Topology of implicit curves and surfaces

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The problem

Given a polynomial $f(x, y)$ (resp. $f(x, y, z)$),

- Compute the **topology** of the curve (resp. the surface) defined by $f(x, y) = 0$, (resp. $f(x, y, z) = 0$).

- Compute an **approximation** of the curve (resp. surface) within the precision $\epsilon$ (Hausdorff distance).

- Compute an **approximation** of the curve (resp. surface) within the precision $\epsilon$, with good **geometric/numerical properties**.
Why?

- Implicit surfaces modellers yield more synthetic, compact models.
- Manipulation of parametric surfaces leads to implicit problems.
- Blending, morphing, smoothing technics are easier with implicit surfaces ...
- Parameterised objects/rational numbers vs implicit objects/algebraic numbers.
- Implicit is everywhere, no need to explicit it.

What?

- Sampling methods
- Subdivision methods
- Projection methods
- Intersection methods
Sampling methods
1. Compute **enough points** on the surface.

2. Connect them **geometrically**.

- **Parametric curve/surface**
  
  Easy to generate points.

  Difficulties in detecting self intersection.

- **General implicit curve/surface**
  
  Difficulties in generating the points.

  Singularities are easier to localise.
Self-intersection points

1. Sample the surface.
2. Segment it according to different information.
3. Bound the regions with the same coding.
4. Intersect the images of these regions by subdivision.

<table>
<thead>
<tr>
<th>(3, 3)</th>
<th>Sampling</th>
<th>Segmentation</th>
<th>Intersections</th>
</tr>
</thead>
<tbody>
<tr>
<td>1000 × 1000</td>
<td>0.15 s</td>
<td>0.02 s</td>
<td>0.5 s</td>
</tr>
</tbody>
</table>
Subdivision solver

- **Bernstein basis**: \( f(x) = \sum_{i=0}^{d} b_i B_d^i(x) \), where \( B_d^i(x) = \binom{d}{i} x^i (1 - x)^{d-i} \). 

\( b = [b_i]_{i=0,...,d} \) are called the **control coefficients**.

- \( f(0) = b_0, f(1) = b_d \),

- \( f'(x) = \sum_{i=0}^{d-1} \Delta(b)_i B_{d-1}^i(x) \) where \( \Delta(b)_i = b_{i+1} - b_i \).

- **Subdivision by De Casteljau algorithm**: 
  
  \[
  b_i^0 = b_i, \quad i = 0, \ldots, d, \\
  b_i^r(t) = (1 - t) b_{i}^{r-1}(t) + t b_{i+1}^{r-1}(t), \quad i = 0, \ldots, d - r.
  \]

  - The control coefficients \( b^-(t) = (b_0^0(t), b_1^1(t), \ldots, b_d^0(t)) \) and \( b^+(t) = (b_0^d(t), b_1^{d-1}(t), \ldots, b_d^0(t)) \) describe \( f \) on \([0, t] \) and \([t, 1]\).

  - For \( t = \frac{1}{2} \), \( b_i^r = \frac{1}{2}(b_{i}^{r-1} + b_{i+1}^{r-1}) \); use of adapted arithmetic.

  - Number of arithmetic operations bounded by \( O(d^2) \), memory space \( O(d) \). Indeed, asymptotic complexity \( O(d \log(d)) \).
 Proposition: (Descartes rule) \( \#\{f(x) = 0; x \in [0, 1]\} = V(b) - 2p, \ p \in \mathbb{N}. \)

Algorithm: isolation of the roots of \( f \) on the interval \([a, b]\)

**INPUT:** A representation \((b, [a, b])\) associate with \( f \) and \( \epsilon \).
- If \( V(b) > 1 \) and \(|b - a| > \epsilon\), subdivide;
- If \( V(b) = 0 \), remove the interval.
- If \( V(b) = 1 \), output interval containing one and only one root.
- If \(|b - a| \leq \epsilon\) and \( V(b) > 0 \) output the interval and the multiplicity.

**OUTPUT:** list of isolating intervals in \([a, b]\) for the real roots of \( f \) or the \( \epsilon \)-multiple root.

- Multiple roots (and their multiplicity) computed within a precision \( \epsilon \).
- \( x := t/(1 - t) \) : Uspensky method.
- Complexity: \( O\left(\frac{1}{2}d(d + 1) r \left( \lceil \log_2 \left( \frac{1 + \sqrt{3}}{2s} \right) \rceil - \log_2(r) + 4 \right) \right) \) \[MVY02+\]
- Natural extension to B-splines.
Benchmarks

Pentium III 933Mhz.

