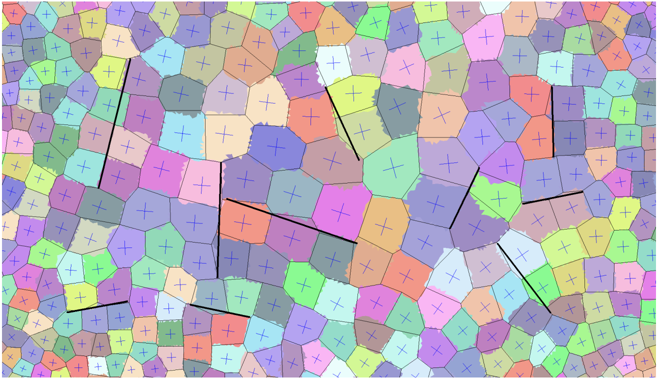
$\mathcal{L}_\infty^\mathcal{R}$ Relaxation: Iteration 20



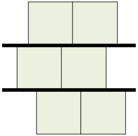
$\mathcal{L}_\infty^R$ Relaxation: Observation



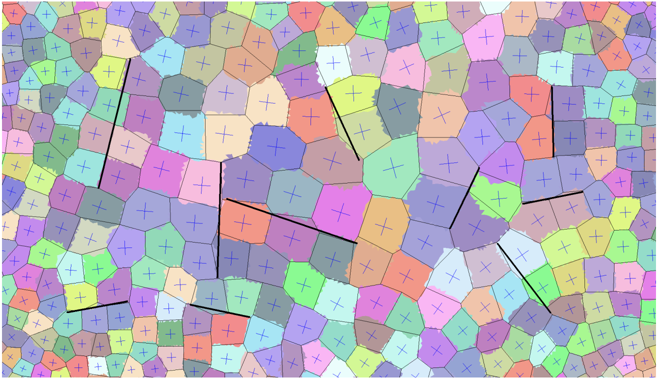
$\mathcal{L}^{\mathcal{R}}_{\infty}$ Relaxation: Observation



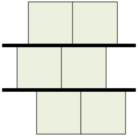
General configuration:



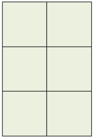
$\mathcal{L}_\infty^R$ Relaxation: Observation



General configuration:

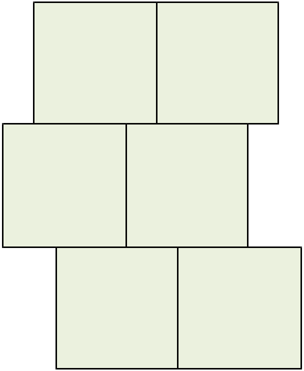


Target: 2-conforming



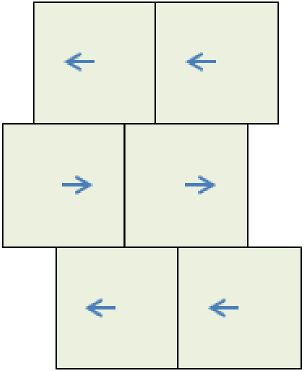
How To Identify 1-conforming Directions?

Shift centroid



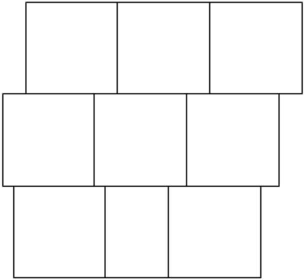
How To Identify 1-conforming Directions?

Shift centroid



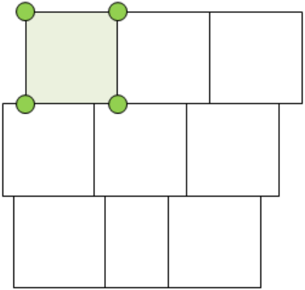
How To Identify 1-conforming Directions?

Shift centroid



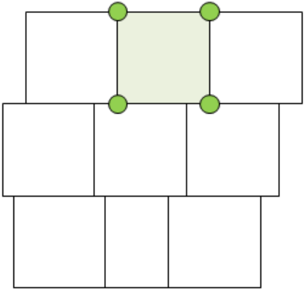
How To Identify 1-conforming Directions?

Shift centroid



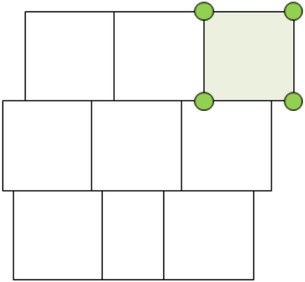
How To Identify 1-conforming Directions?

Shift centroid



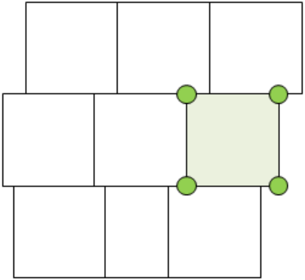
How To Identify 1-conforming Directions?

Shift centroid



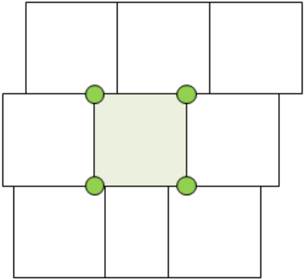
How To Identify 1-conforming Directions?

Shift centroid



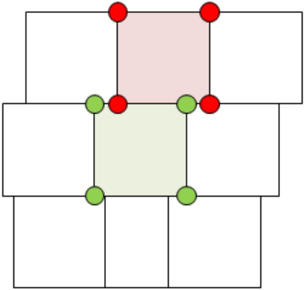
How To Identify 1-conforming Directions?

Shift centroid



How To Identify 1-conforming Directions?

