Resultant(s)

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Determinant, resultant

$$\left\{ \begin{array}{lcl} f_0(\mathbf{x}) & = & \sum_{j=0}^{n} c_{0,j}\, x_j = 0 \\ & \vdots & \\ f_n(\mathbf{x}) & = & \sum_{j=0}^{n} c_{n,j}\, x_j = 0 \end{array} \right.$$

has a non-trivial solution
iff
$\det(c_{i,j}) = 0$.

- ☐ Elimination of the variables $\mathbf{x} = (x_0, \ldots, x_n)$.
- ☐ Projection on the space of coefficients $\mathbf{c} = (c_{i,j})$.

Resultant in one variable

Let $f_0 = c_{0,0} + \cdots + c_{0,d_0}\, x^{d_0}$, $f_1 = c_{1,0} + \ldots + c_{1,d_1}\, x^{d_1}$ (avec $d_0 \leq d_1$).

Sylvester (1840)

$$
\overbrace{\begin{array}{cccc|cccc} f_0 & \cdots & x^{d_1-1} f_0 & & f_1 & \cdots & x^{d_0-1} f_1 & \end{array}}^{d_0+d_1}
$$

$$
\left[\begin{array}{ccc|ccc}
c_{0,0} & & \mathbf{0} & c_{1,0} & & \mathbf{0} \\
\vdots & \ddots & & \vdots & \ddots & \\
\vdots & & c_{0,0} & \vdots & & c_{1,0} \\
c_{0,d_0} & & \vdots & c_{1,d_1} & & \vdots \\
 & \ddots & \vdots & & \ddots & \vdots \\
\mathbf{0} & & c_{0,d_0} & \mathbf{0} & & c_{1,d_1}
\end{array}\right]
\qquad
\left.\begin{array}{l} 1 \\ x \\ . \\ x^{d_1-1} \\ . \\ x^{d_0+d_1-1} \end{array}\right\} d_0 + d_1
$$

Bézout (1779)

$$
\Theta_{f_0,f_1}(x,y) := \frac{f_1(x)\, f_0(y) - f_1(y)\, f_0(x)}{y - x} = \sum_{i=0}^{d_1-1} \theta_{f_0,f_1,i}(x)\, y^i = \sum_{i=0}^{d_1-1} \sum_{j=0}^{d_1-1} \theta_{i,j}\, x^i\, y^j .
$$

The Bézout matrix is $\mathtt{B}_{f_0,f_1} = (\theta_{i,j})_{0 \leq i,i \leq d_1}$.

Theorem : $R(c_{i,j}) := det(S)$ **vanishes iff** $f_0 = 0, f_1 = 0$ **has a common root.**

Resultant theory

Condition on $\mathbf{c} = (c_{i,j})$ such that the system has a **solution** in the projective **variety** X of dimension n:

$$\mathbf{f_c} \begin{cases} f_0(\mathbf{x}) & = & \sum_{j=0}^{k_0} c_{0,j}\, \kappa_{0,j}(\mathbf{x}) \\ & \vdots & \\ f_n(\mathbf{x}) & = & \sum_{j=0}^{k_n} c_{n,j}\, \kappa_{n,j}(\mathbf{x}) \end{cases}$$

⇨ $f_i \in V_i = \langle \kappa_{i,0}(\mathbf{x}), \ldots, \kappa_{i,k_i}(\mathbf{x}) \rangle$, global section of a **line bundle** $\mathcal{L}_i$.

⇨ V_i is **very ample** on X iff:

the elements of V_i separate the points on X.

the elements of V_i separate the tangent vectors on X.

⇨ **Incidence variety**: $W_X = \{(\mathbf{c}, \mathbf{x}) \in \mathbb{P}^{k_0} \times \cdots \times \mathbb{P}^{k_n} \times X, \mathbf{f}_\mathbf{c}(\mathbf{x}) = 0\}$.

$$Z \subset \prod_i \mathbb{P}^{k_i} \xleftarrow{\pi_1} W_X \xrightarrow{\pi_2} X$$

⇨ **Base point:** $\{\mathbf{x} \in X, \kappa_{i,j}(\mathbf{x}) = 0\}$.

Theorem: **If V_i is very ample almost every where on X with no base point, then $Z = \pi_1(W_X)$ is an hypersurface defined by an equation $\mathrm{Res}_X(\mathrm{f}) = 0$.**

- $\mathrm{Res}_X(\mathbf{f}) = 0$ iff $\exists \mathbf{x} \in X$ with $\mathbf{f}(\mathbf{x}) = 0$.
- $\mathrm{Res}_X(\mathbf{f})$ is a multi-homogeneous polynomial of degree in f_i $\int \prod_{j \neq i} c_1(\mathcal{L}_j) = \#^g\{\mathbf{x}, f_j(\mathbf{x}) = 0, j \neq i\}$.

Corollary: **If V_i is very ample almost every where on X, then for $\tilde{X}$ the blowup of X along the base points, the projection $\tilde{Z} = \pi_1(W_{\tilde{X}})$ is an hypersurface defined by an equation $\mathrm{Res}_{\tilde{X}}(\mathrm{f}) = 0$.**

Example: the projective resultant

- $X = \mathbb{P}^n$.
- $V_i = \langle x^\alpha ; |\alpha| = d_i \rangle$
- V_i very ample iff $d_i > 0$.
- No base point.
- $\mathrm{Res}_{\mathbb{P}^n}(\mathbf{f}) = 0$ iff $\exists \mathbf{x} \in \mathbb{P}^n$ with $\mathbf{f}(\mathbf{x}) = 0$.
- $\deg_{\mathbf{f}_i}(\mathrm{Res}_{\mathbb{P}^n}(\mathbf{f})) = \prod_{j \neq i} d_j$.

In practice

We often consider affine situations:

$$f_i = \sum_j c_{i,j}\, \kappa_{i,j}(\mathbf{t})$$

where $\kappa_{i,j}(\mathbf{t})$ is a polynomial in $\mathbf{t} = (t_1, \ldots, t_n)$.

⇨ We assume the $\mathcal{K}_i = \langle \kappa_{i,j}(\mathbf{t}) \rangle_j$ separates the points and tangents.

□ Homogenize f_i over $\mathbb{P}^n$ in $\overline{f_i}$.

Define the corresponding V_i and the base points.

Blowup the base points and construct $\mathrm{Res}_{\tilde{\mathbb{P}^n}}(\mathbf{f})$.

□ Let U be the open subset such that $[\kappa_{1,0}(\mathbf{t}), \ldots, \kappa_{i,k_i}(\mathbf{t})] \neq 0$.

Define $\sigma(\mathbf{t}) = (\sigma_0(\mathbf{t}), \ldots, \sigma_N(\mathbf{t}))$ and $\psi_{i,j}$ s. t. $\kappa_{i,j} = \psi_{i,j} \circ \sigma$, $\deg(\psi_{i,j}) = d_i$.

Define $X = \overline{\sigma(\mathbf{t})}$ and construct $\mathrm{Res}_X(\mathbf{f})$.

Example: the toric resultant

- $\kappa_{i,j}(t) = t^{\alpha_{i,j}}$, $\alpha_{i,j} \in A_i$.
- $X = \mathcal{T} := \sigma((\mathbb{K}^*)^n)$ where $\sigma : \mathbf{t} \mapsto (\mathbf{t}^{\alpha_0 + \cdots + \alpha_n})_{\alpha_i \in A_i}$
- $\psi_{i,j} = x_{\beta_{i,j}}$.
- If $\dim(A_i) = n$ and $\sum_i \mathbb{Z}\, A_i = \mathbb{Z}^n$, then the resultant $\mathrm{Res}_{\mathcal{T}}(\mathbf{f})$ exists.
- $\mathrm{Res}_{\mathcal{T}}(\mathbf{f}) = 0$ iff $\exists \mathbf{x} \in \mathcal{T}$ with $\mathbf{f}(\mathbf{x}) = 0$.
- The degree of $\mathrm{Res}_{\mathcal{T}}(\mathbf{f})$ in f_i is $MV(A_0, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$ [BKK'75].