The number of equations per s. (C++ with 64-bit floats; $\epsilon = 0.000001$):

<table>
<thead>
<tr>
<th>degree</th>
<th>8</th>
<th>9</th>
<th>12</th>
<th>16</th>
<th>18</th>
<th>20</th>
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<tbody>
<tr>
<td></td>
<td>25 000</td>
<td>20-22 000</td>
<td>12-13 000</td>
<td>7.5-8 000</td>
<td>5.9-6.2 000</td>
<td>5.4 000</td>
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</tbody>
</table>

Equations per s. (precision bits vs. degree; $\epsilon = 0.000001$) using GMP library:

<table>
<thead>
<tr>
<th></th>
<th>16</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>128-bit</td>
<td>96</td>
<td>62.5</td>
<td>25.4</td>
<td>12.5</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>192-bit</td>
<td>83.3</td>
<td>53.2</td>
<td>21.5</td>
<td>10.8</td>
<td>4.0</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>256-bit</td>
<td>73.5</td>
<td>47.2</td>
<td>18.9</td>
<td>9.5</td>
<td>3.6</td>
<td>1.8</td>
<td>–</td>
</tr>
<tr>
<td>384-bit</td>
<td>60.2</td>
<td>37.7</td>
<td>15.2</td>
<td>7.6</td>
<td>2.9</td>
<td>1.4</td>
<td>0.8</td>
</tr>
<tr>
<td>512-bit</td>
<td>51</td>
<td>31.2</td>
<td>12.2</td>
<td>6.1</td>
<td>2.3</td>
<td>1.2</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Compare favorably with other efficient solvers (Aberth method, mpsolve).
Example

\[
(8c + 4)x^2y^2z^2 - c^4(x^4y^2 + y^4z^2 + x^2z^4) \\
+ c^2(x^2y^4 + y^2z^4 + x^4z^2) - \frac{2c+1}{4}(x^2 + y^2 + z^2 - 1)^2 = 0,
\]

\[
c = \frac{1+\sqrt{5}}{2}
\]

\~5s (sampling 10^5 points), \~1min (meshing).
Subdivision methods
Solving by subdivision methods

**Rectangular patches:** \( f(x, y) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} b_{j,i} B_{d_1}^i(x) B_{d_2}^j(y) \) associated with the box \([0, 1] \times [0, 1]\).

- **Subdivision** by row or by column, similar to the univariate case.

- Arithmetic **complexity** of a subdivision bounded by \( \mathcal{O}(d^3) \) \((d = \text{max}(d_1, d_2))\), memory space \( \mathcal{O}(d^2) \).

**Triangular patches:** \( f(x, y) = \sum_{i+j+k=d} b_{i,j,k} \frac{d!}{i!j!k!} x^i y^j (1 - x - y)^k \) associated with the representation on the 2d **simplex**.

- Subdivision at a **new point**. Arithmetic complexity \( \mathcal{O}(d^3) \), memory space \( \mathcal{O}(d^2) \).

- Combined with **Delaunay triangulations**.

- Extension to A-patches.

Joint work with D. Amar & M. Yvinec
Approximating an implicit curve

Algorithm: Representation of the implicit curve \( f(x, y) = 0 \)

**INPUT:** A triangular representation of \( f \) \( L := ((A, B, C), b) \) and a precision \( \epsilon \).
- If at least one of the triangle edges is bigger that \( \epsilon \), split the triangle and insert the new triangles in \( L \):
  - when the number of sign changes of some row (column or diagonal) is \( \geq 2 \),
  - or when the coefficients of \( f'_x \) (or \( f'_y, f'_z \)) have not the same sign.
- Remove the triangle from \( L \) if the coefficients of \( f \) have the same sign.
- Save it
  - when all the edges of the triangle are smaller than \( \epsilon \),
  - or when the total number of sign changes on the border sides is 2 and \( f'_x \) or \( f'_y, f'_z \), has a constant sign. Isolate the roots.

**OUTPUT:** A list of segments approximating the curve \( f(x, y) = 0 \).

Joint work with D. Amar & M. Yvinec
• Insertion of the circumcenter (barycenter), in order to break the *bad triangle*.
• No specific directions/axes used.
• New edges are constructed, no tangency problem.
• Number of triangles related to the complexe local feature size.
• Application to the intersection of curves, surfaces.

