

# Resultant(s)

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30th July 2002

## Determinant, resultant

$$\begin{cases} f_0(\mathbf{x}) = \sum_{j=0}^n c_{0,j} x_j = 0 \\ \vdots \\ f_n(\mathbf{x}) = \sum_{j=0}^n c_{n,j} x_j = 0 \end{cases}$$

has a non-trivial solution  
iff

$$\det(c_{i,j}) = 0.$$

- Elimination of the variables  $\mathbf{x} = (x_0, \dots, x_n)$ .
- Projection on the space of coefficients  $\mathbf{c} = (c_{i,j})$ .

## Resultant in one variable

Let  $f_0 = c_{0,0} + \dots + c_{0,d_0} x^{d_0}$ ,  $f_1 = c_{1,0} + \dots + c_{1,d_1} x^{d_1}$  (avec  $d_0 \leq d_1$ ).

**Sylvester (1840)**

$$\left[ \begin{array}{ccc|ccc}
 \overbrace{f_0 \dots x^{d_1-1} f_0}^{d_0+d_1} & & & \overbrace{f_1 \dots x^{d_0-1} f_1} & & \\
 c_{0,0} & & & c_{1,0} & & 0 \\
 \vdots & \ddots & & \vdots & \ddots & \\
 \vdots & & c_{0,0} & \vdots & & c_{1,0} \\
 c_{0,d_0} & & \vdots & c_{1,d_1} & & \vdots \\
 \vdots & \ddots & \vdots & & \ddots & \vdots \\
 0 & & c_{0,d_0} & 0 & & c_{1,d_1}
 \end{array} \right] \left. \begin{array}{l} 1 \\ x \\ \cdot \\ x^{d_1-1} \\ \cdot \\ x^{d_0+d_1-1} \end{array} \right\} d_0 + d_1$$

**Bézout (1779)**

$$\Theta_{f_0, f_1}(x, y) := \frac{f_1(x) f_0(y) - f_1(y) f_0(x)}{y - x} = \sum_{i=0}^{d_1-1} \theta_{f_0, f_1, i}(x) y^i = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_1-1} \theta_{i, j} x^i y^j.$$

The Bézout matrix is  $B_{f_0, f_1} = (\theta_{i, j})_{0 \leq i, j \leq d_1}$ .

**Theorem :**  $R(c_{i, j}) := \det(S)$  vanishes iff  $f_0 = 0, f_1 = 0$  has a common root.

# Resultant theory

**Condition on  $\mathbf{c} = (c_{i,j})$**  such that the system has a **solution** in the projective **variety  $X$**  of dimension  $n$ :

$$\mathbf{f}_{\mathbf{c}} \begin{cases} f_0(\mathbf{x}) & = & \sum_{j=0}^{k_0} c_{0,j} \kappa_{0,j}(\mathbf{x}) \\ & \vdots & \\ f_n(\mathbf{x}) & = & \sum_{j=0}^{k_n} c_{n,j} \kappa_{n,j}(\mathbf{x}) \end{cases}$$

$\Rightarrow f_i \in V_i = \langle \kappa_{i,0}(\mathbf{x}), \dots, \kappa_{i,k_i}(\mathbf{x}) \rangle$ , global section of a **line bundle  $\mathcal{L}_i$** .

$\Rightarrow V_i$  is **very ample** on  $X$  iff:

the elements of  $V_i$  separate the points on  $X$ .

the elements of  $V_i$  separate the tangent vectors on  $X$ .

⇒ **Incidence variety:**  $W_X = \{(\mathbf{c}, \mathbf{x}) \in \mathbb{P}^{k_0} \times \dots \times \mathbb{P}^{k_n} \times X, \mathbf{f}_{\mathbf{c}}(\mathbf{x}) = 0\}$ .

$$Z \subset \prod_i \mathbb{P}^{k_i} \xleftarrow{\pi_1} W_X \xrightarrow{\pi_2} X$$

⇒ **Base point:**  $\{\mathbf{x} \in X, \kappa_{i,j}(\mathbf{x}) = 0\}$ .

**Theorem:** If  $V_i$  is very ample almost everywhere on  $X$  with no base point, then  $Z = \pi_1(W_X)$  is an hypersurface defined by an equation  $\text{Res}_X(\mathbf{f}) = 0$ .

- $\text{Res}_X(\mathbf{f}) = 0$  iff  $\exists \mathbf{x} \in X$  with  $\mathbf{f}(\mathbf{x}) = 0$ .
- $\text{Res}_X(\mathbf{f})$  is a multi-homogeneous polynomial of degree in  $f_i$   
 $\int \prod_{j \neq i} c_1(\mathcal{L}_j) = \#^g \{\mathbf{x}, f_j(\mathbf{x}) = 0, j \neq i\}$ .

**Corollary:** If  $V_i$  is very ample almost everywhere on  $X$ , then for  $\tilde{X}$  the blowup of  $X$  along the base points, the projection  $\tilde{Z} = \pi_1(W_{\tilde{X}})$  is an hypersurface defined by an equation  $\text{Res}_{\tilde{X}}(\mathbf{f}) = 0$ .

## Example: the projective resultant

- $X = \mathbb{P}^n$ .
- $V_i = \langle x^\alpha; |\alpha| = d_i \rangle$
- $V_i$  very ample iff  $d_i > 0$ .
- No base point.
- $\text{Res}_{\mathbb{P}^n}(\mathbf{f}) = 0$  iff  $\exists \mathbf{x} \in \mathbb{P}^n$  with  $\mathbf{f}(\mathbf{x}) = 0$ .
- $\deg_{\mathbf{f}_i}(\text{Res}_{\mathbb{P}^n}(\mathbf{f})) = \prod_{j \neq i} d_j$ .

## In practice

We often consider affine situations:

$$f_i = \sum_j c_{i,j} \kappa_{i,j}(\mathbf{t})$$

where  $\kappa_{i,j}(\mathbf{t})$  is a polynomial in  $\mathbf{t} = (t_1, \dots, t_n)$ .

⇒ We assume the  $\mathcal{K}_i = \langle \kappa_{i,j}(\mathbf{t}) \rangle_j$  separates the points and tangents.

□ Homogenize  $f_i$  over  $\mathbb{P}^n$  in  $\overline{f}_i$ .

Define the corresponding  $V_i$  and the base points.

Blowup the base points and construct  $\text{Res}_{\tilde{\mathbb{P}}^n}(\mathbf{f})$ .

□ Let  $U$  be the open subset such that  $[\kappa_{1,0}(\mathbf{t}), \dots, \kappa_{i,k_i}(\mathbf{t})] \neq 0$ .

Define  $\sigma(\mathbf{t}) = (\sigma_0(\mathbf{t}), \dots, \sigma_N(\mathbf{t}))$  and  $\psi_{i,j}$  s. t.  $\kappa_{i,j} = \psi_{i,j} \circ \sigma$ ,  $\deg(\psi_{i,j}) = d_i$ .

Define  $X = \overline{\sigma(\mathbf{t})}$  and construct  $\text{Res}_X(\mathbf{f})$ .

