Regularity Criteria for the Topology of Algebraic Curves and Surfaces

B. Mourrain
GALAAD INRIA Sophia Antipolis

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Motivations

Given a polynomial equation $f = 0$ and a domain $D \subset \mathbb{R}^n$, can we deduce information on the real solutions $f = 0$ from information on the boundary ∂D, the representation of f, ...?

- If $D = [a, b] \subset \mathbb{R}$, $f(a)\, f(b) < 0$, there is at least one root of f in $[a, b]$.
- If $D = [a, b] \subset \mathbb{R}$, $\partial_x f(x) \neq 0$ on $]a, b[$ and $f(a)\, f(b) < 0$, then f has exactly one root in $[a, b]$.
- If $D = [0, +\infty[$, $f(x) = a_0 + a_1\, x + \cdots + a_d\, x^d$, and $v := V(a_0, \dots, a_d) \leq 1$, then f has exactly v root in $[a, b]$.
- If $D = [a, b] \subset \mathbb{R}$, $f(x) = c_0 B_d^0(x; a, b) + \cdots + c_d\, B_d^d(x; a, b)$ and $v := V(c_0, \dots, c_d) \leq 1$, then f has exactly v root in $[a, b]$.

Solvers

Univariate solver

- Bernstein representation of $f(x) \in \mathbb{Q}[x]$ of degree d:
  $f(x) = \sum_{i=0}^{d} b_i \, B_d^i(x; a, b)$, where
  $B_d^i(x; a, b) = \binom{d}{i} (x-a)^i (b-x)^{d-i}$.
- $V(\mathbf{b})$ = number of sign variations of $\mathbf{b} = [b_0 \ldots, b_d]$.

Algorithm (isolation of the roots of f on the interval $[a, b]$)

INPUT: *A polynomial $f := (\mathbf{b}, [a, b])$ with simple real roots (and ϵ).*

If $V(\mathbf{b}) > 1$ and $|b - a| > \epsilon$, subdivide;

If $V(\mathbf{b}) = 0$, remove the interval.

If $V(\mathbf{b}) = 1$, output interval containing one and only one root.

(If $|b - a| \leq \epsilon$ and $V(\mathbf{b}) > 0$ output the interval).

OUTPUT: *list of isolating intervals in $[a, b]$ for the real roots of f (or the ϵ-multiple root).*

❒ Complexity: $\mathcal{O}_B(d^6\tau^2)$ [MVY02],[RZ03],[MRR04],[EKMW05]; $\tilde{\mathcal{O}}_B(d^4\tau^2)$ [KM05], [ESY06], [EMT06].

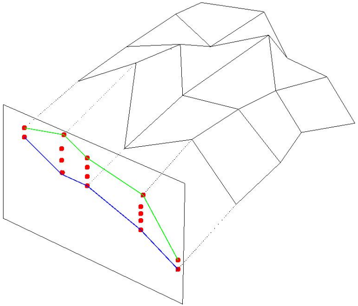
Multivariate subdivision solver [P'04], [MP'05]

$$\begin{cases} f_1(\mathbf{u}) = \sum_{i_1,\dots,i_n} b^1_{i_1,\dots,i_n} B^{d_1,\dots,d_n}_{i_1,\dots,i_n}(u_1,\dots,u_n), \\ \vdots \\ f_s(\mathbf{u}) = \sum_{i_1,\dots,i_n} b^s_{i_1,\dots,i_n} B^{d_1,\dots,d_n}_{i_1,\dots,i_n}(u_1,\dots,u_n), \end{cases}$$

Algorithm

1. *preconditioning on the equations;*
2. *reduction of the domain;*
3. *if the reduction ratio is too small, subdivision of the domain.*

Reduction



$$m_j(f;x_j) = \sum_{i_j=0}^{d_j} \min_{\{0\le i_k\le d_k, k\neq j\}} b_{i_1,\dots,i_n} B_{d_j}^{i_j}(x_j;a_j,b_j)$$

$$M_j(f;x_j) = \sum_{i_j=0}^{d_j} \max_{\{0\le i_k\le d_k, k\neq j\}} b_{i_1,\dots,i_n} B_{d_j}^{i_j}(x_j;a_j,b_j).$$