Joint work with D. Amar & M. Yvinec
272x^2 + 96xy + 192xz + 32y^2 + 64yz + 64z^2 - 571.2x - 142.4y - 252.8z + 323.64 = 0
128x^2 + 1152y^2 - 1024yz + 256z^2 - 144x - 886.4y + 358.4z + 220.12 = 0
64x^2 + 256y^2 + 128z^2 - 64x - 288y - 160z + 143 = 0

Product of the 3 equations: 931 boxes in 0.18 s (i686 2.2 GHz, 256 M).
Projection methods
Resultant in one variable

Let $f_0 = c_{0,0} + \cdots + c_{0,d_0} x^{d_0}$, $f_1 = c_{1,0} + \cdots + c_{1,d_1} x^{d_1}$ (with $d_0 \leq d_1$).

Sylvester (1840)

\[
\begin{bmatrix}
  c_{0,0} & \cdots & x^{d_1-1} f_0 \\
  \vdots & \ddots & \vdots \\
  c_{0,d_0} & \cdots & 0 \\
  0 & \cdots & c_{0,d_0}
\end{bmatrix}
\begin{bmatrix}
  c_{1,0} & \cdots & x^{d_0-1} f_1 \\
  \vdots & \ddots & \vdots \\
  c_{1,d_1} & \cdots & 0 \\
  0 & \cdots & c_{1,d_1}
\end{bmatrix}
\begin{bmatrix}
  1 \\
  x \\
  x^{d_1-1} \\
  x^{d_0+d_1-1}
\end{bmatrix}
\]

Sylvester (1840)

\[
\begin{bmatrix}
  c_{0,0} & \cdots & x^{d_1-1} f_0 \\
  \vdots & \ddots & \vdots \\
  c_{0,d_0} & \cdots & 0 \\
  0 & \cdots & c_{0,d_0}
\end{bmatrix}
\begin{bmatrix}
  c_{1,0} & \cdots & x^{d_0-1} f_1 \\
  \vdots & \ddots & \vdots \\
  c_{1,d_1} & \cdots & 0 \\
  0 & \cdots & c_{1,d_1}
\end{bmatrix}
\begin{bmatrix}
  1 \\
  x \\
  x^{d_1-1} \\
  x^{d_0+d_1-1}
\end{bmatrix}
\]

Bézout (1779)

\[
\Theta_{f_0,f_1}(x, y) := \frac{f_1(x) f_0(y) - f_1(y) f_0(x)}{y - x} = \sum_{i=0}^{d_1-1} \theta_{f_0,f_1,i}(x) y^i = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_1-1} \theta_{i,j} x^i y^j.
\]

The Bézout matrix is $B_{f_0,f_1} = (\theta_{i,j})_{0 \leq i,j \leq d_1}$.

**Theorem**: $R(c_{i,j}) := det(S)$ vanishes iff $f_0 = 0, f_1 = 0$ has a common root.
Resultants

Condition on $c = (c_{i,j})$ such that the system has a solution in the projective variety $X$ of dimension $n$:

$$
\begin{align*}
    f_0(x) &= \sum_{j=0}^{k_0} c_{0,j} \kappa_{0,j}(x) \\
    &\vdots \\
    f_n(x) &= \sum_{j=0}^{k_n} c_{n,j} \kappa_{n,j}(x)
\end{align*}
$$

\[ \Rightarrow \text{Projection} \] on the space of coefficients: hypersurface $\text{Res}_X(c) = 0$.

\[ \Rightarrow \text{Explicit formula for the degree in the coefficients of each } f_i. \]

\[ \Rightarrow \text{Explicit construction as maximal minor of the matrix of a map such as } \]

\[ S : \langle x^{E_0} \rangle \times \cdots \langle x^{E_n} \rangle \to \langle x^F \rangle \]

\[ (q_0, \ldots, q_n) \mapsto \sum_{i=0}^{n} q_i f_i \]
**Projective resultant:** \( \{\kappa_{i,j}(x)\} = \{x^{\alpha_j}; |\alpha_j| = d_i\}. X = \mathbb{P}^n. \)

Sylvester-like matrix. Ratio of two Determinants. Determinant of the Koszul complex. [Mac1902], [J91].