## Example: the toric resultant

- $\kappa_{i,j}(t) = t^{\alpha_{i,j}}, \alpha_{i,j} \in A_i$ .
- $X = \mathcal{T} := \sigma((\mathbb{K}^*)^n)$  where  $\sigma : \mathbf{t} \mapsto (\mathbf{t}^{\alpha_0 + \dots + \alpha_n})_{\alpha_i \in A_i}$
- $\psi_{i,j} = x_{\beta_{i,j}}$ .
- If  $\dim(A_i) = n$  and  $\sum_i \mathbb{Z} A_i = \mathbb{Z}^n$ , then the resultant  $\text{Res}_{\mathcal{T}}(\mathbf{f})$  exists.
- $\text{Res}_{\mathcal{T}}(\mathbf{f}) = 0$  iff  $\exists \mathbf{x} \in \mathcal{T}$  with  $\mathbf{f}(\mathbf{x}) = 0$ .
- The degree of  $\text{Res}_{\mathcal{T}}(\mathbf{f})$  in  $f_i$  is  $MV(A_0, \dots, A_{i-1}, A_{i+1}, \dots, A_n)$  [BKK'75].



**How to compute it**

## Do linear algebra

□ By Gröbner basis computation using an elimination order (too costly).

□ As the gcd of the maximal minors of

$$\begin{aligned} \mathbf{S} : \langle \mathbf{x}^{E_0} \rangle \times \cdots \times \langle \mathbf{x}^{E_n} \rangle &\rightarrow \langle \mathbf{x}^E \rangle \\ (q_0, \dots, q_n) &\mapsto \sum_{i=0}^n q_i f_i \end{aligned}$$

□ As ratio of two minors of such a map.

□ As the determinant of a complex

$$0 \rightarrow K_s \xrightarrow{\delta_s} K_{s-1} \rightarrow \cdots \rightarrow K_1 \xrightarrow{\delta_1} K_0 \rightarrow 0$$

$$\text{Res}_X(\mathbf{f}) = \frac{\det(M_s) \det(M_{s-2}) \cdots}{\det(M_{s-1}) \det(M_{s-3}) \cdots}$$

⇒ Matrix formulation for a control of the objects, of the error when dealing with approximate coefficients, so that structure can be exploited.

## Constructions

**Projective resultant:**  $\{\kappa_{i,j}(\mathbf{x})\} = \{\mathbf{x}^{\alpha_j}; |\alpha_j| = d_i\}$ .  $X = \mathbb{P}^n$ .

Sylvester-like matrix. Ratio of two Determinants. Determinant of the Koszul complex. [Mac1902], [J91].

**Toric resultant:**  $\{\kappa_{i,j}(\mathbf{t})\} = \{\mathbf{t}^{\alpha_j}; \alpha_j \in A_i\}$ ,  $\mathbf{t} \in (\mathbb{K} - \{0\})^n$ ,  $X = \mathcal{T}_{A_0 \oplus \dots \oplus A_n}$ .

Polytope geometry. Sylvester-like matrix. Maximal minors. Ratio of two Determinants [BKK75, GKZ91, PSCE93, DA01].

**Resultant over a parameterised variety:**  $\{\kappa_{i,j}(\mathbf{t})\}$  associated with the parametrisation of  $X = \overline{\sigma(U)}$ .

Bezoutian matrix. Maximal minors. A multiple of  $\text{Res}_X(\mathbf{f})$ . [EM98, BEM00].

**Residual resultant:**  $\kappa_{i,j}(\mathbf{x}) \in (g_1(\mathbf{x}), \dots, g_k(\mathbf{x}))$ .  $X$  is the **blow-up** of  $\mathbb{P}^n$  along  $\mathcal{Z}(g_1, \dots, g_k)$ .

Explicit resolution of  $(F : G)$ . Gcd of the maximal minors. Degree formula. Ratio of determinants. [BKM75, BEM01, B01].

# Projective resultant

Sylvester-like matrix. Ratio of two Determinants. Determinant of the Koszul complex [Mac1902], [J91], [C95] [DA02].

**Example:** Macaulay resultant matrix of a generic linear form  $f_0 = u_0 + u_1x_1 + u_2x_2$  and  $f_1, f_2$ .

>  $S := \text{mresultant}([u[0] + u[1]*x[1] + u[2]*x[2], f_1, f_2], [x[1], x[2]]);$

$$\left[ \begin{array}{cccc|ccc|ccc} u_0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -\frac{1}{6} \\ u_2 & u_0 & 0 & 0 & 2 & 0 & -8 & 0 & -\frac{1}{6} & 0 \\ u_1 & 0 & u_0 & 0 & 0 & 2 & -8 & -\frac{1}{6} & 0 & -1 \\ 0 & u_1 & u_2 & u_0 & -8 & -8 & 8 & 0 & -1 & 1 \\ \hline 0 & 0 & u_1 & 0 & 0 & -8 & 13 & -1 & 0 & 1 \\ 0 & u_2 & 0 & 0 & -8 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 13 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_1 & 13 & 8 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & u_2 & 8 & 4 & 0 & 0 & 1 & 0 \end{array} \right].$$

$$E_2 = \{1, x_1, x_2\},$$

$$E_1 = \{1, x_1, x_2\},$$

$$E_0 = \{1, x_1, x_2, x_1x_2\},$$

$$F = \{1, x_1, x_2, x_1x_2, x_1^2, x_1^3, x_1^2x_2, x_2^2, x_1x_2^2, x_2^3\}.$$

## The algorithm:

$$\square \nu = \sum_{i=0}^n (d_i - 1) + 1.$$

$$\square F = \{\mathbf{t}^\alpha; |\alpha| \leq \nu\}.$$

$$\square t_n^{d_n} \mathbf{t}^{E_n} = \{\mathbf{t}^\alpha \in \mathbf{t}^F \text{ divisible by } t_n^{d_n}\}.$$

$$t_{n-1}^{d_{n-1}} \mathbf{t}^{E_{n-1}} = \{\mathbf{t}^\alpha \in \mathbf{t}^F - t_n^{d_n} \mathbf{t}^{E_n} \text{ divisible by } t_{n-1}^{d_{n-1}}\}$$

⋮

$$\mathbf{t}^{E_0} = \mathbf{t}^F - t_n^{d_n} \mathbf{t}^{E_n} - \dots - t_1^{d_1} \mathbf{t}^{E_1}.$$

□ Construct the square matrix of  $\mathbf{S}$  in the monomial basis, where

$$\begin{aligned} \mathbf{S} : \langle \mathbf{x}^{E_0} \rangle \times \dots \times \langle \mathbf{x}^{E_n} \rangle &\rightarrow \langle \mathbf{x}^E \rangle \\ (q_0, \dots, q_n) &\mapsto \sum_{i=0}^n q_i f_i \end{aligned}$$

□  $\# E_0 = \prod_{i=1}^n d_i.$

□ Generically,  $\mathbf{t}^{E_0}$  is a basis of  $\mathbb{K}[\mathbf{t}]/(f_1, \dots, f_n).$

## Toric resultant

Polytope geometry. Sylvester-like matrix. Maximal minors. Ratio of two Determinants [BKK75, GKZ94, PS93, CE93, DA01].