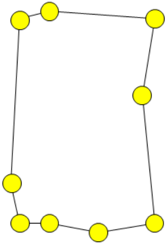
Shift centroid



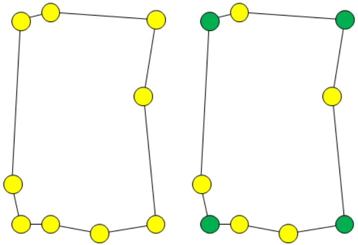
$\mathcal{L}_\infty^\mathcal{R}$ Conforming Relaxation

begin
  while *no convergence* **do**
    Discrete partitioning
    Relocate generators to **shifted** centroids

$\mathcal{L}_\infty^\mathcal{R}$ Conforming Relaxation



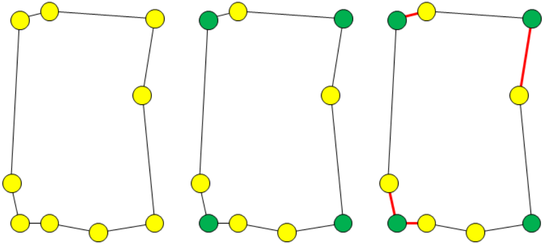
$\mathcal{L}_\infty^\mathcal{R}$ Conforming Relaxation



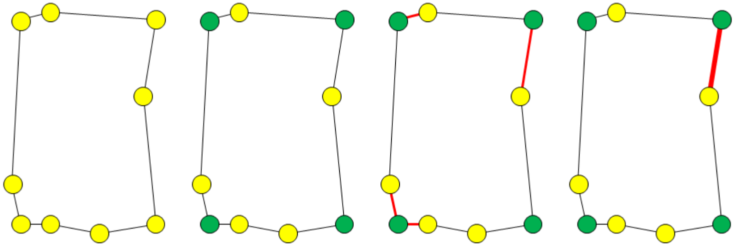
$v_1, \ldots, v_p$: meta-vertices (ordered)

$$\mathcal{E} = \max_{\mathbf{a} \subset \{\mathbf{u},\mathbf{v}\} \{c_1,c_2,c_3,c_4\} \subset \{v_1,\ldots,v_p\}} \left[\min \left(|(c_2 - c_1) \cdot \mathbf{a}|, |(c_3 - c_2) \cdot \mathbf{a}^{90}|, |(c_4 - c_3) \cdot \mathbf{a}|, |(c_1 - c_4) \cdot \mathbf{a}^{90}| \right) \right]$$

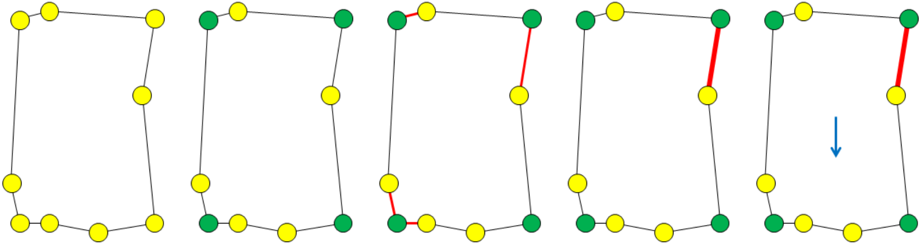
$\mathcal{L}_\infty^\mathcal{R}$ Conforming Relaxation



$\mathcal{L}_\infty^\mathcal{R}$ Conforming Relaxation

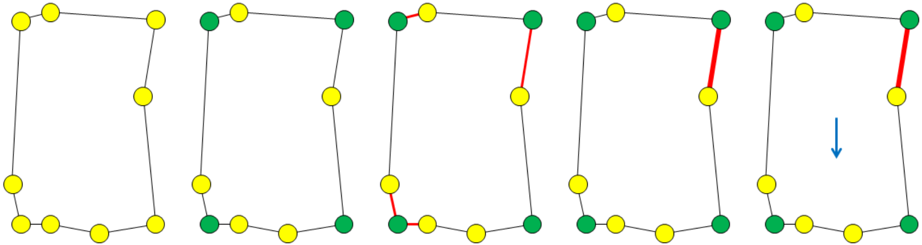


$\mathcal{L}^{\mathcal{R}}_{\infty}$ Conforming Relaxation

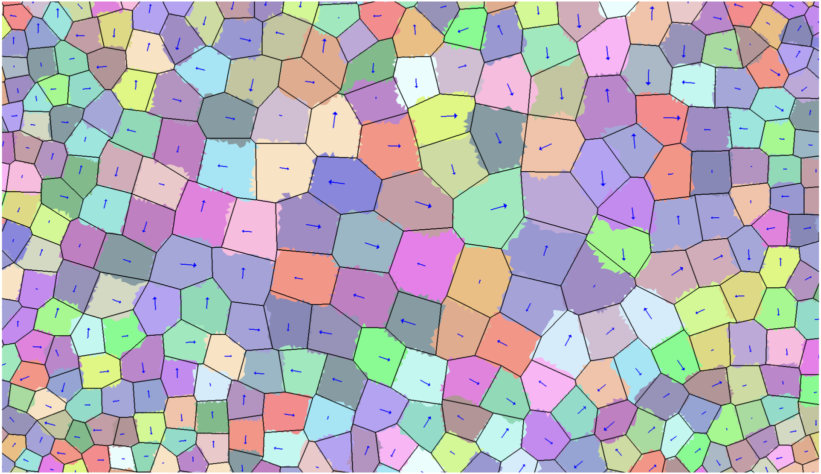


$\mathcal{L}^{\mathcal{R}}_{\infty}$ Conforming Relaxation

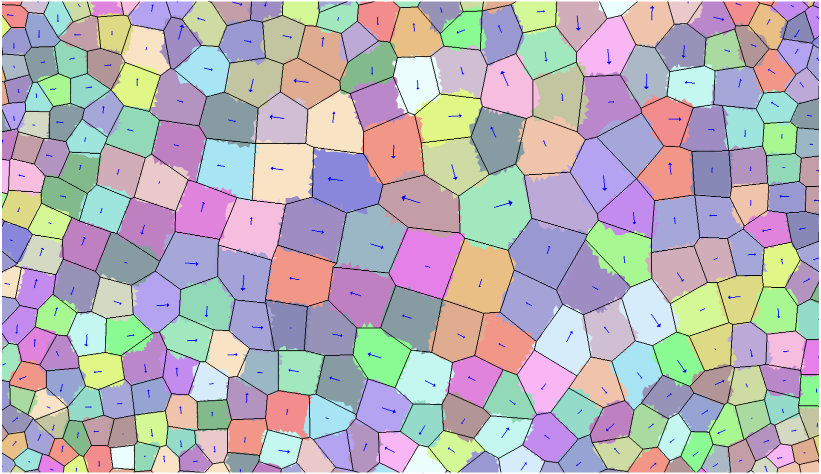
STEP / CRITERIA	*Size*	*Shape*	*Orientation*	*Degree*	*Regularity*
❸ Conforming Relaxation	○	○	○	•	•



