An introduction to algebraic methods for solving polynomial equations

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Arrangement of surfaces

Constructions

- ☐ Intersection points of curves, surfaces.
- ☐ Approximation of curves of intersection.
- ☐ Offsets, Median of curves, surfaces.

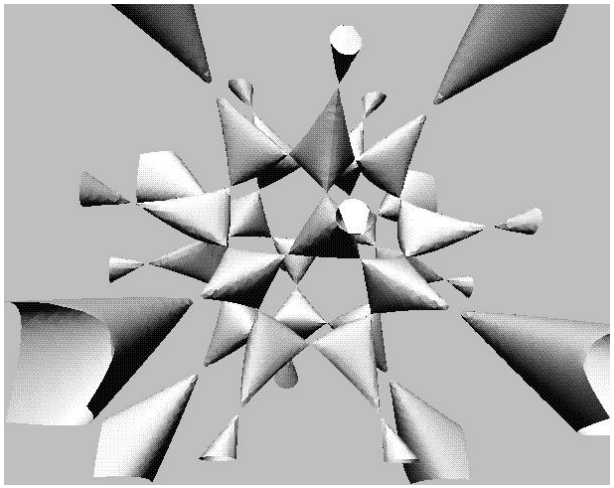
$\Rightarrow$ **fast solveurs, control on the error, refinement procedures.**

Predicats

- ☐ Sorting points on a curve.
- ☐ Connectivity. Topological coherence.
- ☐ Geometric predicats on the constructed points, curves, . . .

$\Rightarrow$ **fast tests (μs), filtering technics, polynomial formula/algebraic numbers. Algebraic manipulations, resultants.**

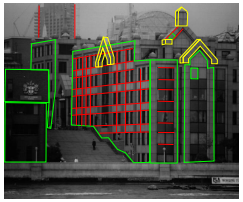
Drawing implicit surfaces

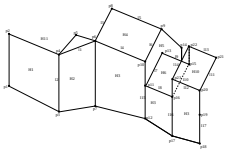


$$(8c+4)x^2y^2z^2 - c^4(x^4y^2 + y^4z^2 + x^2z^4) + c^2(x^2y^4 + y^2z^4 + x^4z^2) - \frac{2c+1}{4}(x^2+y^2+z^2-1)^2 = 0,\ c = \frac{1+\sqrt{5}}{2}$$

Modelisation from images

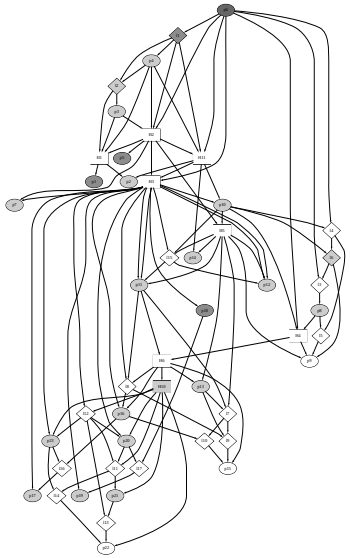
Extraction of points, lines, planes, . . .





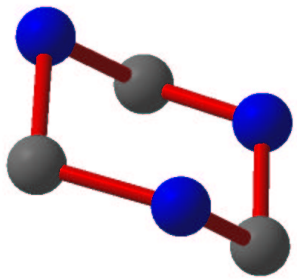
Symbolic treatement of the geometric constraints.

- **algebra**, rewritting, simplification.
- **proof**, **automatic** discovering of properties.



Numerical ajustment of the 3D model.

The cyclohexan

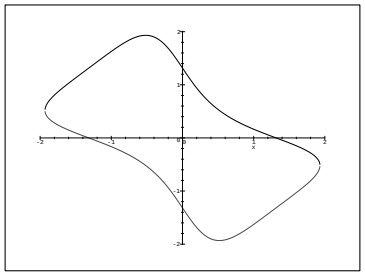


$$f_1 = -\frac{\sqrt{3}}{2} + \frac{1}{2}\,t_2^2 + \frac{1}{2}\,t_3^2 + 2\,t_2 t_3 + \frac{\sqrt{3}}{2} t_2^2 t_3^2 \quad = \quad 0$$

$$f_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}\,t_1^2 + \frac{1}{2}\,t_3^2 + 2\,t_1 t_3 + \frac{\sqrt{3}}{2} t_1^2 t_3^2 \quad = \quad 0$$

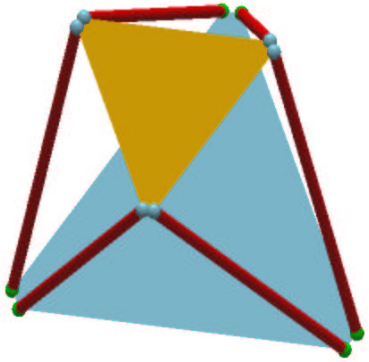
$$f_3 = -\frac{\sqrt{3}}{2} + \frac{1}{2}\,t_1^2 + \frac{1}{2}\,t_2^2 + 2\,t_1 t_2 + \frac{\sqrt{3}}{2} t_1^2 t_2^2 \quad = \quad 0$$

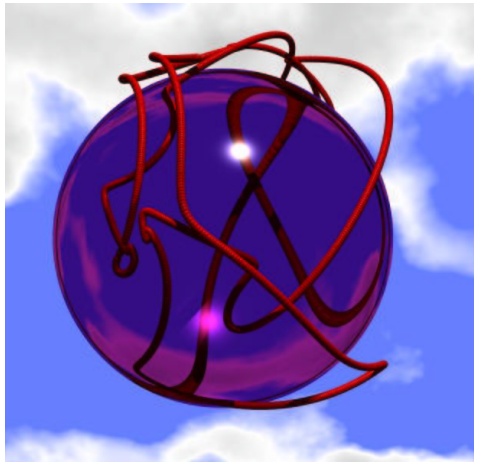
A curve of degree 4 + 2 isolated points + 6 embedded points



t_1	t_2	t_3
0.5176444559	0.5176444559	0.5176444563
-0.5176444567	-0.5176444567	-0.5176444555
0.5176444559	-1.931851652	0.5176444563
-0.5176444567	1.931851652	-0.5176444555
1.931851652	-0.5176444567	-0.5176444555
-1.931851652	0.5176444559	0.5176444563
0.5176444561	0.5176444561	-1.931851652
-0.5176444561	- 0.5176444561	1.931851652

A robotic problem





Equations: $\|R\,Y_i + T - X_i\|^2 - d_i^2 = 0, i = 1, \ldots, 6$,

$$R = \frac{1}{a^2+b^2+c^2+d^2}\begin{bmatrix} a^2-b^2-c^2+d^2 & 2\,ab-2\,cd & 2\,ac+2\,bd \\ 2\,ab+2\,cd & -a^2+b^2-c^2+d^2 & 2\,bc-2\,ad \\ 2\,ac-2\,bd & 2\,ad+2\,bc & -a^2-b^2+c^2+d^2 \end{bmatrix}, T = \begin{bmatrix} u/z \\ v/z \\ w/z \end{bmatrix}$$

Solutions: Generically 40 solutions: [RV93], [M94], [L93].
Fast and accurate solveurs: $\simeq$1s, error 10^{-6}.

Solving polynomial equations

Solving polynomial equations

The context:

- The equations $f_1 = \cdots = f_m = 0$, $f_i \in R = \mathbb{C}[x_1, \ldots, x_n]$.
- The solutions $\zeta \in \mathbb{C}^n$ st. $f_1(\zeta) = \cdots = f_m(\zeta) = 0$.

The problems:

Count/compute an approximation of all the (real) solutions (in a box).

The objectives:

- **Numerical stability.**
- **Certification et control.**
- **Efficiency via structure.**

Solvers

- **Analytic solvers:** exploit the value of f and its derivatives.

  Newton like methods, Minimisation methods, Weierstrass method.

- **Subdivision solvers:** use an exclusion criterion to isolate the roots.

  Taylor exclusion function, interval arithmetic, Descartes rule.

- **Algebraic solvers:** exploit the known relation between the unkowns.

  Gröbner basis, normal form computations. Reduction to univariate or eigenvalue problems.

- **Homotopic solvers:** deform a system with known roots into the system to solve.

  Projective, toric, flat, deformation.

- **Geometric solvers:** project the problem onto a smaller subspace.

  Resultant-based methods. Reduction to univariate or eigenvalue problems.

Analytic methods

Subdivision methods

A Univariate polynomial solver

□ Usual representation of a $f(x)$ of degree d: $f(x) = \sum_{i=0}^{d} a_i\, x^i$.

□ Bernstein basis: $f(x) = \sum_{i=0}^{d} b_i\, B_d^i(x)$, where $B_d^i(x) = \binom{d}{i} x^i (1-x)^{d-i}$.

$\mathbf{b} = [b_i]_{i=0,\ldots,d}$ are called the *control coefficients.*

Properties:

- $f(0) = b_0, f(1) = b_d$,
- $f'(x) = \sum_{i=0}^{d-1} \Delta(\mathbf{b})_i\, B_{d-1}^i(x)$ where $\Delta(\mathbf{b})_i = b_{i+1} - b_i$.

Proposition: **The number of sign changes $V(\mathbf{b})$ of $\mathbf{b} = [b_i]_{i=0,\ldots,d}$ bounds the number of real roots of f on $[0,1]$ and is equal to it modulo 2.**

If $V(\mathbf{b}) = 0$, the number of real root on $[0,1]$ is 0, and

If $V(\mathbf{b}) = 1$, the number of real root on $[0,1]$ is 1.