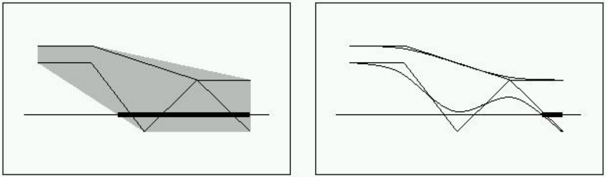
Proposition (PS93)

The intersection of the convex hull of the control polygon with the axis contains the projection of the zeroes of $\mathbf{f}(\mathbf{u}) = 0$.

Proposition

For any $\mathbf{u} = (u_1, \ldots, u_n) \in \mathcal{D}$, *and any* $j = 1, \ldots, n$, *we have*

$$m_j(f; u_j) \leq f(\mathbf{u}) \leq M_j(f; u_j).$$



Use the roots of $m_j(f, u_j) = 0$**,** $M_j(f, u_j) = 0$ **to reduce the domain of search.**

Preconditioning (for square systems)

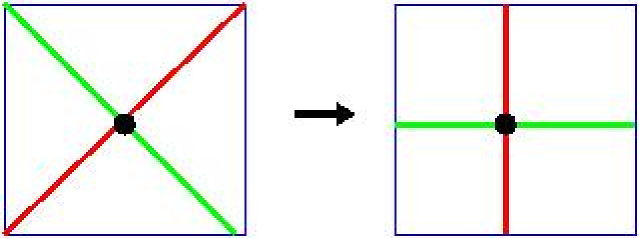
Transform $\mathbf{f}$ into $\tilde{\mathbf{f}} = M\,\mathbf{f}$

a) Optimize the distance between the equations:

$$||f||^2 = \sum_{0\le i_1\le d_1,\ldots,0\le i_n\le d_n} |\mathbf{b}(f)_{i_1,\ldots,i_n}|^2,$$

by taking for M, the matrix of eigenvectors of $Q = (\langle f_i|f_j\rangle)_{1\le i,j\le s}$.

b) $M = J_{\mathbf{f}}^{-1}(\mathbf{u}_0)$ for $\mathbf{u}_0 \in \mathcal{D}$.



Theorem (Multivariate Vincent theorem)

If $f(\mathbf{x})$ has no root in the complex polydisc $D(1/2, 1/2)^n$, then the coefficients of f in the Bernstein basis of $[0, 1]^n$ are of the same sign.

- **Convergence of the control polygon:**

Theorem

There exists $\kappa_2(f)$ such that for $\mathcal{D}$ of size ϵ small enought,

$$\forall \mathbf{x} \in \mathcal{D};\ |f(\mathbf{x}) - \mathbf{b}(f; \mathbf{x})| \leq \kappa_2(f)\, \epsilon^2.$$

- **Local quadratic convergence of the reduction:**

Proposition

Let $\mathcal{D}$ a domain of size ϵ containing a simple root of $\mathbf{f}$. There exists $\kappa_{\mathbf{f}} > 0$, such that for ϵ small enought

$$|\tilde{M}_j(\tilde{\mathbf{f}}; u_j) - \tilde{m}_j(\tilde{\mathbf{f}}; u_j)| \leq \kappa_{\mathbf{f}} \epsilon^2.$$

Algorithm (Rational Univariate Representation)

INPUT: *polynomial equations* $f_1, \ldots, f_s \in \mathbb{Q}[x_1, \ldots, x_n]$ *defining a zero-dimensional ideal.*

- *Compute the structure of* $\mathcal{A} = \mathbb{Q}[x_1, x_2, \ldots]/(f_1, f_2, \ldots)$.
- *Compute the multiplication by a generic linear form* $l(\mathbf{x}) = u_0 + u_1\, x_1 + \cdots + u_n\, x_n$ *in* $\mathcal{A}$.
- *Compute the square-free part of its characteristic polynomial* $d_0(t)$ *and its derivative* $d_i(t)$ *with respect to* u_i.
- *Isolate the real roots of* $d_0(t)$*, compute the image of the isolating interval by the map*

$$t \mapsto \left(\frac{d_1(t)}{d_0'(t)}, \ldots, \frac{d_n(t)}{d_0'(t)}\right)$$

  and refine the intervals until their image are disjoint.

