Topology of implicit curves and surfaces

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The problem

Given a polynomial $f(x, y)$ (resp. $f(x, y, z)$),

□ Compute the **topology** of the curve (resp. the surface) defined by $f(x, y) = 0$, (resp. $f(x, y, z) = 0$).

□ Compute an **approximation** of the curve (resp. surface) within the precision ϵ (Hausdorff distance).

□ Compute an **approximation** of the curve (resp. surface) within the precision ϵ, with good **geometric/numerical properties**.

Why ?

⇨ Implicit surfaces modellers yield more synthetic, compact models.

⇨ Manipulation of parametric surfaces leads to implicit problems.

⇨ Blending, morphing, smoothing technics are easier with implicit surfaces ...

⇨ Parameterised objects/rational numbers vs implicit objects/algebraic numbers.

⇨ Implicit is everywhere, no need to explicit it.

What ?

- **Sampling methods**
- **Subdivision methods**
- **Projection methods**
- **Intersection methods**

Sampling methods

Outline

1. Compute **enough points** on the surface.

2. Connect them **geometrically**.

□ **Parametric curve/surface**

Easy to generate points.

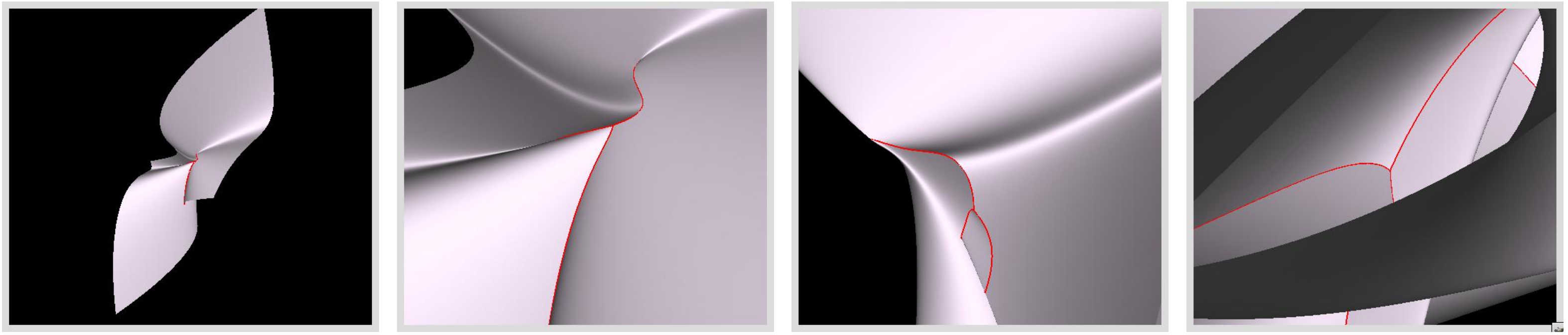
Difficulties in detecting self intersection.

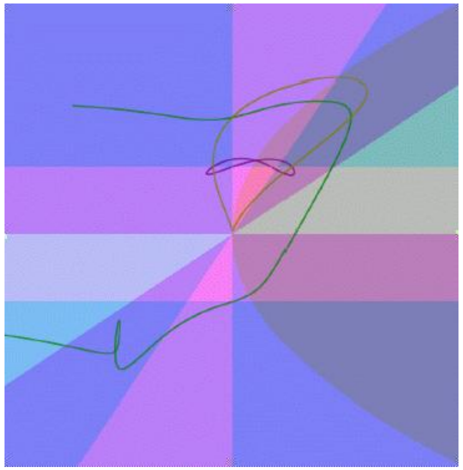
□ **General implicit curve/surface**

Difficulties in generating the points.

Singularities are easier to localise.

Self-intersection points





- Sample the surface.
- Segment it according to diff. information.
- Bound the regions with the same coding.
- Intersection the image of these regions by subdivision.

$(3,3)$	Sampling	Segmentation	Intersections
1000×1000	0.15 s	0.02s	0.5 s

Subdvision solver

☐ **Bernstein basis**: $f(x) = \sum_{i=0}^{d} b_i\, B_d^i(x)$, where $B_d^i(x) = \binom{d}{i} x^i (1-x)^{d-i}$.

$\mathbf{b} = [b_i]_{i=0,\ldots,d}$ are called the ***control coefficients***.

- $f(0) = b_0, f(1) = b_d$,
- $f'(x) = \sum_{i=0}^{d-1} \Delta(\mathbf{b})_i\, B_{d-1}^i(x)$ where $\Delta(\mathbf{b})_i = b_{i+1} - b_i$.

☐ **Subdivision by De Casteljau algorithm:**

$$b_i^0 = b_i,\ \ i = 0, \ldots, d,$$
$$b_i^r(t) = (1-t)\, b_i^{r-1}(t) + t\, b_{i+1}^{r-1}(t),\ \ i = 0, \ldots, d-r.$$

- The control coefficients $\mathbf{b}^-(t) = (b_0^0(t), b_0^1(t), \ldots, b_0^d(t))$ and $\mathbf{b}^+(t) = (b_0^d(t), b_1^{d-1}(t), \ldots, b_d^0(t))$ describe f on $[0,t]$ and $[t,1]$.