**Toric resultant:** \( \{\kappa_{i,j}(t)\} = \{t^{\alpha_j}; \alpha_j \in A_i\}, t \in (K - \{0\})^n, X = T_{A_0 \oplus \cdots \oplus A_n}. \)

Polytope geometry. Sylvester-like matrix. Maximal minors. Ratio of two Determinants [BKK75, GKZ91, PSC93, DA01].

**Resultant over a parameterised variety:** \( \{\kappa_{i,j}(t)\} \) associated with the parametrisation of \( X = \overline{\sigma(U)} \).

Bezoutian matrix. Maximal minors. A multiple of \( \text{Res}_X(c) \). [EM98, BEM00].

**Residual resultant:** \( \kappa_{i,j}(x) \in (g_1(x), \ldots, g_k(x)). X \) is the blow-up of \( \mathbb{P}^n \) along \( Z(g_1, \ldots, g_k) \).

Explicit resolution of \((F : G)\). Gcd of the maximal minors. Degree formula. Ratio of determinants. [BKM75, BEM01, B01].
Curves

Algorithm: Topology of an implicit curve

- Compute the critical value for the projection along the $y$-abcisses.
- Above each point, compute the $y$-value, with their multiplicity.
- Between two critical points, compute the number of branches.
- Connect the points between two consecutive levels by $y$-order, the multi-branches being at the multiple point.

⇒ Rationnal representation of the singular $y$ in terms of the $x$.
⇒ Descartes rule to detect the multiple point among the regular ones.
Algorithm: Topology of an implicit surface

- Project onto the plane.
- Compute the arrangement of the contour, singularity curves in the plane.
- Take a point inside each cell and compute the number of sheets above.
- Connect the regular sheets along the border of the contour, singular curves.

⇒ Tangent curves in the projection.

⇒ Degree, numeric problems inflated by projection.

See [Col75], [GHS01]
Intersection methods
Solvers

- **Analytic solvers:** exploit the value of $f$ and its derivatives.
  Newton like methods, Minimisation methods, Weierstrass method.

- **Homotopic solvers:** deform a system with known roots into the system to solve.
  Projective, toric, flat, deformation.

- **Subdivision solvers:** use an exclusion criterion to isolate the roots.
  Taylor exclusion function, interval arithmetic, Descartes rule.

- **Algebraic solvers:** exploit the known relation between the unknowns.
  Gröbner basis, normal form computations. Reduction to univariate or eigenvalue problems.

- **Geometric solvers:** project the problem onto a smaller subspace.
  Resultant-based methods. Reduction to univariate or eigenvalue problems.
The quotient algebra $A$

- The polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$.
- The equations $f_1 = 0, \ldots, f_m = 0$ to solve, with $f_i \in R$.
- The ideal $I = (f_1, \ldots, f_m) = \{\sum_i h_i f_i; h_i \in R\}$.
- The quotient algebra $A = R/I$ of polynomials modulo $I$: $a \equiv a'$ iff $a - a' \in I$.
  (cf. polynomial functions on the set of solutions.)
- **How to represent and exploit effectively the structure of $A$?**
  - A basis for $A$.
  - The multiplicative tables.
Multiplication operators

We assume that $\mathcal{Z}(I) = \{\zeta_1, \ldots, \zeta_d\} \leftrightarrow \mathcal{A}$ of finite dimension $D$ over $\mathbb{K}$.

$$M_a : \mathcal{A} \to \mathcal{A} \quad u \mapsto a u$$

$$M_a^t : \hat{\mathcal{A}} \to \hat{\mathcal{A}} \quad \Lambda \mapsto a \cdot \Lambda = \Lambda \circ M_a$$

Theorem:

□ The eigenvalues of $M_a$ are $\{a(\zeta_1), \ldots, a(\zeta_d)\}$.

□ The eigenvectors of all $(M_a^t)_{a \in \mathcal{A}}$ are (up to a scalar) $1_{\zeta_i} : p \mapsto p(\zeta_i)$.