How to compute it

Do linear algebra

□ By Gröbner basis computation using an elimination order (too costly).

□ As the gcd of the maximal minors of

$$\begin{array}{rcl} \mathtt{S} : \langle \mathbf{x}^{E_0} \rangle \times \cdots \langle \mathbf{x}^{E_n} \rangle & \longrightarrow & \langle \mathbf{x}^{E} \rangle \\ (q_0, \ldots, q_n) & \mapsto & \sum_{i=0}^{n} q_i\, f_i \end{array}$$

□ As ratio of two minors of such a map.

□ As the determinant of a complex

$$0 \to K_s \xrightarrow{\delta_s} K_{s-1} \to \cdots K_1 \xrightarrow{\delta_1} K_0 \to 0$$

$$\mathrm{Res}_X(\mathbf{f}) = \frac{\det(M_s)\det(M_{s-2})\cdots}{\det(M_{s-1})\det(M_{s-3})\cdots}$$

⇨ Matrix formulation for a control of the objects, of the error when dealing with approximate coefficients, so that structure can be exploited.

Constructions

Projective resultant: $\{\kappa_{i,j}(\mathbf{x})\} = \{\mathbf{x}^{\alpha_j};\ |\alpha_j| = d_i\}$. $X = \mathbb{P}^n$.

Sylvester-like matrix. Ratio of two Determinants. Determinant of the Koszul complex. [Mac1902], [J91].

Toric resultant: $\{\kappa_{i,j}(\mathtt{t})\} = \{\mathtt{t}^{\alpha_j};\ \alpha_j \in A_i\}$, $\mathtt{t} \in (\mathbb{K} - \{0\})^n$, $X = \mathcal{T}_{A_0 \oplus \cdots \oplus A_n}$.

Polytope geometry. Sylvester-like matrix. Maximal minors. Ratio of two Determinants [BKK75, GKZ91, PSCE93, DA01].

Resultant over a parameterised variety: $\{\kappa_{i,j}(\mathtt{t})\}$ associated with the parametrisation of $X = \overline{\sigma(U)}$.

Bezoutian matrix. Maximal minors. A multiple of $\mathrm{Res}_X(\mathbf{f})$. [EM98, BEM00].

Residual resultant: $\kappa_{i,j}(\mathbf{x}) \in (g_1(\mathbf{x}), \ldots, g_k(\mathbf{x}))$. X is the **blow-up** of $\mathbb{P}^n$ along $\mathcal{Z}(g_1, \ldots, g_k)$.

Explicit resolution of $(F : G)$. Gcd of the maximal minors. Degree formula. Ratio of determinants. [BKM75, BEM01, B01].

Projective resultant

Sylvester-like matrix. Ratio of two Determinants. Determinant of the Koszul complex [Mac1902], [J91], [C95] [DA02].

Example: Macaulay resultant matrix of a generic linear form $f_0 = u_0 + u_1 x_1 + u_2 x_2$ and f_1, f_2.

```
> S := mresultant([u[0]+u[1]*x[1]+u[2]*x[2],f1,f2],[x[1],x[2]]);
```

$$\left[\begin{array}{cccc|ccc|ccc}
u_0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -\frac{1}{6} \\
u_2 & u_0 & 0 & 0 & 2 & 0 & -8 & 0 & -\frac{1}{6} & 0 \\
u_1 & 0 & u_0 & 0 & 0 & 2 & -8 & -\frac{1}{6} & 0 & -1 \\
0 & u_1 & u_2 & u_0 & -8 & -8 & 8 & 0 & -1 & 1 \\
\hline
0 & 0 & u_1 & 0 & 0 & -8 & 13 & -1 & 0 & 1 \\
0 & u_2 & 0 & 0 & -8 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 13 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u_1 & 13 & 8 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & u_2 & 8 & 4 & 0 & 0 & 1 & 0
\end{array}\right].$$

$$\begin{aligned}
E_2 &= \{1, x_1, x_2\}, \\
E_1 &= \{1, x_1, x_2\}, \\
E_0 &= \{1, x_1, x_2, x_1 x_2\}, \\
F &= \{1, x_1, x_2, x_1\, x_2, x_1^2, x_1^3, x_1^2 x_2, x_2^2, x_1 x_2^2, x_2^3\}.
\end{aligned}$$

The algorithm:

- $\nu = \sum_{i=0}^{n}(d_i - 1) + 1.$
- $F = \{\mathbf{t}^{\alpha}; |\alpha| \leq \nu\}.$
- $t_n^{d_n}\,\mathbf{t}^{E_n} = \{\mathbf{t}^{\alpha} \in \mathbf{t}^{F} \text{ divisible by } t_n^{d_n}\}.$

  $t_{n-1}^{d_{n-1}}\,\mathbf{t}^{E_{n-1}} = \{\mathbf{t}^{\alpha} \in \mathbf{t}^{F} - t_n^{d_n}\mathbf{t}^{E_n} \text{divisible by } \mathbf{t}_{n-1}^{d_n}\}$

  $\vdots$

  $\mathbf{t}^{E_0} = \mathbf{t}^{F} - t_n^{d_n}\mathbf{t}^{E_n} - \cdots - t_1^{d_1}\mathbf{t}^{E_1}.$
- Construct the square matrix of $\mathbf{S}$ in the monomial basis, where

$$\begin{array}{rcl} \mathbf{S} : \langle \mathbf{x}^{E_0} \rangle \times \cdots \langle \mathbf{x}^{E_n} \rangle & \longrightarrow & \langle \mathbf{x}^{E} \rangle \\ (q_0, \ldots, q_n) & \mapsto & \sum_{i=0}^{n} q_i\, f_i \end{array}$$

- $\# E_0 = \prod_{i=1}^{n} d_i$.
- Generically, $\mathbf{t}^{E_0}$ is a basis of $\mathbb{K}[\mathbf{t}]/(f_1, \ldots, f_n)$.

Toric resultant

Polytope geometry. Sylvester-like matrix. Maximal minors. Ratio of two Determinants [BKK75, GKZ94, PS93, CE93, DA01].

Example: Consider the system

$$\left\{\begin{array}{l} f_0 = c_{0,0} t_1 t_2 + c_{0,1} t_1 + c_{0,2} t_2 + c_{0,3}, \\ f_1 = c_{1,0} t_1 t_2 + c_{1,1} t_1 + c_{1,2} t_2 + c_{1,3}, \\ f_2 = c_{2,0},\, {t_1}^2 + c_{2,1} {t_2}^2 + c_{2,1} t_1 + c_{2,2} t_2 + c_{2,3}. \end{array}\right.$$

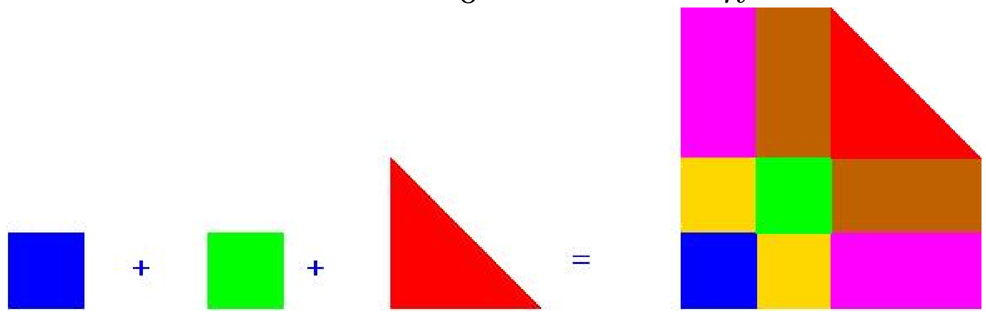
```
> S:= spresultant([f0,f1,f2],[t[1],t[2]]);
```