- For $t = \frac{1}{2}$, $b_i^r = \frac{1}{2}(b_i^{r-1} + b_{i+1}^{r-1})$.; use of adapted arithmetic.
- Number of arithmetic operations bounded by $\mathcal{O}(d^2)$, memory space $\mathcal{O}(d)$. Indeed, asymptotic complexity $\mathcal{O}(d \log(d))$.

□ Isolation of real roots

Proposition: **(Descartes rule)** $\#\{f(x)=0; x \in [0,1]\} = V(\mathbf{b}) - 2p, p \in \mathbb{N}$.

Algorithm: isolation of the roots of f on the interval $[a,b]$

INPUT: *A representation $(\mathbf{b}, [a,b])$ associate with f and ϵ.*
- *If $V(\mathbf{b}) > 1$ and $|b-a| > \epsilon$, subdivide;*
- *If $V(\mathbf{b}) = 0$, remove the interval.*
- *If $V(\mathbf{b}) = 1$, output interval containing one and only one root.*
- *If $|b-a| \leq \epsilon$ and $V(\mathbf{b}) > 0$ output the interval and the multiplicity.*

OUTPUT: *list of isolating intervals in $[a,b]$ for the real roots of f or the ϵ-multiple root.*

- Multiple roots (and their multiplicity) computed within a precision ϵ.
- $x := t/(1-t)$: Uspensky method.
- Complexity: $\mathcal{O}(\frac{1}{2}d(d+1)\, r \left(\lceil \log_2\left(\frac{1+\sqrt{3}}{2s}\right)\rceil - \log_2(r) + 4\right))$ [MVY02+]
- Natural extension to B-splines.

Benchmarks

Pentium III 933Mhz.

The number of equations per s. (C++ with 64-bit floats; $\epsilon = 0.000001$):

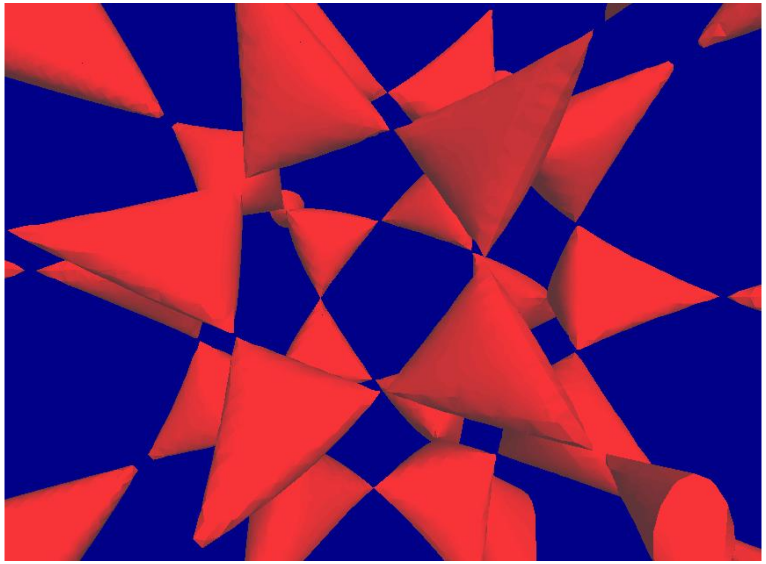
degree	8	9	12	16	18	20
	25 000	20-22 000	12-13 000	7.5-8 000	5.9-6.2 000	5.4 000

Equations per s. (precision bits vs. degree; $\epsilon = 0.000001$) using GMP library:

	16	20	30	40	60	80	100
128-bit	96	62.5	25.4	12.5	–	–	–
192-bit	83.3	53.2	21.5	10.8	4.0	–	–
256-bit	73.5	47.2	18.9	9.5	3.6	1.8	–
384-bit	60.2	37.7	15.2	7.6	2.9	1.4	0.8
512-bit	51	31.2	12.2	6.1	2.3	1.2	0.7

Compare favorably with other efficient solvers (Aberth method, `mpsolve`).

Example



$$(8c+4)x^2y^2z^2 - c^4(x^4y^2 + y^4z^2 + x^2z^4)$$
$$+c^2(x^2y^4 + y^2z^4 + x^4z^2) - \frac{2c+1}{4}(x^2+y^2+z^2-1)^2 = 0,$$
$$c = \frac{1+\sqrt{5}}{2}$$

~5s (sampling 10^5 points), ~1min (meshing).

Subdivision methods

Solving by subdivision methods

Rectangular patches: $f(x,y) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} b_{j,i} B_{d_1}^i(x) B_{d_2}^j(y)$ associated with the box $[0,1] \times [0,1]$.

• **Subdivision** by row or by column, similar to the univariate case.

• Arithmetic **complexity** of a subdivision bounded by $\mathcal{O}(d^3)$ $(d = max(d_1, d_2))$, memory space $\mathcal{O}(d^2)$.

Triangular patches: $f(x,y) = \sum_{i+j+k=d} b_{i,j,k} \frac{d!}{i!j!k!} x^i\, y^j\, (1-x-y)^k$ associated with the representation on the 2d **simplex**.