OUTPUT: *A set of boxes isolating the real roots of* $f_1 = 0, \ldots, f_s = 0$.

Topology of planar curves

Topology of a planar curve

☞ $f(x, y) \in \mathbb{Q}[x, y]$ square-free, defining $\mathcal{C}$.

☞ Determine a planar graph (points and segments) isotopic to $\mathcal{C}$ in $\mathcal{D} = [a, b] \times [c, d]$.

Interesting points:

❒ x-critical points of $\mathcal{C}$: $f(x, y) = \partial_y f(x, y) = 0$.
❒ Singular points of $\mathcal{C}$: $f(x, y) = \partial_x f(x, y) = \partial_y f(x, y) = 0$.
❒ Extremal points of f: $\partial_x f(x, y) = \partial_y f(x, y) = 0$.

☞ **Deduce the topology from the isolation of some specific points.**

☞ **Separate the point isolation problem from the proper topology computation.**

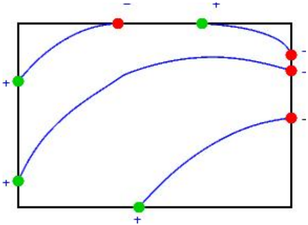
Basic tools:

- Univariate solvers to isolate/approximate roots of polyomials.
- Multivariate solvers of zero-dimensional systems:
  - Algebraic solvers,
  - Subdivision solvers,

  which output isolating boxes containing a root, within a precision ϵ.

No x-critical point in a domain $\mathcal{D}$

How is the curve in the box ?



Proposition

If $\mathcal{C}$ is x-regular in $\mathcal{D}$, the topology of $\mathcal{C}$ in $\mathcal{D}$ is uniquely determined by its intersection $\mathcal{C} \cap \partial\mathcal{D}$.

x-index of p on $\mathcal{C} \cap \partial\mathcal{D}$: $sign(T_p^i(\mathcal{C}), e_1)$.

Algorithm

Connect and remove recursively two consecutive points p, q of $\mathcal{C} \cap \partial\mathcal{D}$ with x-index$(p) = 1$, x-index$(q) = -1$ and $x_p < x_q$.

Topological degree in $\mathbb{R}^2$

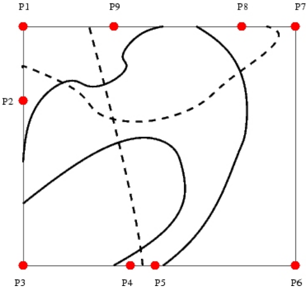
Definition

Number of times $(f_1(x,y), f_2(x,y))$ goes around $(0,0)$ when (x,y) goes around ∂D (counter-clockwise).

Definition

For $\mathbf{f} = (f_1, f_2) : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}^2$,

$$\deg(\mathbf{f}, \mathcal{D}, o) = \lim_{\epsilon \to o} \sum_{p \in \mathbf{f}^{-1}(\epsilon)} sign(J_{\mathbf{f}}(p))$$



Proposition

Let (p_i) be a counter-clockwise subdivision of $\partial\mathcal{D}$ ($p_{s+1} = p_1$), such that on $[p_i, p_{i+1}]$, $f_{\sigma(i)}$ ($\sigma(i) \in \{1, 2\}$) is of constant sign. Then

$$\deg(\mathbf{f}, \mathcal{D}, 0) = \frac{1}{8} \sum_{i=1}^{s} (-1)^{\sigma(i)} \det(sign(\mathbf{f}(p_i)), sign(\mathbf{f}(p_{i+1})))(*)$$

Algorithm

INPUT: $f_1, f_2 \in \mathbb{Q}[x, y]$ *with no common root on* $\partial\mathcal{D}$.