$$\left[\begin{array}{cccc|ccccc|ccc}
c_{0,3} & 0 & 0 & 0 & c_{1,3} & 0 & 0 & 0 & 0 & c_{2,3} & 0 & 0 \\
c_{0,2} & c_{0,3} & 0 & 0 & c_{1,2} & c_{1,3} & 0 & 0 & 0 & c_{2,2} & 0 & 0 \\
0 & 0 & c_{0,3} & 0 & 0 & 0 & c_{1,3} & 0 & 0 & 0 & c_{2,3} & 0 \\
c_{0,1} & 0 & c_{0,2} & c_{0,3} & c_{1,1} & 0 & c_{1,2} & c_{1,3} & 0 & c_{2,1} & c_{2,2} & c_{2,3} \\
0 & 0 & c_{0,1} & 0 & 0 & 0 & c_{1,1} & 0 & c_{1,3} & 0 & c_{2,1} & 0 \\
c_{0,0} & c_{0,1} & 0 & c_{0,2} & c_{1,0} & c_{1,1} & 0 & c_{1,2} & 0 & 0 & c_{2,1} & c_{2,2} \\
0 & c_{0,0} & 0 & 0 & 0 & c_{1,0} & 0 & 0 & 0 & 0 & 0 & c_{2,1} \\
0 & c_{0,2} & 0 & 0 & 0 & c_{1,2} & 0 & 0 & 0 & c_{2,1} & 0 & 0 \\
0 & 0 & c_{0,0} & c_{0,1} & 0 & 0 & c_{1,0} & c_{1,1} & c_{1,2} & c_{2,0} & 0 & c_{2,1} \\
0 & 0 & 0 & c_{0,0} & 0 & 0 & 0 & c_{1,0} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{1,1} & 0 & c_{2,0} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{1,0} & 0 & 0 & c_{2,0}
\end{array}\right]$$

There are 4 columns in f_0, which is $\#_g Z(f_1 = 0, f_2 = 0)$.

The algorithm:

- ☐ Construct a regular subdivision of $A_0 \oplus \cdots \oplus A_n$.



- ☐ For each cell $C = F_0 \oplus F_{i_0} \oplus \cdots \oplus F_n$ with $\dim(F_{i_0}) = 0$ and i_0 maximal, add the monomials of $(F_0 \oplus \cdots F_{i_0-1} \oplus F_{i_0+1} \oplus \cdots \oplus F_n)^\delta$ to E_{i_0}.

- ☐ Construct the square matrix of $\mathbf{S}$ in the monomial basis, where

$$\begin{array}{rcl} \mathbf{S} : \langle \mathbf{x}^{E_0} \rangle \times \cdots \langle \mathbf{x}^{E_n} \rangle & \rightarrow & \langle \mathbf{x}^{E} \rangle \\ (q_0, \ldots, q_n) & \mapsto & \sum_{i=0}^n q_i\, f_i \end{array}$$

- ☐ $\# E_0 = MV(A_1, \ldots, A_n)$.

- ☐ Generically, $\mathbf{t}^{E_0}$ is a basis of $\mathbb{K}[\mathbf{t}]/(f_1, \ldots, f_n)$.

Resultant over a parameterised variety

Bezoutian:

- $X_{(0)} = (x_1, \ldots, x_n)$, $X_{(1)} = (y_1, x_2 \ldots, x_n), \ldots,$ $X_{(n)} = (y_1, \ldots, y_n)$.
- $\theta_i(P) = \frac{P(X_{(i)}) - P(X_{(i-1)})}{y_i - x_i}$.

$$\Theta_{f_0, f_1, \ldots, f_n} = \begin{vmatrix} f_0(X_{(0)}) & \theta_1(f_0) & \cdots & \theta_n(f_0) \\ \vdots & \vdots & & \vdots \\ f_n(X_{(0)}) & \theta_1(f_n) & \cdots & \theta_n(f_n) \end{vmatrix} = \sum_{\alpha,\beta} b_{\alpha,\beta}\, \mathbf{x}^{\alpha} \mathbf{y}^{\beta}$$

Bezoutian matrix:

$$\mathtt{B}_{f_0, \ldots, f_n} = (b_{\alpha,\beta})_{\alpha,\beta}.$$

Theorem: **[EM98, BEM00] Any maximal of $B_{f_0, \ldots, f_n}$ is a multiple of $\mathrm{Res}_X(\mathbf{c})$.**

```
> Theta([u[0]+u[1]*x[1]+u[2]*x[2],f1,f2],[x[1],x[2]]);
```

$$-4\,u_0 y_1 y_2 + 5\,u_0 {y_1}^2 + \left(4\,u_0 - \frac{2}{3}\,u_1\right) y_2 + \left(8\,u_0 - \frac{10}{3}\,u_1 + \frac{25}{6}\,u_2\right) y_1 + \left(\frac{4}{3}\,u_1 - 8\,u_0 - \frac{10}{3}\,u_2\right)$$
$$+ \left(-4\,u_1 y_1 y_2 + 5\,u_1 {y_1}^2 - 4\,u_0 y_2 + (8\,u_1 + 5\,u_2 + 5\,u_0)\,y_1 + \left(8\,u_0 + \tfrac{25}{6}\,u_2\right)\right) x_1$$
$$+ \left(-4\,u_2 y_1 y_2 + 5\,u_2 {y_1}^2 + (4\,u_2 - 4\,u_0)\,y_2 + (8\,u_1 - 4\,u_0)\,y_1 + \left(\tfrac{10}{3}\,u_2 + 12\,u_0 - \tfrac{2}{3}\,u_1\right)\right) x_2$$
$$+ (-4\,u_2 y_2 - 4\,u_1 y_1 - 4\,u_0)\,x_1 x_2 + (-4\,u_2 y_2 - 4\,u_1 y_1 - 4\,u_0)\,{x_2}^2$$

```
> mbezout([u[0]+u[1]*x[1]+u[2]*x[2],f1,f2],[x[1],x[2]]);
```

$$\begin{bmatrix} -4\,u_0 & 5\,u_0 & 4\,u_0 - \frac{2}{3}\,u_1 & 8\,u_0 - \frac{10}{3}\,u_1 + \frac{25}{6}\,u_2 & 4/3\,u_1 - 8\,u_0 - \frac{10}{3}\,u_2 \\ -4\,u_1 & 5\,u_1 & -4\,u_0 & 8\,u_1 + 5\,u_2 + 5\,u_0 & 8\,u_0 + \frac{25}{6}\,u_2 \\ -4\,u_2 & 5\,u_2 & -4\,u_0 + 4\,u_2 & 8\,u_1 - 4\,u_0 & -\frac{2}{3}\,u_1 + \frac{10}{3}\,u_2 + 12\,u_0 \\ 0 & 0 & -4\,u_2 & -4\,u_1 & -4\,u_0 \\ 0 & 0 & -4\,u_2 & -4\,u_1 & -4\,u_0 \end{bmatrix}$$

```
> factor(det(submatrix(",1..4,2..5)));
```

$$\frac{5}{11664}\left(u_0 + \frac{1}{3}\,u_1 + \frac{7}{6}\,u_2\right)^2 \left(u_0 - \frac{1}{3}\,u_1 + \frac{5}{6}\,u_2\right)^2$$

Example: Consider the three following polynomials

$$\begin{cases} f_0 = c_{0,0} + c_{0,1}t_1 + c_{0,2}t_2 + c_{0,3}({t_1}^2 + {t_2}^2) \\ f_1 = c_{1,0} + c_{1,1}t_1 + c_{1,2}t_2 + c_{1,3}({t_1}^2 + {t_2}^2) + c_{1,4}({t_1}^2 + {t_2}^2)^2 \\ f_2 = c_{2,0} + c_{2,1}t_1 + c_{2,2}t_2 + c_{2,3}({t_1}^2 + {t_2}^2) + c_{2,4}({t_1}^2 + {t_2}^2)^2. \end{cases}$$

□ The projective and toric resultant of these polynomials is zero.