• Subdivision at a **new point**. Arithmetic complexity $\mathcal{O}(d^3)$, memory space $\mathcal{O}(d^2)$.

• Combined with **Delaunay triangulations**.

• Extension to A-patches.

Approximating an implicit curve

Algorithm: Representation of the implicit curve $f(x, y) = 0$

INPUT: *A triangular representation of* f $L := ((A, B, C), \mathbf{b})$ *and a precision* ϵ.

• *If at least one of the triangle edges is bigger that* ϵ, *split the triangle and insert the new triangles in* L:

- when the number of sign changes of some row (column or diagonal) is ≥ 2,

- or when the coefficients of f'_x *(or* f'_y, f'_z*) have not the same sign.*

• *Remove the triangle from* L *if the coefficients of* f *have the same sign.*

• *Save it*

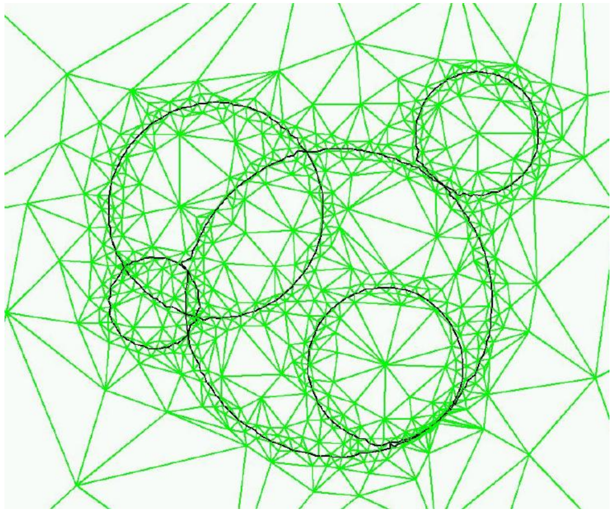
- when all the edges of the triangle are smaller than ϵ,

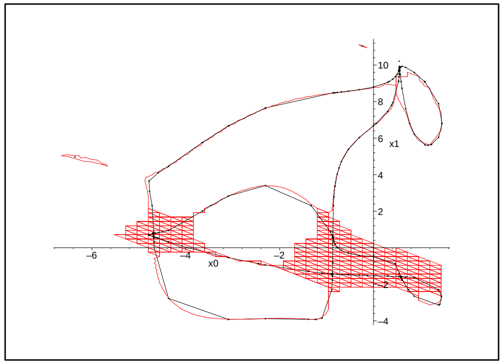
- or when the total number of sign changes on the border sides is 2 *and* f'_x *or* f'_y, f'_z, *has a constant sign. Isolate the roots.*

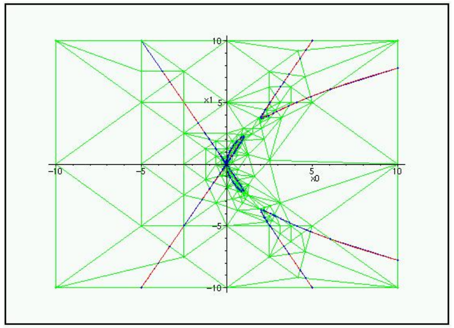
OUTPUT: *A list of segments approximating the curve* $f(x, y) = 0$.

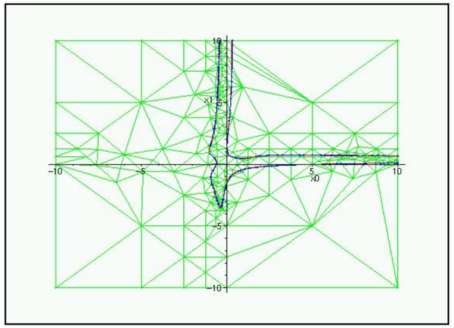
- Insertion of the circumcenter (barycenter), in order to break the *bad triangle*.
- No specific directions/axes used.
- New edges are constructed, no tangency problem.
- Number of triangles related to the complexe local feature size.
- Application to the intersection of curves, surfaces.

Examples

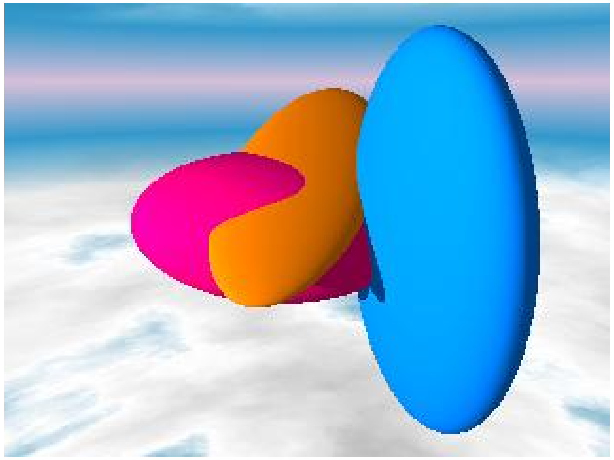








Example in 3D



$$272x^2 + 96xy + 192xz + 32y^2 + 64yz + 64z^2 - 571.2x - 142.4y - 252.8z + 323.64 = 0$$
$$128x^2 + 1152y^2 - 1024yz + 256z^2 - 144x - 886.4y + 358.4z + 220.12 = 0$$
$$64x^2 + 256y^2 + 128z^2 - 64x - 288y - 160z + 143 = 0$$

Product of the 3 equations: 931 boxes in 0.18 s (i686 2.2 GHz, 256 M).