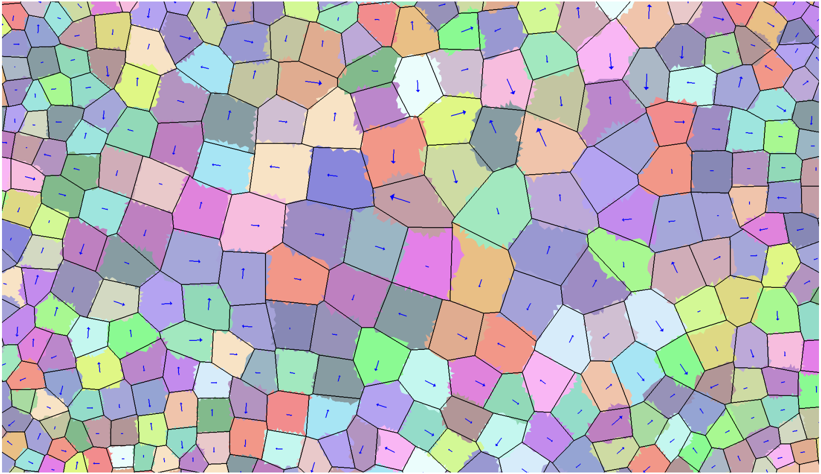
$\mathcal{L}^{\mathcal{R}}_{\infty}$ Conforming Relaxation: Initial



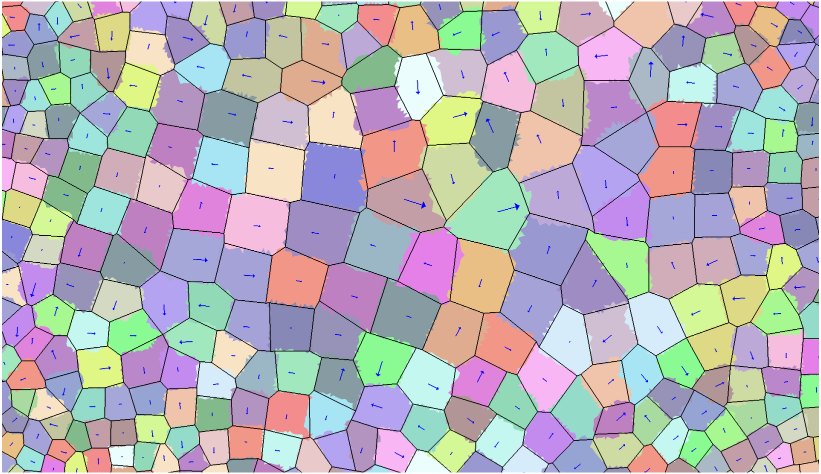
$\mathcal{L}_\infty^\mathcal{R}$ Conforming Relaxation: Iteration 1



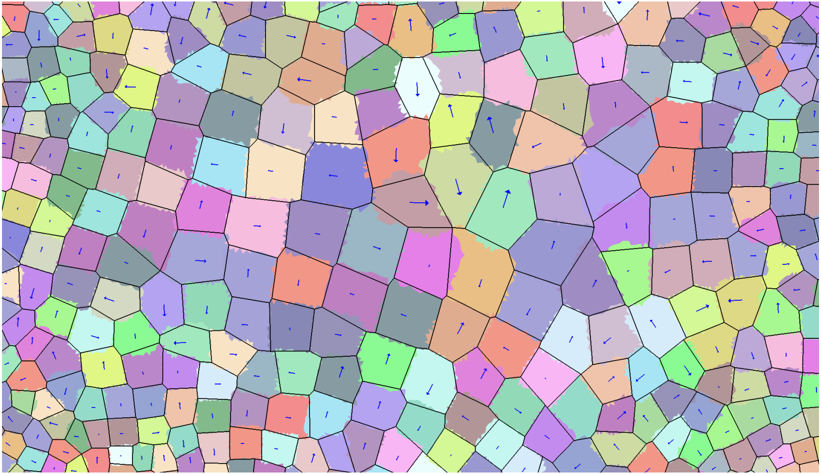
$\mathcal{L}_{\infty}^{\mathcal{R}}$ Conforming Relaxation: Iteration 2



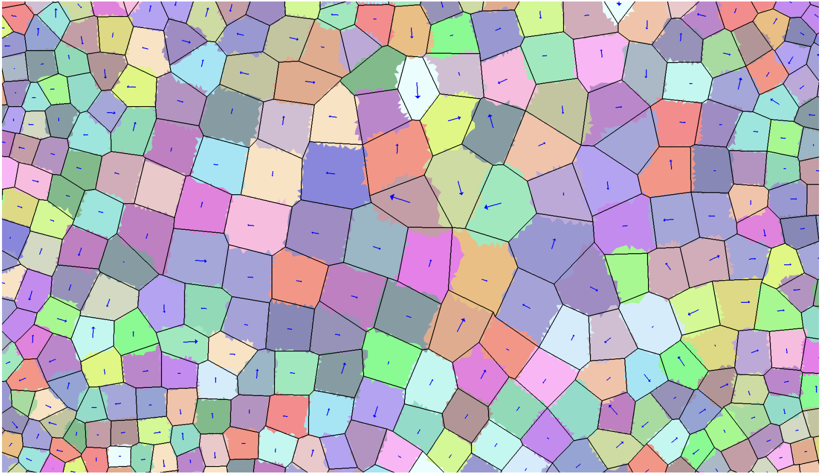
$\mathcal{L}_\infty^\mathcal{R}$ Conforming Relaxation: Iteration 3