**Example:** Consider the system

$$\begin{cases} f_0 = c_{0,0}t_1t_2 + c_{0,1}t_1 + c_{0,2}t_2 + c_{0,3}, \\ f_1 = c_{1,0}t_1t_2 + c_{1,1}t_1 + c_{1,2}t_2 + c_{1,3}, \\ f_2 = c_{2,0}t_1^2 + c_{2,1}t_2^2 + c_{2,1}t_1 + c_{2,2}t_2 + c_{2,3}. \end{cases}$$

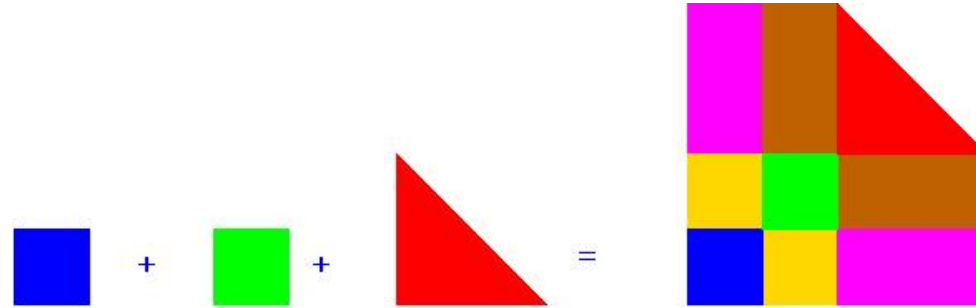
>  $S := \text{sresultant}([f_0, f_1, f_2], [t[1], t[2]]);$

$$\begin{bmatrix} c_{0,3} & 0 & 0 & 0 & c_{1,3} & 0 & 0 & 0 & 0 & c_{2,3} & 0 & 0 \\ c_{0,2} & c_{0,3} & 0 & 0 & c_{1,2} & c_{1,3} & 0 & 0 & 0 & c_{2,2} & 0 & 0 \\ 0 & 0 & c_{0,3} & 0 & 0 & 0 & c_{1,3} & 0 & 0 & 0 & c_{2,3} & 0 \\ c_{0,1} & 0 & c_{0,2} & c_{0,3} & c_{1,1} & 0 & c_{1,2} & c_{1,3} & 0 & c_{2,1} & c_{2,2} & c_{2,3} \\ 0 & 0 & c_{0,1} & 0 & 0 & 0 & c_{1,1} & 0 & c_{1,3} & 0 & c_{2,1} & 0 \\ c_{0,0} & c_{0,1} & 0 & c_{0,2} & c_{1,0} & c_{1,1} & 0 & c_{1,2} & 0 & 0 & c_{2,1} & c_{2,2} \\ 0 & c_{0,0} & 0 & 0 & 0 & c_{1,0} & 0 & 0 & 0 & 0 & 0 & c_{2,1} \\ 0 & c_{0,2} & 0 & 0 & 0 & c_{1,2} & 0 & 0 & 0 & c_{2,1} & 0 & 0 \\ 0 & 0 & c_{0,0} & c_{0,1} & 0 & 0 & c_{1,0} & c_{1,1} & c_{1,2} & c_{2,0} & 0 & c_{2,1} \\ 0 & 0 & 0 & c_{0,0} & 0 & 0 & 0 & c_{1,0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{1,1} & 0 & c_{2,0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{1,0} & 0 & 0 & c_{2,0} \end{bmatrix}$$

There are 4 columns in  $f_0$ , which is  $\#_g Z(f_1 = 0, f_2 = 0)$ .

## The algorithm:

- Construct a regular subdivision of  $A_0 \oplus \dots \oplus A_n$ .



- For each cell  $C = F_0 \oplus F_{i_0} \oplus \dots \oplus F_n$  with  $\dim(F_{i_0}) = 0$  and  $i_0$  maximal, add the monomials of  $(F_0 \oplus \dots \oplus F_{i_0-1} \oplus F_{i_0+1} \oplus \dots \oplus F_n)^\delta$  to  $E_{i_0}$ .
- Construct the square matrix of  $\mathbf{S}$  in the monomial basis, where

$$\mathbf{S} : \langle \mathbf{x}^{E_0} \rangle \times \dots \times \langle \mathbf{x}^{E_n} \rangle \rightarrow \langle \mathbf{x}^E \rangle$$

$$(q_0, \dots, q_n) \mapsto \sum_{i=0}^n q_i f_i$$

- $\# E_0 = MV(A_1, \dots, A_n)$ .
- Generically,  $\mathbf{t}^{E_0}$  is a basis of  $\mathbb{K}[\mathbf{t}]/(f_1, \dots, f_n)$ .



# Resultant over a parameterised variety

## Bezoutian:

- $X_{(0)} = (x_1, \dots, x_n), X_{(1)} = (y_1, x_2, \dots, x_n), \dots, X_{(n)} = (y_1, \dots, y_n).$
- $\theta_i(P) = \frac{P(X_{(i)}) - P(X_{(i-1)})}{y_i - x_i}.$

$$\Theta_{f_0, f_1, \dots, f_n} = \begin{vmatrix} f_0(X_{(0)}) & \theta_1(f_0) & \cdots & \theta_n(f_0) \\ \vdots & \vdots & \cdots & \vdots \\ f_n(X_{(0)}) & \theta_1(f_n) & \cdots & \theta_n(f_n) \end{vmatrix} = \sum_{\alpha, \beta} b_{\alpha, \beta} \mathbf{x}^\alpha \mathbf{y}^\beta$$

## Bezoutian matrix:

$$B_{f_0, \dots, f_n} = (b_{\alpha, \beta})_{\alpha, \beta}.$$

**Theorem:** [EM98, BEM00] Any maximal of  $B_{f_0, \dots, f_n}$  is a multiple of  $\text{Res}_X(\mathbf{c})$ .

> Theta([u[0]+u[1]\*x[1]+u[2]\*x[2],f1,f2],[x[1],x[2]]);

$$\begin{aligned}
 & -4u_0y_1y_2 + 5u_0y_1^2 + \left(4u_0 - \frac{2}{3}u_1\right)y_2 + \left(8u_0 - \frac{10}{3}u_1 + \frac{25}{6}u_2\right)y_1 + \left(\frac{4}{3}u_1 - 8u_0 - \frac{10}{3}u_2\right) \\
 & + \left(-4u_1y_1y_2 + 5u_1y_1^2 - 4u_0y_2 + (8u_1 + 5u_2 + 5u_0)y_1 + \left(8u_0 + \frac{25}{6}u_2\right)\right)x_1 \\
 & + \left(-4u_2y_1y_2 + 5u_2y_1^2 + (4u_2 - 4u_0)y_2 + (8u_1 - 4u_0)y_1 + \left(\frac{10}{3}u_2 + 12u_0 - \frac{2}{3}u_1\right)\right)x_2 \\
 & + (-4u_2y_2 - 4u_1y_1 - 4u_0)x_1x_2 + (-4u_2y_2 - 4u_1y_1 - 4u_0)x_2^2
 \end{aligned}$$