Projection methods

Resultant in one variable

Let $f_0 = c_{0,0} + \cdots + c_{0,d_0}\, x^{d_0}$, $f_1 = c_{1,0} + \ldots + c_{1,d_1}\, x^{d_1}$ (with $d_0 \leq d_1$).

Sylvester (1840)

$$
\overbrace{\begin{array}{ccc|ccc} f_0 & \cdots & x^{d_1-1} f_0 & f_1 & \cdots & x^{d_0-1} f_1 \end{array}}^{d_0+d_1}
$$

$$
\left[\begin{array}{ccc|ccc}
c_{0,0} & & \mathbf{0} & c_{1,0} & & \mathbf{0} \\
\vdots & \ddots & & \vdots & \ddots & \\
\vdots & & c_{0,0} & \vdots & & c_{1,0} \\
c_{0,d_0} & & \vdots & c_{1,d_1} & & \vdots \\
 & \ddots & \vdots & & \ddots & \vdots \\
\mathbf{0} & & c_{0,d_0} & \mathbf{0} & & c_{1,d_1}
\end{array}\right]
\quad
\left.\begin{array}{l} 1 \\ x \\ . \\ x^{d_1-1} \\ . \\ x^{d_0+d_1-1} \end{array}\right\} d_0 + d_1
$$

Bézout (1779)

$$
\Theta_{f_0,f_1}(x,y) := \frac{f_1(x)\, f_0(y) - f_1(y)\, f_0(x)}{y - x} = \sum_{i=0}^{d_1-1} \theta_{f_0,f_1,i}(x)\, y^i = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_1-1} \theta_{i,j}\, x^i\, y^j .
$$

The Bézout matrix is $\mathsf{B}_{f_0,f_1} = (\theta_{i,j})_{0 \leq i,i \leq d_1}$.

Theorem : $R(c_{i,j}) := det(S)$ **vanishes iff** $f_0 = 0, f_1 = 0$ **has a common root.**

Resultants

Condition on $\mathbf{c} = (c_{i,j})$ such that the system has a solution in the projective variety X of dimension n:

$$\begin{cases} f_0(\mathbf{x}) & = & \sum_{j=0}^{k_0} c_{0,j}\, \kappa_{0,j}(\mathbf{x}) \\ & \vdots & \\ f_n(\mathbf{x}) & = & \sum_{j=0}^{k_n} c_{n,j}\, \kappa_{n,j}(\mathbf{x}) \end{cases}$$

⇨ **Projection** on the space of coefficients: hypersurface $\mathrm{Res}_X(\mathbf{c}) = 0$.

⇨ Explicit formula for the degree in the coefficients of each f_i.

⇨ Explicit construction as maximal minor of the matrix of a map such as

$$\begin{array}{rcl} \mathcal{S} : \langle \mathbf{x}^{E_0} \rangle \times \cdots \langle \mathbf{x}^{E_n} \rangle & \to & \langle \mathbf{x}^{F} \rangle \\ (q_0, \ldots, q_n) & \mapsto & \sum_{i=0}^{n} q_i\, f_i \end{array}$$

Projective resultant: $\{\kappa_{i,j}(\mathbf{x})\} = \{\mathbf{x}^{\alpha_j};\ |\alpha_j| = d_i\}$. $X = \mathbb{P}^n$.

Sylvester-like matrix. Ratio of two Determinants. Determinant of the Koszul complex. [Mac1902], [J91].

Toric resultant: $\{\kappa_{i,j}(\mathtt{t})\} = \{\mathtt{t}^{\alpha_j};\ \alpha_j \in A_i\}$, $\mathtt{t} \in (\mathbb{K} - \{0\})^n$, $X = \mathcal{T}_{A_0 \oplus \cdots \oplus A_n}$.

Polytope geomtry. Sylvester-like matrix. Maximal minors. Ratio of two Determinants [BKK75, GKZ91, PSCE93, DA01].

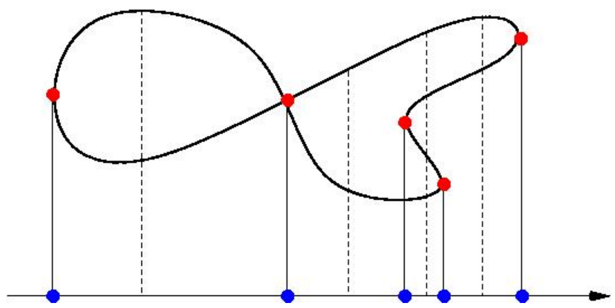
Resultant over a parameterised variety: $\{\kappa_{i,j}(\mathtt{t})\}$ associated with the parametrisation of $X = \overline{\sigma(U)}$.

Bezoutian matrix. Maximal minors. A multiple of $\mathrm{Res}_X(\mathbf{c})$. [EM98, BEM00].