De Casteljau subdivision algorithm

$$b_i^0 = b_i, \;\; i = 0, \ldots, d,$$
$$b_i^r(t) = (1-t)\, b_i^{r-1}(t) + t\, b_{i+1}^{r-1}(t), \;\; i = 0, \ldots, d-r.$$

- The control coefficients $\mathbf{b}^-(t) = (b_0^0(t), b_0^1(t), \ldots, b_0^d(t))$ and $\mathbf{b}^+(t) = (b_0^d(t), b_1^{d-1}(t), \ldots, b_d^0(t))$ describe f on $[0,t]$ and $[t,1]$.

  A polynomial = a sequence of coefficients + an interval $[a,b]$

- For $t = \frac{1}{2}$, $b_i^r = \frac{1}{2}(b_i^{r-1} + b_{i+1}^{r-1})$.
- Number of arithmetic operations bounded by $\mathcal{O}(d^2)$, memory space $\mathcal{O}(d)$.
- Indeed, asymptotic complexity $\mathcal{O}(d \log(d))$.

Isolation algorithm

Algorithm: isolation of the roots of f on the interval $[a, b]$

INPUT: *A representation $(\mathbf{b}, [a, b])$ associate with f.*

- *Compute the number of sign changes $V(\mathbf{b})$.*
- *If $V(\mathbf{b}) > 1$ and its size is greater than ϵ, subdivide the representation into two subrepresentations $\mathbf{b}^-, \mathbf{b}^+$, corresponding to the two halves of the input interval and apply recursively the algorithm to them.*
- *If $V(\mathbf{b}) = 0$, remove the interval.*
- *If $V(\mathbf{b}) = 1$, the interval contains one root, that can be isolated within the precision ϵ.*

OUTPUT: *list of subintervals of $[a, b]$ containing exactly one real root of f.*

- Isolation and approximation of the roots on an interval.
- Multiple roots (and their multiplicity) computed within a precision ϵ
- All the real roots of $f(x)$, with $x := t/(1-t)$, $t \in [0, 1[$ and $x = -x$.
- With this change of variable, also called Uspensky algorithm.
- Natural extension to B-splines.

Theorem: [MVY01] Assume that $f(x) = 0$ has simple roots.

1. **An upper bound of the number of recursion steps of the** isolation **algorithm is**

$$l = \lceil \log_2 \left(\frac{5d}{2s} \right) \rceil.$$

where d is the degree of f and $s = \min_{\{x_i \neq x_j \text{ roots of f}\}} |x_i - x_j|$.

2. **An upper bound of the number of arithmetic operations of the procedure** isolation **is**

$$v = \frac{1}{2} d(d+1)\, r \left(\lceil \log_2 \left(\frac{5d}{2s} \right) \rceil - \log_2(r) + 4 \right),$$

where r is the number of sign changes of the sequence $\mathrm{b} = (b_k)_{0,\ldots,d}$.

Extension to highter dimension

Rectangular patches:

$$f(x,y) = \sum_{i=0}^{d_1} \sum_{j=0}^{d_2} b_{j,i} B^i_{d_1}(x) B^j_{d_2}(y).$$

associated with the box $[0,1] \times [0,1]$.

Triangular patches:

$$f(x,y) = \sum_{i+j+k=d} b_{i,j,k} \frac{d!}{i!j!k!} x^i y^j (1-x-y)^k.$$

associated with the representation on the 2d simplex.

Properties of rectangular patches

• We associate to f the **matrix of control coefficients** $\mathbf{b} = (b_{i,j})_{0 \le i \le d_1, 0 \le j \le d_2}$ and **the box** $[0,1] \times [0,1]$.

• The subsequences

$$\begin{cases} \mathbf{b}^S := (b_{0,0}, \ldots, b_{0,d_1}) \\ \mathbf{b}^N := (b_{d_2,0}, \ldots, b_{d_2,d_1}) \\ \mathbf{b}^E := (b_{0,0}, \ldots, b_{d_2,0}) \\ \mathbf{b}^W := (b_{0,d_1}, \ldots, b_{d_2,d_1}) \end{cases}$$

represents the function $f(x,0)$ (resp. $f(x,1)$, $f(0,y)$, $f(1,y)$ on $[0,1]$.

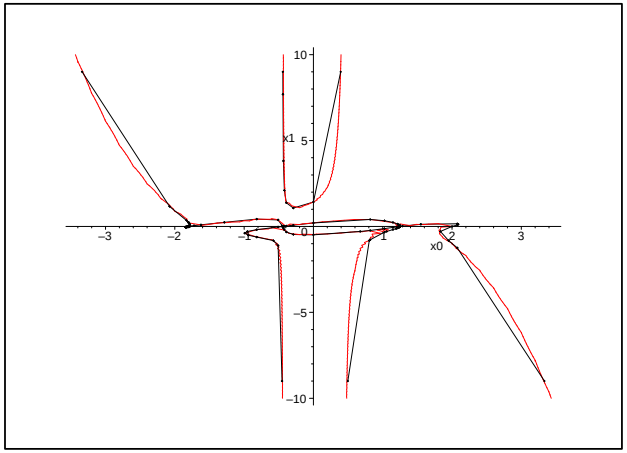
• **Subdivision** by row or by column, similar to the univariate case.

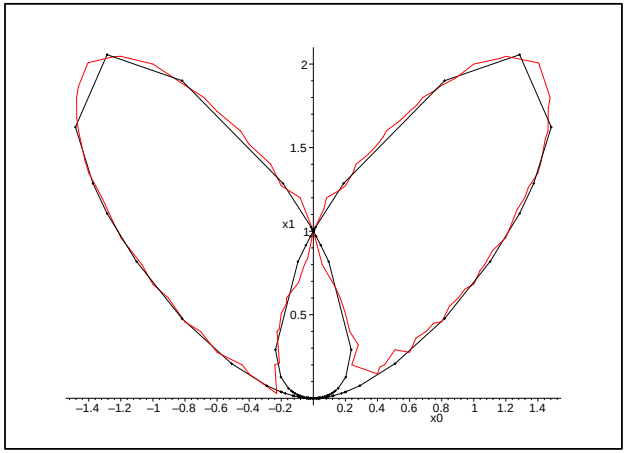
• Arithmetic complexity of a subdivision bounded by $\mathcal{O}(d^3)$ ($d = max(d_1, d_2)$), memory space $\mathcal{O}(d^2)$.

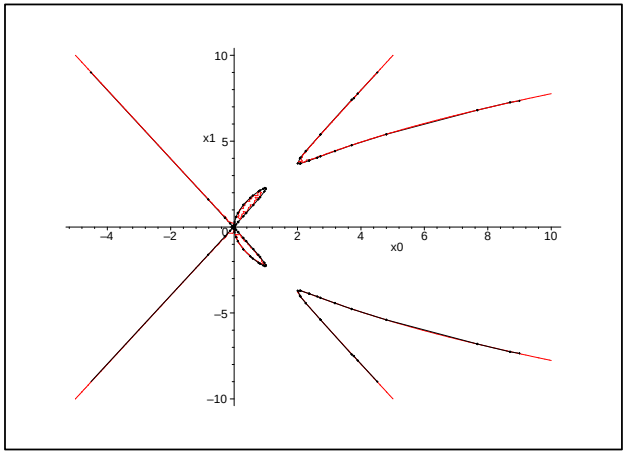
Applications (in progress)

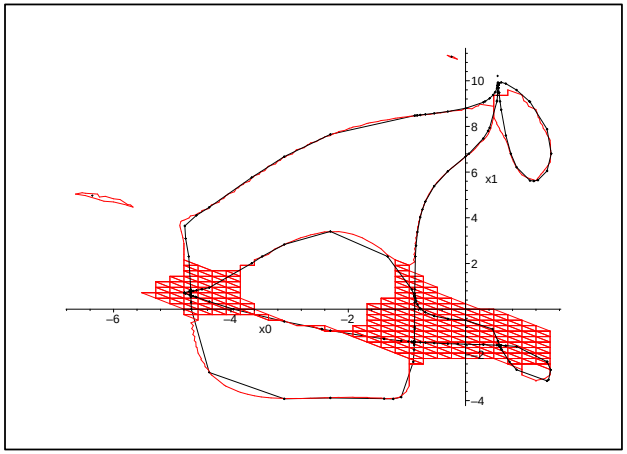
- Compute an **approximation** of the curve (resp. surface) within the precision ϵ, with good **geometric/numerical properties**.
- Computation of the topological degree.
- Isolation of complex roots, resp. real roots of several functions.
- Intersection of curves, surfaces.
- Arrangement of curves, surface patches.

(still in progress)


10
5
x1
−3
−2
−1
0
1
2
3
x0
−5
−10


2
1.5
x1
1
0.5
−1.4
−1.2
−1
−0.8
−0.6
−0.4
−0.2
0.2
0.4
0.6
0.8
1
1.2
1.4
x0


10
x1
5
−4
−2
0
2
4
6
8
10
x0
−5
−10


10
8
6
x1
4
2
−6
−4
−2
x0
−4

Algebraic methods

Ingredients

- Relations/equations/**constraints** satisfied by the **unknowns**.

- Quotient algebra. Computation modulo the constraints.

  Do as if the unknowns are known .

- Compte the roots, by reducing to **linear algebra problems**

  Duality, eigenvalue computation.