□ A maximal minor of this $B(f_0, f_1, f_2)$ (rank 10), yields $\Delta = q_1 q_2 (q_3)^2 \rho$, containing 207805 monomials:

$$\begin{array}{rcl} q_1 & = & -c_{0,2}c_{1,3}c_{2,4} + c_{0,2}c_{1,4}c_{2,3} + c_{1,2}c_{0,3}c_{2,4} - c_{2,2}c_{0,3}c_{1,4} \\ q_2 & = & c_{0,1}c_{1,3}c_{2,4} - c_{0,1}c_{1,4}c_{2,3} - c_{1,1}c_{0,3}c_{2,4} + c_{2,1}c_{0,3}c_{1,4} \\ q_3 & = & {c_{0,3}}^2{c_{1,1}}^2{c_{2,4}}^2 - 2{c_{0,3}}^2c_{1,1}c_{2,1}c_{2,4}c_{1,4} + {c_{0,3}}^2{c_{2,4}}^2{c_{1,2}}^2 + \cdots \\ \rho & = & {c_{2,0}}^4{c_{1,4}}^4{c_{0,2}}^4 + {c_{2,0}}^4{c_{1,4}}^4{c_{0,1}}^4 + {c_{1,0}}^4{c_{2,4}}^4{c_{0,2}}^4 + {c_{1,0}}^4{c_{2,4}}^4{c_{0,1}}^4 + \cdots \end{array}$$

The resultant is ρ (2495 monomials, degree 4 in f_i).

Residual resultant of a complete intersection

We consider generic systems of equations of the form

$$\left\{ \begin{array}{l} f_0 = \sum_{j=1}^{k} h_{j,0}(\mathbf{x})\, g_j(\mathbf{x}) \\ \vdots \\ f_n = \sum_{j=1}^{k} h_{j,n}(\mathbf{x})\, g_j(\mathbf{x}) \end{array} \right.$$

with $\deg(f_i) = d_i$, $\deg(g_j) = e_i$, $G = (g_1, \ldots, g_k)$ a complete intersection.

⇨ The **residual resultant** is the condition on the the coefficients of $h_{i,j}$ such that $V(F : G) \neq \emptyset$ **(or equiv** $F^{sat} \neq G^{sat}$**) .**

⇨ It is of degree $S_{r_i}(e_1, \ldots, e_d) = \frac{VdM_{r_i}}{VdM_1}(e_1, \ldots, e_d)$ in the coefficients of f_i where

$$r_i(T) = \sigma_n(\mathbf{d}) + \sum_{l=k}^{n} \sigma_{n-k}(\mathbf{d})\, T^l \text{ and } \mathbf{d} = (d_0, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n).$$

Construction:

Let Δ_I be the $k \times k$ minor of H, with column indices in I.

$$\partial_{1,s} : \left(\prod_{I, 0 \leq i_1 < \ldots < i_n \leq m} R_{[s-d_{i_1}-\ldots-d_{i_n}+\sum_{i=1}^n k_i]} \right) \times R_{[s-d_0]} \times \cdots \times R_{[s-d_m]} \longrightarrow R_{[s]}$$

$$((q_I)_I, (q_0, \ldots, q_m)) \mapsto \sum_I q_I \Delta_I + q_0 f_0 + \cdots + q_m f_m$$

Theorem: **For $s \geq \nu_{\mathbf{d},\mathbf{e}} := \sum_{i=0}^n d_i - n - (n-k+2)\, e_n$,**

- **$\partial_{1,s}$ is surjective iff $V(F:G) \neq \emptyset$; iff the resultant is not zero.**
- **The gcd of the maximal minors of $\partial_{1,s}$ is the resultant.**

Example: Consider the system of generic cubics of $\mathbb{P}^3$ containing the umbilic:

$$\left\{\begin{array}{lcl} f_0 & = & (a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3)(x_0^2 + x_1^2 + x_2^2) \\ & & +(a_4x_0^2 + a_5x_1^2 + a_6x_2^2 + a_7x_3^2 + a_8x_0x_1 + a_9x_0x_2 + a_{10}x_0x_3 + a_{11}x_1x_2 + a_{12}x_1x_3 + a_{13}x_2x_3)x_3 \\ f_1 & = & (b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3)(x_0^2 + x_1^2 + x_2^2) \\ & & +(b_4x_0^2 + b_5x_1^2 + b_6x_2^2 + b_7x_3^2 + b_8x_0x_1 + b_9x_0x_2 + b_{10}x_0x_3 + b_{11}x_1x_2 + b_{12}x_1x_3 + b_{13}x_2x_3)x_3 \\ f_2 & = & (c_0x_0 + c_1x_1 + c_2x_2 + c_3x_3)(x_0^2 + x_1^2 + x_2^2) \\ & & +(c_4x_0^2 + c_5x_1^2 + c_6x_2^2 + c_7x_3^2 + c_8x_0x_1 + c_9x_0x_2 + c_{10}x_0x_3 + c_{11}x_1x_2 + c_{12}x_1x_3 + c_{13}x_2x_3)x_3 \\ f_3 & = & (d_0x_0 + d_1x_1 + d_2x_2 + d_3x_3)(x_0^2 + x_1^2 + x_2^2) \\ & & +(d_4x_0^2 + d_5x_1^2 + d_6x_2^2 + d_7x_3^2 + d_8x_0x_1 + d_9x_0x_2 + d_{10}x_0x_3 + d_{11}x_1x_2 + d_{12}x_1x_3 + d_{13}x_2x_3)x_3 \end{array}\right.$$

We set $\mathbf{g} = (x_3, x_0^2 + x_1^2 + x_2^2)$. The previous construction gives $N_0 = N_1 = N_2 = N_3 = 15$. The size of the matrix $\mathtt{M}_\nu$ of ∂_ν is a 84×200.

A maximal minor of rank 84 whose determinant has degree 15 in the coefficients of f_0 yields a non-zero-multiple of the residual resultant.

Notice that the projective and toric resultants are identically 0 in this case.

Residual resultant of a local complete intersection of codimension 2

$$(f_0(\mathbf{x}),\dots,f_n(\mathbf{x})) = (g_1(\mathbf{x}),\dots,g_k(\mathbf{x}))\,H.$$

⇨ If $G=(g_1,\dots,g_k)\subset R=\mathbb{K}[\mathbf{c}][\mathbf{x}]$ of codimension 2, then by Hilbert-Burch theorem

$$0\to\bigoplus_{i=1}^{k-1}R[-l_i]\xrightarrow{\psi}\bigoplus_{i=1}^{k}R[-k_i]\xrightarrow{\mathbf{g}}G\to 0$$

and

$$\bigoplus_{i=1}^{k-1}R[-l_i]\oplus\bigoplus_{i=0}^{n}R[-d_i]\xrightarrow{\psi\oplus H}\bigoplus_{i=1}^{k}R[-k_i]\xrightarrow{\mathbf{g}}G/F\to 0$$

so that $Ann_R(G/F)=(F:G)=\Delta_k(\psi\oplus H)$.

⇨ The correspondind Eagon-Northcott complex $\mathcal{E}$ resolves $(F:G)$.

Theorem: $\mathrm{Res}_{\mathbb{P}^n_G}(\mathbf{f})\neq 0$ **iff** $V(F:G)=\emptyset$ **iff** $F^{sat}=G^{sat}$ **iff** $\delta_{\mathcal{E},1,\nu}$ **is surjective** ($\nu\geq d_0+\cdots+d_n-s\,\min(k_l+1)$)**.**

How to use it

- Analyse the geometry of the solutions of problem to solve (**preprocessing step**).
- Deduce an adequate resultant formulation to treat it (**preprocessing step**).
- Construction a matrix resultant formulation of the problem

  and generate the corresponding code (**preprocessing step**).
- Instanciate the parameters and run the solver (**run time**).
- Be able to detect the singular situation, in order to switch to a different scheme.