- *For each side segment* I *of* $\mathcal{D}$,
  - *compute the restriction* g_i *of* f_i *on* I, *expressed in the Bernstein basis on* I.
  - *If* $V(g_1, I) = 0$ *or* $V(g_2, I) = 0$, *store the end points with the sign of* f_1, f_2.
  - *Otherwise split* I *and proceed recursively on the subintervals.*
- *For the list of stored points, compute the topological degree by formula* $(*)$.

Theorem (Khimshiashvili)

Let $\mathbf{x}$ *be the single singular point of* $f(x,y)=0$ *in* $\mathcal{D}$*:*

$$N_{br}(f,\mathcal{D}) = 2(1-\deg(\nabla f,\mathcal{D},0)).$$

Theorem

Let $\mathbf{x}$ *be the single singular point of* $f=0$ *on* $\mathcal{D}$ *and suppose it isolated in* $f=g=0$*:*

$$N_{br}(f=0,g>0,\mathbf{x}) - N_{br}(f=0,g<0,\mathbf{x}) = 2\deg((f,J_{f,g}),\mathcal{D},0).$$

- $g=(\partial_x f)^2+(\partial_y f)^2$: number of branches of f at the singular point $\mathbf{x}$.

Proposition

Let $\mathcal{D}$ be a convex domain with

- *a unique singular point s of $\mathcal{C}$,*
- *no other extremal point of f in $\mathcal{D}$,*
- $N_{br}(f, \mathcal{D}) = \sharp(\partial\mathcal{D} \cap \mathcal{C})$.

Then $\mathcal{C} \cap \mathcal{D}$ can be deformed in $s \star (\partial\mathcal{D} \cap \mathcal{C})$.

Algorithm (Regular domains for singular points)

INPUT: $f(x, y) \in \mathbb{Q}[x, y]$.

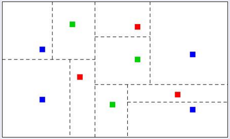
- *Isolate the roots of $\partial_x f = \partial_y f$.*
- *Refine the isolating boxes $\mathcal{D}$ of the extremal points $\in \mathcal{C}$, until $\sharp(\partial\mathcal{D} \cap \mathcal{C}) = N_{br}(f, \mathcal{D})$.*

OUTPUT: *set of isolating boxes for the singular points of $\mathcal{C}$, with a conic structure.*

The "Minnesota" algorithm

Algorithm (Topology of planar curves)

- *Isolate the singular points, the smooth x-critical points and smooth y-critical points.*

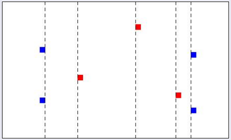


- *Decompose the domain into boxes containing at most one of these points.*
- *For boxes with no singular point, apply the connection algorithm for regular domains.*
- *For boxes with one singular point, split along a side of the isolating box and apply the regular connection algorithm.*

The "Shadoc" algorithm

Algorithm (Topology of planar curves)

- *Isolate the x-critical points in regular domains st.* $\sharp(\mathcal{C} \cap \partial\mathcal{D}) = N(f, \mathcal{D})$ *or* 2.



- *Split along a vertical side of the isolating boxes.*
- *Apply the connection algorithm for regular domains outside the boxes.*
- *Connect inside the boxes by a conic structure.*

[DEMO?]

Topology of 3D curves

Subdivision method for 3d curves [CMP'06]

❒ $D = [a, b] \times [c, d] \times [e, f] \subset \mathbb{R}^3$
❒ $\mathcal{C}$ be defined by $f(x, y, z) = 0, g(x, y, z) = 0$, with $f, g \in \mathbb{Q}[x, y, z]$, $\sqrt{(f, g)} = (f, g)$.

☞ $\mathcal{C}$ is **regular** in $D \subset \mathbb{R}^n$ if its topology is uniquely determined from $\mathcal{C} \cap \partial D$.