> mbezout([u[0]+u[1]\*x[1]+u[2]\*x[2],f1,f2],[x[1],x[2]]);

$$\begin{bmatrix}
 -4u_0 & 5u_0 & 4u_0 - \frac{2}{3}u_1 & 8u_0 - \frac{10}{3}u_1 + \frac{25}{6}u_2 & 4/3u_1 - 8u_0 - \frac{10}{3}u_2 \\
 -4u_1 & 5u_1 & -4u_0 & 8u_1 + 5u_2 + 5u_0 & 8u_0 + \frac{25}{6}u_2 \\
 -4u_2 & 5u_2 & -4u_0 + 4u_2 & 8u_1 - 4u_0 & -\frac{2}{3}u_1 + \frac{10}{3}u_2 + 12u_0 \\
 0 & 0 & -4u_2 & -4u_1 & -4u_0 \\
 0 & 0 & -4u_2 & -4u_1 & -4u_0
 \end{bmatrix}$$

> factor(det(submatrix(",1..4,2..5))));

$$\frac{5}{11664} \left(u_0 + \frac{1}{3}u_1 + \frac{7}{6}u_2\right)^2 \left(u_0 - \frac{1}{3}u_1 + \frac{5}{6}u_2\right)^2$$

**Example:** Consider the three following polynomials

$$\begin{cases} f_0 = c_{0,0} + c_{0,1}t_1 + c_{0,2}t_2 + c_{0,3}(t_1^2 + t_2^2) \\ f_1 = c_{1,0} + c_{1,1}t_1 + c_{1,2}t_2 + c_{1,3}(t_1^2 + t_2^2) + c_{1,4}(t_1^2 + t_2^2)^2 \\ f_2 = c_{2,0} + c_{2,1}t_1 + c_{2,2}t_2 + c_{2,3}(t_1^2 + t_2^2) + c_{2,4}(t_1^2 + t_2^2)^2. \end{cases}$$

- The projective and toric resultant of these polynomials is zero.
- A maximal minor of this  $B(f_0, f_1, f_2)$  (rank 10), yields  $\Delta = q_1q_2(q_3)^2\rho$ , containing 207805 monomials:

$$\begin{aligned} q_1 &= -c_{0,2}c_{1,3}c_{2,4} + c_{0,2}c_{1,4}c_{2,3} + c_{1,2}c_{0,3}c_{2,4} - c_{2,2}c_{0,3}c_{1,4} \\ q_2 &= c_{0,1}c_{1,3}c_{2,4} - c_{0,1}c_{1,4}c_{2,3} - c_{1,1}c_{0,3}c_{2,4} + c_{2,1}c_{0,3}c_{1,4} \\ q_3 &= c_{0,3}^2c_{1,1}^2c_{2,4}^2 - 2c_{0,3}^2c_{1,1}c_{2,1}c_{2,4}c_{1,4} + c_{0,3}^2c_{2,4}^2c_{1,2}^2 + \dots \\ \rho &= c_{2,0}^4c_{1,4}^4c_{0,2}^4 + c_{2,0}^4c_{1,4}^4c_{0,1}^4 + c_{1,0}^4c_{2,4}^4c_{0,2}^4 + c_{1,0}^4c_{2,4}^4c_{0,1}^4 + \dots \end{aligned}$$

The resultant is  $\rho$  (2495 monomials, degree 4 in  $f_i$ ).

## Residual resultant of a complete intersection

We consider generic systems of equations of the form

$$\begin{cases} f_0 = \sum_{j=1}^k h_{j,0}(\mathbf{x}) g_j(\mathbf{x}) \\ \vdots \\ f_n = \sum_{j=1}^k h_{j,n}(\mathbf{x}) g_j(\mathbf{x}) \end{cases}$$

with  $\deg(f_i) = d_i$ ,  $\deg(g_j) = e_j$ ,  $G = (g_1, \dots, g_k)$  a complete intersection.

$\Leftrightarrow$  The **residual resultant** is the condition on the coefficients of  $h_{i,j}$  such that  $V(F : G) \neq \emptyset$  (or equiv  $F^{sat} \neq G^{sat}$ ).

$\Leftrightarrow$  It is of degree  $S_{r_i}(e_1, \dots, e_d) = \frac{VdM_{r_i}}{VdM_1}(e_1, \dots, e_d)$  in the coefficients of  $f_i$  where

$$r_i(T) = \sigma_n(\mathbf{d}) + \sum_{l=k}^n \sigma_{n-k}(\mathbf{d}) T^l \text{ and } \mathbf{d} = (d_0, \dots, d_{i-1}, d_{i+1}, \dots, d_n).$$

## Construction:

Let  $\Delta_I$  be the  $k \times k$  minor of  $H$ , with column indices in  $I$ .

$$\partial_{1,s} : \left( \prod_{I, 0 \leq i_1 < \dots < i_n \leq m} R_{[s-d_{i_1}-\dots-d_{i_n}+\sum_{i=1}^n k_i]} \right) \times R_{[s-d_0]} \times \dots \times R_{[s-d_m]} \longrightarrow R_{[s]}$$

$$((q_I)_I, (q_0, \dots, q_m)) \mapsto \sum_I q_I \Delta_I + q_0 f_0 + \dots + q_m f_m$$

**Theorem:** For  $s \geq \nu_{d,e} := \sum_{i=0}^n d_i - n - (n - k + 2) e_n$ ,

- $\partial_{1,s}$  is surjective iff  $V(F : G) \neq \emptyset$ ; iff the resultant is not zero.
- The gcd of the maximal minors of  $\partial_{1,s}$  is the resultant.

**Example:** Consider the system of generic cubics of  $\mathbb{P}^3$  containing the umbilic:

$$\left\{ \begin{array}{l} f_0 = (a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3)(x_0^2 + x_1^2 + x_2^2) \\ \quad + (a_4x_0^2 + a_5x_1^2 + a_6x_2^2 + a_7x_3^2 + a_8x_0x_1 + a_9x_0x_2 + a_{10}x_0x_3 + a_{11}x_1x_2 + a_{12}x_1x_3 + a_{13}x_2x_3)x_3 \\ f_1 = (b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3)(x_0^2 + x_1^2 + x_2^2) \\ \quad + (b_4x_0^2 + b_5x_1^2 + b_6x_2^2 + b_7x_3^2 + b_8x_0x_1 + b_9x_0x_2 + b_{10}x_0x_3 + b_{11}x_1x_2 + b_{12}x_1x_3 + b_{13}x_2x_3)x_3 \\ f_2 = (c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3)(x_0^2 + x_1^2 + x_2^2) \\ \quad + (c_4x_0^2 + c_5x_1^2 + c_6x_2^2 + c_7x_3^2 + c_8x_0x_1 + c_9x_0x_2 + c_{10}x_0x_3 + c_{11}x_1x_2 + c_{12}x_1x_3 + c_{13}x_2x_3)x_3 \\ f_3 = (d_0x_0 + d_1x_1 + d_2x_2 + d_3x_3)(x_0^2 + x_1^2 + x_2^2) \\ \quad + (d_4x_0^2 + d_5x_1^2 + d_6x_2^2 + d_7x_3^2 + d_8x_0x_1 + d_9x_0x_2 + d_{10}x_0x_3 + d_{11}x_1x_2 + d_{12}x_1x_3 + d_{13}x_2x_3)x_3 \end{array} \right.$$

We set  $\mathbf{g} = (x_3, x_0^2 + x_1^2 + x_2^2)$ . The previous construction gives  $N_0 = N_1 = N_2 = N_3 = 15$ . The size of the matrix  $M_\nu$  of  $\partial_\nu$  is a  $84 \times 200$ .