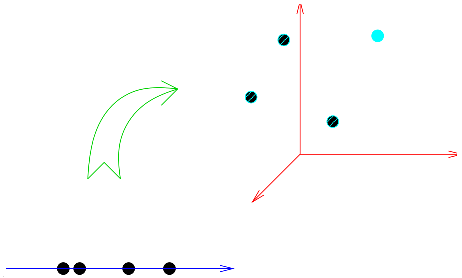
The U-resultant

- ☐ Take $f_0 = u_0 + u_1\, t_1 + \ldots + u_n\, t_n$.
- ☐ Compute the resultant $\mathrm{Res}_X(f_0, f_1, \ldots, f_n)$.
- ☐ For a system $f_1, \ldots, f_n$ generic for the resultant formulation, it factorises as

$$\mathrm{Res}_X(f_0, f_1, \ldots, f_n) = \star \prod_{\zeta \in V_X(f_1=0,\ldots,f_n=0)} (u_0 + u_1\zeta_1 + \cdots + u_n\zeta_n)^{\mu_\zeta} = \star\, \Delta(\mathbf{u}).$$

- ☐ Exploit its properties to recover the roots $\zeta \in V_X(f_1 = 0, \ldots, f_n = 0)$.
- ☐ The construction extends naturally to residual resultant.

Rational Univariate Representation of the roots



Algorithm: Rational Univariate Representation.

1. *Compute a* multiple *of the Chow form* $\Delta(\mathbf{u})$ *and its square free part* $d(\mathbf{u})$*.*
2. *Choose a generic* $t \in \mathbb{K}^{n+1}$ *and compute the first coefficients of*

$$d(t+u) = d_0(u_0) + u_1\, d_1(u_0) + \cdots + u_n\, d_n(u_0) + \cdots$$

3. *A non minimal rational univariate representation of the roots is given by* $\zeta_1 = \frac{d_1(u_0)}{d_0'(u_0)}, \ldots,$ $\zeta_n = \frac{d_n(u_0)}{d_0'(u_0)}$, $d_0(u_0) = 0$.
4. *Factorize* $d_0(u_0)$ *and keep the good factors for a minimal representation.*

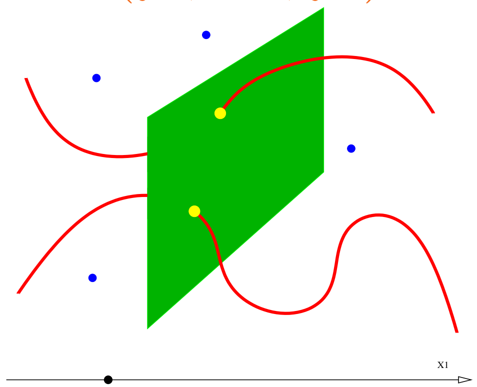
Remark: t is generic iff $gcd(d_0(u_0), d_0'(u_0)) = 1$.

Geometric decomposition

Theorem: **Any nonzero maximal minor $\Delta(u)$ of the Bezoutian matrix B_u of the polynomials $f_0 := u_0 + u_1 z_1 + \cdots + u_n z_n, f_1, \ldots, f_n \in \mathbb{K}(u)[z]$ is divisible by the Chow form $\Delta_0(u)$ of the isolated points of $V_{\mathbb{A}^n}(f_1, \ldots, f_n)$.**

⇨ By hiding a variable x_i, we consider the generic fibers of the projection onto the x_i axis.

⇨ Curves become points, surfaces become curves,. . .



Algorithm: Geometric decomposition of a variety.

Let $f_1, \ldots, f_m$ be m equations, in n variables, with coefficients in $\mathbb{K}$.
1. If $n = 0$, then stop.
2. Otherwise compute a rational representation of the isolated roots of n random linear combination of the f_i.
3. Choose one variable (say z_1, or a random combination of them) as a parameter, and go to 1, with $n := n - 1$ and $\mathbb{K} := \mathbb{K}(z_1)$.

Solving by eigenvector computation

☐ Take for f_0 a generic polynomial.

☐ Compute a resultant matrix of $f_0, f_1, \ldots, f_n$ and decompose is as

$$\mathbb{S} = \begin{array}{c} E_0 \\ \\ F' \end{array} \overset{\begin{array}{cc} E_0 & E_1 \cdots E_n \end{array}}{\left[\begin{array}{c|c} A & B \\ \hline C & D \\ & \end{array}\right]}$$

☐ Compute the Schur complement $M_{f_0} = A - B\, D^{-1}\, C$.

☐ If $f_1, \ldots, f_n$ are generic for the resultant formulation, M_{f_0} is the matrix of multiplication by f_0 in the basis $\mathbf{t}^{E_0}$ of $\mathbb{K}[\mathbf{t}]/(f_1, \ldots, f_m)$.

☐ Deduce the roots from M_{f_0} by eigen computation.

Multiplication operators

We assume that $\mathcal{Z}(I) = \{\zeta_1, \ldots, \zeta_d\} \Leftrightarrow \mathcal{A}$ of finite dimension D over $\mathbb{K}$.

$$\begin{array}{rcl} M_a : \mathcal{A} & \rightarrow & \mathcal{A} \\ u & \mapsto & a\,u \end{array} \qquad \begin{array}{rcl} M_a^{\mathtt{t}} : \widehat{\mathcal{A}} & \rightarrow & \widehat{\mathcal{A}} \\ \Lambda & \mapsto & a \cdot \Lambda = \Lambda \circ M_a \end{array}$$

Theorem:

- ☐ **The eigenvalues of M_a are $\{a(\zeta_1), \ldots, a(\zeta_d)\}$.**
- ☐ **The eigenvectors of all $(M_a^{\mathtt{t}})_{a \in \mathcal{A}}$ are (up to a scalar) $\mathbf{1}_{\zeta_i} : p \mapsto p(\zeta_i)$.**

Theorem: **In a basis of $\mathcal{A}$, all the matrices M_a $(a \in \mathcal{A})$ are of the form**

$$\mathtt{M}_a = \begin{bmatrix} \mathtt{N}_a^1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathtt{N}_a^d \end{bmatrix} \text{ with } \mathtt{N}_a^i = \begin{bmatrix} a(\zeta_i) & & \star \\ & \ddots & \\ \mathbf{0} & & a(\zeta_i) \end{bmatrix}$$

Corollary: (Chow form)
$\Delta(\mathbf{u}) = \det(u_0 + u_1\, \mathtt{M}_{x_1} + \cdots + u_n\, \mathtt{M}_{x_n}) = \prod_{\zeta \in \mathcal{Z}(I)} (u_0 + u_1\zeta_1 + \cdots + u_n\zeta_n)^{\mu_\zeta}.$

Implicit inverse power method

Algorithm: Compute the root which minimizes $|f_0|$

1. Compute the resultant matrix $\mathtt{S}$ *of* $f_0, \ldots, f_n$.
2. Choose $\mathbf{w}_0$ *at random.*
3. Solve $\mathtt{S}^t \begin{bmatrix} \mathbf{w}_{n+1} \\ \mathbf{v}_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{w}_n \\ \mathbf{0} \end{bmatrix}$
(so that $\mathbf{w}_{n+1} = \frac{1}{\|\mathbf{w}_n\|}(\mathtt{M}^{\mathtt{t}}_{f_0})^{-1}\mathbf{w}_n$*) until* $\|\mathbf{w}_{n+1} - \mathbf{w}_n\| < \epsilon$.
The root is $\zeta = (\frac{w_{x_1}}{w_1}, \ldots, \frac{w_{x_n}}{w_1})$.