The quotient algebra $\mathcal{A}$

- The polynomial ring $R = \mathbb{K}[x_1, \ldots, x_n]$.
- The ideal $I = (f_1, \ldots, f_m) = \{\sum_i h_i f_i; h_i \in R\}$.
- The quotient algebra $\mathcal{A} = R/I$ of polynomials modulo I: $a \equiv a'$ iff $a - a' \in I$.

  (cf. polynomial functions on the set of solutions.)
- **How to represent and exploit effectively the structure of $\mathcal{A}$?**
  - ☐ A basis for $\mathcal{A}$.
  - ☐ The multiplicative tables.

The quotient algebra

Let I be the ideal of $R = \mathbb{K}[x_1, x_2]$ generated by

```
> f1 := 13*x[1]^2+8*x[1]*x[2]+4*x[2]^2-8*x[1]-8*x[2]+2:
> f2 := x[1]^2+x[1]*x[2]-x[1]-1/6:
```

The quotient ring $\mathcal{A} = \mathbb{K}[x_1, x_2]/I$ is a vector space of dimension 4. A basis of $\mathcal{A}$ is $1, x_1, x_2, x_1x_2$.We have

```
> expand(x[1]*x[1] -f2);
```

$$x_1^2 \equiv -x_1x_2 + x_1 + \frac{1}{6}.$$

```
> expand(x[1]*(x[1]*x[2])+ 1/9*x[1]*f1-(5/9+13/9*x[1]+4/9*x[2])*f2);
```

$$x_1^2x_2 \equiv -x_1x_2 + \frac{55}{54}\,x_1 + \frac{2}{27}\,x_2 + \frac{5}{54}.$$

More generally, any polynomial in $\mathbb{K}[x_1, x_2]$ can be reduced, modulo the polynomials f_1, f_2, to a linear combination of the monomials $1, x_1, x_2, x_1\,x_2$.

The univariate case

Consider $f = f_d\, x^d + \cdots + f_1 x + f_0$ $(f_d \neq 0)$, with simple roots $\zeta_1, \ldots, \zeta_d$.

- The idempotents are the Lagrange interpolation polynomials :
$$\mathbf{e}_i(x) = \prod_{j=1, j\neq i}^{d} \frac{x-\zeta_j}{\zeta_i - \zeta_j}.$$

□ $\mathbf{e}_i^2 \equiv \mathbf{e}_i$, $\mathbf{e}_i\, \mathbf{e}_j \equiv 0$, $i \neq j$, $\mathbf{e}_1 + \cdots + \mathbf{e}_n \equiv 1$.

□ $\mathcal{A} = \mathbb{C}[x]/(f) = \mathbb{K}\,\mathbf{e}_1 \oplus \cdots \oplus \mathbb{K}\,\mathbf{e}_d$; $\quad \forall a \in \mathcal{A}, a \equiv a(\zeta_1)\,\mathbf{e}_1 + \ldots + a(\zeta_d)\,\mathbf{e}_d$.

□ The dual basis of $(\mathbf{e}_i)$ is $(\mathbf{1}_{\zeta_i})$.

□ Another basis of $\mathcal{A}$ is $(1, x, \ldots, x^{d-1})$. The matrix of multiplication by x in this basis is

$$\mathtt{M}_x = \begin{pmatrix} 0 & \cdots & 0 & -\frac{f_0}{f_d} \\ 1 & \ddots & \vdots & \vdots \\ & \ddots & 0 & \vdots \\ 0 & & 1 & -\frac{f_{d-1}}{f_d} \end{pmatrix}.$$

The structure of $\mathcal{A} = R/I$

Theorem: $\mathcal{A}$ **finite dimensional vector space iff** $\mathcal{Z}_{\mathbb{C}}(I) = \{\zeta_1, \ldots, \zeta_d\}$ **is finite.**

$$\mathcal{A} = \mathbb{C}[x_1, \ldots, x_n]/I = \mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_d,$$

with

- $\mathcal{A}_i = \mathbf{e}_i\,\mathcal{A}$,
- $\mathbf{e}_i^2 = \mathbf{e}_i$, $\mathbf{e}_i\,\mathbf{e}_j = 0$ if $i \neq j$,
- $\mathbf{e}_1 + \cdots + \mathbf{e}_d = 1$.

□ Primary decomposition of : $I = Q_1 \cap \cdots \cap Q_d$ where Q_i is $\mathbf{m}_{\zeta_i}$-primary.
□ $\mathcal{A}_i \sim R/Q_i$.

Duality

- **The dual de R** is $\widehat{R} = \{\Lambda : R \to \mathbb{K}, \text{ linear } \}$.
- Let $(\mathbf{d}^\alpha)_{\alpha\in\mathbb{N}^n}$ be the dual basis of $(\mathbf{x}^\alpha)_{\alpha\in\mathbb{N}^n}$.
$$\Lambda = \sum_{\alpha\in\mathbb{N}^n} \Lambda(\mathbf{x}^\alpha)\mathbf{d}^\alpha \in \mathbb{K}[[\mathbf{d}_1, \ldots, \mathbf{d}_n]].$$

Example: $\mathbf{1}_\zeta : p \mapsto p(\zeta)$ is of the form $\mathbf{1}_\alpha = \sum_{\alpha\in\mathbb{N}^n} \zeta^\alpha \mathbf{d}^\alpha = \frac{1}{\prod_{i=1}^n (1-\zeta_i \mathbf{d}_i)}$.

- **The R-module structure:** $\forall a \in R, \forall \Lambda \in \widehat{R}$, $a \cdot \Lambda : b \mapsto a \cdot \Lambda(b) = \Lambda(ab)$.

Example: $x_1 \cdot \mathbf{d}_1^{\alpha_1}\mathbf{d}^{\alpha_2}\cdots\mathbf{d}^{\alpha_n} = \mathbf{d}_1^{\alpha_1-1}\mathbf{d}^{\alpha_2}\cdots\mathbf{d}^{\alpha_n}$ if $\alpha_1 > 0$ and 0 otherwise.

- **The dual of $\mathcal{A} = R/I$** is $\widehat{\mathcal{A}} = \{\Lambda : R \to \mathbb{K} \text{ st. } \Lambda(I) = 0\} = I^\perp$.

Example:

For $\zeta \in \mathcal{Z}(I)$, $\mathbf{1}_\zeta : a \mapsto a(\zeta)$.

$a \mapsto$ the coefficient of $\mathbf{x}^\alpha$ in the normal form of a.

The dual

The following computation gives the value of the linear form $1+\mathbf{d}_1+\mathbf{d}_1\mathbf{d}_2+{\mathbf{d}_3}^2$ on the polynomial $1+x_1+x_1x_2$:

```
> apply((1+d[1]+d[1]*d[2]+d[3]^2),(1+x[1]+x[1]*x[2]));
```

$$3$$

Let us now illustrate the structure of *module*:

```
> (1+x[1]+x[1]*x[2]) &.(1+d[1]+d[1]*d[2]+d[3]^2);
```

$$3+\mathbf{d}_1+\mathbf{d}_1\mathbf{d}_2+{\mathbf{d}_3}^2+\mathbf{d}_2$$

We check that the constant term of this expansion is the value of the linear form $1+\mathbf{d}_1+\mathbf{d}_1\mathbf{d}_2+{\mathbf{d}_3}^2$ at the polynomial $1+x_1+x_1x_2$.

Multiplication operators

We assume that $\mathcal{Z}(I) = \{\zeta_1, \ldots, \zeta_d\} \Leftrightarrow \mathcal{A}$ of finite dimension D over $\mathbb{K}$.

$$\begin{array}{rcl} M_a : \mathcal{A} & \to & \mathcal{A} \\ u & \mapsto & a\,u \end{array} \qquad \begin{array}{rcl} M_a^{\mathtt{t}} : \widehat{\mathcal{A}} & \to & \widehat{\mathcal{A}} \\ \Lambda & \mapsto & a \cdot \Lambda = \Lambda \circ M_a \end{array}$$

Theorem:

- **The eigenvalues of M_a are $\{a(\zeta_1), \ldots, a(\zeta_d)\}$.**
- **The eigenvectors of all $(M_a^{\mathtt{t}})_{a\in\mathcal{A}}$ are (up to a scalar) $\mathbf{1}_{\zeta_i} : p \mapsto p(\zeta_i)$.**

Theorem: **In a basis of $\mathcal{A}$, all the matrices M_a $(a \in \mathcal{A})$ are of the form**

$$\mathtt{M}_a = \begin{bmatrix} \mathtt{N}_a^1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathtt{N}_a^d \end{bmatrix} \text{ with } \mathtt{N}_a^i = \begin{bmatrix} a(\zeta_i) & & \star \\ & \ddots & \\ \mathbf{0} & & a(\zeta_i) \end{bmatrix}$$

Corollary: **(Chow form)**
$\Delta(\mathbf{u}) = \det(u_0 + u_1\,\mathtt{M}_{x_1} + \cdots + u_n\,\mathtt{M}_{x_n}) = \prod_{\zeta\in\mathcal{Z}(I)}(u_0 + u_1\zeta_1 + \cdots + u_n\zeta_n)^{\mu_\zeta}$.