A maximal minor of rank 84 whose determinant has degree 15 in the coefficients of  $f_0$  yields a non-zero-multiple of the residual resultant.

Notice that the projective and toric resultants are identically 0 in this case.

## Residual resultant of a local complete intersection of codimension 2

$$(f_0(\mathbf{x}), \dots, f_n(\mathbf{x})) = (g_1(\mathbf{x}), \dots, g_k(\mathbf{x})) H.$$

$\Leftrightarrow$  If  $G = (g_1, \dots, g_k) \subset R = \mathbb{K}[\mathbf{c}][\mathbf{x}]$  of codimension 2, then by Hilbert-Burch theorem

$$0 \rightarrow \bigoplus_{i=1}^{k-1} R[-l_i] \xrightarrow{\psi} \bigoplus_{i=1}^k R[-k_i] \xrightarrow{\mathbf{g}} G \rightarrow 0$$

and

$$\bigoplus_{i=1}^{k-1} R[-l_i] \oplus \bigoplus_{i=0}^n R[-d_i] \xrightarrow{\psi \oplus H} \bigoplus_{i=1}^k R[-k_i] \xrightarrow{\mathbf{g}} G/F \rightarrow 0$$

so that  $\text{Ann}_R(G/F) = (F : G) = \Delta_k(\psi \oplus H)$ .

$\Leftrightarrow$  The corresponding Eagon-Northcott complex  $\mathcal{E}$  resolves  $(F : G)$ .

**Theorem:**  $\text{Res}_{\mathbb{P}_G^n}(\mathbf{f}) \neq 0$  iff  $V(F : G) = \emptyset$  iff  $F^{\text{sat}} = G^{\text{sat}}$   
 iff  $\delta_{\mathcal{E}, 1, \nu}$  is surjective ( $\nu \geq d_0 + \dots + d_n - s \min(k_l + 1)$ ).

# How to use it



- Analyse the geometry of the solutions of problem to solve (**preprocessing step**).
- Deduce an adequate resultant formulation to treat it (**preprocessing step**).
- Construction a matrix resultant formulation of the problem and generate the corresponding code (**preprocessing step**).
- Instanciate the parameters and run the solver (**run time**).
- Be able to detect the singular situation, in order to switch to a different scheme.

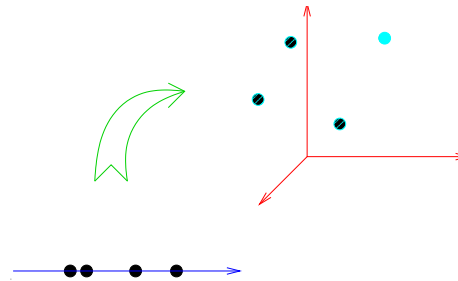
## The U-resultant

- Take  $f_0 = u_0 + u_1 t_1 + \dots + u_n t_n$ .
- Compute the resultant  $\text{Res}_X(f_0, f_1, \dots, f_n)$ .
- For a system  $f_1, \dots, f_n$  generic for the resultant formulation, it factorises as

$$\text{Res}_X(f_0, f_1, \dots, f_n) = \star \prod_{\zeta \in V_X(f_1=0, \dots, f_n=0)} (u_0 + u_1 \zeta_1 + \dots + u_n \zeta_n)^{\mu_\zeta} = \star \Delta(\mathbf{u}).$$

- Exploit its properties to recover the roots  $\zeta \in V_X(f_1 = 0, \dots, f_n = 0)$ .
- The construction extends naturally to residual resultant.

# Rational Univariate Representation of the roots



## Algorithm: Rational Univariate Representation.

1. Compute a multiple of the Chow form  $\Delta(\mathbf{u})$  and its square free part  $d(\mathbf{u})$ .
2. Choose a generic  $t \in \mathbb{K}^{n+1}$  and compute the first coefficients of

$$d(t + u) = d_0(u_0) + u_1 d_1(u_0) + \cdots + u_n d_n(u_0) + \cdots$$

3. A non minimal rational univariate representation of the roots is given by  $\zeta_1 = \frac{d_1(u_0)}{d'_0(u_0)}, \dots,$   
 $\zeta_n = \frac{d_n(u_0)}{d'_0(u_0)}, d_0(u_0) = 0.$
4. Factorize  $d_0(u_0)$  and keep the good factors for a minimal representation.

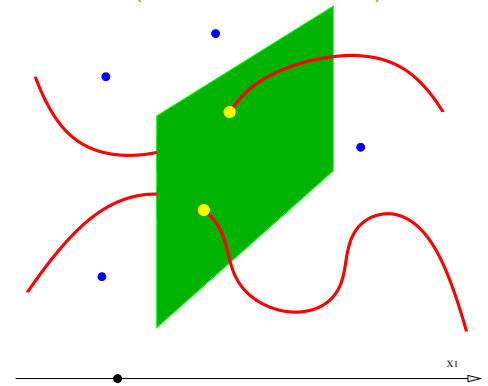
**Remark:**  $t$  is generic iff  $\gcd(d_0(u_0), d'_0(u_0)) = 1$ .

# Geometric decomposition

**Theorem:** Any nonzero maximal minor  $\Delta(u)$  of the Bezoutian matrix  $B_u$  of the polynomials  $f_0 := u_0 + u_1 z_1 + \cdots + u_n z_n, f_1, \dots, f_n \in \mathbb{K}(u)[z]$  is divisible by the Chow form  $\Delta_0(u)$  of the isolated points of  $V_{\mathbb{A}^n}(f_1, \dots, f_n)$ .

$\Rightarrow$  By hiding a variable  $x_i$ , we consider the generic fibers of the projection onto the  $x_i$  axis.

$\Rightarrow$  Curves become points, surfaces become curves, . . .



## Algorithm: Geometric decomposition of a variety.

Let  $f_1, \dots, f_m$  be  $m$  equations, in  $n$  variables, with coefficients in  $\mathbb{K}$ .

1. If  $n = 0$ , then stop.

2. Otherwise compute a rational representation of the isolated roots of  $n$  random linear combination of the  $f_i$ .

3. Choose one variable (say  $z_1$ , or a random combination of them) as a parameter, and go to 1, with  $n := n - 1$  and  $\mathbb{K} := \mathbb{K}(z_1)$ .