	N	S	D	n	k	T
s44	36	138	16	2	7	0.050s
s442	165	821	32	3	6	0.151s
s4422	715	3704	64	4	8	1.179s
s455	364	1664	100	3	6	2.331s
s2445	1820	8795	160	4	8	4.323s
s22445	8568	41942	320	5	8	28.213s
sq4	126	585	16	4	5	0.313s
sq5	462	2175	32	5	44	2.135s
sing	210	4998	21	2	14	0.438s
kruppa	792	15822	1	5	1	0.698s

Solving $f_1 = \cdots = f_n = 0$ by hiding a variable

1. Construct the resultant matrix $\mathtt{S}(x_n)$ of $f_1, \ldots, f_n$ as polynomials in $x_1, \ldots, x_{n-1}$ with coefficients in $\mathbb{K}[x_n]$.

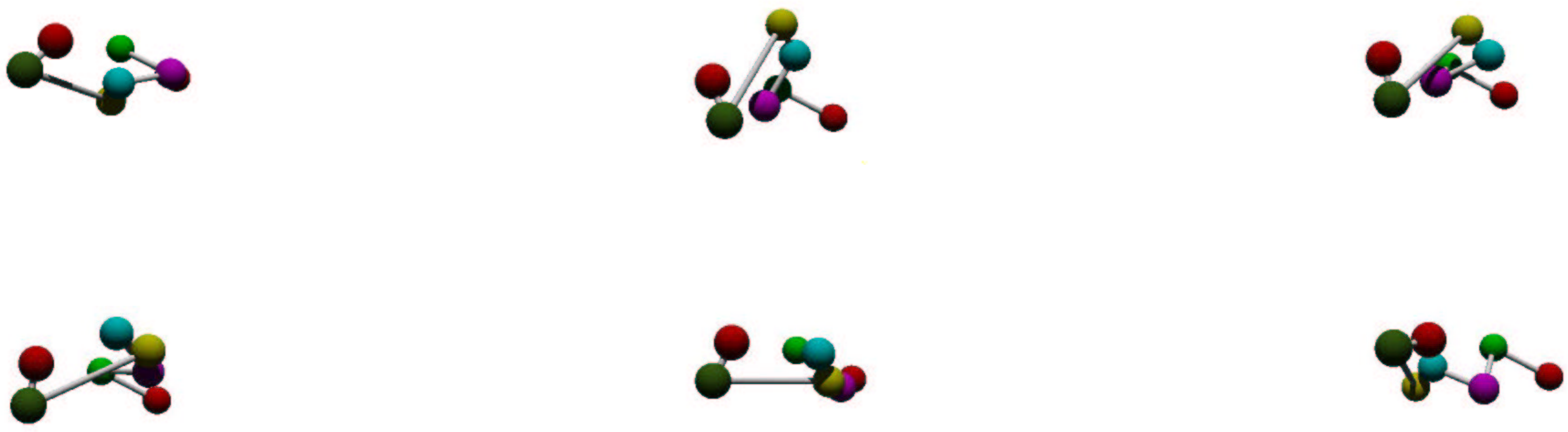
2. Solve $\mathtt{S}(x_n)^t \mathbf{w} = 0$.

   - Either by solving $\det(\mathtt{S}(x_n)) = 0$ and by deducing the corresponding $\mathbf{w}$.
   - or by reducing it to an eigenproblem:
$$(\mathtt{S}_d^t\, x_n^d + \mathtt{S}_{d-1}^t x_n^{d-1} + \cdots + \mathtt{S}_0^t)\, \mathbf{w} = 0$$
   or
$$\left(\begin{bmatrix} 0 & \mathbb{I} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \mathbf{0} & \mathbb{I} \\ \mathtt{S}_0^t & \cdots & \mathtt{S}_{d-2}^t & \mathtt{S}_{d-1}^t \end{bmatrix} - x_n \begin{bmatrix} \mathtt{S}_d^t & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathtt{S}_d^t & 0 \\ 0 & \cdots & 0 & \mathtt{S}_d^t \end{bmatrix} \right) \underline{\mathbf{w}} = 0,$$

3. Deduce the other coordinates of the roots from $\mathbf{w}$.

Conformation of molecules

Problem: Compute the possible conformations of a molecule when the position and orientation of the extremity is fixed.

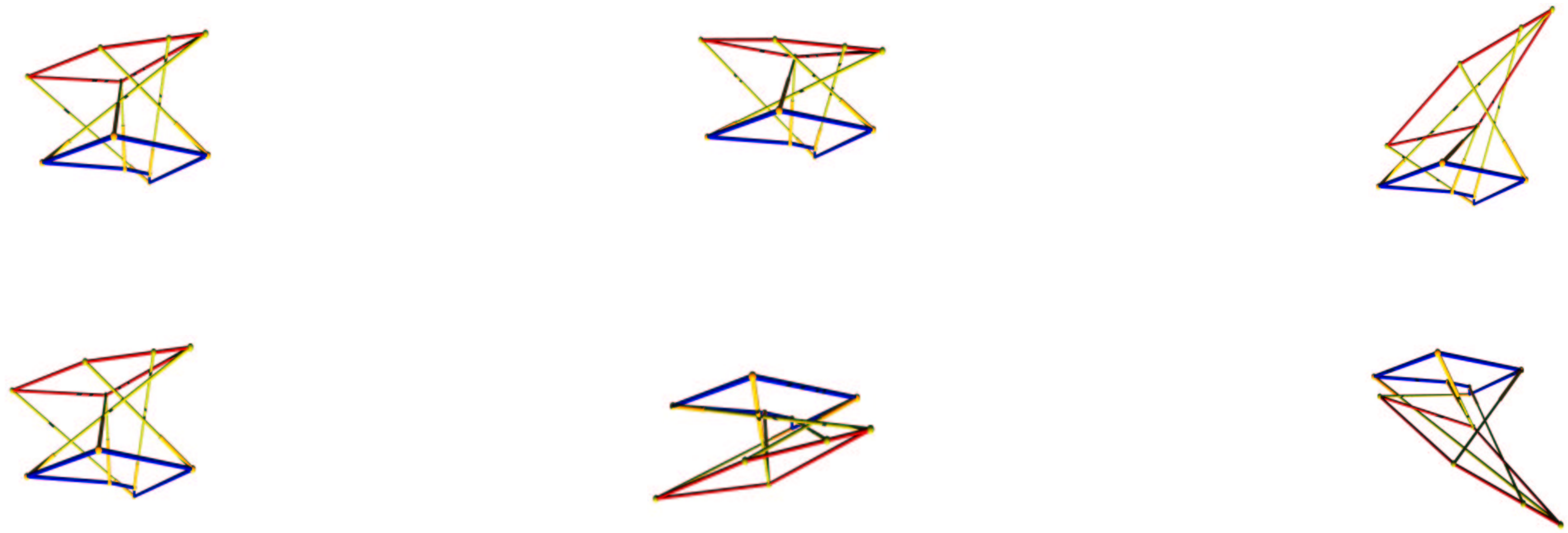


Error: $\|A_1 \circ \cdots \circ A_6 - H\| < 10^{-6}$ **Time:** $0.090s$

Joint work with O. Ruatta

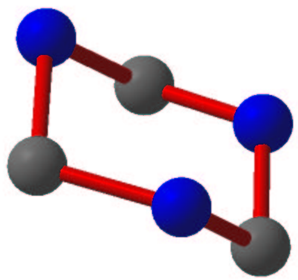
Direct kinematic problem of a parallel robot:

Problem: Compute the position of the platform for fixed lengths of the arms.