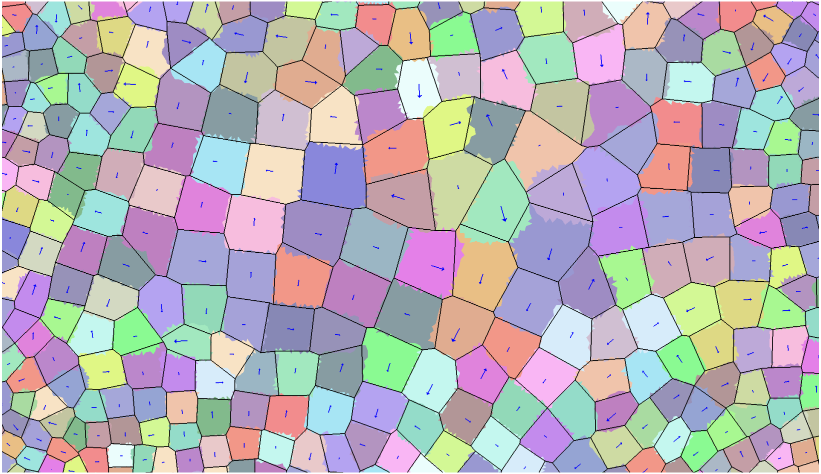
$\mathcal{L}_\infty^\mathcal{R}$ Conforming Relaxation: Iteration 4



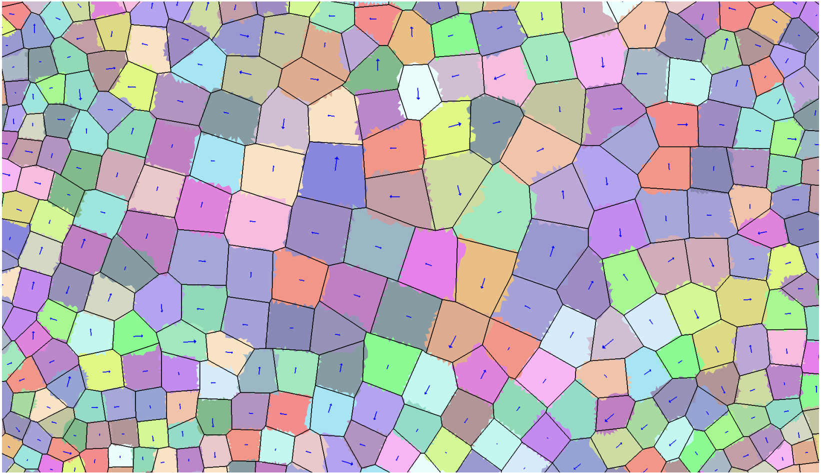
$\mathcal{L}_\infty^\mathcal{R}$ Conforming Relaxation: Iteration 5



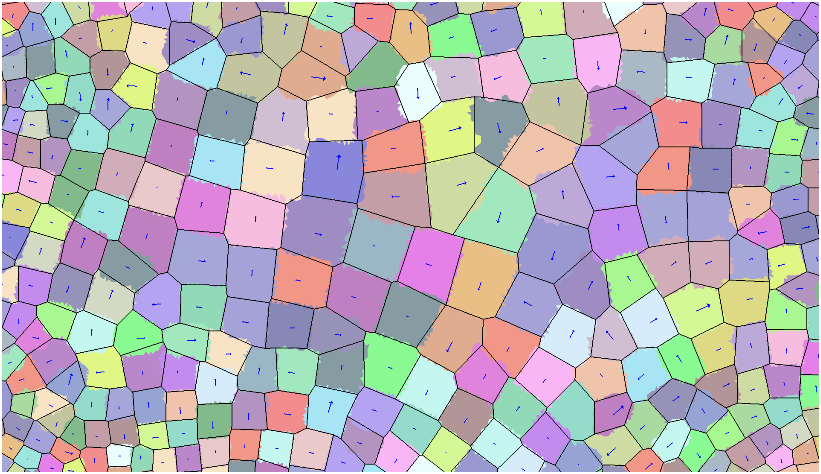
$\mathcal{L}_{\infty}^{\mathcal{R}}$ Conforming Relaxation: Iteration 6



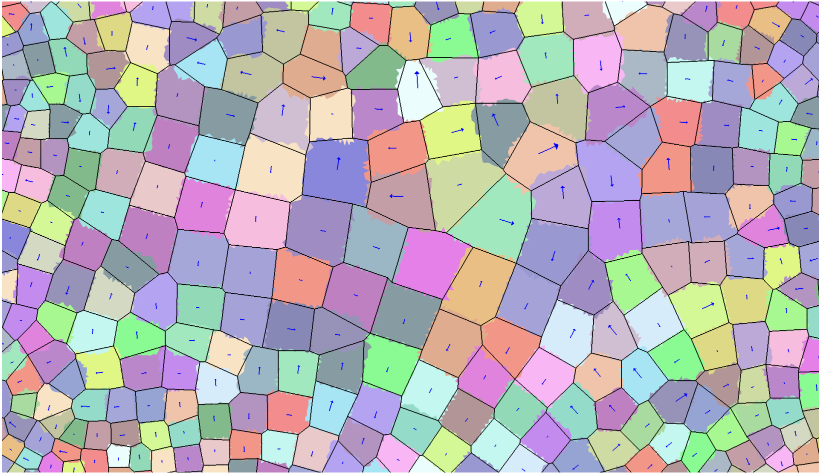
$\mathcal{L}_\infty^\mathcal{R}$ Conforming Relaxation: Iteration 7



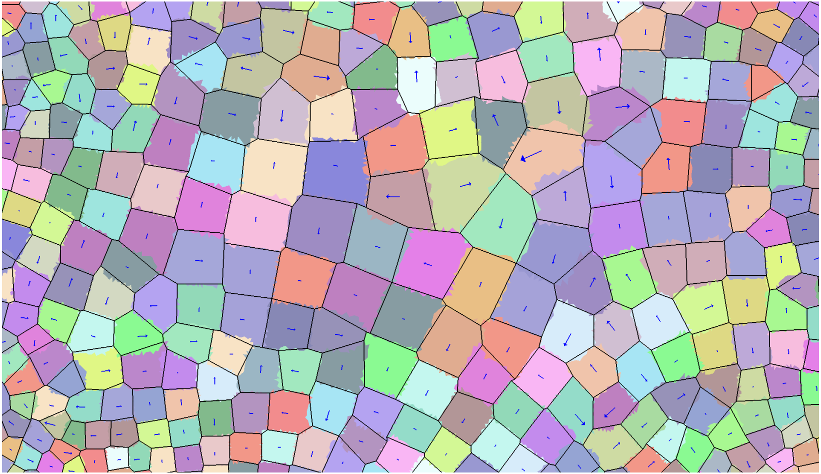
$\mathcal{L}_\infty^\mathcal{R}$ Conforming Relaxation: Iteration 8