Residual resultant: $\kappa_{i,j}(\mathbf{x}) \in (g_1(\mathbf{x}), \ldots, g_k(\mathbf{x}))$. X is the **blow-up** of $\mathbb{P}^n$ along $\mathcal{Z}(g_1, \ldots, g_k)$.

Explicit resolution of $(F : G)$. Gcd of the maximal minors. Degree formula. Ratio of determinants. [BKM75, BEM01, B01].

Curves



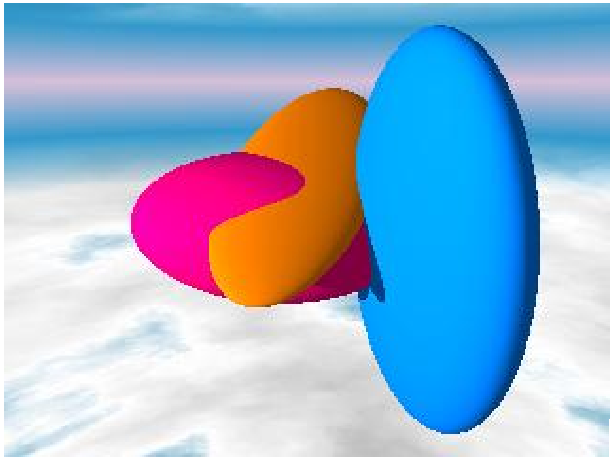
Algorithm: Topology of an implicit curve

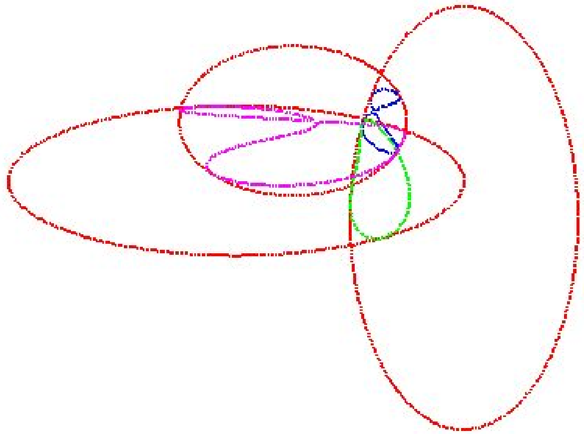
- *Compute the critical value for the projection along the y-abcisses.*
- *Above each point, compute the y-value, with their multiplicity.*
- *Between two critical points, compute the number of branches.*
- *Connect the points between two consecutive levels by y-order, the multi-branches beeing at* **the** *multiple point.*

⇨ Rationnal representation of the singular y in terms of the x.

⇨ Descartes rule to detect the multiple point among the regular ones.

Surfaces





Algorithm: Topology of an implicit surface

- *Project onto the plane.*
- *Compute the arrangement of the contour, singularity curves in the plane.*
- *Take a point inside each cell and compute the number of sheets above.*
- *Connect the regular sheets along the border of the contour, singular curves.*

⇨ Tangent curves in the projection.

⇨ Degree, numeric problems inflated by projection.

See [Col75], [GHS01]

Intersection methods

Solvers

- **Analytic solvers: exploit the value of f and its derivatives.**

  Newton like methods, Minimisation methods, Weierstrass method.

- **Homotopic solvers: deform a system with known roots into the system to solve.**

  Projective, toric, flat, deformation.

- **Subdivision solvers: use an exclusion criterion to isolate the roots.**

  Taylor exclusion function, interval arithmetic, Descartes rule.

- **Algebraic solvers: exploit the known relation between the unkowns.**

  Gröbner basis, normal form computations. Reduction to univariate or eigenvalue problems.

- **Geometric solvers: project the problem onto a smaller subspace.**

  Resultant-based methods. Reduction to univariate or eigenvalue problems.

The quotient algebra $\mathcal{A}$

- The polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$.
- The equations $f_1 = 0, \ldots, f_m = 0$ to solve, with $f_i \in R$.
- The ideal $I = (f_1, \ldots, f_m) = \{\sum_i h_i f_i; h_i \in R\}$.
- The quotient algebra $\mathcal{A} = R/I$ of polynomials modulo I: $a \equiv a'$ iff $a - a' \in I$.

  (cf. polynomial functions on the set of solutions.)
- **How to represent and exploit effectively the structure of $\mathcal{A}$?**
  - ☐ **A basis for $\mathcal{A}$.**
  - ☐ **The multiplicative tables.**

Multiplication operators

We assume that $\mathcal{Z}(I) = \{\zeta_1, \ldots, \zeta_d\} \Leftrightarrow \mathcal{A}$ of finite dimension D over $\mathbb{K}$.

$$\begin{array}{rcl} M_a : \mathcal{A} & \rightarrow & \mathcal{A} \\ u & \mapsto & a\,u \end{array} \qquad \begin{array}{rcl} M_a^{\mathtt{t}} : \widehat{\mathcal{A}} & \rightarrow & \widehat{\mathcal{A}} \\ \Lambda & \mapsto & a \cdot \Lambda = \Lambda \circ M_a \end{array}$$

Theorem:

☐ **The eigenvalues of M_a are $\{a(\zeta_1), \ldots, a(\zeta_d)\}$.**

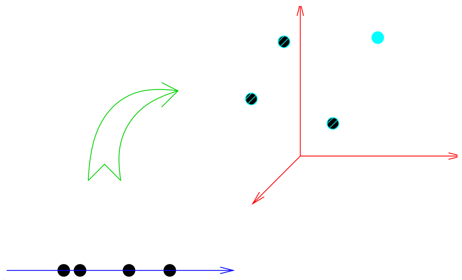
☐ **The eigenvectors of all $(M_a^{\mathtt{t}})_{a \in \mathcal{A}}$ are (up to a scalar) $\mathbf{1}_{\zeta_i} : p \mapsto p(\zeta_i)$.**

Theorem: **In a basis of $\mathcal{A}$, all the matrices M_a $(a \in \mathcal{A})$ are of the form**

$$\mathtt{M}_a = \begin{bmatrix} \mathtt{N}_a^1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathtt{N}_a^d \end{bmatrix} \text{ with } \mathtt{N}_a^i = \begin{bmatrix} a(\zeta_i) & & \star \\ & \ddots & \\ \mathbf{0} & & a(\zeta_i) \end{bmatrix}$$

Corollary: (Chow form)
$\Delta(\mathbf{u}) = \det(u_0 + u_1\,\mathtt{M}_{x_1} + \cdots + u_n\,\mathtt{M}_{x_n}) = \prod_{\zeta \in \mathcal{Z}(I)} (u_0 + u_1\zeta_1 + \cdots + u_n\zeta_n)^{\mu_\zeta}.$

Rational Univariate Representation of the roots



Algorithm: Rational Univariate Representation.

1. *Compute a* multiple *of the Chow form $\Delta(\mathbf{u})$ and its square free part $d(\mathbf{u})$.*
2. *Choose a generic $t \in \mathbb{K}^{n+1}$ and compute the first coefficients of*

$$d(t+u) = d_0(u_0) + u_1\, d_1(u_0) + \cdots + u_n\, d_n(u_0) + \cdots$$

3. *A non minimal rational univariate representation of the roots is given by $\zeta_1 = \frac{d_1(u_0)}{d_0'(u_0)}, \ldots,$ $\zeta_n = \frac{d_n(u_0)}{d_0'(u_0)}$, $d_0(u_0) = 0$.*
4. *Factorize $d_0(u_0)$ and keep the good factors for a minimal representation.*

Remark: t is generic iff $gcd(d_0(u_0),\, d_0'(u_0)) = 1$.

Normal form computation

Compute the projection of $\mathbb{K}[\mathbf{x}]$ onto a vector space B, modulo the ideal $I = (f_1, \ldots, f_m)$.

⇨ Grobner basis [CLO92, F99].

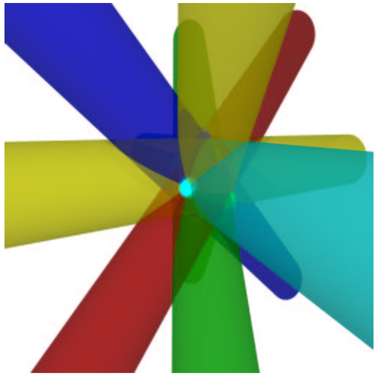
Compatibility with a monomial ordering but numerical instability.

⇨ Generalisation [M99, MT00, MT02].

No monomial ordering required. Linear algebra *with column pivoting* ; better numerical behavior of the basis.

Linear algebra on sparse matrices. Generic Sparse LU decomposition.

Cylinders throught 4 and 5 points

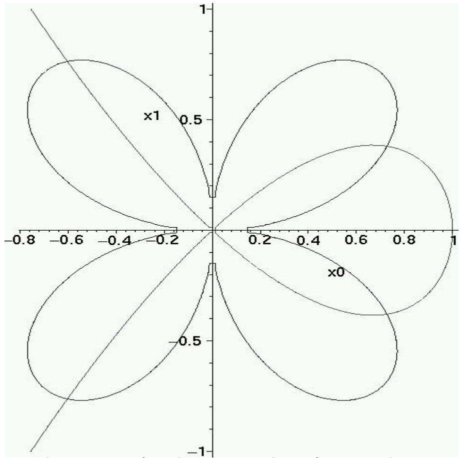


- Cylinders throught 4 points: curve of degree 3.
- Cylinders throught 5 points: $6 = 3 \times 3 - 3$.
- Cylinders throught 4 points and fixed radius: $12 = 3 \times 4$.
- Line tangent to 4 unit balls: 12.
- Cylinders throught 4 points and extremal radius: $18 = 3 \times 10 - 3 \times 4$.