Multiplication operators

Let us compute the matrix of multiplication by x_1 in the basis $(1, x_1, x_2, x_1x_2)$ of $\mathcal{A} = \mathbb{K}[x_1, x_2]/(f_1, f_2)$, where f_1, f_2 are the polynomials. We multiply these monomials by x_1 and reduce them to a normal form. According to the computations of the previous example, we have:

$$1 \times x_1 \equiv x_1,\ x_1 \times x_1 \equiv -x_1x_2 + x_1 + \frac{1}{6},$$
$$x_2 \times x_1 \equiv x_1x_2,\ x_1x_2 \times x_1 \equiv -x_1x_2 + \frac{55}{54}\,x_1 + \frac{2}{27}\,x_2 + \frac{5}{54}.$$

```
> M1 := matrixof([x[1],  x[1]*x[1] -f2,  x[1]*x[2],
>     x[1]*(x[1]*x[2])+ 1/9*x[1]*f1-(5/9+13/9*x[1]+4/9*x[2])*f2],
>     [[1,x[1],x[2],x[1]*x[2]]]);
```

$$M_1 = \begin{bmatrix} 0 & \frac{1}{6} & 0 & \frac{5}{54} \\ 1 & 1 & 0 & \frac{55}{54} \\ 0 & 0 & 0 & \frac{2}{27} \\ 0 & -1 & 1 & -1 \end{bmatrix}.$$

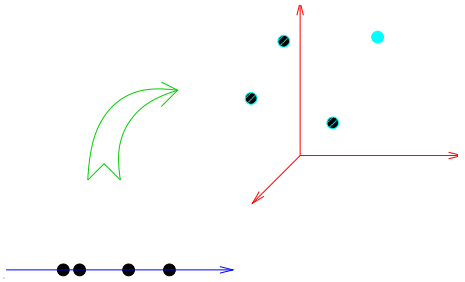
We compute the eigenvalues, their multiplicity, and the corresponding normalised eigenvector of the transposed of the matrix of multiplication by x_1:

```
> neigenvects(transpose(M1),1);
```

$$[-\frac{1}{3}, 2, \left\{[1, -\frac{1}{3}, \frac{5}{6}, -\frac{5}{18}]\right\}], [\frac{1}{3}, 2, \left\{[1, \frac{1}{3}, \frac{7}{6}, \frac{7}{18}]\right\}].$$

As the basis chosen for the computation is $(1, x_1, x_2, x_1x_2)$, the previous theorem tells us that the solutions of the system can be read off, from the 2^{nd} and the 3^{rd} coordinates of the normalised eigenvectors: $\zeta_1 = (-\frac{1}{3}, \frac{5}{6})$ and $\zeta_2 = (\frac{1}{3}, \frac{7}{6})$. Moreover, the 4^{th} coordinate of these vectors is the product of the 2^{nd} by the 3^{rd} coordinates.

Rational Univariate Representation of the roots



Algorithm: Rational Univariate Representation.

1. *Compute a* multiple *of the Chow form* $\Delta(\mathbf{u})$ *and its square free part* $d(\mathbf{u})$.
2. *Choose a generic* $t \in \mathbb{K}^{n+1}$ *and compute the first coefficients of*

$$d(t+u) = d_0(u_0) + u_1\, d_1(u_0) + \cdots + u_n\, d_n(u_0) + \cdots$$

3. *A non minimal rational univariate representation of the roots is given by* $\zeta_1 = \frac{d_1(u_0)}{d_0'(u_0)}$, $\ldots$, $\zeta_n = \frac{d_n(u_0)}{d_0'(u_0)}$, $d_0(u_0) = 0$.
4. *Factorize* $d_0(u_0)$ *and keep the good factors for a minimal representation.*

Remark: t is generic iff $gcd(d_0(u_0),\, d_0'(u_0)) = 1$.

Chow form

We compute the Chow form of the variety $I = (f_1, f_2)$, using the matrices of multiplication by x_1 and x_2, computed previously.

```
> factor(det(u[0]+ u[1]*M1+ u[2]*M2));
```

$$\left(u_0 + \frac{1}{3}\,u_1 + \frac{7}{6}\,u_2\right)^2 \left(u_0 - \frac{1}{3}\,u_1 + \frac{5}{6}\,u_2\right)^2$$

We check that it is a product of linear forms, whose coefficients yield the roots $\zeta_1 = (-\frac{1}{3}, \frac{5}{6})$ and $\zeta_2 = (\frac{1}{3}, \frac{7}{6})$. The exponents yield the multiplicity of the roots (here 2). As here the roots are rational, we can easily factorise this polynomial as a product of linear forms. But usually, this factorisation is possible only on an algebraic extension of the coefficient field.

From the previous Chow form, we deduce:

$$\xi_1 = -\frac{1}{6\,(1+u_0)},\ \xi_2 = \frac{11+12\,u_0}{12\,(1+u_0)},\ \left(u_0 + \frac{3}{2}\right)\left(u_0 + \frac{1}{2}\right) = 0.$$

which reduces to the constant representations

$$\begin{cases} u_0 = -3/2, x_1 = 1/3, x_2 = 5/6 \\ u_0 = -1/2, x_1 = -1/3, x_2 = 7/6 \end{cases}$$

Linear and quadratic forms

Let Λ be the linear form

$$\Lambda = 2 \times \mathbf{1}_{(-1/3,5/6)} + 2 \times \mathbf{1}_{(1/3,7/6)}.$$

We check that $(-1/3, 5/6)$ and $(1/3, 7/6)$ are in $\mathcal{Z}(f_1, f_2)$. The matrix of Q_Λ in the basis $\{1, x_1, x_2, x_1\, x_2\}$ of $\mathcal{A}$ is

$$[Q_\Lambda] = \begin{bmatrix} \Lambda(1) & \Lambda(x_1) & \Lambda(x_2) & \Lambda(x_1\, x_2) \\ \Lambda(x_1) & \Lambda(x_1^2) & \Lambda(x_1 x_2) & \Lambda(x_1^2\, x_2) \\ \Lambda(x_2) & \Lambda(x_1\, x_2) & \Lambda(x_2^2) & \Lambda(x_1\, x_2^2) \\ \Lambda(x_1\, x_2) & \Lambda(x_1^2\, x_2) & \Lambda(x_1 x_2^2) & \Lambda(x_1^2 x_2^2) \end{bmatrix} = \begin{bmatrix} 4 & 0 & 4 & \frac{2}{9} \\ 0 & \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ 4 & \frac{2}{9} & \frac{37}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9} & \frac{37}{81} \end{bmatrix}$$

Trace

$$\mathrm{Tr}(x_1) := trace(M_{x_1}) = trace\left(\begin{bmatrix} 0 & \frac{1}{6} & 0 & \frac{5}{54} \\ 1 & 1 & 0 & \frac{55}{54} \\ 0 & 0 & 0 & \frac{2}{27} \\ 0 & -1 & 1 & -1 \end{bmatrix}\right) = 0$$

By a direct computation, we get $Tr(1) = 4$, $Tr(x_1) = 0$, $Tr(x_2) = 4, Tr(x_1x_2) = \frac{2}{9}$.

By by using the transposed operators $\mathtt{M}_{x_i}^{\mathtt{t}}$ we get:

```
> T0  := evalm([4,0,4,2/9]):
> T1  := evalm(transpose(M1)&*T0):  T2:= evalm(transpose(M2)&*T0):
> T11 := evalm(transpose(M1)&*T1): T12:= evalm(transpose(M2)&*T1):
> T112:= evalm(transpose(M2)&*T11):
> Q1  := matrix(4,4,[T0,T1,T2,T12]);
> Qx1 := matrix(4,4,[T1,T11,T12,T112]);
```

$$Q_1 = \begin{pmatrix} 4 & 0 & 4 & \frac{2}{9} \\ 0 & \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ 4 & \frac{2}{9} & \frac{37}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9} & \frac{37}{81} \end{pmatrix}, \quad Q_{x_1} = \begin{pmatrix} 0 & \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{4}{9} & 0 & \frac{4}{9} & \frac{2}{81} \\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9} & \frac{37}{81} \\ \frac{4}{9} & \frac{2}{81} & \frac{37}{81} & \frac{4}{81} \end{pmatrix}$$

Real roots

Let $f_i \in \mathbb{R}[\mathbf{x}], i = 1, \ldots, m$ and $\mathcal{Z}_{\mathbb{C}} = \{\zeta \in \mathbb{C}^n, f_1(\zeta) = \cdots = f_m(\zeta) = 0\}$.

Consider the **linear form**:

$$Tr : a \mapsto trace(M_a) = \sum_{\zeta \in \mathcal{Z}} \mu_\zeta \, a(\zeta)$$

and the associated **quadratic form**:

$$Q_h : (a, b) \mapsto Tr(h\, a\, b).$$

Its matrix in the basis $(\mathbf{x}^\alpha)_{\alpha \in E}$ is $\left(Tr(h\, \mathbf{x}^{\alpha+\beta})\right)_{\alpha,\beta \in E}$.

Theorem: (Hermite) Let $h \in \mathbb{R}[\mathbf{x}]$.

1. **The rank of Q_h is the number of (complex) roots $\zeta \in \mathcal{Z}_{\mathbb{C}}$ st. $h(\zeta) \neq 0$.**

2. **The signature of Q_h is**
   $\#\{\, \zeta$ real root such that $h(\zeta) > 0 \,\} - \#\{\, \zeta$ real root such that $h(\zeta) < 0 \,\}$.