Theorem: In a basis of $\mathcal{A}$, all the matrices $M_a$ ($a \in \mathcal{A}$) are of the form

$$M_a = \begin{bmatrix} N_a^1 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & N_a^d \end{bmatrix} \text{ with } N_a^i = \begin{bmatrix} a(\zeta_i) & \cdots & * \\ \cdots & \cdots & \cdots \\ 0 & \cdots & a(\zeta_i) \end{bmatrix}$$

Corollary: (Chow form)

$\Delta(u) = \det(u_0 + u_1 M_{x_1} + \cdots + u_n M_{x_n}) = \prod_{\zeta \in \mathcal{Z}(I)} (u_0 + u_1 \zeta_1 + \cdots + u_n \zeta_n)^{\mu_\zeta}$. 

B. Mourrain

25
Rational Univariate Representation of the roots

**Algorithm: Rational Univariate Representation.**

1. Compute a multiple of the Chow form \( \Delta(u) \) and its square free part \( d(u) \).
2. Choose a generic \( t \in \mathbb{K}^{n+1} \) and compute the first coefficients of
   \[
d(t + u) = d_0(u_0) + u_1 d_1(u_0) + \cdots + u_n d_n(u_0) + \cdots
   \]
3. A non minimal rational univariate representation of the roots is given by \( \zeta_1 = \frac{d_1(u_0)}{d_0'(u_0)}, \ldots, \)
   \[
   \zeta_n = \frac{d_n(u_0)}{d_0'(u_0)}, d_0(u_0) = 0.
   \]
4. Factorize \( d_0(u_0) \) and keep the good factors for a minimal representation.

**Remark:** \( t \) is generic iff \( \gcd(d_0(u_0), d_0'(u_0)) = 1 \).
Normal form computation

Compute the projection of \( \mathbb{K}[x] \) onto a vector space \( B \), modulo the ideal \( I = (f_1, \ldots, f_m) \).

\[ \Rightarrow \] Grobner basis [CLO92, F99].

Compatibility with a monomial ordering but numerical instability.

\[ \Rightarrow \] Generalisation [M99, MT00, MT02].

No monomial ordering required. Linear algebra \textit{with column pivoting} ; better numerical behavior of the basis.

Linear algebra on sparse matrices. Generic Sparse LU decomposition.
Cylinders through 4 and 5 points

- Cylinders through 4 points: curve of degree 3.
- Cylinders through 5 points: $6 = 3 \times 3 - 3$.
- Cylinders through 4 points and fixed radius: $12 = 3 \times 4$.
- Line tangent to 4 unit balls: 12.
- Cylinders through 4 points and extremal radius: $18 = 3 \times 10 - 3 \times 4$.

| Problem                                      | time  | max($|f_i|$) |
|----------------------------------------------|-------|-------------|
| Cylinders through 5 points                  | 0.03s | $5 \cdot 10^{-9}$ |
| Parallel cylinders through 2×4 points        | 0.03s | $5 \cdot 10^{-9}$ |
| Cylinders through 4 points, extremal radius  | 2.9s  | $10^{-6}$   |

Computations performed on an Intel PII 400 128 Mo of Ram

joint work with O.Devillers, F. Preparata, Ph. Trebuchet
Comparison

\[
f_1 = x^6 + 3x^4y^2 + 3x^2y^4 + y^6 - 4x^2y^2 \\
f_2 = y^2 - x^2 + x^3
\]

\[
f_1 = x^9 + y^9 - 1 \\
f_2 = x^{10} + y^{10} - 1
\]

\[
f = y^2 - 2y(x^{10} + 0.5x^9y^2 - \frac{1}{8}x^8y^4 + \frac{1}{16}x^7y^6 - \frac{1}{28}x^6y^8 + \frac{7}{256}x^5y^{10} - \frac{21}{1024}x^4y^{12} + \frac{23}{2048}x^3y^{14} - \frac{429}{32768}x^2y^{16} + \frac{715}{65536}xy^{18} - \frac{2431}{262144}y^{20}) + x^{20} + x^{19}y^2
\]
- **Resultant in** $x_1$

<table>
<thead>
<tr>
<th>Example</th>
<th>Degree of the variables</th>
<th>Evaluation</th>
<th>Time</th>
<th>Number of real roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$f_1, f_2$</td>
<td>$6,3$</td>
<td>$6,2$</td>
<td>$10^{-9}$</td>
</tr>
<tr>
<td>11</td>
<td>$f_1, f_2$</td>
<td>$9,10$</td>
<td>$9,10$</td>
<td>$10^{-2}$</td>
</tr>
<tr>
<td>15</td>
<td>$f_1, f_2$</td>
<td>$6,4$</td>
<td>$6,4$</td>
<td></td>
</tr>
</tbody>
</table>