## Solving by eigenvector computation

- Take for  $f_0$  a generic polynomial.
- Compute a resultant matrix of  $f_0, f_1, \dots, f_n$  and decompose it as

$$S = \begin{array}{c} E_0 \\ F' \end{array} \left[ \begin{array}{c|ccc} & E_0 & E_1 & \cdots & E_n \\ \hline A & & & & B \\ \hline C & & & & D \end{array} \right]$$

- Compute the Schur complement  $M_{f_0} = A - B D^{-1} C$ .
- If  $f_1, \dots, f_n$  are generic for the resultant formulation,  $M_{f_0}$  is the matrix of multiplication by  $f_0$  in the basis  $\mathbf{t}^{E_0}$  of  $\mathbb{K}[\mathbf{t}]/(f_1, \dots, f_m)$ .
- Deduce the roots from  $M_{f_0}$  by eigen computation.

## Multiplication operators

We assume that  $\mathcal{Z}(I) = \{\zeta_1, \dots, \zeta_d\} \Leftrightarrow \mathcal{A}$  of finite dimension  $D$  over  $\mathbb{K}$ .

$$M_a : \mathcal{A} \rightarrow \mathcal{A} \quad M_a^\dagger : \widehat{\mathcal{A}} \rightarrow \widehat{\mathcal{A}}$$

$$u \mapsto au \quad \Lambda \mapsto a \cdot \Lambda = \Lambda \circ M_a$$

**Theorem:**

- **The eigenvalues of  $M_a$  are  $\{a(\zeta_1), \dots, a(\zeta_d)\}$ .**
- **The eigenvectors of all  $(M_a^\dagger)_{a \in \mathcal{A}}$  are (up to a scalar)  $\mathbf{1}_{\zeta_i} : p \mapsto p(\zeta_i)$ .**

**Theorem:** In a basis of  $\mathcal{A}$ , all the matrices  $M_a$  ( $a \in \mathcal{A}$ ) are of the form

$$M_a = \begin{bmatrix} N_a^1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & N_a^d \end{bmatrix} \quad \text{with } N_a^i = \begin{bmatrix} a(\zeta_i) & & \star \\ & \ddots & \\ \mathbf{0} & & a(\zeta_i) \end{bmatrix}$$

**Corollary: (Chow form)**

$$\Delta(\mathbf{u}) = \det(u_0 + u_1 M_{x_1} + \dots + u_n M_{x_n}) = \prod_{\zeta \in \mathcal{Z}(I)} (u_0 + u_1 \zeta_1 + \dots + u_n \zeta_n)^{\mu_\zeta}.$$

# Implicit inverse power method

**Algorithm: Compute the root which minimizes  $|f_0|$**

1. Compute the resultant matrix  $S$  of  $f_0, \dots, f_n$ .
  2. Choose  $\mathbf{w}_0$  at random.
  3. Solve  $S^t \begin{bmatrix} \mathbf{w}_{n+1} \\ \mathbf{v}_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{w}_n \\ \mathbf{0} \end{bmatrix}$   
 (so that  $\mathbf{w}_{n+1} = \frac{1}{\|\mathbf{w}_n\|} (M_{f_0}^t)^{-1} \mathbf{w}_n$ ) until  $\|\mathbf{w}_{n+1} - \mathbf{w}_n\| < \epsilon$ .
- The root is  $\zeta = \left( \frac{w_{x_1}}{w_1}, \dots, \frac{w_{x_n}}{w_1} \right)$ .

	N	S	D	n	k	T
s44	36	138	16	2	7	0.050s
s442	165	821	32	3	6	0.151s
s4422	715	3704	64	4	8	1.179s
s455	364	1664	100	3	6	2.331s
s2445	1820	8795	160	4	8	4.323s
s22445	8568	41942	320	5	8	28.213s
sq4	126	585	16	4	5	0.313s
sq5	462	2175	32	5	44	2.135s
sing	210	4998	21	2	14	0.438s
kruppa	792	15822	1	5	1	0.698s

## Solving $f_1 = \dots = f_n = 0$ by hiding a variable

1. Construct the resultant matrix  $\mathbf{S}(x_n)$  of  $f_1, \dots, f_n$  as polynomials in  $x_1, \dots, x_{n-1}$  with coefficients in  $\mathbb{K}[x_n]$ .

2. Solve  $\mathbf{S}(x_n)^t \mathbf{w} = 0$ .

- Either by solving  $\det(\mathbf{S}(x_n)) = 0$  and by deducing the corresponding  $\mathbf{w}$ .
- or by reducing it to an eigenproblem:

$$(\mathbf{S}_d^t x_n^d + \mathbf{S}_{d-1}^t x_n^{d-1} + \dots + \mathbf{S}_0^t) \mathbf{w} = 0$$

or

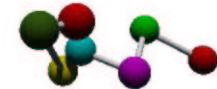
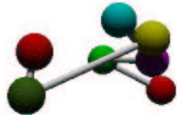
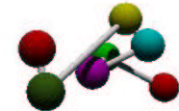
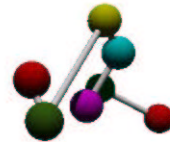
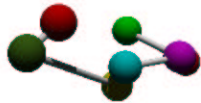
$$\left( \begin{bmatrix} 0 & \mathbb{I} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \mathbf{0} & \mathbb{I} \\ \mathbf{S}_0^t & \cdots & \mathbf{S}_{d-2}^t & \mathbf{S}_{d-1}^t \end{bmatrix} - x_n \begin{bmatrix} \mathbf{S}_d^t & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{S}_d^t & 0 \\ 0 & \cdots & 0 & \mathbf{S}_d^t \end{bmatrix} \right) \underline{\mathbf{w}} = 0,$$

3. Deduce the other coordinates of the roots from  $\mathbf{w}$ .



## Conformation of molecules

**Problem:** Compute the possible conformations of a molecule when the position and orientation of the extremity is fixed.

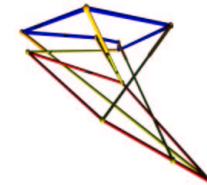
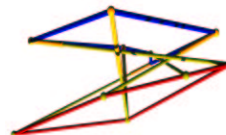
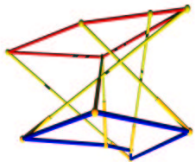
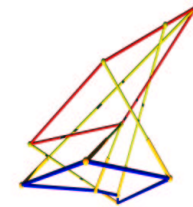
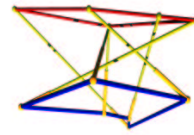
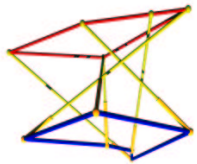


**Error:**  $\|A_1 \circ \dots \circ A_6 - H\| < 10^{-6}$

**Time:** 0.090s

## Direct kinematic problem of a parallel robot:

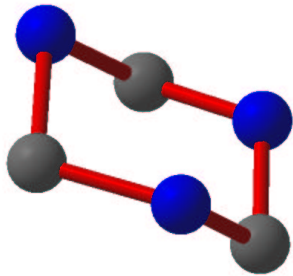
**Problem:** Compute the position of the platform for fixed lengths of the arms.