Example (continued)

$$Q_1 = \begin{pmatrix} 4 & 0 & 4 & \frac{2}{9} \\ 0 & \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ 4 & \frac{2}{9} & \frac{37}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9} & \frac{37}{81} \end{pmatrix}, \quad Q_{x_1} = \begin{pmatrix} 0 & \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{4}{9} & 0 & \frac{4}{9} & \frac{2}{81} \\ \frac{2}{9} & \frac{4}{9} & \frac{4}{9} & \frac{37}{81} \\ \frac{4}{9} & \frac{2}{81} & \frac{37}{81} & \frac{4}{81} \end{pmatrix}$$

The rank and the signature of the quadratic forms Q_1, Q_{x_1} are

```
> rank(Q1), signature(Q1), rank(Qx1), signature(Qx1);
```

$$2, \quad [2,0], \quad 2, \quad [1,1],$$

which tell us (without computing these roots) that there are 2 real roots, one with $x_1 < 0$ and one with $x_1 > 0$.

Computing the quotient structure

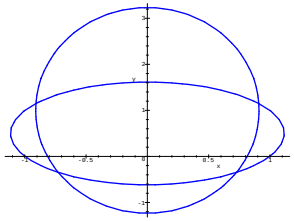
Gröbner basis methods

- A total **monomial ordering** $\prec$, compatible with the monomial multiplication.
- The generators $f_i = m - m_1 - m_2 - \cdots$ of I are seen as rewriting rules
$$m \to m_1 + m_2 + \cdots.$$
- The Gröbner basis computation is a **completion** procedure st.
$$f \in I \text{ iff } f \xrightarrow{*} 0.$$
- A **basis** of $\mathcal{A}$ is the set of monomials which are not divisible by the leading monomials of the rewriting rules.
- The **multiplication table** M_a:
  - multiplication of the monomial basis by a.
  - normalisation.

GB are going boink, don't they ?

A system:

$$\begin{cases} p_1 := a\,x_1^2 + b\,x_2^2 + l_1(x_1, x_2) \\ p_2 := c\,x_1^2 + d\,x_2^2 + l_2(x_1, x_2) \end{cases}$$

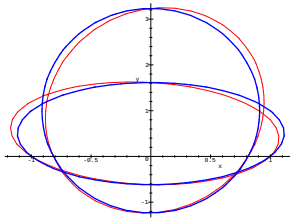


Basis of $\mathcal{A} = \mathbb{K}[\mathbf{x}_1, x_2]/(p_1, p_2)$:

$(1, x_1, x_2, x_1\,x_2)$.

A small perturbation:

$$\begin{cases} \tilde{p}_1 := p_1 + \epsilon_1\,x_1x_2 \\ \tilde{p}_2 := p_2 + \epsilon_2\,x_1x_2 \end{cases}$$



Basis of $\tilde{\mathcal{A}} = \mathbb{K}[x_1, x_2]/(\tilde{p}_1, \tilde{p}_2)$:

$(1, x_1, x_2, x_2^2)$.

Catastroph !

- A new set of monomials for the basis.
- Big coefficients (in $\frac{1}{\epsilon_i}$) appear.

Perturbation of Grobner basis

```
> f1 := x[1]^2+x[2]^2 -x[1]+x[2]-2;
> f2 := x[1]^2-x[2]^2 + 2*x[2]-3;
> gbasis([f1,f2],tdeg(x[1],x[2]));
```

$$[2\,{x_2}^2 - x_1 - x_2 + 1, 2\,{x_1}^2 - x_1 + 3\,x_2 - 5]$$

Leading monomials: x_1^2, x_2^2. Basis of $\mathcal{A}$: $\{1, x_1, x_2, x_1\,x_2\}$

A small perturbation:

```
> gbasis([f1,f2+1./10000000*x[1]*x[2]],tdeg(x[1],x[2]));
```

$$\begin{aligned}[0.0000001\,x_1x_2 - 2\,{x_2}^2 + x_1 + x_2 - 1, {x_1}^2 + {x_2}^2 - x_1 + x_2 - 2,\\ {x_2}^3 - 10000000.999999999999950000000000000125\,{x_2}^2\\ +5000000.250000012499999374999968750001562500078125\,x_1\\ +5000000.750000037499993124999906250017187500234375\,x_2\\ -5000000.250000062499999374999843750015625003906250\end{aligned}$$

Leading monomials: $x_1\,x_2, x_1^2, x_2^3$. Basis of $\mathcal{A}$ is $\{1, x_1, x_2, x_2^2\}$.

What we would like to have:

- The basis $(1, x_1, x_2, x_1\,x_2)$
- Rewriting rules of the form:

$$
\begin{aligned}
&x_1^2 \equiv -0.00000005\,x_1x_2 + 1/2\,x_1 - 3/2\,x_2 + 5/2\\
&x_2^2 \equiv +0.00000005\,x_1x_2 + 1/2\,x_1 + 1/2\,x_2 - 1/2\\
&x_2{x_1}^2 \equiv 0.49999999\,x_1x_2 - 0.74999998\,x_1 + 1.75000003\,x_2 + 0.74999994,\\
&x_1{x_2}^2 \equiv 0.49999999\,x_1x_2 - 0.25000004\,x_1 - 0.74999991\,x_2 + 1.25000004]
\end{aligned}
$$

This set of relations yields directly the matrices of multiplication by x_1, x_2 in $\mathcal{A}$.

Equations with approximate coefficients

- We consider the **neighbourhood** of a given system.
- Family of systems depending on parameters, of the same "shape".
- Around a **regular** value of the parameters,
  - **continuity** of the solution set.
  - **continuity** of the algebraic structure.
- At a **singular** value of the parameters, all sort of bad things may happen.

How to proceed ?

- Analyse the class of systems that we have to solve.
- Apply tuned methods for generic systems of this class.

A new normal form criterion

Proposition: **If B is a basis of $\mathcal{A}$ for some regular value of the parameters, it will be the case in a neightborhood.**

□ We fix the vector space $\langle B \rangle$ on which, we want to normalize. polynomials

□ $B^+ = B \cup x_1 B \cup \cdots \cup x_n B$

Hypothesis:

• B is is a set of monomials of $R = \mathbb{K}[\mathbf{x}]$, connected to 1: $\forall m = x_{i_1} \cdots x_{i_k} \in B$, $\forall l = 0, \ldots, k$, $x_{i_1} \cdots x_{i_l} \in B$.

• A projection $N : \langle B^+ \rangle \to \langle B \rangle$, with $N^2 = N$.

Question: Is N a normal form modulo $I = (Ker(N))$?

Theorem: **Let $M_i : \langle B \rangle \to \langle B \rangle$ such that $M_i(b) = N(x_i b)$.**
$M_i \circ M_j = M_j \circ M_i, i, j = 1, \ldots, n \Leftrightarrow B$ **basis of** $\mathcal{A} = R/I$.

Algorithm of normal form

Algorithm: Normal form for $\mathcal{A}$.

INPUT: *Let $f_1, \ldots, f_m \in R$, $I = (f_1, \ldots, f_m)$ and assume that $\mathcal{A} = R/I$ is zero-dimensional. Let L be a finite vector space of R containing $f_1, \ldots, f_m$ and connected to 1.*

(1) $K_0 = \langle f_1, \ldots, f_m \rangle$;

(2) While $K_n \neq K_{n-1}$, compute $K_{n+1} = K_n^+ \cap L$, replace n by $n+1$;

(3) Compute a supplementary vector space B of $K_*(= K_{n_0})$, which is connected to 1;

(4) If $B^+ \not\subset L$, replace L by L^+ and go to step (2). Otherwise stop.

OUTPUT: *A basis of K_*.*

- The projection for B^+ along K^* on B is a normal form.
- Linear algebra on vector spaces of polynomials (sparse matrices, superlu).
- Applications: Gröbner basis as a special case, Laurent polynomials, local rings

. . .

Timing for square systems (normal form + eigenvectors):

Degree	Nb Var	Nb Sol	Machine	Coefficients	Time	$Max(\|f_i\|)$
2	3	8	ultra1	[-1,1]	0.05(0.02)s	1e-11
2	3	8	ultra1	[-1000,1000]	0.05(0.02)s	1e-10
2	3	8	PII933	[-1,1]	"0.01"s	1e-11
2	3	8	PII933	[-1000,1000]	"0.01"s	1e-10
3	3	27	PIII933	[-1,1]	0.02(0.01)s	1e-11
3	3	27	PIII933	[-1000,1000]	0.02s	1e-7
3	3	27	ultra1	[-1,1]	0.22(0.04)s	1e-11
3	3	27	ultra1	[-1000,1000]	0.22(0.04)s	1e-7
4	4	256	ultra1	[-1,1]	186.36(1.31)s	1e-3
4	5	1024	PIII933	[-1,1]	–(0.15)s	–

Time: mean over 10 runs. "": most of the time spent for disk access etc...

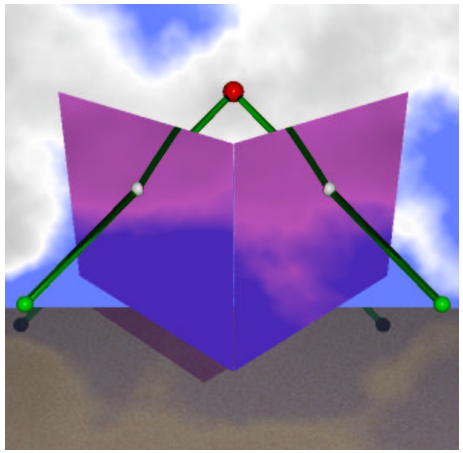
Numerical degradation: eigenvectors computation.