- **Resultant + Univariate solving**

<table>
<thead>
<tr>
<th>Example</th>
<th>Evaluation</th>
<th>Time</th>
<th>Number of real roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>$10^{-16}$</td>
<td>0.084</td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td>$10^{-15}$</td>
<td>3.489</td>
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</tr>
<tr>
<td>15</td>
<td>$10^{-9}$</td>
<td>0.151</td>
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- **Subdivision**

<table>
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<tr>
<th>Example</th>
<th>$\varepsilon$</th>
<th>Evaluation</th>
<th>Number of intervals</th>
<th>Time</th>
<th>Number of real roots</th>
</tr>
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<tbody>
<tr>
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<td>$10^{-5}$</td>
<td>5</td>
<td>0.030</td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td>$10^{-5}$</td>
<td>$10^{-4}$</td>
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</tr>
<tr>
<td>15</td>
<td>$10^{-5}$</td>
<td>$10^{-4}$</td>
<td>4</td>
<td>0.016</td>
<td>4</td>
</tr>
</tbody>
</table>

- **Normal form**

<table>
<thead>
<tr>
<th>Example</th>
<th>Time / $\gamma$</th>
<th>Evaluation</th>
<th>Number of real roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>dinvlex / mac</td>
<td>$10^{-6}$</td>
<td>5</td>
</tr>
<tr>
<td>11</td>
<td>0.01</td>
<td>$10^{-6}$</td>
<td>5</td>
</tr>
<tr>
<td>15</td>
<td>0.02</td>
<td>$10^{-6}$</td>
<td>4</td>
</tr>
</tbody>
</table>

Experimentations by A. Salles.
• Examples with kastura(n), modular arithmetic:

<table>
<thead>
<tr>
<th>n</th>
<th>mac</th>
<th>random</th>
<th>dlex</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>0.17s</td>
<td>0.28s</td>
<td>0.58s</td>
</tr>
<tr>
<td>7</td>
<td>0.95s</td>
<td>5.07s</td>
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<td>10</td>
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<td>635s</td>
</tr>
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<td>11</td>
<td>1412s</td>
<td>∞</td>
<td>4591.43s</td>
</tr>
</tbody>
</table>

• Katsura(6), and floating point arithmetic :

<table>
<thead>
<tr>
<th>choice function</th>
<th>number of bits</th>
<th>time</th>
<th>max(∥f_i∥_∞)</th>
</tr>
</thead>
<tbody>
<tr>
<td>dlex</td>
<td>128</td>
<td>1.48s</td>
<td>10^{-28}</td>
</tr>
<tr>
<td>dinvlex</td>
<td>128</td>
<td>4.35s</td>
<td>10^{-24}</td>
</tr>
<tr>
<td>mac</td>
<td>128</td>
<td>1s</td>
<td>10^{-30}</td>
</tr>
<tr>
<td>dinvlex</td>
<td>80</td>
<td>3.98s</td>
<td>10^{-15}</td>
</tr>
<tr>
<td>mac</td>
<td>80</td>
<td>0.95s</td>
<td>10^{-19}</td>
</tr>
<tr>
<td>dlex</td>
<td>80</td>
<td>1.35s</td>
<td>10^{-20}</td>
</tr>
<tr>
<td>dlex</td>
<td>64</td>
<td></td>
<td></td>
</tr>
<tr>
<td>dinvlex</td>
<td>64</td>
<td></td>
<td></td>
</tr>
<tr>
<td>mac</td>
<td>64</td>
<td>0.9s</td>
<td>10^{-11}</td>
</tr>
</tbody>
</table>
• Parallel robot, approximate coefficients.

| choice function | number of bits | time  | $max(||f_i||_{\infty})$ |
|-----------------|----------------|-------|-------------------------|
| dlex            | 250            | 11.16s| $0.42 \times 10^{-63}$  |
| mac             | 250            | 11.62s| $0.46 \times 10^{-63}$  |
| dinvlex         | 250            | 13.8s | $0.135 \times 10^{-60}$ |
| dlex            | 128            | 9.13s | $0.3 \times 10^{-24}$   |
| dinvlex         | 128            | 11.1s | $0.3 \times 10^{-23}$   |
| mac             | 128            | 9.80s | $0.1 \times 10^{-24}$   |
| dlex            | 80             |       | -                       |
| dinvlex         | 80             |       | -                       |
| mac             | 80             | 6.80s | $10^{-12}$              |

• Parallel robot, rational coefficients.