**Error:**  $|||R Y_i + T - X_i||^2 - d_i^2| < 10^{-6}$

**Time:** 0.5s

# The cyclohexan

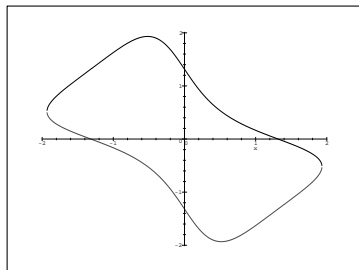


$$f_1 = -\frac{\sqrt{3}}{2} + \frac{1}{2}t_2^2 + \frac{1}{2}t_3^2 + 2t_2t_3 + \frac{\sqrt{3}}{2}t_2^2t_3^2 = 0$$

$$f_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}t_1^2 + \frac{1}{2}t_3^2 + 2t_1t_3 + \frac{\sqrt{3}}{2}t_1^2t_3^2 = 0$$

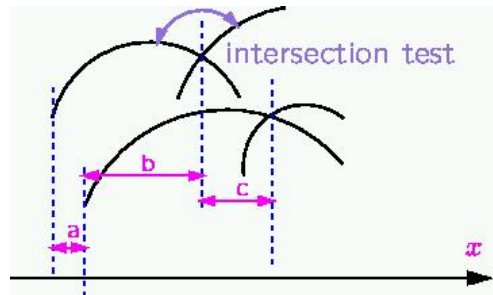
$$f_3 = -\frac{\sqrt{3}}{2} + \frac{1}{2}t_1^2 + \frac{1}{2}t_2^2 + 2t_1t_2 + \frac{\sqrt{3}}{2}t_1^2t_2^2 = 0$$

A curve of degree 4 + 2 isolated points + 6 embedded points



$t_1$	$t_2$	$t_3$
0.5176444559	0.5176444559	0.5176444563
-0.5176444567	-0.5176444567	-0.5176444555
0.5176444559	-1.931851652	0.5176444563
-0.5176444567	1.931851652	-0.5176444555
1.931851652	-0.5176444567	-0.5176444555
-1.931851652	0.5176444559	0.5176444563
0.5176444561	0.5176444561	-1.931851652
-0.5176444561	-0.5176444561	1.931851652

# Predicates



- Resultant formulations, in terms of a translation parameter.
- Sign of polynomials, of degree at most 12.
- Filtering technics.

		circle arcs		
		first	polynomial	polynomial
$\mu S$		Interval + L_real	static +semi-static +Interval +GMP	static +semi-static +Interv. first +GMP
left right		2.48	0.36	0.36
		67	24.3	6.8
		2170	129	128
with LEDA 4.2				
Geometric Predicates				

## Implicitisation

Compute the implicit equations of the surface, image of the map

$$\sigma : \mathbf{t} = (t_1, t_2) \in U \subset \mathbb{K}^2 \mapsto \left( \frac{f_1(\mathbf{t})}{f_0(\mathbf{t})}, \dots, \frac{f_3(\mathbf{t})}{f_0(\mathbf{t})} \right) \in \mathbb{K}^3$$

ie. the (irreducible) polynomial  $f(x, y, z)$  of minimal degree such that  $f \circ \sigma = 0$ .

□ Homogeneisation  $\sigma^h : \bar{\mathbf{t}} = (t_0 : t_1 : t_2) \mapsto (f_0^h(\bar{\mathbf{t}}) : f_1^h(\bar{\mathbf{t}}) : f_2^h(\bar{\mathbf{t}}) : f_3^h(\bar{\mathbf{t}}))$  with  $\deg(f_i^h) = d$ .

□ Degree of  $f = d^2 - \sum_{b \text{ base point}} \mu_b$ .

⇒ Eliminate  $(t_1, t_2)$  between

$$F_1 = f_1(\mathbf{t}) - x f_0(\mathbf{t}), F_2 = f_2(\mathbf{t}) - y f_0(\mathbf{t}), F_3 = f_3(\mathbf{t}) - z f_0(\mathbf{t}).$$

## Applications

⇒ Detection of self-intersection and singularities.

⇒ Compute the inverse image of a point on the surface.

⇒ Ray-tracing reduces to the eigenproblem  $(S^t(A) + t S^t(B)) \mathbf{w} = 0$ .

⇒ Intersection of parametric curves and implicit surfaces reduce to the eigenproblem  $(S_d^t t^d + \dots + S_0^t) \mathbf{w} = 0$ , or

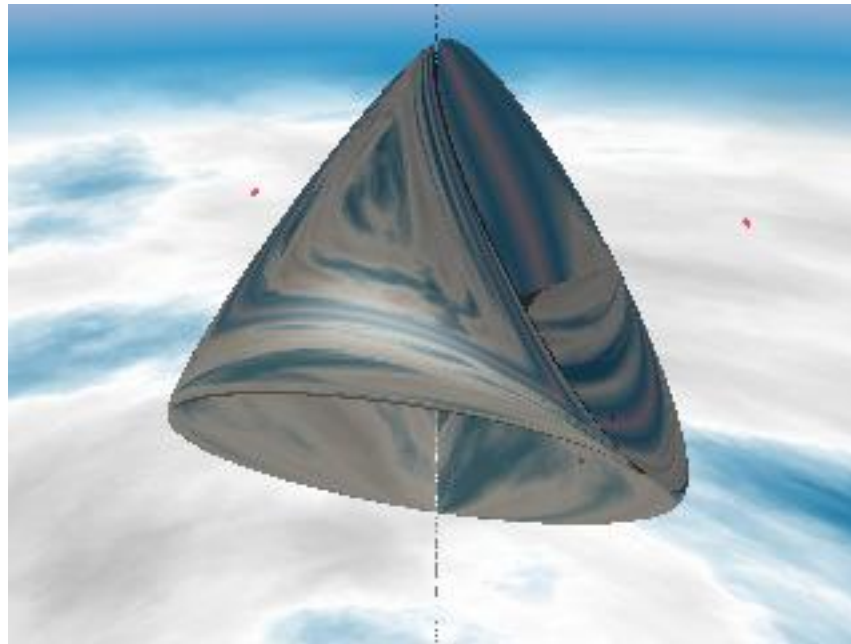
$$\left( \begin{bmatrix} 0 & \mathbb{I} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \mathbf{0} & \mathbb{I} \\ S_0^t & \cdots & S_{d-2}^t & S_{d-1}^t \end{bmatrix} - t \begin{bmatrix} S_d^t & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & S_d^t & 0 \\ 0 & \cdots & 0 & S_d^t \end{bmatrix} \right) \underline{\mathbf{w}} = 0,$$

⇒ Bounding volume using the dual surface equations.

## Example of a Steiner surface

$$\sigma^h : \bar{\mathbf{t}} = (t_0 : t_1 : t_2) \mapsto (q_0(\bar{\mathbf{t}}) : q_1(\bar{\mathbf{t}}) : q_2(\bar{\mathbf{t}}) : q_3(\bar{\mathbf{t}}))$$

where the  $q_i$  are of degree 2, with no base points.