$\mathcal{L}_{\infty}^{\mathcal{R}}$ Conforming Relaxation: Iteration 9

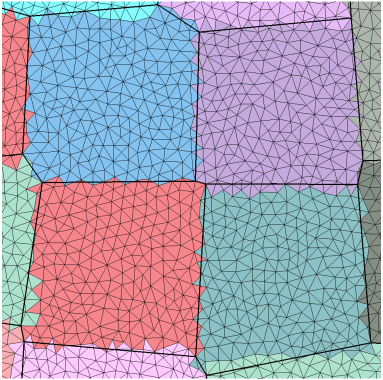


$\mathcal{L}^{\mathcal{R}}_{\infty}$ Conforming Relaxation: Iteration 10

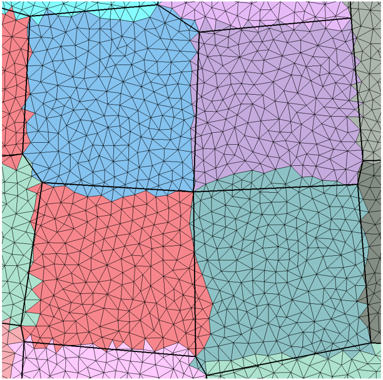


$\mathcal{L}_\infty^\mathcal{R}$ Conforming Relaxation: Observation

General configuration: butterfly

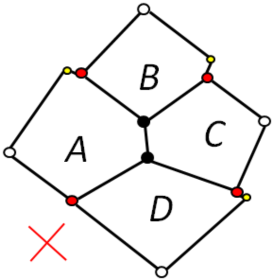


$\mathcal{L}_\infty^\mathcal{R}$ Conforming Relaxation: Observation



Local Parameterizations [Floater & Hormann]

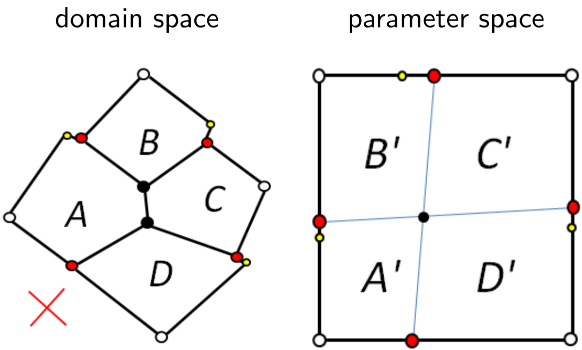
domain space



Two steps

1. Parameterize four tiles over unit square
2. Re-index tile triangles

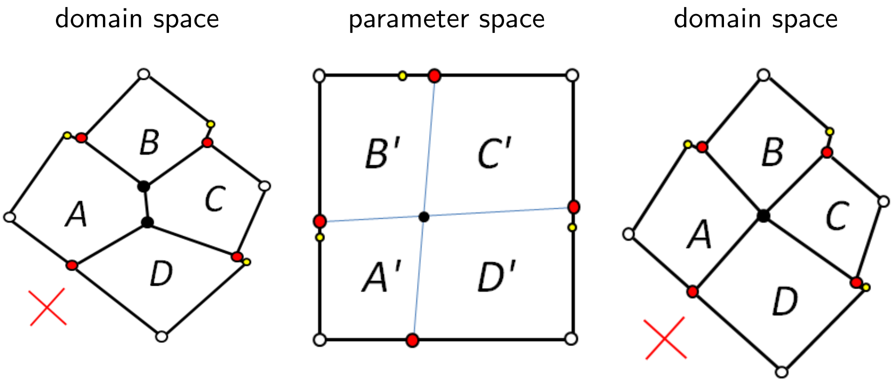
Local Parameterizations [Floater & Hormann]



Two steps

1. Parameterize four tiles over unit square
2. Re-index tile triangles

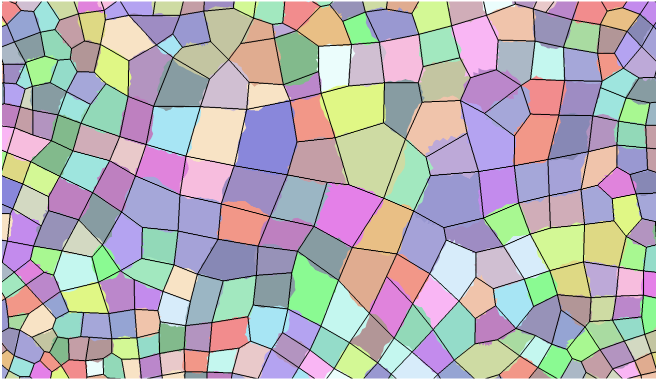
Local Parameterizations [Floater & Hormann]



Two steps

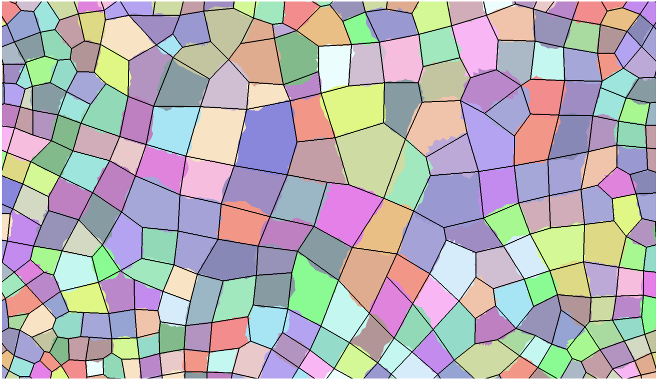
1. Parameterize four tiles over unit square
2. Re-index tile triangles