Error: $| \| R\, Y_i + T - X_i \|^2 - d_i^2 | < 10^{-6}$ **Time:** $0.5s$

The cyclohexan

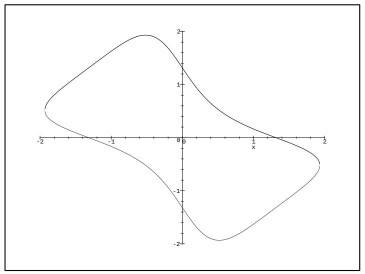


$$f_1 = -\frac{\sqrt{3}}{2} + \frac{1}{2}\,t_2^2 + \frac{1}{2}\,t_3^2 + 2\,t_2 t_3 + \frac{\sqrt{3}}{2} t_2^2 t_3^2 \quad = \quad 0$$

$$f_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}\,t_1^2 + \frac{1}{2}\,t_3^2 + 2\,t_1 t_3 + \frac{\sqrt{3}}{2} t_1^2 t_3^2 \quad = \quad 0$$

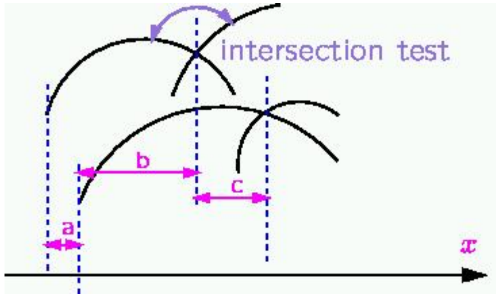
$$f_3 = -\frac{\sqrt{3}}{2} + \frac{1}{2}\,t_1^2 + \frac{1}{2}\,t_2^2 + 2\,t_1 t_2 + \frac{\sqrt{3}}{2} t_1^2 t_2^2 \quad = \quad 0$$

A curve of degree 4 $\quad + \, 2$ isolated points $+ \, 6$ embedded points

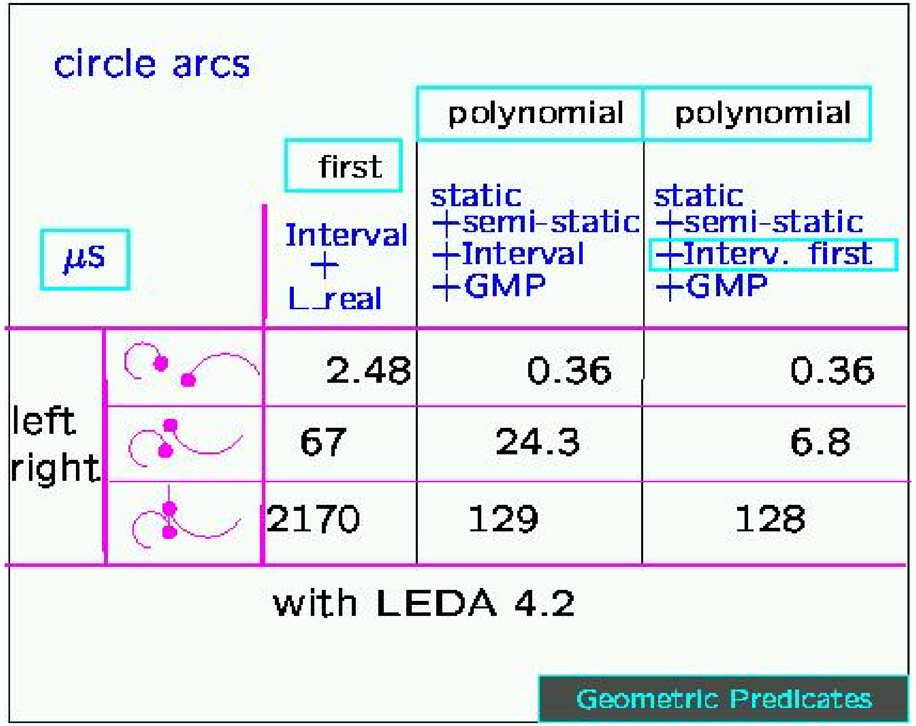


t_1	t_2	t_3
0.5176444559	0.5176444559	0.5176444563
-0.5176444567	-0.5176444567	-0.5176444555
0.5176444559	-1.931851652	0.5176444563
-0.5176444567	1.931851652	-0.5176444555
1.931851652	-0.5176444567	-0.5176444555
-1.931851652	0.5176444559	0.5176444563
0.5176444561	0.5176444561	-1.931851652
-0.5176444561	- 0.5176444561	1.931851652

Predicates



- Resultant formulations, in terms of a translation parameter.
- Sign of polynomials, of degree atmost 12.
- Filtering technics.



circle arcs

μs		first Interval + L_real	polynomial static +semi-static +Interval +GMP	polynomial static +semi-static +Interv. first +GMP
left right		2.48	0.36	0.36
		67	24.3	6.8
		2170	129	128

with LEDA 4.2

Geometric Predicates

Implicitisation

Compute the implicit equations of the surface, image of the map

$$\sigma : \mathbf{t} = (t_1, t_2) \in U \subset \mathbb{K}^2 \quad \mapsto \quad (\frac{f_1(\mathbf{t})}{f_0(\mathbf{t})}, \ldots, \frac{f_3(\mathbf{t})}{f_0(\mathbf{t})}) \in \mathbb{K}^3$$

ie. the (irreducible) polynomial $f(x, y, z)$ of minimal degree such that $f \circ \sigma = 0$.

□ Homogeneisation $\sigma^h : \bar{\mathbf{t}} = (t_0 : t_1 : t_2) \mapsto (f_0^h(\bar{\mathbf{t}}) : f_1^h(\bar{\mathbf{t}}) : f_2^h(\bar{\mathbf{t}}) : f_3^h(\bar{\mathbf{t}}))$ with $\deg(f_i^h) = d$.

□ Degree of $f = d^2 - \sum_{b \text{ base point}} \mu_b$.

⇨ Eliminate (t_1, t_2) between

$$F_1 = f_1(\mathbf{t}) - x f_0(\mathbf{t}), F_2 = f_2(\mathbf{t}) - y\, f_0(\mathbf{t}), F_3 = f_3(\mathbf{t}) - z\, f_0(\mathbf{t}).$$

Applications

⇨ Detection of self-intersection and singularities.

⇨ Compute the inverse image of a point on the surface.

⇨ Ray-tracing reduces to the eigenproblem $(\mathbb{S}^t(A) + t\,\mathbb{S}^t(B))\,\mathbf{w} = 0$.

⇨ Intersection of parametric curves and implicit surfaces reduce to the eigenproblem $(\mathbb{S}_d^t t^d + \cdots + \mathbb{S}_0^t)\,\mathbf{w} = 0$, or

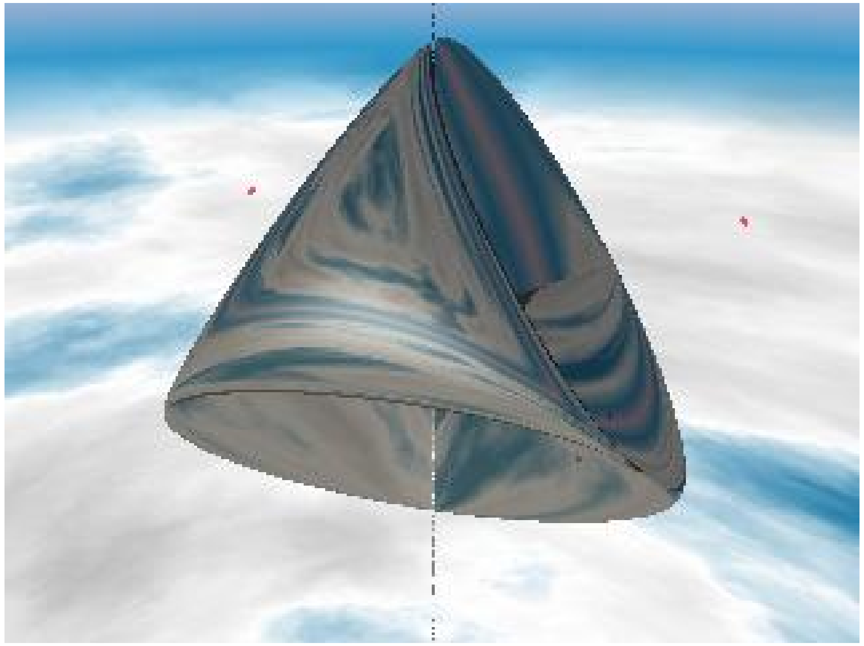
$$\left(\begin{bmatrix} 0 & \mathbb{I} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \mathbf{0} & \mathbb{I} \\ \mathbb{S}_0^t & \cdots & \mathbb{S}_{d-2}^t & \mathbb{S}_{d-1}^t \end{bmatrix} - t \begin{bmatrix} \mathbb{S}_d^t & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbb{S}_d^t & 0 \\ 0 & \cdots & 0 & \mathbb{S}_d^t \end{bmatrix} \right) \underline{\mathbf{w}} = 0,$$

⇨ Bounding volume using the dual surface equations.