Macaulay revisited

Size of the linear system(s) to invert:

n	5	6	7	8	9	10	11
	5	6	7	8	9	10	11
	20	30	42	56	72	90	110
	30	**60**	105	168	252	360	495
	20	**60**	**140**	**280**	504	840	1320
	5	30	105	**280**	**630**	**1260**	2310
		6	42	168	504	**1260**	**2772**
			7	56	252	840	2310
				8	72	360	1320
					9	90	495
						10	110
							11
Σ	80	192	448	1024	2304	5120	11264
M	**430**	**1 652**	**6 307**	**24 054**	**91 866**	**351 692**	**1 350 030**
D	32	64	128	256	512	1024	2048

Autocalibration Problem, in computer vision



- Camera, pine hole model
- if **m,m'** two image of $M \in \mathbb{R}^3$ in two photos **m** F**m'**=0

where F is the Fundamental matrix.

Problem: Compute the intrinsic parameter matrix, using Kruppa equations:

$$F\,X\,F^t = \lambda\,T_e\,X\,T_{e'}$$

□ 6 quadratic homogeneous equations in 6 variables ($0.38s$, Alpha 500Mhz).

Exact root	Computed root
1.049401330318981	1.049378730793354
4.884653820635368	4.884757558650871
6.011985256613766	6.011985146332036
.1726009605860270	.1725610425715577
1.727887086410446	1.727898150468536

Blind identification in Signal Processing

□ Transmition of an input signal $\mathbf{x}(n)$ of information of size p depending on the discrete time n into a channel of length L. The output is $\mathbf{y}(n)$.

□ We want to compute the impulse response matrix $H(n)$ s.t.

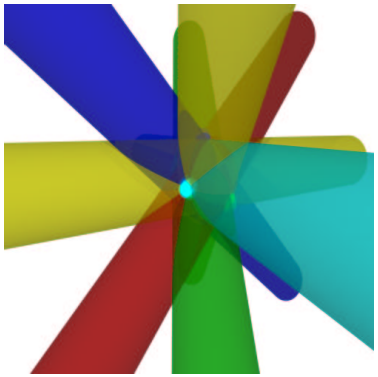
$$\mathbf{y}(n) = \Sigma_{m=0}^{L-1} H(m)\mathbf{x}(n-m) \ + \ \mathbf{b}(\mathbf{n}), \ \mathbf{b}(n) \ \text{ is the noise.}$$

If $\mathbf{b}(n)$ is Gaussian centred, a statistic analysis yields

$$\Sigma_{m=0}^{L-1}\Sigma_{i=1}^{p} h_{\alpha,i}(m)h_{\beta,i}(m)(-1)^{n-m} = E(y_\alpha(n)y_\beta(n-l)).$$

	A real root
x0	-1.803468527372455
x1	-5.162835380624794
x2	-7.568759900599482
x3	-6.893354578266418
x4	-3.998807562745594
x5	-1.164422870375179
Error $= 10^{-8}$, Time $= 0.76s$	

Cylinders throught 4 and 5 points



- Cylinders throught 4 points: curve of degree 3.
- Cylinders throught 5 points: $6 = 3 \times 3 - 3$.
- Cylinders throught 4 points and fixed radius: $12 = 3 \times 4$.
- Line tangent to 4 unit balls: 12.
- Cylinders throught 4 points and extremal radius: $18 = 3 * \times 10 - 3 \times 4$.

$Problem$	$time$	$max(\|f_i\|)$
cylinders through 5 points	$0.03s$	$5 \cdot 10^{-9}$
parallel cylinders through 2×4 points	$0.03s$	$5 \cdot 10^{-9}$
cylinders through 4 points, extremal radius	$2.9s$	10^{-6}

Computations performed on an Intel PII 400 128 Mo of Ram

Controlled iterative methods

We consider the elements in $\mathcal{A}$ as multiplicative operators. Induction in the quotient $\mathcal{A}$.

- $u_{n+1} = f_0 u_n$ (Bernoulli).
- $u_{n+1} \equiv u_n^2$ (Sebastio e Sylva).
- $u_{n+1} \equiv \frac{1}{2}(u_n \pm u_n^{-1})$ (Joukovski).

• Convergence to one (or a subsum) of the "eigenvectors" or **idempotents** associated with the roots.

• Convergence from the beginning and quadratic for Sebastio and Joukovski.

Applications:

- Compute the root(s) which maximize(s), minimize(s) $|h|$.
- Count/compute the roots which are almost real.
- Count/compute the roots in a box.

Iterative methods in one variable

- **Allows to select the root of $f(x) = 0$ st. $|u_0(\zeta)|$ maximum, minimum, $Re(u_0(\zeta)) > 0$, almost real, in a box, ...**

- (Fast and controlled) convergence toward a combination of idempotents of $\mathcal{A}$.

Exemple: Sebastio e Sylva, by H. Prieto in
http://www.inria.fr/galaad/logiciels/ALP/ .

	d	x0	k	T
`double`	10^4	0.0	8	22.03s
`double`	10^5	0.0	8	346.52s
`complex`	10^4	(0.4,0.6)	6	16.86s
`complex`	10^5	(0.4,0.6)	6	292.02s
`complex`	10^6	(0.4,0.6)	7	2520s

Let us consider a system with two real and two complex roots:

```
> f1:= x[1]^2+2*x[1]*x[2]-x[1]-1;
> f2:= x[1]^2+x[2]^2-8*x[1];
```

Approximation of the roots are

ζ_1	ζ_2	ζ_3	ζ_4
6.8200982	$-0.19395427 + 0.20520688\,\mathbf{i}$	$-0.19395427 - 0.20520688\,\mathbf{i}$	0.36781361
-2.8367388	$-0.61937124 - 1.3895199\,\mathbf{i}$	$-0.61937124 + 1.3895199\,\mathbf{i}$	1.6754769

.

We illustrate the Sebastio e Sylva method by computing first, the root for which $|x_1|$ is maximal. We start with $u_0 = x_1$. After 4 iterations, we obtain

$$u_4 = 7.6055995 + 7.7975926x_1 - 0.46159096x_2 - 15.740471x_1x_2.$$

By multiplying it by x_1 and x_2 in $\mathcal{A}$, we obtain $\zeta_1 = (6.820095, -2.836734)$. If we start with

$$u_0 \equiv (x_1 - \frac{1}{2})^{-1} \equiv -\frac{78}{35} - \frac{228}{35}\,x_1 - \frac{32}{35}\,x_2 - \frac{16}{7}\,x_1x_2,$$

the algorithm should converge to the root for which x_1 is the closest to $\frac{1}{2}$. Indeed, after 4 iterations, we obtain

$$u_4 = 0.15292071 + 0.89409187x_1 + 0.16270766x_2 + 0.29923055x_1x_2,$$

which yields the root $\zeta_4 = (0.3678148, 1.675476)$.

Homotopic methods

Weierstrass in one variable

$$P(x) = \prod_{i=1}^{d} (x - \zeta_i) = x^d + \sum_{i=0}^{d-1} \sigma_{d-i}(\zeta)\, x^i = x^d + a_1 x^{d-1} + \ldots + a_d$$
$$\Updownarrow \tag{1}$$
$$(\Sigma) : \begin{cases} \sigma_i(\zeta) - a_i \\ i \in \{1, \ldots, d\} \end{cases}$$

Apply Newton's method to the system $(\Sigma) \Rightarrow$ Weierstrass, Durand-Kerner's method.

Advantages:

- No inversion of the jacobian matrix. An explicit formula for the iteration

$$I\left(\mathbf{z}^{(n+1)}\right)_i = \mathbf{z}_i^{(n)} - \frac{P\left(\mathbf{z}_i^{(n)}\right)}{\prod_{j \neq i} \mathbf{z}_i^{(n)} - \mathbf{z}_j^{(n)}} \tag{2}$$

- Unproved global convergence.

Relations between roots and coefficients

We consider $\{f_1, \ldots, f_n\} \subset \mathbb{K}[\mathbf{x}]$, and $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$.

- $\mathcal{Z}(f_1, \ldots, f_n) = \left\{\mathbf{z} \in \overline{\mathbb{K}}^n | f_1(\mathbf{z}) = \cdots = f_n(\mathbf{z}) = 0\right\} = \{\mathbf{z}_1, \ldots, \mathbf{z}_D\}$.
- A basis $\mathbf{x}^E = (\mathbf{x}^\alpha)_{\alpha \in E}$ of $\mathcal{A}$ as $\mathbb{K}$-vector space ($E = \{\alpha_1, \ldots, \alpha_D\} \subset \mathbb{N}^n$).
- Decomposition of $\mathcal{A} = \mathbb{K}\mathbf{e}_{\zeta_1} \oplus \cdots \oplus \mathbb{K}\mathbf{e}_{\zeta_D}$.
- **idempotent**:

$$\mathbf{e}_{z_i}(\mathbf{z}, \mathbf{x}) = (-1)^i \begin{vmatrix} \mathbf{x}^{\alpha_1} & \ldots & \mathbf{x}^{\alpha_D} \\ \mathbf{z}_1^{\alpha_1} & \ldots & \mathbf{z}_1^{\alpha_D} \\ \vdots & & \vdots \\ \mathbf{z}_{i-1}^{\alpha_1} & \ldots & \mathbf{z}_{i-1}^{\alpha_D} \\ \mathbf{z}_{i+1}^{\alpha_1} & \ldots & \mathbf{z}_{i+1}^{\alpha_D} \\ \vdots & & \vdots \\ \mathbf{z}_D^{\alpha_1} & \ldots & \mathbf{z}_D^{\alpha_D} \end{vmatrix} / V_E(\mathbf{z}) \text{ with } V_E(\mathbf{z}) = \begin{vmatrix} \mathbf{z}_1^{\alpha_1} & \ldots & \mathbf{z}_1^{\alpha_D} \\ \vdots & \ddots & \vdots \\ \mathbf{z}_D^{\alpha_1} & \ldots & \mathbf{z}_D^{\alpha_D} \end{vmatrix}.$$

- $R_Q(\mathbf{z},\mathbf{x}) = \begin{vmatrix} Q(\mathbf{x}) & \mathbf{x}^{\alpha_1} & \dots & \mathbf{x}^{\alpha_D} \\ Q(\mathbf{z}_1) & \mathbf{z}_1^{\alpha_1} & \dots & \mathbf{z}_1^{\alpha_D} \\ \vdots & \dots & \vdots & \vdots \\ Q(\mathbf{z}_D) & \mathbf{z}_D^{\alpha_1} & \dots & \mathbf{z}_D^{\alpha_D} \end{vmatrix}$
- $F_Q(\zeta,\mathbf{x}) = Q(\mathbf{x}) - \frac{R_Q(\mathbf{z},\mathbf{x})}{V_E(\zeta)}$

Proposition:

- $R_Q(\mathbf{z},\mathbf{z}_i) = 0,\ \forall i \in \{1,\dots,D\}$,
- $F_Q(\mathbf{x})$ **is the normal form of** Q **in the basis** $\mathbf{x}^E$ **of** $\mathcal{A}$.