## Resultant matrix:

- $T(\mathbf{t}, \mathbf{y}) = \det(\partial_{t_0}\sigma^h, \partial_{t_1}\sigma^h, \partial_{t_2}\sigma^h, \mathbf{y})$ .
- $\tau_i(\mathbf{y}) = \partial_{t_i}(T(\mathbf{t}, \mathbf{y}))$ .
- $S(x, y, z) = [q_1(\mathbf{t}) - x_1 q_0(\mathbf{t}), q_2(\mathbf{t}) - x_2 q_0(\mathbf{t}), q_y(\mathbf{t}) - x_3 q_0(\mathbf{t}), \tau_1(\mathbf{y}), \tau_2(\mathbf{y}), \tau_3(\mathbf{y})]$ .

$\Rightarrow S(x, y, z)$  is a  $6 \times 6$  matrix.

$\Rightarrow \det(S(x, y, z))$  is the implicit equation (of degree 4).

$\Rightarrow \dim(\ker(S(a_0, b_0, c_0)))$  is the number of inverse image of  $(a_0, b_0, c_0)$ .

$\Rightarrow$  If  $K$  is  $r \times 6$  st.  $\text{Span}(K) = \ker(S(a_0, b_0, c_0))$ , the coordinates of the inverse images can be computed from the eigenvectors of

$$(K_1 - \lambda K_0) \mathbf{w} = 0.$$

where  $K_i$  are  $r \times r$  submatrices of  $K$ .



## Computation of $f(x, y, z)$

- Compute a Gröbner basis  $(F) + (w f_0 - 1)$  for any ordering and apply a change of ordering to get a linear relation of smaller degree between the  $y^\alpha$ .
- Combine a Bezoutian part and a Sylvester part of the polynomials  $f_i(\mathbf{t}) - y_i \lambda^d$  in degree  $2d - 2$ . Morley/Dixon Matrix [J91, AS01].
- Construction of the Bezoutian matrices of  $F_i$  and maximal minor [BEM00].
- Residual resultants as determinant of the Hilbert-Burch complex [B01].
- Moving lines or quadrics:  $P(\mathbf{t}, x, y, z)$  with  $P(\mathbf{t}, \sigma(\mathbf{t})) \equiv 0$  [CSC98].
- Use an inversion formula to express  $\mathbf{t}$  in terms of  $\mathbf{y}$  on the surface, in order to deduce its equation [PDSS'02, SE'02].

## Experimentation

$$\begin{cases} h_1(s, t) &= 3t + 3s^2t - t^3 \\ h_2(s, t) &= 3s + 3st^2 - s^3 \\ h_3(s, t) &= 3t^2 - 3s^2 \\ h_0(s, t) &= 1 \end{cases}$$

$d = 3$ ,  $\deg(f) = 9$ , no base point, proper parametrisation.

Groebner	1.7s
Grobner imp.	0.690
Inversion	0.110s
Equation (no fact.)	0.410s
Equation (with fact.)	0.640s
Macaulay	0.479s
Morley	0.09s
Moving surfaces	1.6s
Bezout	0.071 (-)

$$\begin{cases} h_1(s, t) = 2s^3 + 13 + st^2 + 7ts^2 \\ h_2(s, t) = 2t^3 + s^2 + 15 + 4s^2t + 6t + s^3 \\ h_3(s, t) = 2t^2 + s + 17 + 9s^2t + 7t + s^2 \\ h_0(s, t) = 2t^2 + s + 19 + 4s^2t + 7t + s^3 \end{cases}$$

$d = 3$ ,  $\deg(f) = 9$ , no base point, proper parametrisation.

Groebner	> 3600s
Grobner imp.	10.690s
Inversion	> 1500s
Equation (no fact.)	????
Equation (with fact.)	????
Macaulay	0/0
Morley	1.510s
Moving surfaces	5.120s
Bezout	7.771s (0.001s)

$$\left\{ \begin{array}{l} h_1(s, t) = 32s^2t - 56st + 24t + 16s^3 - 120s^2 + 128s - 24 \\ h_2(s, t) = 16t^3 - 16t^2 + 16st^2 + 48s^2t - 456st + 392t + 8s^2 + 384s - 392 \\ h_3(s, t) = 64t^3 - 48t^2 + 48st^2 + 32s^2t - 1328st + 1232t + 16s^3 + 192s^2 + 1040s - 12 \\ h_0(s, t) = -6 + 4s - 6st + 2s^2 + 6t \end{array} \right.$$

$d = 3$ ,  $\deg(f) = 4$ , base points, proper parametrisation.

Groebner	theoric failure
Grobner imp.	0.100s
Inversion	69s
Equation (no fact.)	> 2200s
Equation (with fact.)	163s
Macaulay	theoric failure
Morley	0.250s (-)
Moving surfaces	theoric failure
Bezoutian	1.280s (-)
Residual resultant	0.42s (0.069s)

$$\begin{cases} h_1(s, t) = s^3 + t^3 + 5st^2 + 2st - 1 \\ h_2(s, t) = 2s^2 + 2t^3 + 3st - 2 \\ h_3(s, t) = 5s^3 + 5t^2 + 3st^2 + 2s^2t + 5st - 5 \\ h_0(s, t) = s^4 + t^4 - 1 \end{cases}$$

$d = 4$ ,  $\deg(f) = 10$ , with base points, proper parametrisation.

Groebner simple	theoric failure
Grobner imp.	38.250s
Inversion	> 5000s
Equation (no fact.)	????
Equation (with fact.)	????
Macaulay	theoric failure
Morley	7.919s (-)
Moving surfaces	theoric failure
Bezoutian	113.640s (0.010s)
Residual resultant	105.440s (0.189s)

$$\left\{ \begin{array}{l} h_1(s, t) = 3t(t-1)^2 + (s-1)^3 + 3s \\ h_2(s, t) = 3s(s-1)^2 + t^3 + 3t \\ h_3(s, t) = -3s(s^2 - 5s + 5)t^3 - 3(s^3 + 6s^2 - 9s + 1)t^2 + t(6s^3 + 9s^2 - 18s + 3) - 3s \\ h_0(s, t) = 1 \end{array} \right.$$

$d = (3, 3)$ ,  $\deg(f) = 18$ , *no base point*, proper parametrisation.

Groebner	theoric failure
Grobner imp.	399.55s
Inversion	> 3600s
Equation (no fact.)	????
Equation (with fact.)	????
Macaulay	theoric failure
Morley	330s (0.5500s)
Moving surfaces	465s
Bezout	146s (-)
Residual resultant	> 3 h (0.445s)

## Acknowledgements

- L. Busé (Residual resultant).
- M. Elkadi (Residue).
- I. Emeris (Toric resultant).
- S. Bis (Implicitisation).
- O. Ruatta (Implicitisation).
- J.P. Tecourt (Steiner surfaces).
- M. Teillaud (Geometric predicates).
- Ph. Trebuchet (Solver).

## Softwares

- **multires**: <http://www-sop.inria.fr/galaad/logiciels/multires/>
- **synaps**: <http://www-sop.inria.fr/galaad/logiciels/synaps/>