Example of a Steiner surface

$$\sigma^h : \bar{\mathbf{t}} = (t_0 : t_1 : t_2) \mapsto (q_0(\bar{\mathbf{t}}) : q_1(\bar{\mathbf{t}}) : q_2(\bar{\mathbf{t}}) : q_3(\bar{\mathbf{t}}))$$

where the q_i are of degree 2, with no base points.



Resultant matrix:

- $T(\mathbf{t},\mathbf{y}) = \det(\partial_{t_0}\sigma^h, \partial_{t_1}\sigma^h, \partial_{t_2}\sigma^h, \mathbf{y})$.
- $\tau_i(\mathbf{y}) = \partial_{t_i}(T(\mathbf{t},\mathbf{y})$.
- $S(x,y,z) = [q_1(\mathbf{t}) - x_1\, q_0(\mathbf{t}), q_2(\mathbf{t}) - x_2\, q_0(\mathbf{t}), q_y(\mathbf{t}) - x_3\, q_0(\mathbf{t}), \tau_1(\mathbf{y}), \tau_2(\mathbf{y}), \tau_3(\mathbf{y})]$.

⇨ $S(x,y,z)$ is a 6×6 matrix.

⇨ $\det(S(x,y,z))$ is the implicit equation (of degree 4).

⇨ $\dim(\ker(S(a_0,b_0,c_0)))$ is the number of inverse image of (a_0,b_0,c_0).

⇨ If K is $r\times 6$ st. $Span(K) = \ker(S(a_0,b_0,c_0))$, the coordinnates of the inverse images can be computed from the eigenvectors of

$$(K_1 - \lambda\, K_0)\,\mathbf{w} = 0.$$

where K_i are $r\times r$ submatrices of K.

Computation of $f(x, y, z)$

☐ Compute a Gröbner basis $(F) + (w\, f_0 - 1)$ for any ordering and apply a change of ordering to get a linear relation of smaller degree between the $\mathbf{y}^{\alpha}$.

☐ Combine a Bezoutian part and a Sylvester part of the polynomials $f_i(\mathbf{t}) - y_i \lambda^d$ in degree $2\,d - 2$. Morley/Dixon Matrix [J91, AS01].

☐ Construction of the Bezoutian matrices of F_i and maximal minor [BEM00].

☐ Residual resultants as determinant of the Hilbert-Burch complex [B01].

☐ Moving lines or quadrics: $P(\mathbf{t}, x, y, z)$ with $P(\mathbf{t}, \sigma(\mathbf{t})) \equiv 0$ [CSC98].

☐ Use an inversion formula to express $\mathbf{t}$ in terms of $\mathbf{y}$ on the surface, in order to deduce its equation [PDSS'02,SE'02].

Experimentation

$$\begin{cases} h_1(s,t) & = & 3t + 3s^2t - t^3 \\ h_2(s,t) & = & 3s + 3st^2 - s^3 \\ h_3(s,t) & = & 3t^2 - 3s^2 \\ h_0(s,t) & = & 1 \end{cases}$$

$d = 3$, $\deg(f) = 9$, no base point, proper parametrisation.

Groebner	1.7s
Grobner imp.	0.690
Inversion Equation (no fact.) Equation (with fact.)	0.110s 0.410s 0.640s
Macaulay	0.479s
Morley	0.09s
Moving surfaces	1.6s
Bezout	0.071 (-)

by S. Bis with Maple, Magma on a Pentium III 933Mhz 512M

$$\begin{cases} h_1(s,t) &= 2s^3 + 13 + st^2 + 7ts^2 \\ h_2(s,t) &= 2t^3 + s^2 + 15 + 4s^2t + 6t + s^3 \\ h_3(s,t) &= 2t^2 + s + 17 + 9s^2t + 7t + s^2 \\ h_0(s,t) &= 2t^2 + s + 19 + 4s^2t + 7t + s^3 \end{cases}$$

$d = 3$, $\deg(f) = 9$, no base point, proper parametrisation.

Groebner	$>$ 3600s
Grobner imp.	10.690s
Inversion Equation (no fact.) Equation (with fact.)	$>$ 1500s ???? ????
Macaulay	0/0
Morley	1.510s
Moving surfaces	5.120s
Bezout	7.771s (0.001s)

$$\begin{cases} h_1(s,t) &= 32s^2t - 56st + 24t + 16s^3 - 120s^2 + 128s - 24 \\ h_2(s,t) &= 16t^3 - 16t^2 + 16st^2 + 48s^2t - 456st + 392t + 8s^2 + 384s - 392 \\ h_3(s,t) &= 64t^3 - 48t^2 + 48st^2 + 32s^2t - 1328st + 1232t + 16s^3 + 192s^2 + 1040s - 12 \\ h_0(s,t) &= -6 + 4s - 6st + 2s^2 + 6t \end{cases}$$

$d = 3$, $\deg(f) = 4$, base points, proper parametrisation.

Groebner	theoric failure
Grobner imp.	0.100s
Inversion Equation (no fact.) Equation (with fact.)	69s > 2200s 163s
Macaulay	theoric failure
Morley	0.250s (-)
Moving surfaces	theoric failure
Bezoutian	1.280s (-)
Residual resultant	0.42s (0.069s)

$$\begin{cases} h_1(s,t) &= s^3 + t^3 + 5st^2 + 2st - 1 \\ h_2(s,t) &= 2s^2 + 2t^3 + 3st - 2 \\ h_3(s,t) &= 5s^3 + 5t^2 + 3st^2 + 2s^2t + 5st - 5 \\ h_0(s,t) &= s^4 + t^4 - 1 \end{cases}$$

$d = 4$, $\deg(f) = 10$, with base points, proper parametrisation.

Groebner simple	theoric failure
Grobner imp.	38.250s
Inversion Equation (no fact.) Equation (with fact.)	$>$ 5000s ???? ????
Macaulay	theoric failure
Morley	7.919s (-)
Moving surfaces	theoric failure
Bezoutian	113.640s (0.010s)
Residual resultant	105.440s (0.189s)

by S. Bis with Maple, Magma on a Pentium III 933Mhz 512M

$$\begin{cases} h_1(s,t) &= 3t(t-1)^2 + (s-1)^3 + 3s \\ h_2(s,t) &= 3s(s-1)^2 + t^3 + 3t \\ h_3(s,t) &= -3s(s^2-5s+5)t^3 - 3(s^3+6s^2-9s+1)t^2 + t(6s^3+9s^2-18s+3) - 3s \\ h_0(s,t) &= 1 \end{cases}$$

$d = (3,3)$, $\deg(f) = 18$, *no base point*, proper parametrisation.

Groebner	theoric failure
Grobner imp.	399.55s
Inversion Equation (no fact.) Equation (with fact.)	$>$ 3600s ???? ????
Macaulay	theoric failure
Morley	330s (0.5500s)
Moving surfaces	465s
Bezout	146s (-)
Residual resultant	$>$ 3 h (0.445s)

Acknowledgements

- L. Busé (Residual resultant).
- M. Elkadi (Residue).
- I. Emeris (Toric resultant).
- S. Bis (Implicitisation).
- O. Ruatta (Implicitisation).
- J.P. Tecourt (Steiner surfaces).
- M. Teillaud (Geometric predicates).
- Ph. Trebuchet (Solver).

Softwares

- `multires`: `http://www-sop.inria.fr/galaad/logiciels/multires/`
- `synaps`: `http://www-sop.inria.fr/galaad/logiciels/synaps/`