Proposition: $\frac{\partial}{\partial z_{i,j}} F_Q(\mathbf{z},\mathbf{x}) = \frac{\frac{\partial R_Q(\mathbf{z},\mathbf{z}_i)}{\partial x_j}}{V_E(\mathbf{z})}\, \mathbf{e}_{\mathbf{z}_i}(\mathbf{z},\mathbf{x})$

Weierstrass's method for multivariate algebraic systems

- $$\mathcal{F}: \begin{cases} (K^n)^D & \longrightarrow & (\mathbb{K}[x_1, \ldots, x_n]_E)^n \\ \mathbf{z} & \longmapsto & \mathcal{F}(\mathbf{z}, \mathbf{x}) = \begin{pmatrix} F_{f_1} \\ \vdots \\ F_{fn} \end{pmatrix} \end{cases}$$

- $$\Delta_{z_i}\mathcal{F}(\mathbf{z}) = \begin{pmatrix} \frac{\partial R_{f_1}\left(\mathbf{z}^{(k)}, \mathbf{z}_i^{(k)}\right)}{\partial x_1} \\ V_E(\mathbf{z}) \\ \vdots \\ \frac{\partial R_{fn}\left(\mathbf{z}^{(k)}, \mathbf{z}_i^{(k)}\right)}{\partial x_1} \\ V_E(\mathbf{z}) \end{pmatrix} \begin{matrix} \cdots \\ \ddots \\ \cdots \end{matrix} \begin{pmatrix} \frac{\partial R_{f_1}\left(\mathbf{z}^{(k)}, \mathbf{z}_i^{(k)}\right)}{\partial x_n} \\ V_E(\mathbf{z}) \\ \vdots \\ \frac{R_{fn}\left(\mathbf{z}^{(k)}, \mathbf{z}_i^{(k)}\right)}{\partial x_n} \\ V_E(\mathbf{z}) \end{pmatrix}.$$

Theorem: **Iteration formula points of Newton's method's applied to $\mathcal{F}$:**

$$\mathbf{z}_i^{(k+1)} = \mathbf{z}_i^{(k)} - \Delta_{z_i}\mathcal{F}(\mathbf{z})^{-1} \begin{pmatrix} f_1\left(\mathbf{z}_i^{(k)}\right) \\ \vdots \\ f_n\left(\mathbf{z}_i^{(k)}\right) \end{pmatrix} \tag{3}$$

Local algorithm

Algorithm: Weierstrass's iteration function

- *Input :* $\mathbf{z}$, E *and* $(f_1, \ldots, f_n)$.
- Step 1 : *Compute the following objects (specialized at* $\mathbf{z}$*) :*

$$MV_E(\mathbf{z}) = \begin{pmatrix} \mathbf{z}_1^{\alpha_1} & \ldots & \mathbf{z}_1^{\alpha_D} \\ \vdots & \ddots & \vdots \\ \mathbf{z}_D^{\alpha_1} & \ldots & \mathbf{z}_D^{\alpha_D} \end{pmatrix}, \; MF(z) = \begin{pmatrix} f_1(\mathbf{z}_1) & \ldots & f_n(\mathbf{z}_1) \\ \vdots & \ddots & \vdots \\ f_1(\mathbf{z}_D) & \ldots & f_n(\mathbf{z}_D) \end{pmatrix}$$

and $dv_{f_k}^{(i)}(\mathbf{x}) = \left(\frac{\partial}{\partial x_i} \mathbf{x}^{\alpha_1}, \ldots, \frac{\partial}{\partial x_i} \mathbf{x}^{\alpha_D}, \frac{\partial}{\partial x_i} f_k(\mathbf{x}) \right)$.

- Step 2 : *Solve* $MV_E(\mathbf{z}) * Y = MF(\mathbf{z})$, *the column of index* k *in* Y *is denoted by* S_{f_k}.
- Step 3 : $dv_{f_j}^{(i)}$ *(specialized at* $\mathbf{z}_i$*) inner product with* $S_{f_k}(\mathbf{z})$ *(normalized by its last coefficient)* $\rightarrow_{j,k} \Delta_{\mathbf{z}_i} \mathcal{F}(\mathbf{z})$
- Step 4 : *For each* $i \in \{1, \ldots, D\}$ *we solve the system* $(f_1(\mathbf{z}_i), \ldots, f_i(\mathbf{z}_i))^t = \Delta_{\mathbf{z}_i} \mathcal{F} * \mathbf{z}'_i$ *where* $\mathbf{z}'_i$ *is the column of index* i *of the output matrix.*
- *Output :* $\mathbf{z}'$.

Proposition: **The arithmetic complexity of a Weierstrass's iteration is bounded by $\mathcal{O}\left(D^3 + n^2D^2 + Dn^3\right)$ arithmetic operations.**

Weierstrass's method with continuation

□ $(g_1, \ldots, g_n)$ a system with **known solutions** $\mathbf{z}^{(0)}$ with $\mathbf{x}^E$ as basis of $\mathcal{A}_{\mathbf{z}^{(0)}}$.

□
$$\begin{cases} [0,1] & \longrightarrow & \mathbb{K}[x_1, \ldots, x_n]^n \\ t & \longmapsto & \mathbf{H}(t) = \begin{pmatrix} tf_1 + (1-t)g_1 \\ \vdots \\ tf_n + (1-t)g_n \end{pmatrix} \end{cases}$$

Algorithm: Weierstrass method with continuation

- *Input :* $\mathbf{z}^{(0)}$, $(f_1, \ldots, f_n)$, $(g_1, \ldots, g_n)$, *the set E, and* $M \in \mathbb{N} - \{0\}$.
- *For* i *from 1 to* M *do*
$$\mathbf{z}^{(i)} = Weierstrass\left(\mathbf{H}\left(i/M\right), \mathbf{z}^{(i-1)}\right)$$
- *Output :* Return $\mathbf{z}^{(M)}$.

Geometric methods

Aim: Project the problem onto a smaller (equivalent) one. Algebraically, we eliminate some of the variables.

- ☐ Analysis of the geometry of the solution (**preprocessing**).
- ☐ Use an adequate resultant formulation (**preprocessing**).
- ☐ Construct a solveur implementing this formulation (**preprocessing**).
- ☐ Instantiate the parameters and solve numerically (**at run-time**).

Resultants

Condition on $\mathbf{c} = (c_{i,j})$ such that the system has a solution in the projective variety X of dimension n:

$$\begin{cases} f_0(\mathbf{x}) &= \sum_{j=0}^{k_0} c_{0,j}\, \kappa_{0,j}(\mathbf{x}) \\ &\vdots \\ f_n(\mathbf{x}) &= \sum_{j=0}^{k_n} c_{n,j}\, \kappa_{n,j}(\mathbf{x}) \end{cases}$$

- Projection on the space of coefficients: hypersurface defined by the resultant
$$\mathrm{Res}_X(\mathbf{c}) = 0$$
.
- Explicit formula for the degree in the coefficients of each f_i.
- Explicit construction as maximal minor of the matrix of a map such as
$$\begin{array}{rcl} \mathcal{S} : \langle \mathbf{x}^{E_0} \rangle \times \cdots \langle \mathbf{x}^{E_n} \rangle & \to & \langle \mathbf{x}^{F} \rangle \\ (q_0, \ldots, q_n) & \mapsto & \sum_{i=0}^{n} q_i\, f_i \end{array}$$

Projective resultant: $\{\kappa_{i,j}(\mathbf{x})\} = \{\mathbf{x}^{\alpha_j};\ |\alpha_j| = d_i\}$. $X = \mathbb{P}^n$.

Sylvester-like matrix. Ratio of two Determinants. Determinant of the Koszul complex. Macaulay (1902), Jouanolou (1990).

Toric resultant: $\{\kappa_{i,j}(\mathtt{t})\} = \{\mathtt{t}^{\alpha_j};\ \alpha_j \in A_i\}$, $\mathtt{t} \in (\mathbb{K} - \{0\})^n$, $X = \mathcal{T}_{A_0 \oplus \cdots \oplus A_n}$.

Polytope geomtry. Sylvester-like matrix. Maximal minors. Ratio of two Determinants (BBK75, GKZ91, PSCE93, DA01).