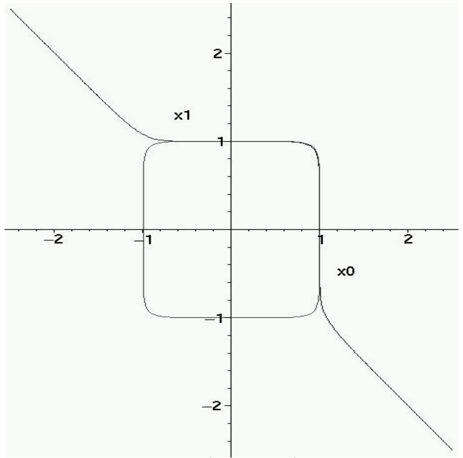
$Problem$	$time$	$max(\|f_i\|)$
Cylinders through 5 points	$0.03s$	$5 \cdot 10^{-9}$
Parallel cylinders through 2×4 points	$0.03s$	$5 \cdot 10^{-9}$
Cylinders through 4 points, extremal radius	$2.9s$	10^{-6}

Computations performed on an Intel PII 400 128 Mo of Ram

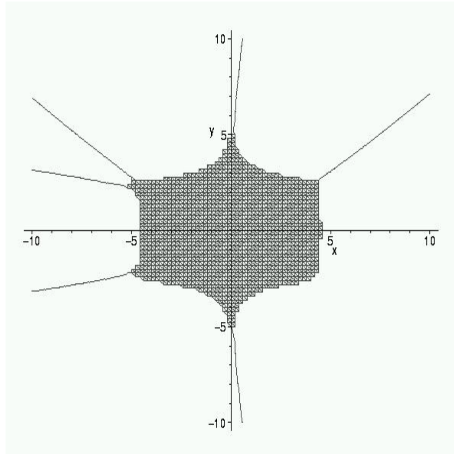
Comparison



$$f_1 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - 4x^2y^2$$
$$f_2 = y^2 - x^2 + x^3$$



$$f_1 = x^9 + y^9 - 1$$
$$f_2 = x^{10} + y^{10} - 1$$



$$f = y^2 - 2y(x^{10} + 0.5x^9y^2 - \frac{1}{8}x^8y^4 + \frac{1}{16}x^7y^6 - \frac{5}{128}x^6y^8 + \frac{7}{256}x^5y^{10} - \frac{21}{1024}x^4y^{1}2 + \frac{23}{2048}x^3y^{1}4 - \frac{429}{32768}x^2y^{16} + \frac{715}{65536}xy^{18} - \frac{2431}{262144}y^{20}) + x^{20} + x^{19}y^2$$

- **Resultant in x_1**

Example	Degree of the variables			Evaluation	Time	Number of real roots
	Function	x_0	x_1			
10	f_1, f_2	6,3	6,2	10^{-9}	0.07	5
11	f_1, f_2	9,10	9,10	10^{-2}	0.63	2
15	f_1, f_2	6,4	6,4			4

- **Resultant + Univariate solving**

Example	Evaluation	Time	Number of real roots
10	10^{-16}	0.084	5
11	10^{-15}	3.489	2
15	10^{-9}	0.151	4

- **Subdvision**

Example	ε	Evaluation	Number of intervals	Time	Number of real roots
10	10^{-5}	10^{-5}	5	0.030	5
11	10^{-5}	10^{-4}	770	79.188	2
15	10^{-5}	10^{-4}	4	0.016	4

- **Normal form**

Example	Time / γ		Evaluation	Number of real roots
	$dinvlex$	mac		
10	0.01	0.01	10^{-6}	5
11	0.03	0.05	10^{-2}	2
15	0.02	0.01	10^{-6}	4

- Examples with kastura(n), modular arithmetic:

n	mac	random	dlex
6	0.17s	0.28s	0.58s
7	0.95s	5.07s	4.66s
10	256.81s	7590.85s	635s
11	1412s	∞	4591.43s

- Katsura(6), and floating point arithmetic :

choice function	number of bits	time	$max(\|\|f_i\|\|_\infty)$
dlex	128	1.48s	10^{-28}
dinvlex	128	4.35s	10^{-24}
mac	128	1s	10^{-30}
dinvlex	80	3.98s	10^{-15}
mac	80	0.95s	10^{-19}
dlex	80	1.35s	10^{-20}
dlex	64		$-$
dinvlex	64		$-$
mac	64	0.9s	10^{-11}

- Parallel robot, approximate coefficients.

choice function	number of bits	time	$max(\|\|f_i\|\|_\infty)$
dlex	250	11.16s	$0.42 * 10^{-63}$
mac	250	11.62s	$0.46 * 10^{-63}$
dinvlex	250	13.8s	$0.135 * 10^{-60}$
dlex	128	9.13s	$0.3 * 10^{-24}$
dinvlex	128	11.1s	$0.3 * 10^{-23}$
mac	128	9.80s	$0.1 * 10^{-24}$
dlex	80	-	-
dinvlex	80	-	-
mac	80	6.80s	10^{-12}

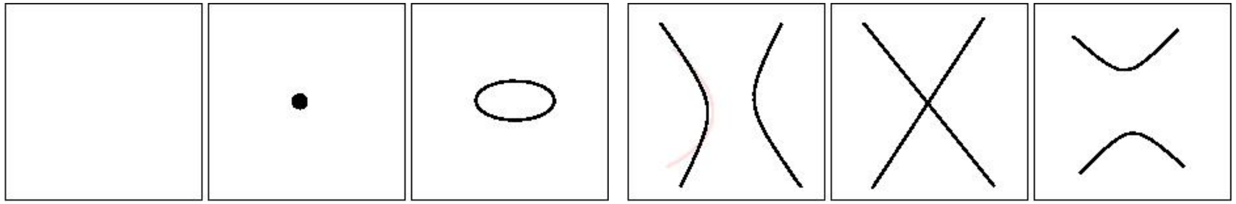
- Parallel robot, rational coefficients.