Local Parameterizations: Results

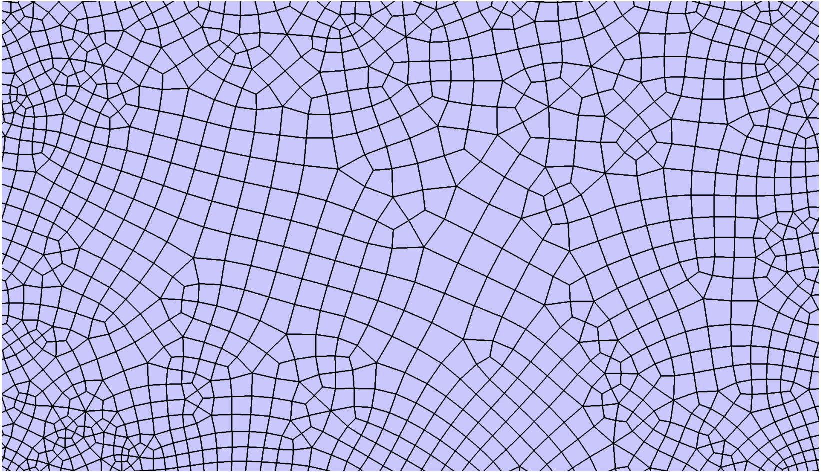


Local Parameterizations: Results

STEP / CRITERIA	*Size*	*Shape*	*Orientation*	*Degree*	*Regularity*
❹ Local parameterizations	◦	◦	◦	•	●

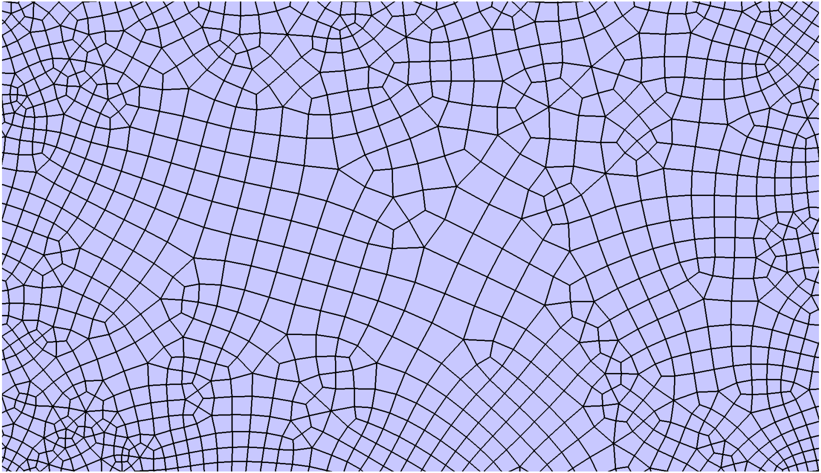


Barycentric Subdivision and Laplacian Smoothing

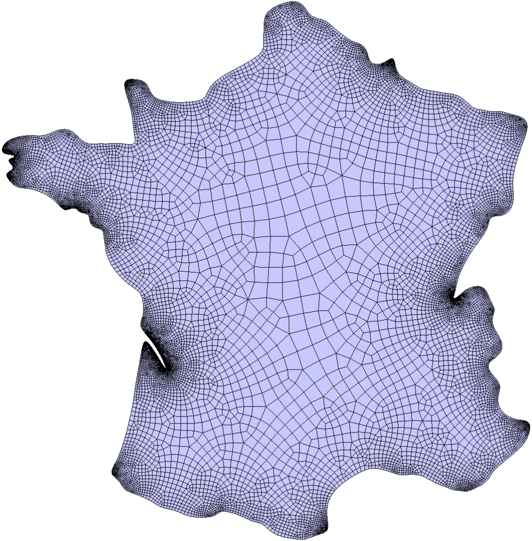


Barycentric Subdivision and Laplacian Smoothing

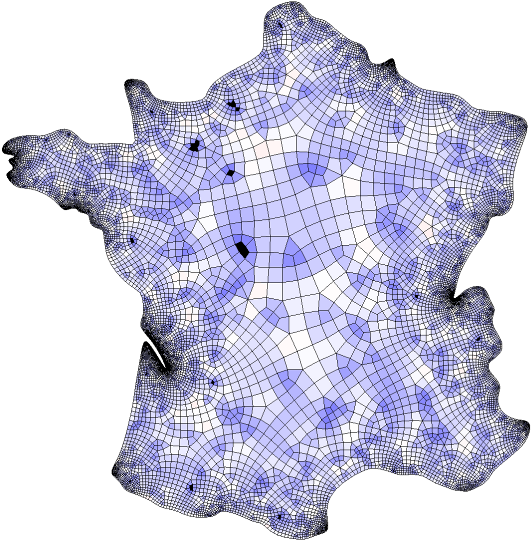
STEP / CRITERIA	*Size*	*Shape*	*Orientation*	*Degree*	*Regularity*
❺ Barycentric subdivision	○	×	○	●	○
❻ Smoothing	○	●	○	○	○