Proposition

If $\mathcal{C}$ has not tangent parallel to (y, z) and no double point in the y and z-directions in D, then $\mathcal{C}$ is regular in D.

❒ Proof: The projection of $\mathcal{C} \cap D$ on the (x, z) (resp. (x, y)) plane is non-singular and has no tangent parallel to the y (resp. z) direction.

There is a branch connecting two consecutive points with x-index $+, -$ for the projection in the y or z-direction.

Regularity criterion

$$\mathbf{t} = \mathbf{t}_x(\mathbf{x})\mathbf{e}_x + \mathbf{t}_y(\mathbf{x})\mathbf{e}_y + \mathbf{t}_z(\mathbf{x})\,\mathbf{e}_z = \bigtriangledown f \wedge \bigtriangledown g = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial_x f & \partial_y f & \partial_z f \\ \partial_x g & \partial_y g & \partial_z g \end{vmatrix}$$

Proposition

The 3D spatial curve $\mathcal{C}$ defined by $f = g = 0$ is regular on D, if

- $\mathbf{t}_x(\mathbf{x}) \neq 0$ *on D, and*
- $\partial_y h \neq 0$ *on z-faces, and $\partial_z h \neq 0$ and it has the same sign on both y-faces of D, for $h = f$ or $h = g$.*

❒ Proof: $\sigma_{x_0} : (y, z) \to (f(x_0, y, z), g(x_0, y, z))$ is injective, because

- its jacobian has a constant sign.
- the topological degree of σ_{x_0} on $D \cap \{x = x_0\}$ is $-1, 0$ or 1.

Algorithm

INPUT: $f(x,y,z) = g(x,y,z) = 0$ *(*$f, g \in \mathbb{Q}[x,y,z]$*) defining the curve* $\mathcal{C}$*; an initial box* $D_0 \subset \mathbb{R}^3$*.*

$\mathcal{L} := \{D_0\}$;

While $\mathcal{L}$ *not empty*

- *pop a box* D *of* L*,*
- *if* $\mathcal{C}$ *is not regular* D*, otherwise compute the graph of* $\mathcal{C}$ *in* D*.*
    - *otherwise if* $|D| > \epsilon$ *split* D *in* $\mathcal{L}$*,*
    - *else treat* D *as a singular box.*

OUTPUT: *A graph of points of* $\mathcal{C}$ *connected by segments.*

For $\epsilon > 0$ small enough, the output graph is isotopic to $\mathcal{C}$.

In practice

❒ Use the Bernstein representation of polynomials on D.

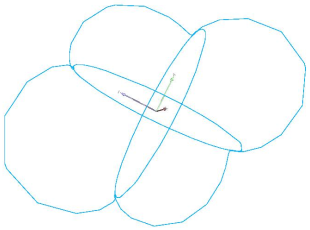
❒ Use de Casteljau algorithm to split the representation.

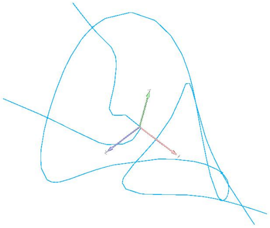
❒ Test sign of a polynomial in a box with this representation.

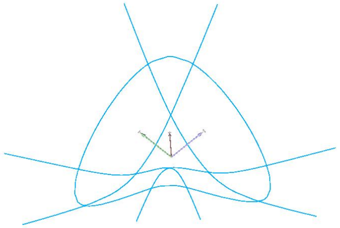
❒ Use enveloping polynomials to garantee sign computation.

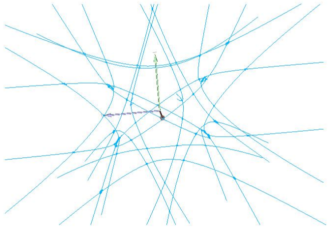
❒ Compute the intersection of $\mathcal{C}$ with the faces of the cube, using 2D Bernstein solver.

Examples









Topology of surfaces

Singular surfaces

A surface $\mathcal{S}$ defined by $f(x, y, z) = 0$, with $f \in \mathbb{Q}[x, y, z]$ square-free.