<table>
<thead>
<tr>
<th>size</th>
<th>mac</th>
<th>minsz</th>
<th>dlex</th>
<th>mix</th>
</tr>
</thead>
<tbody>
<tr>
<td>18M</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30M</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50M</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45M</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Arrangement of quadrics $Q_1, \ldots, Q_n$

Analyse the changes of topology of a section moving in the $z$-direction.

(a) $Q_{i_1} = 0$, $\partial_x(Q_{i_1}) = 0$, $\partial_y(Q_{i_1}) = 0$.

(b) $Q_{i_1} = 0$, $Q_{i_2} = 0$, $(\nabla Q_{i_1} \land \nabla Q_{i_2})_z = 0$.

(c) $Q_{i_1} = 0$, $Q_{i_2} = 0$, $Q_{i_3} = 0$.

Joint work with J.P. Tecourt, M. Teillaud
**Arrangement** = collection of cells of dimension 0,1,2,3 determined by sign conditions and adjacency relations.

**Example:** For a circle of equation \( p(x, y) = x^2 + y^2 - 1 \),

\((\mathcal{E}, p \geq 0), (C, p = 0), (\mathcal{I}, p \leq 0) \) \& \( C \prec \mathcal{I}, C \prec \mathcal{E} \).

---

**Algorithm: Arrangement of quadrics**

- Compute the intersection points corresponding to the events (a), (b), (c).
- Sort them according to the \( z \)-abcissae, by increasing order.
- Compute the lowest arrangement of the conics.
- For each event, determine the cell to which the new critical point belongs and modify the arrangement of the neighbour cells accordingly.
- Connect in the different levels, the cells with the same sign conditions.

⇒ Evaluation of sign conditions of rationnall quantities of \( z \).
272x^2 + 96xy + 192xz + 32y^2 + 64yz + 64z^2 - 571.2x - 142.4y - 252.8z + 323.64 = 0
128x^2 + 1152y^2 - 1024yz + 256z^2 - 144x - 886.4y + 358.4z + 220.12 = 0
64x^2 + 256y^2 + 128z^2 - 64x - 288y - 160z + 143 = 0

[MOVIE]

Joint work with J.P. Tecourt, M. Teillaud
(a) $3 \times 2$ real solutions (0.01s):

(b) $3 \times 8 = 24$ complex solutions; $3 \times 2$ real (0.06s):

(c) 8 complex solutions; 2 real (0.02s):

\[
\begin{align*}
\text{(a)} & \quad [0.825000, 0.700000, 0.287500] \quad \text{C1} \\
\text{(a)} & \quad [0.562500, 0.544649, 0.359835] \quad \text{C1, C2} \\
\text{(a)} & \quad [0.500000, 0.562500, 0.448223] \quad \text{C1, C2, C3} \\
\text{(b)} & \quad [0.498552, 0.561349, 0.448234] \quad \text{C1, C2, C3, C23} \\
\text{(b)} & \quad [0.687835, 0.570199, 0.508852] \quad \text{C1, C2, C3, C23, C13} \\
\text{(b)} & \quad [0.677133, 0.617014, 0.519616] \quad \text{C1, C2, C3, C23, C13, C12} \\
\text{(c)} & \quad [0.676862, 0.612181, 0.521687] \quad \text{C1, C2, C3, C23, C13, C12, C123} \\
\text{(c)} & \quad [0.638126, 0.657542, 0.685372] \quad \text{C1, C2, C3, C23, C13, C12} \\
\text{(b)} & \quad [0.534420, 0.666721, 0.719519] \quad \text{C1, C2, C3, C13, C12} \\
\text{(b)} & \quad [0.662072, 0.686211, 0.723158] \quad \text{C1, C2, C3, C13} \\
\text{(b)} & \quad [0.627783, 0.558545, 0.776837] \quad \text{C1, C2, C3} \\
\text{(a)} & \quad [0.500000, 0.562500, 0.801777] \quad \text{C1, C2} \\
\text{(a)} & \quad [0.562500, 0.780351, 0.890165] \quad \text{C1} \\
\text{(a)} & \quad [0.675000, 0.300000, 0.912500] \\
\end{align*}
\]