	mac	minsz	dlex	mix
size	18M	30M	50M	45M

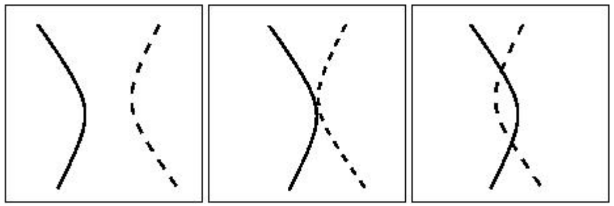
Arrangement of quadrics $Q_1, \ldots, Q_n$

Analyse the changes of topology of a section moving in the z-direction.

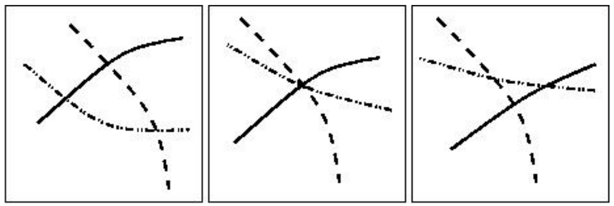
(a) $Q_{i_1} = 0, \partial_x(Q_{i_1}) = 0, \partial_y(Q_{i_1}) = 0$.



(b) $Q_{i_1} = 0, Q_{i_2} = 0, (\nabla Q_{i_1} \bigwedge \nabla Q_{i_2})_z = 0$.



(c) $Q_{i_1} = 0, Q_{i_2} = 0, Q_{i_3} = 0$.



Arrangement $=$ collection of cells of dimension 0,1,2,3 determined by sign conditions and adjacency relations.

Example: For a circle of equation $p(x,y) = x^2 + y^2 - 1$,

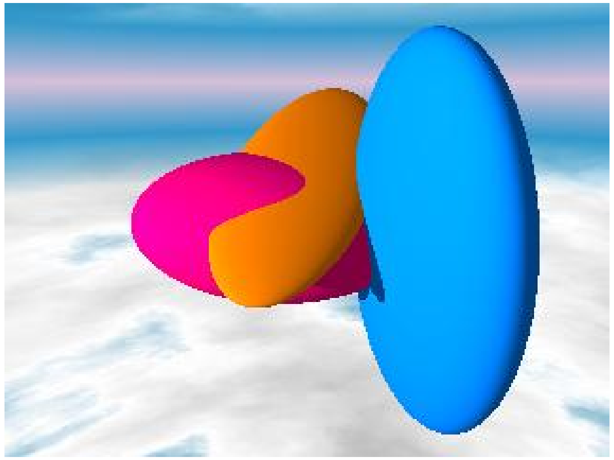
$$(\mathfrak{E}, p \geq 0), (C, p = 0), (\mathfrak{I}, p \leq 0) \ \ \& \ \ C \prec \mathfrak{I}, C \prec \mathfrak{E}.$$

Algorithm: Arrangement of quadrics

- *Compute the intersection points corresponding to the events (a), (b), (c).*
- *Sort them according to the z-abcissae, by increasing order.*
- *Compute the lowest arrangement of the conics.*
- *For each event, determine the cell to which the new critical point belongs and modify the arrangement of the neighbour cells accordingly.*
- *Connect in the different levels, the cells with the same sign conditions.*

⇨ Evaluation of sign conditions of rationnal quantities of z.

Example



$$272x^2 + 96xy + 192xz + 32y^2 + 64yz + 64z^2 - 571.2x - 142.4y - 252.8z + 323.64 = 0$$

$$128x^2 + 1152y^2 - 1024yz + 256z^2 - 144x - 886.4y + 358.4z + 220.12 = 0$$

$$64x^2 + 256y^2 + 128z^2 - 64x - 288y - 160z + 143 = 0$$

[MOVIE]

(a) 3×2 real solutions (0.01s):

(b) $3 \times 8 = 24$ complex solutions; 3×2 real (0.06s):

(c) 8 complex solutions; 2 real (0.02s):

```
(a) [0.825000,0.700000,0.287500]  C1
(a) [0.562500,0.544649,0.359835]  C1, C2
(a) [0.500000,0.562500,0.448223]  C1, C2, C3
(b) [0.498552,0.561349,0.448234]  C1, C2, C3, C23
(b) [0.687835,0.570199,0.508852]  C1, C2, C3, C23, C13
(b) [0.677133,0.617014,0.519616]  C1, C2, C3, C23, C13, C12
(c) [0.676862,0.612181,0.521687]  C1, C2, C3, C23, C13, C12, C123
(c) [0.638126,0.657542,0.685372]  C1, C2, C3, C23, C13, C12
(b) [0.534420,0.666721,0.719519]  C1, C2, C3,      C13, C12
(b) [0.662072,0.686211,0.723158]  C1, C2, C3,      C13
(b) [0.627783,0.558545,0.776837]  C1, C2, C3
(a) [0.500000,0.562500,0.801777]  C1, C2
(a) [0.562500,0.780351,0.890165]  C1
(a) [0.675000,0.300000,0.912500]
```