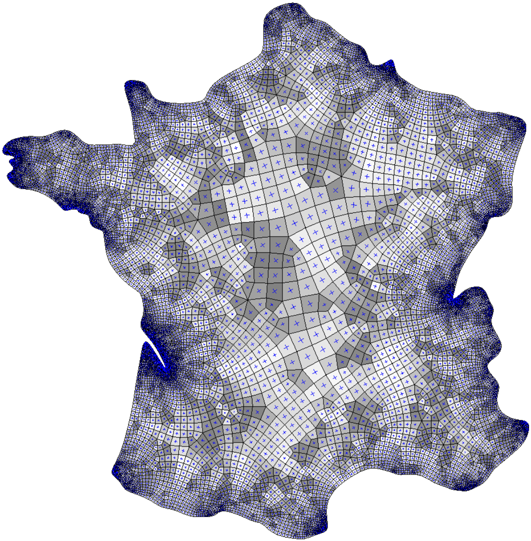
Results: France



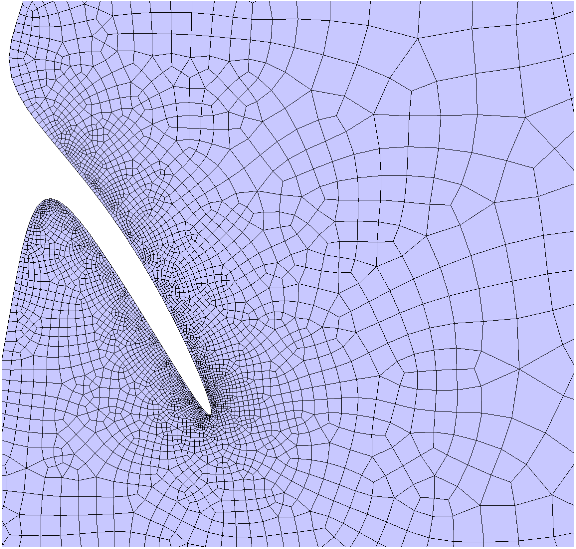
Results: France



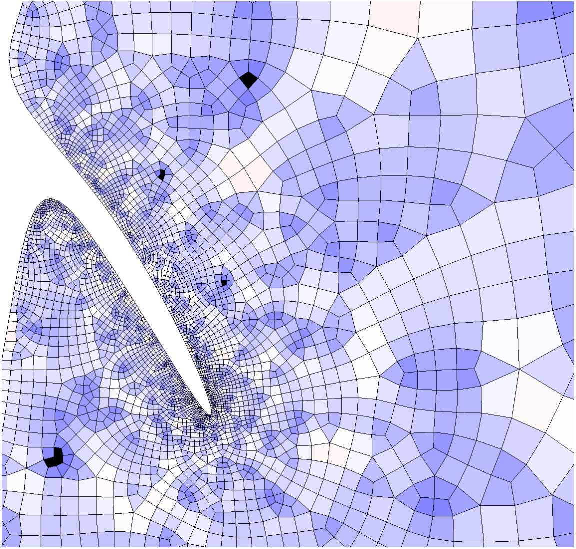
Results: France



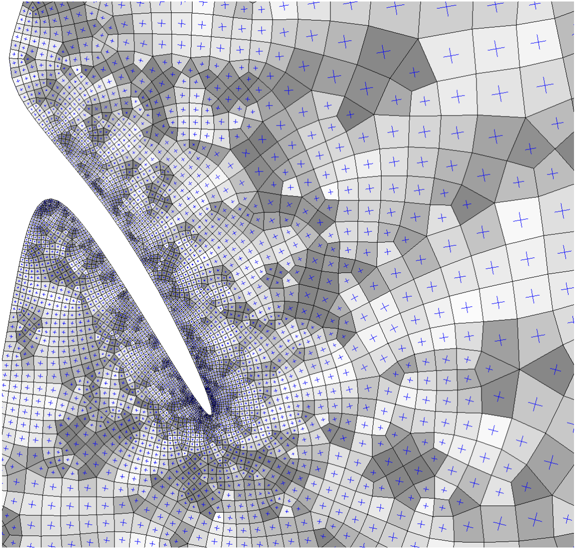
Results: Gironde



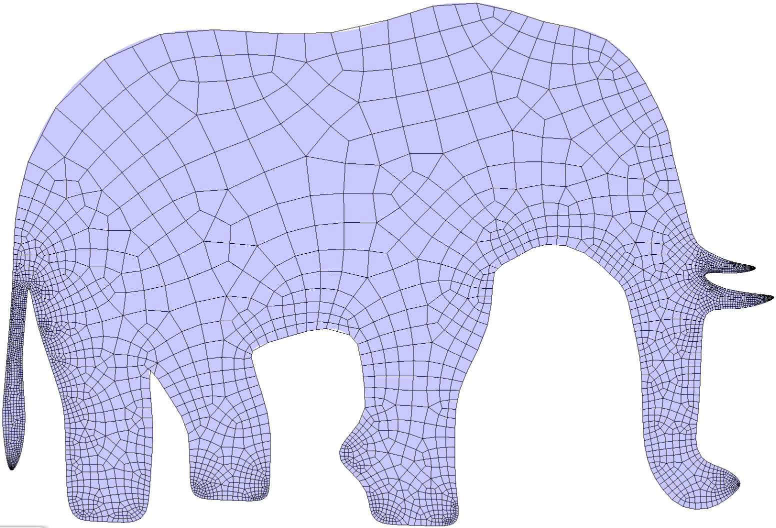
Results: Gironde



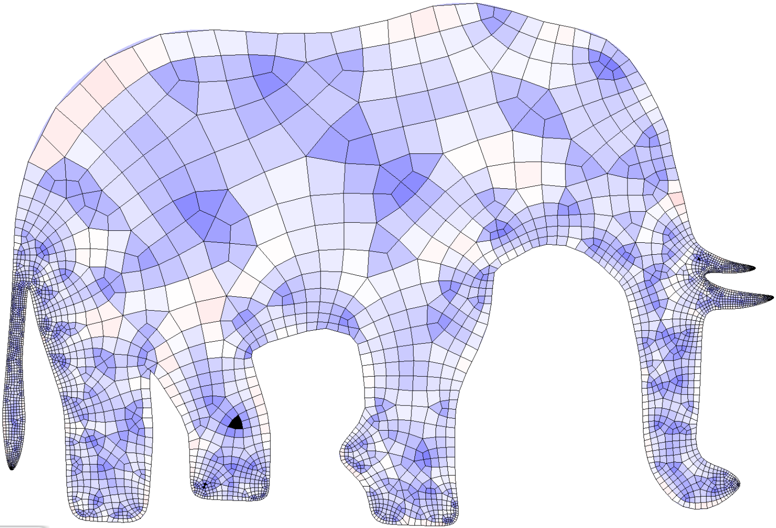
Results: Gironde



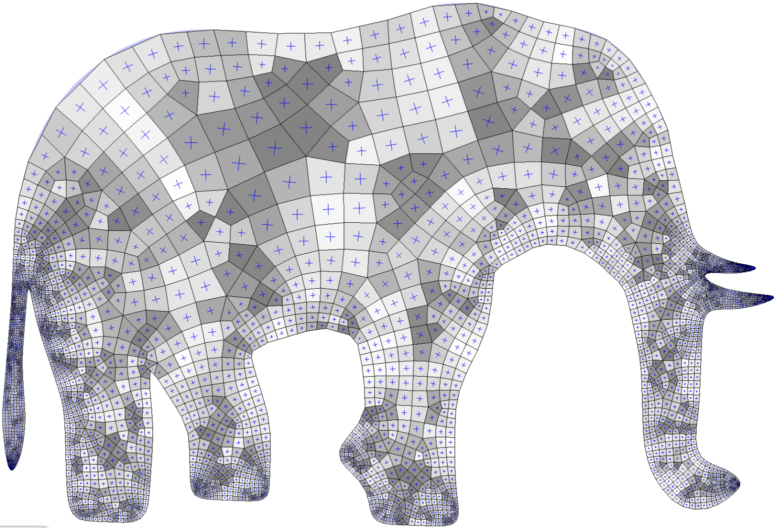
Results: Elephant



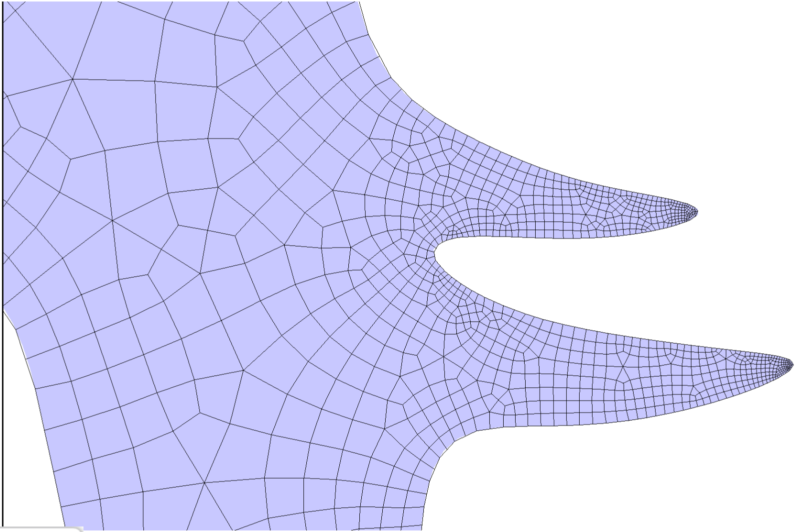
Results: Elephant



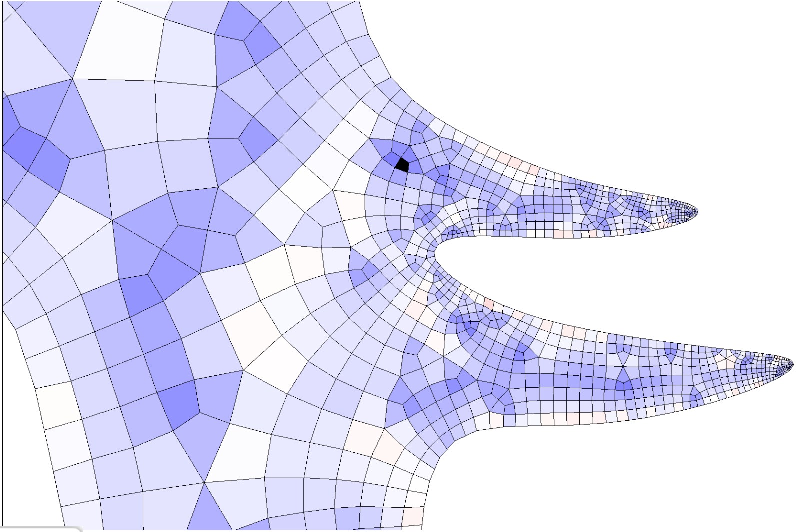
Results: Elephant