Problem: the singular locus $f = \partial_x f(x, y, z) = \partial_y f = \partial_z f = 0$ can be of dimension $0, 1$.

Approach: Compute a (Whitney) **stratification** of $S = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$ such that

- $\mathcal{S}_i$ is smooth of dimension i,
- $\mathcal{S}$ at a point of $\mathcal{S}_2$ is locally like a plane.
- $\mathcal{S}$ near a point of $\mathcal{S}_1$ is locally like a cylinder on the curve $\mathcal{S}_1$.
- $\mathcal{S}$ near a point of $\mathcal{S}_0$ has locally a conic structure centered at this point.

Explicit Whitney stratification

Let $\mathcal{C} = V(f, \partial_z f)$ be the polar variety in the z-direction.

- $\mathcal{S}_0 =$ singular points of $\mathcal{C}$,
- $\mathcal{S}_1 = \mathcal{C} - \mathcal{S}_0$ (smooth points of $\mathcal{C}$),
- $\mathcal{S}_2 = \mathcal{S} - \mathcal{C}$.

Proposition (EHV'92)

$\sqrt{(f, \partial_z f)} = (f, \partial_z f) : Jac(f, \partial_z f).$

❒ Precomputation of generators $g_1, \ldots, g_k$ of $\sqrt{(f, \partial_z f)}$ by algebraic methods.

On $\mathcal{S}_2$

A domain $D = [a, b] \times [c, d] \times [e, f] \subset \mathbb{R}^3$.

Proposition

If $\mathcal{C} \cap D = \emptyset$ (no tangent line parallel to the z-direction) and $\mathcal{S}$ has no tangent parallel to the x or y direction on the z-faces of D, then the topology of $\mathcal{S}$ in D is uniquely determined from its intersection with the edges of D.

❒ Use the Bernstein representation to test sign conditions.

Algorithm (Connection for a regular surface)

- *Compute the set Ξ of intersection points of $\mathcal{S}$ with the edges of D;*
- *Deduce the set Σ of branches of $\mathcal{S} \cap \partial D$ (using 2d-algorithm);*
- *Extract the connected components $\mathcal{P}_1, \ldots, \mathcal{P}_k$ of the graph $\Gamma = (\Xi, \Sigma)$ with vertices Ξ and edges Σ.*

Near $\mathcal{S}_1$

Proposition

If $\mathcal{C}$ is x-regular in D and intersect ∂D in two points, and if $\mathcal{S}$ is regular on the faces of D, then the topology of $\mathcal{S}$ in D is uniquely determined from its intersection with the edges of D and from $\mathcal{C} \cap \partial D$.

❒ Use topological degree on the faces of D where $\mathcal{C}$ intersects to check the local conic structure.

Algorithm (Connection for surface with regular polar variety)

- *On each facet, compute the arcs describing the topology of $\mathcal{S} \cap \partial D$.*
- *Compute the connected components of the remaining set of arcs after removing the end points of $\mathcal{C}$.*
- *For each connected component which starts and ends at the two points of $\mathcal{C}$, add the polar arc and form the corresponding patch.*
- *For each of the remaining connected components, which form cycles, build the corresponding patches.*

Algorithm

- *Subdivide until D contains at most one point of $\mathcal{S}_0$ and $|D| < \epsilon$.*
- *Compute the topology of $\mathcal{S}$ on the faces of ∂D by using the 2d-algorithm.*
- *Compute the cone which link the center the point of $\mathcal{S}$ to $\mathcal{S} \cap \partial D$.*

Proposition

For ϵ small enough, the algorithm compute the topology of $\mathcal{S}$.

[**Implementation in progress**]

❒ Contributors:

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- Chen Liang
- J.P. Pavone
- J. Wintz

❒ Tools:

- SYNAPS: a C++ library for symbolic and numeric computations. GPL, `http://synaps.inria.fr`
- AXEL, an algebraic-geometric modeler, GPL `http://axel.inria.fr`

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