Resultant over a parameterised variety: $\{\kappa_{i,j}(\mathtt{t})\}$ associated with the parametrisation of $X = \overline{\sigma(U)}$.

Bezoutian matrix. Maximal minors. A multiple of $\mathrm{Res}_X(\mathbf{c})$. (EM99, BEM00).

Residual resultant: $\kappa_{i,j}(\mathbf{x}) \in (g_1(\mathbf{x}), \ldots, g_k(\mathbf{x}))$. X is the **blow-up** of $\mathbb{P}^n$ along $\mathcal{Z}(g_1, \ldots, g_k)$.

Explicit resolution of $(F : G)$. Gcd of the maximal minors. Degree formula. Ratio of determinants. (BKM75, BEM00, B01).

Solving $f_1 = \cdots = f_n = 0$ by hiding a variable

1. Construct the resultant matrix $\mathtt{S}(x_n)$ of $f_1, \ldots, f_n$ as polynomials in $x_1, \ldots, x_{n-1}$ with coefficients in $\mathbb{K}[x_n]$.

2. Solve $\mathtt{v}\,\mathtt{S}(x_n) = 0$.

   - Either by solving $\det(S(x_n)) = 0$ and by deducing the corresponding v.
   - or by reducing it to an eigenproblem:

$$\mathtt{v}\,(S_d\,x_n^d + S_{d-1}x_n^{d-1} + \cdots + S_0) = 0 \Rightarrow$$

$$\mathtt{v}\left(\begin{bmatrix} 0 & \cdots & 0 & -S_0S_d^{-1} \\ \mathbb{I} & \ddots & \vdots & \vdots \\ \vdots & \ddots & 0 & -S_{d-2}S_d^{-1} \\ 0 & \cdots & \mathbb{I} & -S_{d-1}S_d^{-1} \end{bmatrix} - x_n\mathbb{I}_{ds}\right) = 0,$$

3. Deduce the other coordinates of the roots from $\mathtt{v}$.

We illustrate this algorithm on the system

$$\begin{cases} f_1 = x_1\,x_2 + x_3 - 2, \\ f_2 = {x_1}^2 x_3 + 2\,x_2\,x_3 - 3, \\ f_3 = x_1\,x_2 + {x_2}^2 + x_2\,x_3 - x_1\,x_3 - 2 \end{cases}$$

We hide x_3 and use the projective resultant formulation. We obtain a 15×15 matrix $\mathtt{S}(x_3)$, and compute its determinant:

```
> S:= mresultant([f1,f2,f3],[x1,x2]): det(S);
```

$$ {x_3}^4 \left(x_3 - 1\right) \left(2\,{x_3}^5 - 11\,{x_3}^4 + 20\,{x_3}^3 - 10\,{x_3}^2 + 10\,x_3 - 27\right). $$

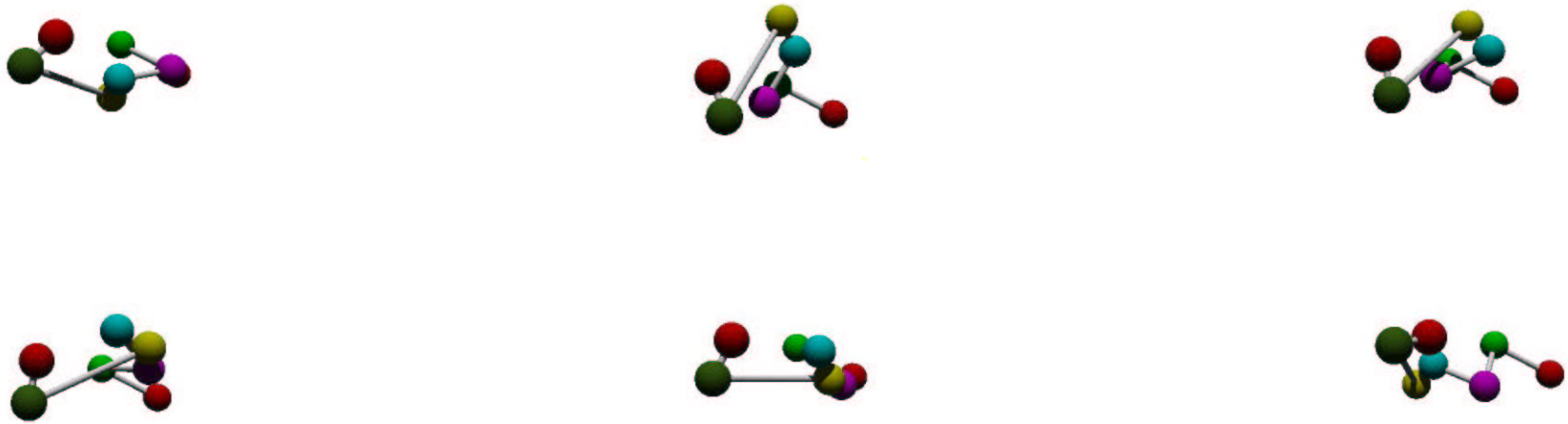
The root $x_3 = 0$ does not yield an affine root of the system $f_1 = f_2 = f_3 = 0$ (the corresponding point is at infinity). Substituting $x_3 = 1$ in $\mathtt{S}(x_3)$, we get a matrix of rank 14. The kernel of $\mathtt{S}(1)^t$ is generated by

$$[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$$

which implies that the corresponding root is $(1, 1, 1)$. For the other eigenvalues (which are the roots of the last factor), we proceed similarly in order to obtain the 5 other (simple) roots of $f_1 = f_2 = f_3 = 0$.

Conformation of molecules

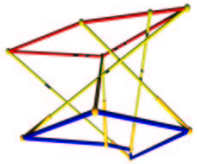
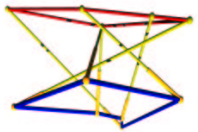
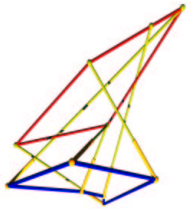
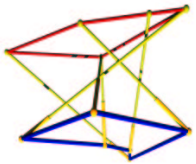
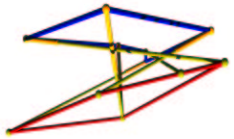
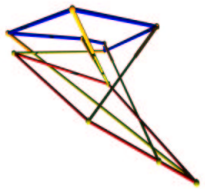
Problem: Compute the possible conformations of a molecule when the position and orientation of the extremity is fixed.



Error: $\|A_1 \circ \cdots \circ A_6 - H\| < 10^{-6}$ **Time:** $0.090s$

Direct kinematic problem of a parallel robot:

Problem: Compute the position of the platform for fixed lengths of the arms.

Error: $|\|R\,Y_i + T - X_i\|^2 - d_i^2| < 10^{-6}$ **Time:** $132s + 0.5s$

Other applications

- Solving overdetermined systems.
- Elimination of variables, projection on subspaces.
- From parameterized to implicit equations.
- Intersection of parameterized curves, surfaces.
- Offset of curves, surfaces.
- Univariate representation of isolated points.
- Geometrical decomposition of varieties.
- Certification of geometric predicates.
- . . .

ALP: an environment for symbolic and numeric computations

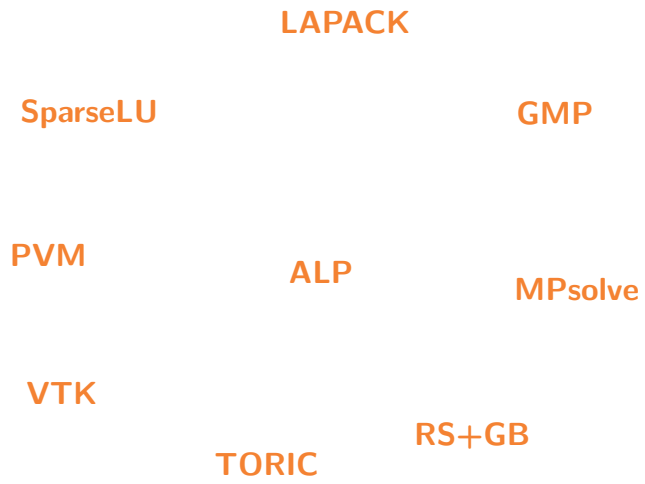
- Basic data structures : **vectors, matrices (dense, Toeplitz, Hankel, sparse, . . .), univariate polynomials, multivariate polynomials.**

- **Genericity of the coefficient type, the internal representation and specialisation**.

- Parametrised type, template expression, instantiation at compile time.

- C++, under LGPL,

- `http://www.inria.fr/galaad/logiciels/ALP/`,



Three level of objects:

□ **Container** : internal representation, associated with iterators, and access/modification methods.

```
typedef array2d<double>  rep1;  typedef lapack<double>   rep2;
```

□ **View** : How we see the container, eg. as a `Vector<R>` or as a univariate polynomial `UPolyDense<R>`.

```
typedef MatDense<lapack<double> >   Mat;
typedef Monom<double, dynamicexp<'x'> > Mon;
typedef MPoly<list<Mon>, Dlex<Mon> >    Pol;
```

Local views sharing datas: `MatRef S = M(Range(3,10),Range(2,9));` **Mat N=S*S;**.

□ **Module** : set of (generic) functions which apply to a category of objects (eg. `VECTOR::Print`, `MATRIX::mult`). Allows specialisation for some data types.

```
template<>
void MATRIX::mult(lapack<double> & r, lapack<double> & a, lapack<double> & b) {
```