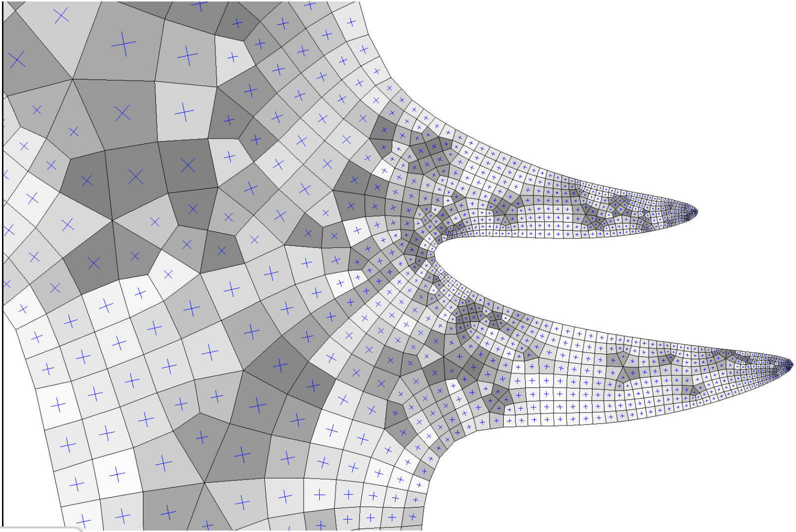
Results: Elephant (zoom)



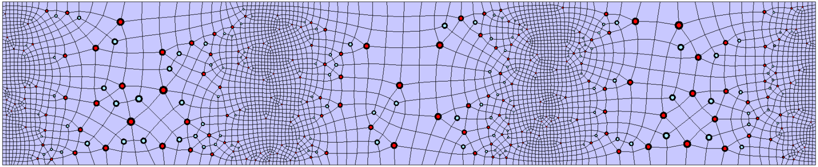
Results: Elephant (zoom)



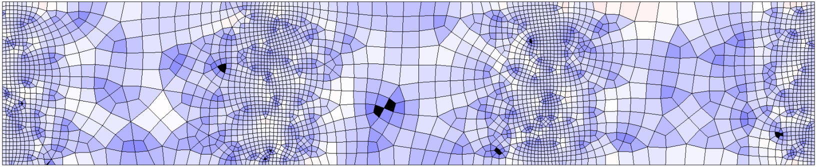
Results: Elephant (zoom)



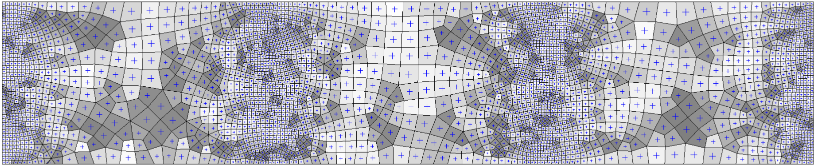
Results: Influence of Sizing Field



Results: Influence of Sizing Field



Results: Influence of Sizing Field



Conclusion

Pure Quadrangle Mesh with Size and Orientation Control

Fully automatic

Label triangles

THANK YOU

International Meshing Roundtable 2011

Paris

www.inria.fr